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Capital market equilibrium with heterogeneous investors

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As a two-parameter model that satisfies stochastic dominance, the mean-extended Gini model is used to build efficient portfolios. The model quantifies risk aversion heterogeneity in capital markets. In a simple Edgeworth box framework, we show how capital market equilibrium is achieved for risky assets. This approach provides a richer basis for analysing the pricing of risky assets under heterogeneous preferences. Our main results are: (1) identical investors, who use the same statistic to represent risk, hold identical portfolios of risky assets equal to the market portfolio; and (2) heterogeneous investors as expressed by the variance or the extended Gini hold different risky assets in portfolios, and therefore no one holds the market portfolio.

Keywords: CAPM; Applied mathematical finance; Market efficiency; Stochastic dominance; Market portfolio

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1. Introduction

We present a two-parameter model as an alternative to the standard capital asset pricing model (CAPM) that has dominated finance since the 1960s. A two-parameter model is convenient and appealing to most investors, practitioners, and financial theoreticians, as it is simple and can present the choice between return and risk in a transparent way. While the contingent markets approach of Arrow and Debreu (1954) provides a theoretical alternative to capital market equilibrium with heterogeneous investors, most financial practitioners prefer to characterize the distribution of risky assets by two summary statistics: one for the mean return, and one for risk. The most popular measure for the latter is the variance.

In the standard two-parameter approach (such as using a mean-variance (MV) utility function), heterogeneity among investors devolves with risk aversion as in the trade-off between risk and mean return and not through the individual's perception of the distribution of asset returns. In fact, heterogeneous MV investors view risky assets homogeneously, as the probability distributions are the same and the correlations identical.

Capital market equilibrium is reached under the CAPM mutual fund separation theorem that asserts investors hold a selection of risky assets known as the market portfolio which is composed of all risky assets and identical for all investors. As the price of risk increases, investors hold a greater proportion of the risk-free asset and reduce their position in the mutual fund of risky assets whose proportions remain unchanged.

Review of actual investors' positions reveals considerable challenge to the market portfolio single equilibrium. Canner *et al.* (1997) are a notable example. They note that popular advice on asset allocation among cash, bonds, and stocks contradicts CAPM and MV financial theory.

We aim to show there is capital market competitive equilibrium in a two-parameter model with the market portfolio but that heterogeneous investors who differ in risk aversion will have to not hold it. Only when investors define risk in exactly the same way can they hold the market portfolio of risky assets. Mean-variance dominates financial theory because of its appeal as a two-parameter approach. Yet its restrictions are so significant that it can provide results consistent with expected utility (EU) maximization in specific cases only. Indeed, normality of stock returns is hardly warranted nowadays when

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skewness and fatter tails are omnipresent in financial data. Quadratic preferences have never been fully justified by consumer behavior. However, the issue of CAPM and the equivalence between mean-variance and EU has been theoretically resolved for a wide range of utilities and distributions by Levy and Markowitz (1979), Chamberlain (1983), Nielsen (1990a), and Berck (1997), to cite a few.

If MV is so widely accepted, why should one bother to look for alternatives? Two answers come to mind. First, MV leads to unreasonable results when risk-averse investors choose EU-maximizing portfolios, as has been demonstrated using second-degree stochastic dominance (SSD) by Hanoch and Levy (1969) and Rothschild and Stiglitz (1970) and using mean-Gini (MG) by Shalit and Yitzhaki (1984). Second, MV leads to objectionable results when one observes the dynamics of capital market equilibrium with heterogeneous investors.

We challenge the existence of the mutual fund separation theorem that claims that, in equilibrium, all investors should hold the same market portfolio, even under heterogeneity. In fact, if investors are heterogeneous in the sense that they perceive the risk of uncertain returns differently, we show that no one at equilibrium should hold the market portfolio of risky assets. The market will clear, in that one set of prices as expressed by mean returns will be revealed, but the proportions of risky assets held by investors will be quite different.

Casual empiricism shows the truth of our assertion. Indeed, investors are made heterogeneous by their personal endowments and risk aversion. Even if one cannot assert that markets are in equilibrium, investors have holdings that differ substantially from the market portfolio. More strongly stated, no one investor holds the market portfolio, or a pension fund, or a hedge fund, or even through the most widely held exchange traded funds.

We use the mean and the extended Gini as the two relevant parameters to represent the probability distribution of risky assets. Yitzhaki (1982, 1983) shows the mean-extended Gini (MEG) approach provides necessary and in some cases sufficient conditions for SSD. The MG approach to finance was developed by Shalit and Yitzhaki (1984), who show the MG-CAPM for homogeneous investors. Bey and Howe (1984), Carroll *et al.* (1992), Okunev (1991), to name a few, have validated, estimated, tested, or contested the theory.

The extended Gini coefficient characterizes increasing risk aversion by stressing the portions of the distribution of returns to which investors are most averse. With one additional parameter, the extended Gini enables the definition of a range of risk aversions from the risk-neutral to the maxi-min while defining the perception of risk. The main issue is how to use and activate the different extended Ginis in the same unsegmented capital market.

The prime question is how we model capital market equilibrium with such heterogeneous participants. We set up the problem in MEG terms and provide a solution using a simple Edgeworth box. Although the discussion is characterized in geometric terms, the results are compelling. Capital market equilibrium with heterogeneous investors reveals that each will hold different efficient portfolios of risky assets but no investor has to hold the market portfolio. Mean-variance implies homogeneity as investors perceive risk similarly. Hence, the only possible equilibrium solution is that each participant holds the same portfolio as the market portfolio.†

In a world of identical expectations on the distribution of asset returns, the MEG approach enables us to differentiate two separate problems.

(i) How is risk perceived and measured?
(ii) How much is one ready to pay to reduce exposure to risk?

The first question is answered by the type of measure risk-averse investors use to capture risk. This measure quantifies and qualifies risk.

The second question as to what price investors are ready to pay to reduce risk is answered by setting in the market one risk price so that the marginal rate of substitution between risk and expected return will be the same for all assets. For homogeneous investors, the marginal rates of substitution are equal when investors hold the same portfolio. Heterogeneous investors, on the other hand, perceive and measure risk differently, even though the return distribution stays the same. In this case the marginal rates of substitution for different investors and different assets can be equal to the price set in the market only if the investors hold different portfolios.

The paper is outlined as follows. We present first the investor's problem using expected utility maximization, and discuss stochastic dominance and the two-parameter MG approach. We then elaborate on the MEG ordering functions. Using an Edgeworth box, we solve the capital market equilibrium—first for homogeneous investors and then for heterogeneous investors—and explain the main results of the paper.

2. The two-parameter investment model

We set the basis for establishing the ranking function in a standard two-period portfolio choice model. Facing N risky assets with random returns r_i for $i=1,\ldots,N$ and initial wealth w_0, the investor chooses a portfolio $\{\alpha_i\}$ such as $\Sigma_{i=1}^{N}\alpha_i = 1$ that maximizes the expected utility of final wealth:

$$\begin{aligned} &\text{Max} \quad E[U(w)], \\ &\text{subject to} \quad w = w_0\left(1+\sum_{i=1}^{N}\alpha_i r_i\right) \quad \text{and} \quad \sum_{i=1}^{N}\alpha_i = 1. \end{aligned} \tag{1}$$

We assume initially that optimal choice of assets generates a distribution of feasible portfolios

†Under a MV framework, both Harris (1980) and Nielsen (1990b) use the Edgeworth box to model capital market equilibrium, the first by analysing the trade-off between risk and return, and the second by characterizing allocation risk.

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solving problem (1). Once feasible portfolios are created, one can compare them by considering increasing and concave utility functions that are known only to the investors. For two portfolios X and Y whose cumulative distributions are given by F and G, the notion of maximum expected utility states that X dominates Y if and only if

$$E_F U(X) \geq E_G U(Y). \tag{2}$$

Since we do not know the utility function, we apply the laws of second-degree stochastic dominance (SSD) in order to determine the set of efficient portfolios. As Hadar and Russell (1969), Hanoch and Levy (1969), and Rothschild and Stiglitz (1970) propose, SSD expresses the conditions under which all risk-averse investors prefer one portfolio over another. SSD states that X dominates Y if and only if

$$\int_{-\infty}^{z} [G(t) - F(t)]\mathrm{d}t \geq 0, \quad \text{for all } z \in (-\infty, \infty). \tag{3}$$

Various methods have been used to apply the conditions expressed by (3). One way to use SSD is to compare the areas under the cumulative distributions of portfolio returns. Alternatively, one can compare the absolute Lorenz curves, which are the cumulative expected returns on the portfolio, following Shorrocks (1983) and Shalit and Yitzhaki (1984). In essence, for all risk-averse investors to prefer one portfolio of assets over another, its Lorenz curve must lie above the Lorenz curve of the alternative.

Neither approach provides practical results in large portfolios, however, as both involve an infinite number of pairwise comparisons of portfolios. SSD also provides researchers with a partial ordering, forcing the imposition of additional restrictions on investor preferences.†

Another way to resolve differences between the EU-SSD approach and a two-parameter approach is to restrict the distribution of returns to two-parameter probability distributions, usually the mean and the variance. Meyer (1987), for example, restricts the distribution to a family that differs only by location and scale parameters. Levy (1989) extends Meyer's results to show the distribution restrictions that guarantee the equivalence of SSD and MV efficient sets. See also Wong and Ma (2008) and Bodurtha and Shen (2004).

Our contribution to the two-parameter approach is to select the parameters from a set of statistics that form the necessary conditions for SSD rules. We thus ensure that the complete ordering of portfolios produced by the two-parameter approach does not contradict the partial ordering produced by SSD rules. In other words, the efficient set generated by the two- parameter approach is guaranteed not to include SSD dominated portfolios.

Unfortunately, MV cannot be considered as a potential model. Indeed, MV is compatible with EU and SSD in limited instances and so the MV-efficient set includes SSD-dominated portfolios.‡

Our goal is to express portfolios such that if one, say X, stochastically dominates the other, say Y, according to SSD, a function built on two parameters of X is greater than a function built on two parameters of Y. For this function we use as parameters the mean and the extended Gini, which Yitzhaki (1982, 1983) shows provide necessary and sometimes sufficient conditions for SSD. In finance theory, Shalit and Yitzhaki (1984) propose the mean-Gini model to analyse risky prospects and construct optimum portfolios.

We construct the ranking function as follows. Let μ_X and Γ_X be the mean and one-half of Gini's mean difference of portfolio X and μ_Y and Γ_Y be the mean and one-half of Gini's mean difference of portfolio Y. If X stochastically dominates Y, then

$$\mu_X \geq \mu_Y \quad \text{and} \quad \mu_X - \Gamma_X \geq \mu_Y - \Gamma_Y \tag{4}$$

are necessary conditions for portfolio X to SSD-dominate portfolio Y. The first inequality in equation (4) compares the mean returns for the two portfolios. The second inequality in quation (4) compares the risk-adjusted mean returns of the two portfolios, where the portfolio Gini's mean difference represents risk for the investor.

Since the SSD criterion provides only a partial ordering, X might not dominate Y and Y might not dominate X. In such a case, one can always find two legitimate utility functions, U^A and U^B, both with non-negative and non-increasing marginal utilities, so that

$$E[U^A(X)] > E[U^A(Y)]$$

and

$$E[U^B(Y)] > E[U^B(X)].$$

SSD criteria are not able to identify which portfolio the investor prefers, no matter how many parameters are used. Because a complete ordering of the feasible distribution is needed, it is reasonable to accept any ordering as legitimate if X does not stochastically dominate Y and Y does not dominate X. We call this property *ranking with accordance to SSD*. Since each of the conditions $\mu_X \geq \mu_Y$ and $(\mu_X - \Gamma_X) \geq (\mu_Y - \Gamma_Y)$ are necessary conditions for X to SSD-dominate Y then the ranking function V, that includes both parameters μ and $\mu - \Gamma$ as variables, will rank alternatives according to SSD. These properties will be explained in length in the next section.

The advantage of using μ, $\mu - \Gamma$ instead of μ, Γ as parameters in the ranking function is that we define them

†Note, however, that there are some results involving portfolios in some limited cases, see Shalit and Yitzhaki (2003) and Zhang (2008). Wong and Chan (2008) compare results on stochastic dominance with results on Prospect theory.

‡To illustrate this issue, assume that X is uniformly distributed between [0, 1] while Y is uniformly distributed between [1000, 2000]. Clearly, all investors prefer Y over X, but both of them are included in the efficient MV set. Consequently, relying on MV to analyse portfolios may produce efficient portfolios that are inconsistent with expected utility theory.

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as positive attributes instead of a combination of positive attribute and detrimental attribute. Hence, the mean μ and the risk-adjusted mean $\mu-\Gamma$ are considered as two 'goods'.† This allows us to borrow without any further adjustment many microeconomic theory results. However, to generate results that are compatible with the classical financial models of risk and return, we also use μ, Γ. Both presentations include the same parameters, and we will use them interchangeably.

3. The mean-extended Gini ordering function

We define the properties of the ranking function $V(\mu, \mu-\Gamma)$, where μ is the mean and Γ one-half of Gini's mean difference of the distribution of risky prospects. Gini's mean difference (hereinafter the Gini) is defined as the expected absolute difference between all realization pairs of the variate w with density $f(w)$ and cumulative distribution $F(w)$, or

$$\Gamma = \frac{1}{2}\int_a^b \int_a^b |W - w| f(W) f(w)\, \mathrm{d}W\, \mathrm{d}w, \tag{5}$$

where a and b are the lower and the upper bounds of the distribution. Alternatively, the Gini can be written as

$$\Gamma = \int_a^b [1 - F(w)]\, \mathrm{d}w - \int_a^b [1 - F(w)]^2\, \mathrm{d}w, \tag{6}$$

or

$$\Gamma = \mu - a - \int_a^b [1 - F(w)]^2\, \mathrm{d}w. \tag{7}$$

The extended Gini coefficient was developed by Yitzhaki (1983) as

$$\Gamma(\nu) = \mu - a - \int_a^b [1 - F(w)]^{\nu}\, \mathrm{d}w, \tag{8}$$

where $\nu \in (1, \infty)$ reflects the investor's aversion toward risk. For a risk-neutral investor, $\nu = 1$ and the Gini is zero. For $\nu = 2$, the standard Gini is obtained. For $\nu \to \infty$, the Gini represents risk as viewed by a maximin investor.

We relate SSD to the mean-Gini model by using the function $\delta(\nu)$ defined as $\delta(\nu) = \mu - \Gamma(\nu)$, or the mean minus the extended Gini. This value can be interpreted as the certainty equivalent of the distribution valued by the type ν investor. The construction of an ordering function that ranks distributions with respect to SSD is based on proposition 1.

Proposition 1: *Conditions $\delta_X(1) \geq \delta_Y(1)$ and $\delta_X(\nu) > \delta_Y(\nu)$ for all $\nu \in (1, \infty)$ are necessary for X to dominate Y according to SSD.*

This proposition was proven by Yitzhaki (1982) for integers ν and in a more general case by Yitzhaki (1983).

Some properties of $\delta(\nu)$ that are needed to pursue our arguments are as follows.‡

(i) $\delta(\nu) = \mu - \Gamma(\nu)$, where μ is the mean of the distribution and $\Gamma(\nu)$ is the extended Gini coefficient. $\delta(\nu)$ may be interpreted as the risk-adjusted mean return (or the certainty equivalent).

(ii) $\partial\delta(\nu)/\partial\nu \leq 0$. That is, $\delta(\nu)$ is a non-increasing function of ν. This property implies that the higher the risk aversion the lower the certainty equivalent of the portfolio.

(iii) The values of $\delta(\nu)$ for a specific ν are

$\delta(0) = b$, since $\Gamma(0) = \mu - a$,
$\delta(1) = \mu$, since $\Gamma(1)=0$,
$\delta(2) = \mu - \Gamma$, where Γ is Gini's mean difference
$\lim_{\nu\to\infty} \delta(\nu) = a$, so that $\lim_{\nu\to\infty} \Gamma(\nu) = \mu - a$.

(iv) If $w = c$ where c is a constant (i.e. the risk-free asset), then
$\delta(\nu) = c$ for all $\nu > 0$, since $\Gamma(\nu) = 0$.

(v) If $w_i = c_0 w_j + c_1$, where $c_0 > 0$, and c_1 are given constants, then
$\delta_i(\nu) = c_0 \delta_j(\nu) + c_1$, since $\Gamma_i(\nu) = c_0 \Gamma_j(\nu)$.

(vi) If $w_3 = c_0 w_1 + c_1 w_2$, where $c_0 > 0$ and $c_1 > 0$ are given constants and if the correlation coefficient between w_1 and w_2 is $-1 \leq \rho_{12} < 1$, then

$$\Gamma_3(\nu) < c_0 \Gamma_1(\nu) + c_1 \Gamma_2(\nu).$$

Properties (iv)–(vi) are similar to the properties of the standard deviation.

(vii) For a portfolio $w = \Sigma_{i=1}^{N} \alpha_i r_i$, where α_i are given constants,

$$\Gamma(\nu) = -\nu \sum_{i=1}^{N} \alpha_i \mathrm{cov}\{r_i, [1 - F(w)]^{\nu-1}\},$$

where $F(w)$ is the cumulative distribution of w. For the case of the Gini, $\nu = 2$, therefore

$$\Gamma_w = 2 \sum_{i=1}^{N} \alpha_i \mathrm{cov}[r_i, F(w)].$$

(viii) Assume that ν are integers such as $\nu = 1, 2, 3, \ldots$, and then $\delta_w(\nu) = E[\min(w_1, \ldots, w_\nu)]$. That is, $\delta_w(\nu)$ is the expected minimum of ν draws from the distribution F_i. This property is useful when estimating the extended Gini, as it relates the Gini to the rank-order statistics.

(ix) With property (viii) and assuming a normal distribution, $\delta_i(\nu) = \mu - C(\nu)\sigma$, where $C(\nu)$ is a constant that depends on ν, and σ is the standard deviation (for $\nu = 2$, $C(\nu) = 1/\sqrt{\pi}$).

(x) The extended Gini of a sum of random variables can be decomposed similarly to the way the variance is decomposed (Schechtman and Yitzhaki 2003).

†The mean and Gini's mean difference are measured in the same units as shown in the next section.
‡These properties are originally presented in Yitzhaki (1983), Shalit and Yitzhaki (1984).

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Property (viii) illustrates the kind of behavior under risk that the use of the Gini certainty equivalent represents. Under expected utility, risk aversion is determined by the difference in utility that investors attach to the gamble's outcomes. The less investors care about the difference between the different possible results, the less they are risk averse. Under the extended Gini paradigm, the more risk averse is the investor the higher is the subjective probability she attaches to bad outcomes. For example, a risk-neutral investor attaches subjective probabilities that are the same as the objective probabilities. With $\nu=2$, investors behave as if the probability of being hurt is the same as being exposed to twice the risk. With $\nu=3$, the behavior is like one who is exposed to three times the risk. As an example, imagine a shark is roaming coastal waters. A risk-neutral swimmer will calculate the swimming benefits by using the objective probability of being struck by a shark. If the swimmer uses $\nu=2$, she will attach as the probability of being struck twice on her entrance into the water, although she will jump only once. If the swimmer uses $\nu\to\infty$, although she intends to enter the water only once, her behavior is as if she will be entering an infinite number of times. That is, if there is a tiny objective probability of having a shark roaming the waters, the behavior of the $\nu\to\infty$ swimmer is as if the shark will strike with a probability of one.

To sum up, one can view $\delta(\nu)$ as the certainty equivalent of a distribution with mean μ where (ν) represents the risk premium. When $\nu\to\infty$, investors using $\delta(\nu)$ evaluate the return to the portfolio in the same way as being evaluated by max–min investors. When $\nu=1$, investors evaluate assets as if they were risk-neutral. In the extreme case of risk lovers (defined as $\nu<1$), $\nu=0$, investors are interested only in the maximum value of a distribution as defined by max–max investors, an attitude that can be translated as "it does not matter how many sharks roam the water, they will never touch me". Given the properties of $\delta(\nu)$, one can construct the ranking function V.

Proposition 2: *Function* $V[\delta(\nu_1), \delta(\nu_2)]$ *with* $\nu_1\geq 1$, $\nu_2>1$, *and* $\partial V/\partial\delta(\nu_1)>0$, $\partial V/\partial\delta(\nu_2)>0$, *ranks risky alternatives with respect to SSD criteria.*

Proof: Assume that $F(w)$ stochastically dominates $G(w)$ according to SSD. Thus, following proposition 1, $\delta_F(\nu_1)\geq\delta_G(\nu_1)$ and $\delta_F(\nu_2)>\delta_G(\nu_2)$; hence, $V[\delta_F(\nu_1), \delta_F(\nu_2)]>V[\delta_G(\nu_1), \delta_G(\nu_2)]$. The term $\delta(\nu)$ is a special case of Yaari's (1987) dual utility function, so the function $V[\delta(\nu_1), \delta(\nu_2)]$ also ranks portfolios with respect to Yaari's utility function. To use $V(,)$ following the MV model, we restrict the discussion to $\nu_1=1$ and $\nu_2>1$, so that V can be written as

$$H[\mu, \mu-\Gamma(\nu)] = V[\delta(1), \delta(\nu)], \quad \text{for } \nu>1.$$

Function H enables us to use μ to represent the mean return and Γ to represent the risk. H ranks distributions as follows. If two distributions have the same certainty equivalent, the one with the higher mean return is preferred. If the two distributions have the same mean return, the one with the higher certainty equivalent is preferred.

We move the investor problem represented by equation (1) into the space (μ,Γ) and now solve the problem with function H. Instead of using a utility function, investors minimize the portfolio's Gini subject to a given mean return. From property (vii), the Gini Γ_w of the portfolio is

$$\Gamma_w = 2\sum_{i=1}^{N}\alpha_i \mathrm{cov}[r_i, F(w)]. \tag{9}$$

In addition to the N risky securities, investors are allowed to borrow or save a risk-free rate asset r_f. Hence, investors choose the portfolio $\{\alpha_i\}$ that minimizes Γ_w subject to a mean return:

$$\mu_w = r_f + \sum_{i=1}^{N}\alpha_i(\mu_i - r_f). \tag{10}$$

Alternatively, investors can choose a portfolio that maximizes $H[\mu_w, \mu_w-\Gamma_w(\nu)]$. The necessary conditions for a maximum are given by

$$(H_1+H_2)(\mu_i-r_f) - H_2\mathrm{d}\Gamma/\mathrm{d}\alpha_i = 0, \quad i=1,\ldots,N. \tag{11}$$

Since the Gini is homogeneous of degree one in α, the Euler theorem states that

$$\Gamma_w = \sum_{i=1}^{N}\alpha_i\partial\Gamma/\partial\alpha_i. \tag{12}$$

Hence, adding the necessary conditions (11) after they are multiplied by their respective α_i leads simply to

$$\frac{(H_1+H_2)}{H_2}(\mu_w-r_f) = \Gamma_w$$

or

$$\frac{(\mu_w-r_f)}{\Gamma_w} = \frac{H_2}{H_1+H_2}, \tag{13}$$

where H_k is the partial derivative of H with respect to the k argument. The solution shows the chosen (optimal) portfolio as the one whose slope equals the slope of function $H[\mu_w, \mu_w-\Gamma_w(\nu)]$ in space $[\mu,\Gamma(\nu)]$. Figure 1 shows the solution on point w^* as unique from the convexity and non-satiation conditions of the indifference curves of H. The slope of the indifference curves is given by

$$\left.\frac{\mathrm{d}\mu}{\mathrm{d}\Gamma}\right|_{H=\text{constant}} = \frac{H_2}{H_1+H_2} > 0. \tag{14}$$

By the maximization of H, the second-order conditions guarantee that

$$-2H_{12}H_1H_2 + H_{22}H_1^2 + H_{11}H_2^2 < 0, \tag{15}$$

where H_{kj} are the second derivatives of H with respect to the k, j arguments. Hence, convexity is obtained by

$$\left.\frac{\mathrm{d}^2\mu}{\mathrm{d}\Gamma^2}\right|_H = \frac{1}{(H_1+H_2)^3}(2H_{12}H_1H_2 - H_{22}H_1^2 - H_{11}H_2^2) > 0. \tag{16}$$

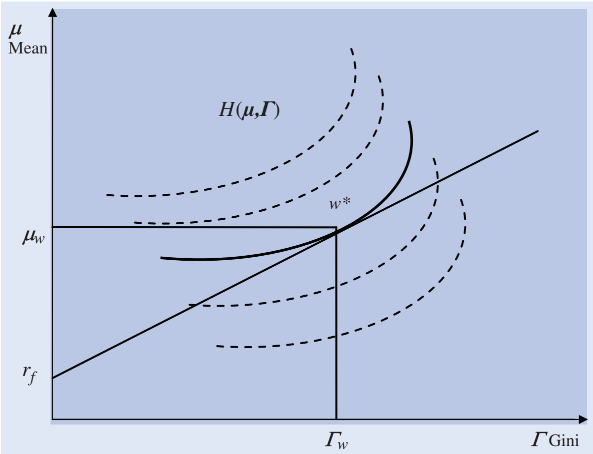

Figure 1. Optimal portfolio in (μ, Γ) space.

From the properties of $\delta(\nu)$, H_1 is the marginal utility produced by increasing the portfolio's mean return, while the certainty equivalent is held constant. Similarly, H_2 is the marginal utility of increasing the certainty equivalent, given a constant mean return. In other words, H_2 expresses the marginal utility of reducing risk along the same mean return. Hence $H_1 + H_2$ is the marginal utility of increasing the portfolio without incurring risk, since adding a constant to the portfolio increases μ and $\delta(\nu)$ by the same amount, as seen from property (v) above.

4. Equilibrium

To demonstrate the existence of a competitive equilibrium in a capital market with heterogeneous investors, we use the basic Edgeworth box. This concept allows us to solve for an exchange economy of heterogeneous agents who have different amounts of risky assets and different preferences toward risk.

The geometric representation of the Edgeworth box requires three components: (i) two types of agents, each with a utility function characterized by convex indifference curves; (ii) an initial distribution of assets to be traded; and (iii) a willingness to trade in order to improve one's utility by bilateral bargaining that leads to efficient allocation and eventually to market equilibrium. For the Edgeworth box to represent a competitive market, it is assumed there are numerous investors for each type of investor.

We adapt the standard Edgeworth box model and consider three assets, two risky ones and one safe. The box consists only of the risky assets that form the box axes. The risk-free rate is treated as a residual investment, which is determined by the budget constraint. Instead of a utility function we use the Gini function obtained when investors minimize the extended Gini of a portfolio $\Gamma_w(\nu)$, subject to the given mean return $\mu_w = r_{\rm f} + \Sigma_{i=1}^N \alpha_i(\mu_i - r_{\rm f})$. The resulting iso-risk indifference curves are a function of only the risky assets that appear in the box. Investor j chooses $\{\alpha_i\}$, such as $\Sigma_{i=1}^N \alpha_i + \alpha_{\rm f} = 1$, where $\alpha_{\rm f}$ is the share of the risk-free asset, to maximize $-\Gamma_w^j(\nu) = -\Sigma_{i=1}^N \alpha_i \operatorname{cov}\{r_i, -\nu[1 - F_w(w)]^{\nu-1}\}$ subject to $\mu_w = r_{\rm f} + \Sigma_{i=1}^N \alpha_i(\mu_i - r_{\rm f})$.

The first-order conditions of that optimization are

$$\frac{\partial \Gamma_w^j(\nu)}{\partial \alpha_i} = \lambda_j(\mu_i - r_{\rm f}), \quad \text{for all } i = 1, \ldots, N, \tag{17}$$

where λ_j is the Lagrangean associated with investor j's mean return constraint. As the Gini is homogeneous of degree one, by the Euler theorem:

$$\Gamma_w^j(\nu) = \lambda_j \sum_{i=1}^N \alpha_i(\mu_i - r_{\rm f}). \tag{18}$$

If we define $1/\lambda_j$ as the price of investor j:

$$\frac{\mu_w^j - r_{\rm f}}{\Gamma_w^j(\nu)} = \frac{1}{\lambda_j},$$

that is also the slope of the tangent in figure 1. Hence, the first-order conditions become

$$\mu_i - r_{\rm f} = \frac{\mu_w^j - r_{\rm f}}{\Gamma_w^j(\nu)} \frac{\partial \Gamma_w^j(\nu)}{\partial \alpha_i}, \quad \text{for all } i = 1, \ldots, N. \tag{19}$$

Under homogeneity, investors have the same attitudes toward risk and the same price of risk, which can be expressed using the market portfolio as $(\mu_{\rm M} - r_{\rm f})/\Gamma_{\rm M}(\nu) = 1/\lambda$. Recall also that

$$\frac{\partial \Gamma_w^j(\nu)}{\partial \alpha_i} = -\nu \operatorname{cov}\{r_i, [1 - F_w^j(r_w)]^{\nu-1}\}.$$

Therefore,

$$\mu_i = r_{\rm f} - (\mu_{\rm M} - r_{\rm f})\nu \operatorname{cov}\{r_i, [1 - F_{\rm M}(r_{\rm M})]^{\nu-1}\}/\Gamma_{\rm M}(\nu),$$

or

$$\mu_i = r_{\rm f} + (\mu_{\rm M} - r_{\rm f})\beta_{\rm M}(\nu). \tag{20}$$

This is the standard MG CAPM when all investors have the same type of risk aversion characterized by ν. The equation prices the mean return into the systematic risk using an identical measure of risk for all risky assets and all investors.

Under heterogeneity, groups of investors have different ν and therefore each type has a different definition of risk. However, relative prices are equal among investors. For each investor j, the first-order conditions are obtained as

$$\frac{\partial \Gamma_w^j(\nu)/\partial \alpha_i}{\partial \Gamma_w^j(\nu)/\partial \alpha_k} = \frac{\mu_i - r_{\rm f}}{\mu_k - r_{\rm f}}, \quad \text{for all } i, k = 1, \ldots, N. \tag{21}$$

Between investors, the equilibrium conditions amount to

$$\mu_i - r_{\rm f} = \frac{1}{\lambda_j} \frac{\partial \Gamma_w^j(\nu_j)}{\partial \alpha_i} = \frac{1}{\lambda_l} \frac{\partial \Gamma_w^l(\nu_l)}{\partial \alpha_i}, \tag{22}$$

for all j, l investors and all i assets.

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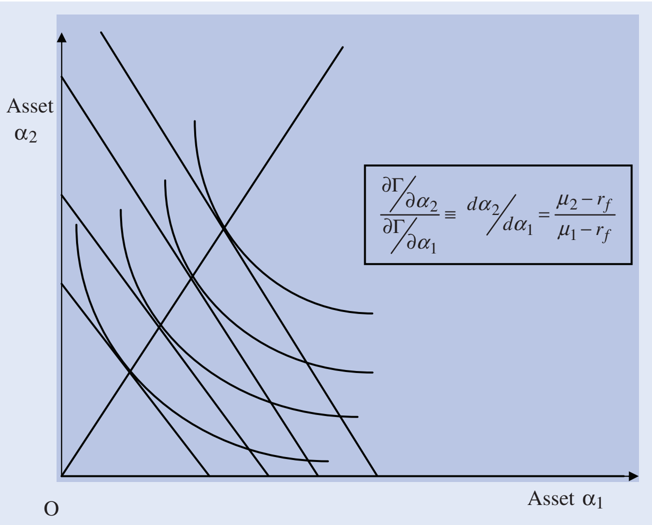

Figure 2. Optimal Gini indifference curves in asset space.

Assuming differentiability, conditions (22) can be expressed as

$$\mu_i - r_{\rm f} = -\frac{1}{\lambda_j}\nu_j \mathrm{cov}\left\{r_i, [1 - F_w^j(r_w)]^{\nu_j - 1}\right\}$$
$$= -\frac{1}{\lambda_l}\nu_l \mathrm{cov}\left\{r_i, [1 - F_w^l(r_w)]^{\nu_l - 1}\right\}.$$

The equilibrium conditions can now be written as

$$\frac{\mathrm{cov}\{r_i, [1 - F_w^j(r_w)]^{\nu_j - 1}}{\mathrm{cov}\{r_k, [1 - F_w^j(r_w)]^{\nu_j - 1}} = \frac{\mathrm{cov}\{r_i, [1 - F_w^l(r_w)]^{\nu_l - 1}}{\mathrm{cov}\{r_k, [1 - F_w^l(r_w)]^{\nu_l - 1}} = \frac{\mu_i - r_{\rm f}}{\mu_k - r_{\rm f}},$$

for all j, l investors and all i, k risky assets.

Second-order conditions are guaranteed by the quasi-convexity of the Gini function.† In the space defined by the risky assets $\{\alpha_i\}$, conditions (21) indicate that the slope of the indifference curves of the Gini function is equal to the ratio of excess asset mean returns. This is a standard solution that occurs when investors choose a portfolio that maximizes $H[\mu_w, \mu_w - \Gamma_w(\nu)]$. The results are expressed in the indifference curves drawn in space (μ, Γ) shown in figure 1.

As the Gini function is quasi-convex and homogeneous of degree one with respect to portfolio weights of the risky assets $\{\alpha_i\}$, the indifference curves are equally spaced convex isoquants, as shown in figure 2 for two risky assets and one risk-free asset.‡ Because of the homogeneity of a given Gini function, the slopes of the isoquants are constant along rays through the origin.

Since we can define an explicit Gini function for a specific ν, conditions (21) construct a distinct linear expansion path that is the locus of Gini-minimization portfolios. As wealth allocated to risky assets increases, the portfolio mean return increases together with the Gini function, which moves to a new isoquant. This is obtained by increasing the shares of the two risky assets and reducing the share of the risk-free asset. Indeed, the wealth shown in the Edgeworth box is only the wealth allocated to the risky assets after deducting the share of wealth held in the risk-free asset. As long as the asset mean returns are constant, the expansion path is a straight line through the origin. The slope of the expansion path defines the ratio of risky assets held by the investor. As the slope depends upon the Gini function, the ratio of risky assets varies with the perception towards risk as expressed by ν.

It is worth mentioning that figure 2 applies to MV investors who use the variance as a measure of risk. In this case, the isoquants represent the variance of the portfolio of risky assets. Hence, we can include MV investors as a special group in the capital market.

We first examine homogeneous investors who have identical perceptions toward risk.§ We claim that homogeneity of risk perception leads investors to hold identical portfolios of risky assets.¶ If, furthermore, portfolios are duplicated under the assumption of constant returns to scale, investors will exhibit identical ratios of risky assets. In classical financial market theory, the 'market portfolio' represents the shares outstanding held by all investors. This is basically the ratio of all risky securities. Thus, all investors hold the identical market portfolio.

This is in essence the basic CAPM result we demonstrate using the Edgeworth box in figure 3. Here we consider a market with only two risky assets that total $\bar{\alpha}_1$ and $\bar{\alpha}_2$. We have two types of investors (A and B), who for the moment are homogeneous in the sense that they have identical perceptions of risk, but differ by their initial endowments of risky assets such that $\alpha_1^{\rm A} + \alpha_1^{\rm B} = \bar{\alpha}_1$ and $\alpha_2^{\rm A} + \alpha_2^{\rm B} = \bar{\alpha}_2$. This initial endowment is shown by point I. As with the standard Edgeworth box geometry, the origin of preferences of type A investors is $\rm O_A$ and of type B investors $\rm O_B$.

The Gini function indifference curves, which are identical for A and B, show that the two types of investors would benefit by trading among those in the same categories until they reach the Pareto-efficient allocation E. At the initial endowment I, the initial mean return ratios are different for the type A and type B investors, and do not allow for trading. Thus, mean returns change until the same price ratio is tangent to the two Gini indifference curves, as shown at point E. This point is the competitive equilibrium located on the

†To show quasi-convexity, consider portfolio $z = \alpha x + (1 - \alpha)y$ where x and y are assets' returns. Then $\mathrm{cov}[z, F(z)] = \alpha\mathrm{cov}[x, F(z)] + 1 - \alpha)\mathrm{cov}[y, F(z)] \le \alpha\mathrm{cov}[x, F(x)] + (1 - \alpha)\mathrm{cov}[y, F(y)]$ or $\Gamma_z \le \alpha\Gamma_x + (1 - \alpha)\Gamma_y$ since $F(x)$ and $F(y)$ are ordered according to x and y, respectively, but not necessarily according to z.

‡Only the shares allocated to risky assets are shown on the axes. The share of the risk-free asset determines the location of the 'budget constraint'; the farther it is from the origin, the lower the share of risk-free asset in total wealth.

§Although investors are homogeneous in the way they perceive risk, they can be heterogeneous in the way they price risk, as reflected by the risk-free-to-market portfolio ratios.

¶Homogeneity of risk perception implies that all MG investors have the same ν, or that all investors are MV investors. See property (ix) of δ.

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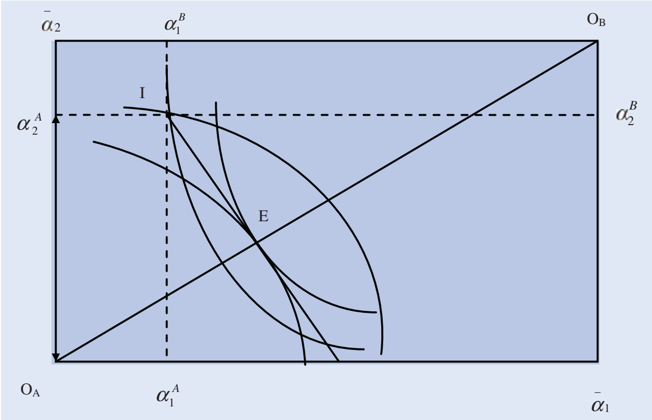

Figure 3. Capital market equilibrium with homogeneous investors.

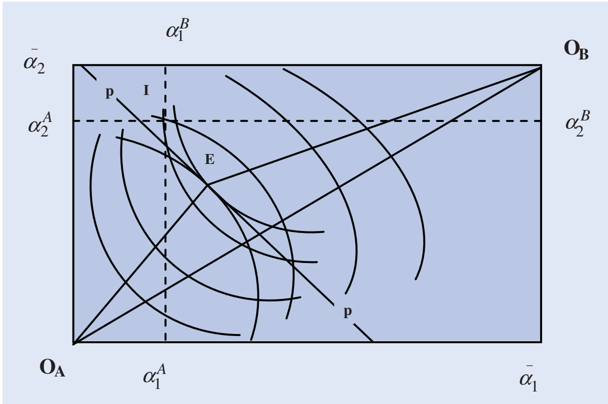

Figure 4. Capital market equilibrium with heterogeneous investors.

diagonal as the expansion paths of the two types of investors are identical.

The market portfolio is the slope of the diagonal $\bar{\alpha}_1/\bar{\alpha}_2$ which represents the same ratio of risky assets held by each type of investor. At equilibrium E, the price ratio as expressed by the unique slope of the tangent is the ratio of mean returns:

$$\frac{\partial\Gamma/\partial\alpha_2}{\partial\Gamma/\partial\alpha_1} = -\frac{\mathrm{d}\alpha_1}{\mathrm{d}\alpha_2} = \frac{\mu_2 - r_f}{\mu_1 - r_f}.$$

For MV investors the indifference curves (like those for the Gini in figure 2) are derived from the variance of a portfolio of risky assets whose covariance matrix is unique and identical for all. Hence, the MV isoquants are the same for type A and type B MV investors; their shape depends on the covariance between α_1 and α_2. Hence, for MV investors, the only equilibrium solution is located on the diagonal of the box, implying that they hold the same market portfolio of risky assets.

The Edgeworth box in figure 3 reflects the capital market equilibrium in the case of MV or for homogeneous mean-Gini investors. Our first result summarizes this equilibrium.

Result 1: At equilibrium, homogeneous investors, either Gini or MV homogeneous investors, hold the market portfolio of risky assets as expressed by the slope of the diagonal of the Edgeworth box.

The problem of equilibrium is different with heterogeneous investors.† Heterogeneity results from the way distributions are taken into account, implying a different quantification of the risk measure. Hence we consider two types of investors with different ν: type A investors and type B investors, each with different endowments, as shown in the Edgeworth box in figure 4.

Because the types have different aversion toward risk, their indifference extended Gini curves are not the same, and they produce different expansion paths. From the initial endowment allocation at point I, investors trade to improve their positions and move to new indifference extended Gini curves. The higher the extended Gini curve, the higher the portfolio's mean return. Hence, investors trade, resulting in changes of the price ratio of mean returns. Extended Gini functions are minimized until investors reach the Pareto-efficient competitive equilibrium at point E where Gini curves are tangent to each other with slopes equal to the mean return ratio as shown by line p–p. To state this formally,

$$\left.\frac{\mathrm{d}\alpha_2}{\mathrm{d}\alpha_1}\right|_A = \left.\frac{\mathrm{d}\alpha_2}{\mathrm{d}\alpha_1}\right|_B = -\frac{\mu_1 - r_f}{\mu_2 - r_f}.$$

This price ratio defines a unique equilibrium. Since the extended Gini curves are not identical, the optimal expansion paths for type A and type B investors are different. Therefore, the ratio of the risky asset optimal portfolio held by each type of investor is different, and no investor will hold the 'market portfolio' that is represented by the slope of the diagonal of the Edgeworth box. In other words, the equilibrium is expressed as

$$\left.\frac{\alpha_1}{\alpha_2}\right|_{\mathrm{A}} \neq \left.\frac{\alpha_1}{\alpha_2}\right|_{\mathrm{B}} \neq \frac{\bar{\alpha}_1}{\bar{\alpha}_2}.$$

This leads us to the second result.

Result 2: Unless risky asset returns are all multivariate-normal, at equilibrium, heterogeneous extended Gini investors hold different portfolios of risky assets and no one has to hold the market portfolio as expressed by the slope of the diagonal of the Edgeworth box.‡

†If returns are multivariate normal, heterogeneity is reduced to homogeneity and the standard MV result is obtained.
‡If returns are multivariate-normally distributed, the MG and MEG models become linearly similar to MV. In this case, the heterogeneous iso-risk curves are identical for all investors regardless of their ν. Needless to say, it is sufficient for one risky asset not to be normally distributed to obtain different iso-risk curves. When there is a large number of investors, it is possible that one or several will hold the market portfolio, but we claim that no one *has to*.

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The contract curve is the locus of all undominated equilibria following various initial endowments. Income distribution comes about in the relative size of the investors' initial endowments. From welfare economics analysis, we draw the following two results.

Result 3: As the extended Gini is homogeneous of degree one in asset shares, the contract curve is either identical to the diagonal of the Edgeworth box or lies on one side of the diagonal.†

This result implies that once a type of investor tends to invest relatively more in one asset, he will continue to do so under all market circumstances. Thus it is possible to identify and relate types of assets with classes of investors.

Result 4: Expected returns on assets depend directly upon the income distribution across types of investors.

In some sense, this result moves us back to traditional microeconomic theory that asserts the significance of income distribution when consumers have different tastes. Yet, our result clearly contradicts the CAPM, which claims that asset returns are determined solely as a function of the demand of a representative investor.

Result 5: Heterogeneous investors who have the same ν will hold an identical portfolio of risky assets.

The mean-extended Gini model has been shown to be richer than the mean variance, in that it enables the researcher to construct infinite numbers of 'capital asset pricing models' for ν homogeneous markets. We show elsewhere that if investors have the same degree of risk aversion, one can estimate capital asset pricing model betas for every ν and also find the average market ν that best fits the data (Shalit and Yitzhaki 1984, 1989). The heterogeneous model with many ν differs considerably from these works as conditions (21) establish specific equilibrium relations between asset returns and risk as viewed by all investors in the market. The next challenge is to generate a reduced form of the model in order to estimate this new representation of the CAPM.

Anyone used to the traditional CAPM may ask how the market aggregates heterogeneous agents in order to achieve equilibrium. Our approach is similar to the way markets aggregate excess demand to reach equilibrium by varying prices and is based on Adam Smith's 'invisible hand' theory. Since each type of investor may hold each type of asset, then each type of investor has an excess demand (or excess supply) at current prices. The market reaches equilibrium when the aggregate excess demand and supply for each asset equals zero. Once we have discarded the notion of identical investors, we have also abandoned the assumption that all investors use an identical definition of risk. As the Edgeworth box indicates, market equilibrium implies that, at equilibrium, the marginal rate of substitution between each two assets is equal among investors relative to their respective different portfolios. Further research is needed to determine cases where an increase in the variability of one asset is viewed by all investors as risk increasing. To see this, note that since each investor holds a different portfolio, and may view risk differently, what matters is whether the increase in variability of the return of one asset increases the risk as viewed by the investor.‡

5. Concluding remarks

To characterize the capital market with heterogeneous risk-averse investors, we use the mean-extended Gini approach as a two-parameter model. As it is compatible with maximizing expected utility, MEG provides the necessary and sometimes sufficient conditions for stochastic dominance theory. Standard capital market equilibrium assumes homogeneous investors with identical perceptions toward risky assets. In these models, heterogeneity comes about with the different trade-offs between the risk-free asset and a portfolio of risky assets.

In our model, we show how homogeneity of risk preferences leads to the mutual fund-portfolio separation results that all investors hold the same market portfolio ratio of risky assets. This is the standard mean-variance result. When there are different perceptions about risk, more general capital market equilibrium emerges.

Heterogeneous investors do not hold the same portfolio of risky assets. Furthermore, no investor must hold the 'market portfolio' in order for capital markets to be in equilibrium. Asset prices are characterized by their mean returns and the various perceptions toward risk. Each group of investors with its unique attitude toward risk defines its positions according to the specific extended Gini.

Although the model is simple, we believe we are the first to propose it in financial economics to characterize the essence of capital market equilibrium with regard to risky assets. Economists have used the Edgeworth box for some time to depict competitive interactions in competitive markets, welfare economics, and international trade, and to show the Walras general equilibrium. The box is so well established in microeconomics that it is quite surprising that it has not been used before to solve the basic issues of capital market equilibrium.

We can offer only one explanation. Financial economics has been captivated by the mean-variance paradigm that is so simple and so intuitive to use. Unfortunately, the Edgeworth box MV equilibrium allows only the diagonal as the contract curve, leading to the identical 'market portfolio' solution.

†This result is derived from the homogeneity property of the isoquants. The contract curve cannot cross the diagonal, as it can only be the diagonal itself or lie on one side of it.

‡This issue can be handled by Marginal Conditional Stochastic Dominance (MCSD) (Shalit and Yitzhaki 2003, Zhang 2008), which can be interpreted as whether all investors agrees on the sign of the beta of the asset. However, MCSD deals with heterogeneous investors holding the same portfolio, while the pricing of risk in the market should deal with each investor holding her optimal portfolio.

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Using the variance to depict risk is known to produce accurate results in choosing risky assets efficiently, but MV has failed to help us understand the true meaning of capital market equilibrium. MV is actually not very useful, nor is it informative in depicting true heterogeneity. The extended Gini allows us to represent investors who have indifference curves with different slopes along different expansion paths and find equilibrium the same way we do in microeconomics.

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