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EVALUATING THE MEAN-GINI APPROACH TO PORTFOLIO SELECTION

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I. Introduction

Gini's mean difference (the Gini) was introduced in 1912 as a statistic for measuring income inequality. Yitzhaki [16], via the development and application of the *mean-Gini* (MG) and the *mean-extended Gini* (MEG), validated the Gini's functionality to risk analysis. Shalit and Yitzhaki [13] further amplified this approach to include finance theory and portfolio analysis.

To characterize the distribution of a 'risky' investment two summary statistics are used: the mean and the Gini. The noted shortcomings of *mean-variance* (MV) analysis, where portfolios of prospects may fail to be ranked consistently according to expected utility maximization, are circumvented by use of the MG analysis that provides necessary conditions for second-degree stochastic dominance (SSD). Therefore, portfolio managers should find the *mean-Gini* approach appealing because, subject to statistical errors, it prevents them from choosing portfolios considered inferior by all investors.

The mean-extended Gini (MEG), similar in structure and application to the MG, utilizes the extended Gini as a measure of dispersion granting the analyst the ability to allow for risk aversion.

The respective merits of the MG and the MEG methods and their relative comparison to MV analysis, through the use of empirical applications, forms the framework of this paper. Bey and Howe [2], Okunev [11], and Yitzhaki and Shalit [18] have all performed similar types of analyses. Bey and Howes' thrust is to compare the empirical properties of the MG efficient set to the MV, mean-semivariance, and stochastic dominance efficient sets. Their study concentrates on testing the performance-levels of prespecified random-select portfolios. Verging from Bey and Howe's study, Okunev's and Yitzhaki and Shalit's work included two additional aspects: efficient sets of portfolios for the MG and MV are obtained minimizing the portfolio risk for given expected rates of return, and, necessary conditions for stochastic dominance are selected in order to construct the MG efficient set (a subset of the SSD efficient set.) Once these efficient sets are compared, the advantages of this procedure become apparent in that only efficient portfolios obtained by each criterion are compared.

Although they rely on different data sets[1] both papers by Yitzhaki and Shalit and Okunev conclude that the MG and MV produce analogous portfolios. This

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1 Moliere and Seatsini [10] show that the mean Gini approach forms necessary conditions for third degree stochastic dominance too.

congruity of results led Okunev to suggest using the MV model as an approximation to the MG model. The Yitzhaki and Shalit and Okunev approach to the construction of MG, MV, and MEG efficient portfolios used in this paper enables the estimation of average risk aversion in the market by comparing the portfolio's optimal structure to the actual market position.

Yitzhaki [17,18], and Shalit and Yitzhaki [13] establish the advantage of the Gini over the variance as a measure of dispersion and risk. First, regardless of the probability distribution of the returns, the MG method is shown to allow for the construction of efficient portfolios that are all included in the set of first, second and third degree stochastic dominance (FSD, SSD and TSD) portfolios.[2] Second, because the Gini is the expected absolute difference between two realizations of the prospect's outcome, it provides an intuitive measure of investment risk. Third, the Gini can be extended into a family of indices which differ from each other by a single parameter ranging from one to infinity. The parameter value of 1 represents risk as viewed by a risk-neutral investor while the other extreme, infinity, represents risk as perceived by a maximin individual.[3] The extension allows for the construction of MEG efficient portfolios that are all included in the first and second degree stochastic dominance efficient sets. Finally, sufficient conditions for stochastic dominance exist when considering the set of probability distributions that intersect at most once (for example, the normal, lognormal, uniform, Gamma, and exponential distributions), because the union of all MEG efficient sets is the SSD efficient set.

Assuming specific values of the extended Gini risk parameter permits the construction of different efficient sets that are all included in the SSD efficient set. Still, it is impossible to conclude without additional data which extended Gini risk parameter will represent the investors' risk aversion. Information on the market portfolio at a given time allows the establishment of the efficient set closest to that portfolio and the evaluation of the extended Gini risk parameter which corresponds to that particular efficient set. This parameter represents the risk aversion coefficient of the average investor in the market.

The purpose of this paper is to evaluate the empirical properties of MG and MEG efficient sets by comparing their composition to the MV efficient set. Three specific questions are addressed: (1) In what ways are the MV and MG efficient sets alike? (2) How similar are the different MEG efficient sets? (3) What is the risk parameter for which the MEG efficient set is closest to the actual market position?

Section II of the paper presents the MG method. In sections III and IV, the data and the composition of the MG and MV portfolios are analyzed. In section V, the risk parameter is evaluated by comparing the actual market portfolio

2 Moliere and Scarsini [10] show that the mean-Gini approach forms necessary conditions for third degree stochastic dominance too.

3 A maximin investor ranks prospects by their worst possible realization and then chooses the last one by its least worst outcome. In that respect, he is a maximal risk averse individual.

position with the various MEG efficient portfolios obtained from the optimization procedure.

II. Mean-Gini Analysis and Efficient Portfolios

The *mean-Gini* (MG) approach to finance analyzes securities according to two indices: their mean return and their Gini statistic, where the latter is used as a measure of risk. In this respect, where the Gini is used as a measure of variability, the analysis is comparable to mean-variance. The two-parameter MG portfolio analysis determines combinations of assets that are efficient in the MG space. Although MG efficient portfolios are alike in construct to MV efficient portfolios, the Gini coefficient has substantive advantages over the variance, therefore the importance of MG cannot be dismissed.

Gini's mean difference is defined as the expected distance between two realizations selected at random from the variable's population:

$$\Gamma = \frac{1}{2} E_R E_r \mid R - r \mid \tag{1}$$

where R and r are two realizations of the returns on the same security. In this respect the Gini resembles the *mean absolute deviation* which is equal to the expected distance between the variate and its mean.

Definition (1) is mainly used in income inequality studies. There the Gini divided by the mean is related to the Lorenz curve of income distribution. In finance theory and its applications, the most commonly applied Gini formula is:[4]

$$\Gamma = 2 \, cov \; [\, R \, , F(R) \,], \tag{2}$$

where cov is the covariance function and $F(R)$ is the cumulative probability distribution of R. This representation of the Gini mirrors the variance except that the cumulative distribution $F(R)$ is used instead of the return itself.

The MG approach appears preferable to the MV for three major reasons. First, by definition of an expected distance between two return realizations, the Gini is an intuitive measure of investment risk. Second, the MG approach is related to first, second, and third degree stochastic dominance, regardless of the normality of the underlying probability distribution or the quadraticity of the utility function. Third, the Gini coefficient can be extended into a family of statistics which recognizes the risk aversion parameter expressed by the investor.

The first of these rationales requires no argument as it is valid by definition. Beginning with the third of these arguments, however, the extended Gini is defined here as:

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4 The various representations of the Gini are developed and presented in Dorfman [4], Stuart and Ord [14], and Shalit and Yitzhaki [13].

$$\Gamma(\nu) = -\nu \; cov \; \{ R \; , [\; 1 - F \; (\; R \;)]\}^{\nu-1} \; , \tag{3}$$

where ν is the extended Gini power parameter ranging from one to infinity.

The link between the *extended Gini power parameter* and the investor's attitude towards risk is explained as follows: for any value of ν, Equation (3) displays a "weighted covariance" between the security's return and its cumulative distribution. As ν rises, the weights attributed to the lower portions of the return's distribution increase. In the extreme case, where ν approaches infinity, the extended Gini reflects the attitude toward risk of an investor who cares only about the lowest return realization. This investor characterizes the distribution solely by its lowest outcome. In other terms, he is the *maximin investor*. If, on the other hand, ν approaches 1, all weights are equal. This is a portrait of a risk-neutral individual.

Additional light on the role of ν is shed by analyzing its effect on the systematic risk of a security in a portfolio of N assets. If one considers a given portfolio of securities, p, whose rate of return is obtained by $R_p = \sum x_i R_i$, where R_i,is the return on security i and x_i the share of security i in the portfolio such that $\sum x_i = 1$, its extended Gini is written as:

$$\Gamma_p(\nu) = -\nu \sum_{i=1}^{N} x_i \; cov \; \left\{ R_i \, , \left[1 - F_p \left(R_p \right) \right]^{\nu-1} \right\} , \tag{4}$$

where F_p is the portfolio's cumulative probability distribution (i.e. a ranking function).[5] Since $1 - F_p$ is smaller than 1, as ν increases, the weight attributable to the portfolios low returns expands. Higher ν means that, given low returns on the portfolio, the performance of security i is emphasized. If ν converges to 1, the Gini index converges to 0, and the mean return becomes the sole criterion for decision; that is, the investor is risk neutral.

5 In the discrete case, with K observations, one uses the rank of the portfolio realization minus .5 and divided by K as the estimator of the cumulative distribution. Then one calculates the covariance between the realization and its cumulative distribution. The additional .5 is not needed in the case of the simple Gini, and it serves as a mid-point estimator of the discrete cumulative distribution. Formally, Let $Rp_i = \sum_{j=1}^{N} x_j R_{ij}$ be the ith realization of the return of the portfolio, and let $Zp_i = \left\{ \frac{[K + .5 - \text{Rank}(R_{pi})]}{K} \right\}^{\nu-1}$ be the estimator of $[1 - F_p]^{\nu-1}$,then the extended Gini is calculated as

$$\Gamma(\nu) = \left(\frac{1}{K}\right) \sum_{i=1}^{K} \left(R_{pi} - \overline{R}_p \, Z_{pi} \right)$$

where $\overline{R}_p$ is the mean return..

An alternative interpretation of ν can be given by utilizing Yaari's [15] theory of risk aversion. Yaari defines a theory of behavior under risk which separates the notion of declining marginal utility of income from behavior under risk. He specifies the certainty equivalent of a distribution as:

$$CE = \int H_u \left[(1 - F(R)) \right] \partial R \tag{5}$$

where H_u is a concave function.

Yitzhaki [17] has shown that the extended Gini is a special case of Yaari's approach. In this case, the certainty equivalent value of a distribution is equal to its mean less the extended Gini or

$$CE(F) = \text{Mean} - \Gamma(\nu).$$

Hence, ν defines the certainty equivalent value of a distribution and it can be shown that for any given cumulative distribution F, the higher the ν the lower the certainty equivalent value of the distribution. In other words, the higher the ν the higher the risk premium required by the investor.

Once a parameter ν is established or chosen, the efficient set of portfolios can be found by solving the following optimization problem.[6] For a stated number of securities, given a required expected rate of return R_0, the analyst must select the mix of investments which minimizes the portfolio's Gini (or extended Gini).

Stated formally the optimization problem is:

$$\min_{x_1,\ldots,x_n} \Gamma(\nu) \tag{6}$$

subject to $$\sum_{i=1}^{N} x_i \overline{R}_1 = \overline{R}_0$$

and $$\sum_{i=1}^{N} x_i = 1;\ x_i \geq 0,$$

where $\Gamma(\nu)$ is the extended Gini for a given parameter ν $(1 < \nu < \infty)$, $\overline{R}_1$ is the expected return on security i, x_i is the share of security i in the portfolio, $\overline{R}_0$ is the

6 This is a standard procedure for portfolios construction. The alternative method is to select arbitrarily a set of portfolios and then apply the efficiency criteria to this set of portfolios (Porter [12]). This method was, however, criticized by Frankfurter and Philips [7].

required expected return on the portfolio, and N is the number of available securities. The requirement $x_i \geq 0$ is optional and is used here because the data has been gathered from the performance of the Tel-Aviv Stock Exchange where short selling is not allowed. The algorithm used in this procedure is presented in the appendix. Other non-linear programming techniques which do not require analytical derivation of the gradient may be used with equal validity.

The second and last of the three major arguments making the MG approach preferable to that of the MV is that, once the set of efficient portfolios is determined, they are related to the second degree stochastic dominance (SSD) efficient set by the following two propositions:

Proposition 1: (Yitzhaki [16]) $\overline{R}_i > R_k$ and $\overline{R}_i - \Gamma_i(v) > \overline{R}_k - \Gamma_k(v)$ are necessary conditions to SSD for portfolio i to dominate portfolio k.

Proposition 2: If i and k are two portfolios with equal expected return whose cumulative distributions $F_i(R)$ and $F_k(R)$ intersect at most once,[7] then $\overline{R}_1 - \Gamma_i(v) > \overline{R}_k - \Gamma k(v)$ for any $v > 1$ is a sufficient condition for i to dominate k. The necessary conditions for SSD rules hold for any probability distribution. The sufficient conditions hold for families of cumulative distributions that intersect at most once, e.g., the normal, lognormal, uniform, and Gamma distributions.

The MEG necessary conditions for SSD require that no other portfolio exists in the feasible set such that Proposition 1 holds true. In theory, to ensure these conditions the proposition must be applied to all possible portfolios and combinations. In practice, only a finite number of efficient portfolios can be calculated for a given v, and the existence of necessary conditions, will be established only with respect to portfolios considered.[8] The MEG approach, although restrictive in the context that it provides only necessary conditions is nevertheless a two-parameter model able to construct efficient portfolios that are all resident in the SSD efficient set.

For cumulative probability distributions that intersect at most once, the union of all the efficient portfolios obtained by the MEG for all v parallels the set of SD efficient portfolios.

If the distributions of all securities are normal, then the MEG and MV efficient sets of the portfolios will be identical. Alternatively, if at least one prospect is not normally distributed, then the MEG efficient sets may differ among themselves as well as from the MV efficient set. The higher the v, the higher will

7 Cumulative distribution F and G intersect once if there exists only one R^* such that for every $R < R^*$, $F(R) < G(R)$ $\{F(R) > G(R)\}$ and for every $R > R^*$, $F(R) > G(R)$ $\{F(R) < G(R)\}$. If there exists no such R^*, F and G do not intersect.

8 The extended Gini is a piece wise linear function of the returns of the single assets. This hints on the possibility of finding all possible portfolios as a linear combinations of the assets in the adjacent corner portfolios. We didn't try to follow such a procedure.

be the weight attributed to lower returns, and, thus, the greater the share of "safe assets" to be found in the efficient set.

III. Data Set and Conditions

The data set consists of eleven asset classes of stocks and bonds traded on the Israeli Stock Exchange in Tel-Aviv from January 1977 to January 1983. Two independent events dictated the choice for this specific time period: the start of publication of stock market asset class indices by the Israel Central Bureau of Statistics in December 1976, and, the reflection of a completely different pattern of market behavior after the crash of the Israeli market in October 1983. Due to the severity of this 1983 crash, the Israeli government intervened to repossess the shares of commercial banks and convert them into CPI-indexed bonds which could not be traded for 5 years. This event completely altered the nature of the commercial bank shares and the structure of the Tel-Aviv Stock Exchange. Asset classes are used because, at that time, individual returns were not available on an unified basis. Nominal rates of return on these classes of assets are computed monthly by the Israeli Central Bureau of Statistics in order to measure the total return on securities. Nominal rates of return are adjusted for monthly inflation. The asset classes considered represent a break-down of the entire Stock Exchange in Israel and include all the stocks and bonds traded in that market. The narrow window of time prevented the performance of certain sensitivity analysis on the data, the main purpose of the study being to conduct experiments with the MEG method for others to follow.

Investors' expectations with regard to future asset performance are assumed to be consistent with past returns. Alternatively, it can also be assumed that investors form their expectations by observing past behavior of returns. This means that investors use all available historical information in selecting their portfolios. This simplified approach allows the use of historical data for the purpose of MG and MV analysis.

Summary statistics of the data are presented in Table 1. This chart shows that the rank of assets according to the Gini and their rank according to the standard deviation are identical, indicating the congruence between the two statistics measuring dispersion. The highest mean return is obtained for Real Estate Firms which also exhibits one of the highest values of risk in terms of standard deviation and Gini. The lowest dispersion according to the two statistics is displayed by Bonds Linked to the Consumer Price Index. Commercial Banks appear to demonstrate lower risk (in terms of standard deviation and Gini) at high mean return, implying that this class of securities will participate in most of the required expected return portfolios.[9] Some classes of assets exhibit negative

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9 It seems that the true risk implicit in these shares was not reflected in their returns during the period of the study since ten months after the study ended, commercial banks shares crashed. 10. The market shares used in the analysis are those of the last period, assuming that the investors had the same information we possessed.

monthly mean real rates of return (approaching zero). This is not surprising as, one of the alternatives of eschewing CPI-linked Bonds is to hold cash at a real loss equal to the rate of inflation.

Table 1

Means of Monthly Real Rates of Return, Standard Deviation, Gini of Securities Traded on the Tel-Aviv Stock Exchange.

Securities	Mean	S.D.	Gini
1. Commercial Banks	1.94	6.41	3.48
2. Mortgage Banks	1.95	14.38	8.07
3. Industrial Financial Institutions	1.23	15.71	8.53
4. Investment Firms	2.84	13.98	7.68
5. Trade and Services	1.65	13.04	7.36
6. Manufacturing	3.29	20.85	10.48
7. Real Estate Firms	3.52	16.40	9.11
8. CPI Linked Bonds	–0.02	2.87	1.60
9. Bonds Traded in Foreign Currency(F.C.)	–0.18	3.99	2.15
10. F.C. Linked Bonds	–0.18	5.48	2.84
11. Bonds convertible into Shares	1.37	10.49	5.45

IV. Mean Variance and Mean-Gini Portfolios

Using quadratic programming, MV efficient portfolios sets are obtained for predetermined expected rates of return. The MV efficient frontier of portfolios is depicted in Figure 1 and the composition of that frontier is reported in Table 2. Commercial Banks participate in almost all portfolios obtained by the optimization procedure from a proportion of 21.1% when the required return is 0.5% to a proportion of 80.8% when the average monthly return is required to be 1.94%. If the required expected rate of return is larger than 1.94% then Manufacturing and Real Estate firms will compose most of these portfolios. This requirement is accompanied by an increase in such portfolios' variance. If lower expected returns and lower variance are required, then CPI-Linked Bonds will provide the bulk of the portfolios' composition. Some assets never participate in the optimal portfolio composition although they are traded on the Israel Stock Exchange. There are three reasons for this anomaly to the theory. First, ex-post

11. Search algorithms for constructing SD efficient portfolios are nonexistent. The only other method to yield SSd efficient portfolios seems to be the mean-semivariance approach, see Bey [1]. Furthermore, as Dybvig and Ross [5] showed, the SSD efficient set is not necessarily convex, implying that a search algorithm to derive SSD efficient portfolio sets will be difficult to construct.

statistics are used whereas investors have ex-ante expectations. Second, not all investors use either MV analysis or other optimization methods; (Although with MG and MEG as shown below, all assets enter, at least for one v, the efficient portfolio). Third, the time parameters of the data may be too small for the analysis, and thus the estimators may be subject to sampling error.

Figure 1
Mean-Variance Efficient Set

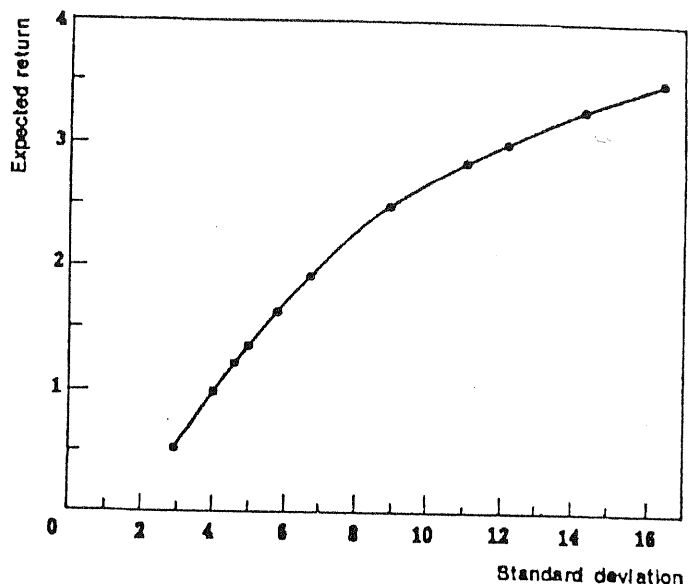


Table 2
Mean-Variance Efficient Set†

Portfolio		Securities Classes										
Mean Return	Std. Dev.	1	2	3	4	5	6	7	8	9	10	11
0.50	2.83	21.2			4.1				66.6	8.1		
1.00	3.87	43.1			2.3		1.6	1.7	51.4			
1.23	4.47	52.6			1.1		2.5	3.0	40.7			
1.37	4.90	58.3			0.4		3.1	3.8	34.4			
1.65	5.66	69.4					4.0	5.0	21.6			
1.94	6.63	80.8					4.9	6.1	8.3			
2.50	8.89	63.2					9.1	27.6				
2.84	11.05	41.2					12.4	46.4				
3.00	12.16	30.9					13.9	55.2				
3.29	14.28	12.1					16.7	71.1				
3.52	16.40							100.0				

† 1: Commercial Banks, 2: Mortgage Banks, 3: Financial Industrial Institutions,4: Investment Firms, 5: Trade and Services, 6: Manufacturing, 7: Real Estate Firms,8: CPI Linked Firms, 9: Foreign Currency Bonds, 10: Foreign Currency Linked Bonds,11: Bonds Convertible into Shares.

The MG efficient set is shown in Table 3 and plotted in Figure 2. In classes of securities with relatively higher expected rate of return and relatively lower risk, i.e. Commercial Banks, the MG efficient portfolios are more concentrated than the MV efficient portfolios. Bey and Howe [2] previously noted this property of MG portfolios.

Table 3
Mean-Gini (ν=2) Efficient Set†

Portfolio		Securities Classes										
Mean Return	Gini Γ(2)	1	2	3	4	5	6	7	8	9	10	11
0.50	2.0	2.2	1.5	1.2	6.5	8.2	0.7	2.6	77.1			
1.00	2.1	53.1							34.8	11.8	0.2	
1.23	2.4	63.1			0.2			0.5	33.9	2.3		
1.37*	2.6	67.1			0.6		0.1	30.4				
1.65	3.0	77.4					1.6	2.8	17.4		0.7	
1.94	3.5	99.4					0.1			0.1		
2.50	4.8	63.3					6.3	30.3				
2.84	6.1	41.8					8.4	49.7				
3.00	6.8	32.1			0.8		1.0	66.1				
3.29	8.0	12.4			0.3		11.0	76.3				
3.52	9.1						0.3	99.7				

* Stochastic Dominant Efficient Set for $R_p \geq 1.37$
† For key see Table 2.

Because of their lower correlation coefficient with Commercial Banks, Manufacturing Firms are in the MV efficient set: this creates better diversification. In the MG efficient set this feature erodes.

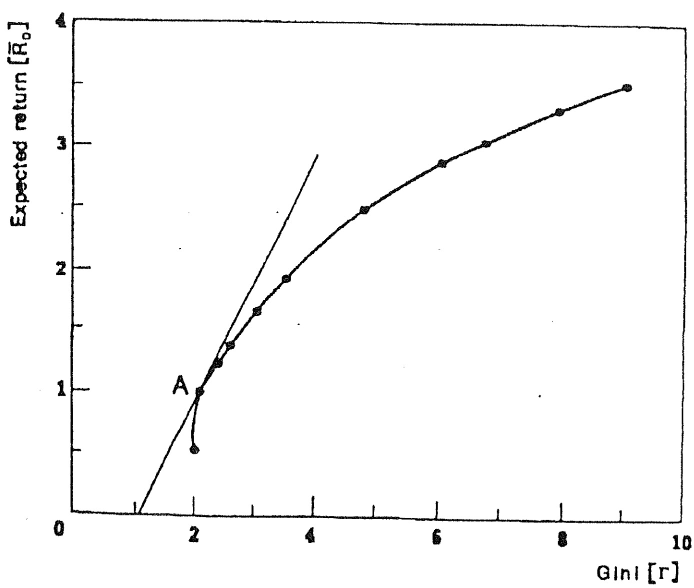
From Proposition 1 the subset of the MG efficient set can be determined since the necessary conditions for SSD are satisfied for all the efficient portfolios in Table 3, where expected rates of return are greater than 1.37%. Indeed, a necessary condition for portfolio i to stochastically dominate portfolio j is that
$Ri - \Gamma i > Rj - \Gamma j$.

Since the efficient set is convex, the efficient MG set which is contained in the SSD efficient can be located by a straight line with a 45 degree slope on the MG efficient set. In Figure 2, the straight line R = Constant + Γ is plotted tangent to the mean-Gini efficient frontier. All portfolios above point A on the frontier are included in the stochastic dominance set because their $R - \Gamma$ are larger than the $R - \Gamma$ of portfolios below A.

Note that the straight line in Figure 2 intersects the horizontal axis. This means that if a 'safe asset' which pays no interest is available, it will be included in the MG and the SSD efficient sets.

In the MEG cases where ν is other than 2, the problem is posed in the following way: given risk preferences expressed by ν and given a required expected rate of return, which portfolios minimize the extended Gini? In this respect, the MEG departs from the traditional two parameter approach and takes into consideration the risk preferences of the investors in the market. Two values, $\nu = 1.3$ and $\nu = 6.0$, are selected: the first to represent a lower risk aversion than that which is implied by the simple Gini coefficient while the second value represents a higher level of risk aversion.

Figure 2
Mean-Gini Efficient Set



The MEG portfolios where $\nu = 1.3$ are shown in Table 4. In general, the distribution of assets is more diversified, although Commercial Banks display the largest proportion in most of the efficient portfolios. On the other hand, the subset of portfolios included in the SSD efficient set is reduced to include the set of efficient portfolios with expected rates of return greater than 2.50%.

The MEG set where $\nu = 6.0$ is presented in Table 5. These portfolios are more likely to be held by risk-averse individuals. It appears that all the various required return configurations are in the SSD efficient set. While the holdings of Commercial Banks and Real Estate Firms are predominant for higher rates of return, it is worth noting the concentration of Commercial Banks and CPI-Linked Bonds for portfolios with a lower expected yield. Since risk aversion is exhibited when there is a relatively larger weight attributed to a lower outcome realization, investors conforming to this behavior tend to prefer efficient portfolios with securities that have a diminished incidence of low outcomes. Comparison of the efficient sets shows that the more significant the v, the larger will be the number of portfolios included in the SSD efficient set.

Table 4
Mean-Extended Gini (ν = 1.3) Efficient Set†

Portfolio		Securities Classes										
Mean Return	Gini. Γ(1.3)	1	2	3	4	5	6	7	8	9	10	11
0.02	0.6	0.1		0.9	0.3	0.5	0.2	0.3	68.6	21.3	7.5	0.3
1.23	1.0	62.5	0.1		0.4		0.3	0.2	36.2	0.2		
1.37	1.1	69.7	0.6		0.4	0.3	0.4	0.7	0.8	26.6	0.4	
1.65	1.3	85.7						0.2	0.6	13.3	0.1	
1.94	1.4	99.7			0.2							
2.50*	1.9	61.1			2.3		9.2	26.8				0.7
2.84	2.4	35.7			13.3		11.0	39.3				
3.00	2.7	30.6			0.2		13.6	55.6				
3.29	3.2	8.4			4.2		23.1	63.8				
3.52	3.7							100.0				

* Stochastic Dominant Efficient Set for Rp ≥ 2.50.
† For key see Table 2.

Table 5
Mean-Extended Gini (ν = 16) Efficient Set†

Portfolio		Securities Classes										
Mean Return	Gini Γ(6)	1	2	3	4	5	6	7	8	9	10	11
0.50	3.8	29.81							32.7	37.5		
1.00	4.9	51.4				0.1		0.4	48.0			
1.23	5.7	56.0						4.5	39.4	0.1		
1.37	6.3	66.1			3.7				24.5	5.8		
1.65	7.2	78.5						3.7	17.8			
1.94	8.3	90.5						5.3	4.2			
2.50*	11.6	64.2					0.2	35.6				
2.84	14.3	43.1						56.9				
3.00	15.7	32.6						67.4				
3.29	18.4	14.						85.9				
3.52	20.5						0.5	99.5				

* Stochastic Dominant Efficient Set for Rp ≥ 2.50.
† For key see Table 2.

V. Risk Aversion and Market Portfolio

The importance of the parameter ν is not restricted to the selection of the optimal efficient set but can also be used for the estimation of systematic risk. Shalit and Yitzhaki [13] show how a variety of ν can be used to construct different Capital Asset Pricing Models (CAPM).

Depending on the parameter ν, the MEG method produces an infinite number of CAPMs. The variability of these CAPMs depends on whether the underlying securities are normally distributed. The task is to determine which risk parameter ν in the efficient portfolio set is closest to the portfolio held by most investors. The solution is obtained by a distance function that measures the extent to which the efficient set approximates the market portfolio. The risk parameter ν is evaluated as the one that best fits the actual stock market data.

Portfolio composition changes substantively with the associated risk parameter. Thus the choice of ν is essential in the identification and characterization of the securities in the portfolio. The following technique proposes to estimate the value of v which will produce a set of efficient portfolios closest to the market portfolio. This method enables the analyst to estimate, from the data, the risk parameter which represents the average investor. The market portfolio is given by the actual position of all the classes of securities held by the public, valued at market prices. The weights are obtained by dividing the values of class shares by the total value of the Stock Exchange. At equilibrium the market position is most desired by all investors, otherwise sales and purchases of individual shares would not only affect the relative distribution but also their values. The weights of the market portfolio with expected return RM are defined as $xM = (x1M,...,xNM)$.

$x(\nu) = [x1(\nu),...,xN(\nu)]$ is the solution of the optimization problem for different values of v and given expected return RM. The distance between the two vectors xM and $x(\nu)$ is defined as:

$$d(\nu) = \left\{ \sum_{i=1}^{N} \left[x_i(\nu) - x_i^M \right] \right\}^{\frac{1}{2}} \tag{7}$$

The distance $d(\nu)$ is used as a measure of goodness of fit and the problem is to find which ν minimizes it.

Since this distance does not necessarily behave monotonically, a minimum for $d(\nu)$ will be found by searching over the entire range of ν.

Table 6 presents the MV portfolio, the actual position of the market, the distance $d(\nu)$, and the efficient portfolio composition for various vs. The value of v that minimizes the distance $d(\nu)$ is around 2.5. Hence, the solution to the optimization problem closest to the actual market position defines investors who are generally more risk averse than investors using the simple Gini index (ν =2) as a measure of risk. The same conclusion is reached when using the variance as a measure of risk. The implication is that investors attach a heavier weight to possible losses than the weight suggested by MV analysis. This finding, although sensitive to the data set, is relevant, because the simple MG allocation, like the MV allocation, does not distinguish such features.

The composition of asset classes in the various MEG efficient sets on Table 6 is quite diverse. In the case of $\nu = 2$, Commercial Banks account for 72% of the portfolio and CPI-Linked Bonds for 17%. However, wherever $\nu = 2.5$, the portfolio breaks out as 38% Commercial Banks and 27% CPI-Linked Bonds. This solution is much closer to the actual market position of 34% Commercial Banks and 20.5% CPI Linked Bonds. Thus, the index of $\nu = 2.5$ not only provides us with the smallest distance of $d(\nu)$ but also with a tighter fit with the securities distribution. Hence, where the CAPM is calculated on the basis $\nu = 2.5$ a better performance is to be expected than where the CAPM is calculated by any other value of ν. Further empirical evidence is needed to establish whether the estimated risk parameter typifies the average Israeli investor in the market.

Table 6
Market Shares, Efficient Portfolios, and Relative Distance†

Extd Gini Parameter ν	Distance $d(\nu)$	Portfolio Mean Return	Securities Classes 1	2	3	4	5	6	7	8	9	10	11
1.2	42.4	1.726	72.3			3.1		3.3	3.7	17.2	0.2		
1.5	48.6	1.724	77.5			3.1		0.8	3.5	9.2	5.9		
1.8	42.6	1.718	24.3		4.0	6.3	0.1	0.1	0.6	30.	2.7	27.9	4.0
2.0	41.8	1.720	71.6			2.9		3.3	4.2	17.2	0.4	0.2	
2.2	42.0	1.719	71.7			2.2			7.7	18.2	0.1		
2.4	24.5	1.716	37.7	0.5	22.0			11.0	9.8	13.3	5.3		
2.5	19.0	1.713	38.1			13.5			17.3	27.0	4.1		
2.6	27.2	1.715	48.9	2.0		0.1	0.4	1.5	19.2	27.8			
3.0	36.8	1.700	19.0			6.7		0.9	32.8	20.2	19.0	1.1	
4.0	49.8	1.710	71.9						9.9	1.4	18.0		
6.0	44.6	1.700	74.8			6.9			1.6	16.7			
8.0	51.1	1.680	79.8			2.0		2.4	0.2	9.8	4.3	1.5	
MV	43.8	1.726	73.0					4.1	5.1	17.8			

† For key see Table 2.

The conclusion that actual investors are more risk averse than the risk aversion implied by the use of the Gini or the variance is in accordance with Bey and Howe's [2] conclusion that "there was a strong tendency for the MG efficient set to include high return/high variance portfolios" (p-337). Bey and Howe reach their conclusion by investigating the properties of prespecified portfolios while the emphasis is placed here on the criteria that most accurately represent the empirical data. Although the questions are different, the conclusion is the same. The MG or MV approach constructs high return/high risk portfolios, while actual investors form lower return/lower risk portfolios. The data implies a higher degree of risk aversion than that which is projected by the use of the Gini or the variance. Friend and Blume [8] also reached the conclusion that investors are more risk averse in reality than those investors hypothesized by the use of a logarithmic utility with constant proportional risk aversion. They find the estimated market price of risk to exceed the marginal utility of wealth expressed by the logarithmic function. The conclusion of this paper stems from portfolio theory and the empirical evidence.

Finally, the analyst can postulate how far $\nu = 2.5$ is from 2. It is hard to give a definitive result because risk aversion is a complex interpretive. However, the following example based on Yaari's [15] use of the certainty equivalence (CE) is illuminating. If a binary lottery yields one dollar or zero with 50% probability, what is the CE of the gamble? For the risk-neutral individual the certainty equivalent value is 50 cents, but for the maximin investor it is zero. The use of the simple Gini index to characterize the investor's utility gives a CE equal to 25 cents while using the extended Gini with $\nu = 2.5$, the CE diminishes to 18 cents. This represents an increase of fifty percent in the risk premium by going from $\nu = 2$ to $\nu = 2.5$. Current literature on the subject does not include any procedure that tests whether this difference in the risk parameter is statistically significant. Yet, the extended Gini being a parameter of the distribution in the population, such statistical tests can be developed.

VI. Conclusions

In this paper, we derive the MG and MEG efficient sets of risky prospects and compare them to MV analysis. Contrary to the Bey and Howe [2] approach, which calculates MG and MV efficient sets for prespecified portfolios, our results are obtained via an optimization algorithm. By creating optimal portfolios based on a given set of security returns, this approach provides a relevant and unique methodology for assessing the performance of market portfolios.

The motivation for using MG and MEG analysis is that it is as simple to compute and as convenient as MV analysis plus it provides necessary and sufficient conditions for stochastic dominance; hence, its importance in portfolio selection whenever the MV analysis might fail. This is especially true whenever assets returns are not normally distributed or their distributions are unknown.

Furthermore, computing and comparing the various MEG efficient sets permits the estimation of the risk parameter most likely to be held by investors. For the data set analyzed, the risk parameter revealed that MV or MG efficient portfolio sets underestimate the risk aversion of most investors in this specific market. Accordingly, given the existing set of returns on the Tel-Aviv Stock Exchange, the Capital Asset Pricing Model using the value of $\nu = 2.5$ will outperform the CAPM for amy other value of ν. An optimal efficient set that fits the position held by most investors was obtained only for the value $\nu = 2.5$. Whether this result will hold true for other time periods or other groups of investors remains the subject of future analyses.

Appendix

The efficient set is found by solving problem (6). Unfortunately, this calculation does not provide a means for expressing the derivatives of $\Gamma(v)$ with respect to the portfolio weights, x_i. Hence, an algorithm which does not require these derivatives must be used. Since $\Gamma(\nu)$ is a piece-wise linear function of the portfolio weights, the problem is solved with either a modified linear programming model or a numerical optimization program. On a personal computer the following algorithm for solving problem (6) is preferred.

$$\underset{x_1,\ldots,x_n}{\text{Min}} \; \Gamma(\nu)\, g_1 \left(\sum_{i=1}^{N} x_i \overline{R}_1 - \overline{R}_0 \right) + g_2 \left(\sum_{i=1}^{N} x_i - 1 \right) + g_3 \left(\sum_{i=1}^{N} x_i^* \right)$$

$$\text{where } x_i^* = \begin{cases} x_i & \text{if } x_i < 0 \\ 0 & \text{if } x_i > 0. \end{cases}$$

and the g_i are penalty values which are found by trial and error. If these values for gi are too low, the solution will not satisfy the constraints. If they are too high the solution will be less than optimal: the objective function will be ignored, and only the constraints will be considered.

This is the regular procedure for solving a constrained optimization problem, except that the values of g_i are selected by the analyst (instead of by the computer), thus reducing computing time. The procedure used to carry out the minimization is the numerical optimization algorithm developed by Daks [3] and is based on the variable metric method of Fletcher [6].

Anyhow, it is worth mentioning that any algorithm which does not require an analytical determination of the derivatives can be used here.

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