The Geometry of Continuum Mechanics

Author

Department of Mechanical Engineering, Ben-Gurion University
P.O.Box 653, Beer-Sheva 84105 Israel
rsegev@bgumail.bgu.ac.il

Abstract. [**Arn74**]The stress-energy tensor of field theory is defined and analyzed in a geometric setting where a metric is not available. The stress is a linear mapping that transforms the 3-form representing the flux of any given property, e.g., charge-current density, to the 3-form representing the flux of energy. The example of the electromagnetic stress-energy tensor is given with the additional structure of a volume element.

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CHAPTER 1

Linear Forms and Generalized Forces
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1.1. The Dual of a Vector Space

1.2. Path Integration and Work

1.3. Alternating Arrays

The presentation below is similar to that in [**dR84**, pp. 17–18]

1.3.1. The Levi-Civita alternating symbol. The Levi-Civita symbol provides a tool for working with alternating quantities such as the local representatives of forms. For a sequence of indices $i_1,\dots,i_r$ we will refer to the switching of positions of two elements as a *transposition*. The alternating symbol is defined by

$$\varepsilon^{i_1\dots i_r}_{j_1\dots j_r} = \begin{cases} +1 & \text{if the indices in the sequence } (i_1,\dots,i_r) \text{are distinct and the sequence } (j_1,\dots,j_r) \text{may be obtained from them by an even number of transpositions,} \\ -1 & \text{if the indices in the sequence } (i_1,\dots,i_r) \text{are distinct and the sequence } (j_1,\dots,j_r) \text{may be obtained from them by an odd number of transpositions,} \\ 0 & \text{otherwise.} \end{cases}$$

We note two particular cases when the Levi-Civita symbol vanishes: the situation when the two sequences do not contain the same elements, and the situation when in one (or both) of the sequences two or more indices are equal (e.g., $i_p = i_q$). We will sometimes use the notation $\mathbf{i}$ for the sequence $i_1,\dots,i_r$ and we can write $\varepsilon^{\mathbf{i}}_{\mathbf{j}}$.

The (somewhat degenerate) case where $r = 1$ is traditionally referred to as the Kronecker symbol (usually denoted by δ rather than ε),

$$\varepsilon^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

In the special case where any one of the sequences contains the numbers $\{1,\dots,r\}$ and the other sequence is the ordered sequence $(1,\dots,r)$ the

$(1,\ldots,r)$-sequence will be omitted in the notation. For example,

$$\varepsilon^{i_1\ldots i_r} = \begin{cases} +1 & \text{if } (i_1,\ldots,i_r) \text{ may be obtained from } (1,\ldots,r) \text{ by an even number of transpositions,} \\ -1 & \text{if } (i_1,\ldots,i_r) \text{ may be obtained from } (1,\ldots,r) \text{ by an odd number of transpositions,} \\ 0 & \text{if } (i_1,\ldots,i_r) \text{ cannot be obtained as a permutation of } \{1,\ldots,r\}, \text{ in particular, if two indices are equal.} \end{cases}$$

Clearly,

$$\varepsilon^{i_1\ldots i_r}_{j_1\ldots j_r} = \begin{cases} \varepsilon^{i_1\ldots i_r}\varepsilon_{j_1\ldots j_r} & \text{if the two sequences contain the same elements,} \\ 0 & \text{otherwise.} \end{cases}$$

Assume that indices range in $\{1,\ldots,m\}$. We list below a number of simple properties of the alternating symbol.

In general, we will use the summation convention for repeated indices. However, in various instances, the summation convention cannot be used or it may cause confusion. In such cases we will explicitly write the summation symbol or warn that the implicit summation is not performed.

We first note that

$$\varepsilon^{ii_1\ldots i_{m-1}}_{ij_1\ldots j_{m-1}} = \varepsilon^{i_1\ldots i_{m-1}}_{j_1\ldots j_{m-1}}. \tag{1.3.1}$$

If either $i_p = i_q$ or $j_p = j_q$ for some $p \neq q$, $p,q = 1,\ldots,m-1$, then both sides vanish. Assume that each of the two sequences contain distinct indices. Then, since the $m-1$ elements in the sequence $(i_1,\ldots,i_{m-1})$ belong to $(1,\ldots m)$, they contain all the numbers $1,\ldots,m$ except for one, say k . Hence, in the sum over the repeated i all terms vanish except for the term for which $i = k$, the missing element. It follows that in a non-vanishing term the elements of the sequence $(j_1,\ldots j_{m-1})$ also contain the elements of the $1,\ldots,m$ except for k. This means that actually there is only one non-vanishing term and its sign depends only on the number of transpositions needed to arrive from the i-sequence to the j-sequence.

Similarly,

$$\varepsilon^{i_1\ldots i_p j_1\ldots j_r}_{i_1\ldots i_p k_1\ldots k_r} = \frac{(m-r)!}{(m-r-p)!}\,\varepsilon^{j_1\ldots j_r}_{k_1\ldots k_r}. \tag{1.3.2}$$

The elements of the given sequence $(j_1,\ldots,j_r)$ determine the values that the repeated indices may assume—the $m-r$ values required to complete them to $\{1,\ldots,m\}$—such that no two superscripts will be equal. This implies that for nonvanishing terms the k-sequence contains the same elements (possibly in different order) as the j-sequence. Thus, each non-vanishing term in the sum on the repeated indices is $\varepsilon^{j_1\ldots j_r}_{k_1\ldots k_r}$, independently of the values of the i-indices. The number of such non-vanishing terms is the number of ways you can assign the $m-r$ remaining values for the p repeated indices (choose p symbols out of $m-r$ symbols), i.e., $(m-r)!/(m-r-p)!$.

In particular, for for $r=0$,

$$\varepsilon^{i_1\dots i_p}_{i_1\dots i_p} = \frac{m!}{(m-p)!}. \tag{1.3.3}$$

From the definition of the alternating symbol we also have

$$\varepsilon^{i_1\dots i_r}_{j_1\cdots j_r}\varepsilon^{j_1\cdots j_r}_{k_1\dots k_r} = r!\,\varepsilon^{i_1\dots i_r}_{k_1\dots k_r}, \qquad \varepsilon^{\mathbf{i}}_{\mathbf{j}}\varepsilon^{\mathbf{j}}_{\mathbf{k}} = r!\,\varepsilon^{\mathbf{i}}_{\mathbf{k}}. \tag{1.3.4}$$

Once the indices $i_1,\dots,i_r$ and $k_1,\dots,k_r$ are given, for nonvanishing values of the alternating symbols, the $j_1,\dots,j_r$ indices should be obtained as permutations of these indices and there $r!$ such permutations that we have to add up. Similar arguements lead to the slightly generalized rule,

$$\varepsilon^{i_1\dots i_r}_{j_1\cdots j_r}\varepsilon^{j_1\cdots j_r n_1\dots n_p}_{k_1\dots\dots k_{r+p}} = r!\,\varepsilon^{i_1\dots i_r n_1\dots n_p}_{k_1\dots\dots k_{r+p}}. \tag{1.3.5}$$

REMARK 1.3.1. It is noted that if one requires that the sequences of indices such as $i_1,\dots,i_r$ are of increasing order, *i.e.*, $i_p < i_{p+1}$, then some of the expressions above assume simpler forms as the sequences cannot be permuted any more. Using parenthesis to indicate sequences that are ordered, *e.g.*, $(\mathbf{i})$, we can write for example

$$\varepsilon^{\mathbf{i}}_{(\mathbf{j})}\varepsilon^{(\mathbf{j})}_{\mathbf{k}} = \varepsilon^{\mathbf{i}}_{\mathbf{k}}. \tag{1.3.6}$$

Permutation mappings. If the sequences $\mathbf{i} = (i_1,\dots,i_r)$ and $\mathbf{j} = (j_1,\dots,j_r)$ contain elements from the set $\{1,\dots,r\}$, then there is a bijection

$$p: \{1,\dots,r\} \longrightarrow \{1,\dots,r\}$$

such that $\mathbf{j} = p(\mathbf{i})$, or $j_q = p(i_q)$. It is noted that a permutation mapping is any bijection on a finite set

$$p: \{a,b,\dots\} \longrightarrow \{a,b,\dots\}. \tag{1.3.7}$$

However, once the elements of the sets are enumerated, the permutation

$$p: \{a_1,\dots,a_r\} \longrightarrow \{a_1,\dots,a_r\}, \tag{1.3.8}$$

may be regarded as a permutation on the set $\{1,\dots,r\}$. Conversely, a permutation $p: \{1,\dots,r\} \to \{1,\dots,r\}$ induces a permutation of a sequence

$$\{j_1,\dots,j_r\} \longmapsto \{j_{p(1)},\dots,j_{p(r)}\}.$$

The sign of the permutation p is defined by

$$\operatorname{sign}(p) = \varepsilon^{p(1)\dots p(r)}_{1\dots\dots r} = \varepsilon^{p(1)\dots p(r)}. \tag{1.3.9}$$

Clearly,

$$\varepsilon^{j_{p(1)}\cdots j_{p(r)}}_{j_1\cdots\cdots j_r} = \operatorname{sign}(p). \tag{1.3.10}$$

In addition, the definition if the permutation symbol is equivalent to

$$\varepsilon^{k_1k_2\dots k_r}_{i_1i_2\dots i_r} = \sum_p \operatorname{sign}(p)\varepsilon^{k_{p(1)}}_{i_1}\varepsilon^{k_{p(2)}}_{i_2}\cdots\varepsilon^{k_{p(r)}}_{i_r} = \sum_p \operatorname{sign}(p)\delta^{k_{p(1)}}_{i_1}\delta^{k_{p(2)}}_{i_2}\cdots\delta^{k_{p(r)}}_{i_r}. \tag{1.3.11}$$

Evidently, on the sum over all permutations of $\{1,\dots,r\}$ above, there at most one permutation for which the product does not vanish. Equation (1.3.11) is usually referred to as the $\varepsilon-\delta$-identity.

If $q\colon \{1,\dots,r\}\to\{1,\dots,r\}$ is another permutation, then

$$\operatorname{sign}(q\circ p) = \varepsilon^{q\circ p(1)\dots q\circ p(r)}_{1\dots\dots r} = \varepsilon^{q(p(1))\dots q(p(r))}_{p(1)\dots\dots p(r)}\varepsilon^{p(1)\dots p(r)}_{1\dots\dots r} = \operatorname{sign}(q)\operatorname{sign}(p).$$

Note that since the inverse permutation mapping p^{-1} involves the same number of transpositions as p, $\operatorname{sign}(p^{-1}) = \operatorname{sign}(p)$.

1.3.2. Alternating Arrays and Anti-Symmetrization. An array of degree r, $\omega_{i_1\dots i_r}$, $i_1,\dots,i_r\in\{1,\dots,m\}$ is *alternating* or *completely antisymmetric* if

$$\omega_{i_1\dots i_r} = \varepsilon^{j_1\cdots j_r}_{i_1\dots i_r}\omega_{j_1\cdots j_r}, \quad \text{no sum on repeated indices,} \tag{1.3.12}$$

or alternatively, $\omega_{p(\mathbf{i})} = \operatorname{sign}(p)\omega_{\mathbf{i}}$ for any permutation p. Clearly, the alternating symbol is an alternating array—the unit alternating array. Thus, the components of an alternating array reverse their sign under any transposition. Using an ordered sequence of indices, the definition may be written as

$$\omega_{\mathbf{j}} = \varepsilon^{(\mathbf{i})}_{\mathbf{j}}\omega_{(\mathbf{i})}. \tag{1.3.13}$$

If we want to use the summation convention for repeated indices, the equation above should be changed to

$$\omega_{i_1\dots i_r} = \frac{1}{r!}\,\varepsilon^{j_1\cdots j_r}_{i_1\dots i_r}\omega_{j_1\cdots j_r}, \quad \text{or} \quad \omega_{\mathbf{i}} = \frac{1}{r!}\,\varepsilon^{\mathbf{j}}_{\mathbf{i}}\omega_{\mathbf{j}}, \tag{1.3.14}$$

as both the alternating symbol and the alternating array change sign under any permutation.

Let $A = (A_{i_1\dots i_r})$ be any array, *i.e.*, not necessarily alternating. The array A induces an alternating array $\operatorname{Alt}A = \left(\operatorname{Alt}A_{j_1\cdots j_r}\right)$ by

$$(\operatorname{Alt}A)_{j_1\cdots j_r} = \frac{1}{r!}\,\varepsilon^{i_1\dots i_r}_{j_1\cdots j_r}A_{i_1\dots i_r} \quad \text{or} \quad (\operatorname{Alt}A)_{\mathbf{i}} = \frac{1}{r!}\,\varepsilon^{\mathbf{i}}_{\mathbf{j}}A_{\mathbf{i}}. \tag{1.3.15}$$

Again, the factor $1/r!$ can be avoided if we use ordered sequences so that

$$(\mathrm{Alt}\,A)_{\mathbf{j}} = \varepsilon^{(\mathbf{i})}_{\mathbf{j}} A_{(\mathbf{i})} \tag{1.3.16}$$

$\mathrm{Alt}\,A$ is indeed alterating as

$$\begin{aligned} \frac{1}{r!}\varepsilon^{\mathbf{k}}_{\mathbf{j}}(\mathrm{Alt}\,A)_{\mathbf{k}} &= \frac{1}{r!}\varepsilon^{\mathbf{k}}_{\mathbf{j}}\frac{1}{r!}\varepsilon^{\mathbf{i}}_{\mathbf{k}}A_{\mathbf{i}}, \\ &= \frac{1}{r!}\varepsilon^{\mathbf{i}}_{\mathbf{j}}A_{\mathbf{i}} \quad \text{(using 1.3.4)}, \\ &= (\mathrm{Alt}\,A)_{\mathbf{j}}. \end{aligned} \tag{1.3.17}$$

In addition, the Alt operation is a projection in the sense that it leaves alternating mapping unchanges. If $\omega_{i_1 \ldots i_r}$ is alternating, then,

$$\begin{aligned} (\mathrm{Alt}\,\omega)_{\mathbf{j}} &= \frac{1}{r!}\varepsilon^{\mathbf{i}}_{\mathbf{j}}\omega_{\mathbf{i}}, \\ &= \omega_{\mathbf{j}} \quad \text{(by 1.3.14)}. \end{aligned} \tag{1.3.18}$$

1.3.3. Spaces of Alternating Arrays. From the definition of an alternating array ω of degree r over a space of dimension $N \geqslant r$, it is clear that its components are not independent and that some of its components vanish identically. Specifically, it is clear that if the components $\omega_{(\mathbf{i})}$ are given for all increasing sequences $\mathbf{i} = \{i_1, \ldots, i_r\}$, $1 \leqslant i_1 \leqslant i_2 \cdots \leqslant i_r \leqslant N$, then all other componetns may be obained by the anti-symmetry condition in 1.3.12.

Let $(\mathbf{i})$ be an increasing sequence of r indices. We will use the notation $e_{\mathbf{i}}$ for the alternating array such that

$$(e^{\mathbf{i}})_{\mathbf{j}} = \varepsilon^{(\mathbf{i})}_{\mathbf{j}}. \tag{1.3.19}$$

Clearly, there are

$$C^2_N = \begin{pmatrix} N \\ r \end{pmatrix} = \frac{N!}{(N-r)!\,r!}$$

such arrays. For the collection of distinct sequences $(\mathbf{i})$, the alternating arrays $\{e^{\mathbf{i}}\}$ are linearly independent. For if $a_{(\mathbf{i})}e^{\mathbf{i}} = 0$, then,

$$\begin{aligned} a_{(\mathbf{j})} &= \varepsilon^{(\mathbf{i})}_{(\mathbf{j})}a_{(\mathbf{i})}, \\ &= (e^{(\mathbf{i})})_{(\mathbf{j})}a_{(\mathbf{i})}, \\ &= (a_{(\mathbf{i})}e^{\mathbf{i}})_{(\mathbf{j})}, \\ &= 0. \end{aligned} \tag{1.3.20}$$

It follows that the C_r^N alternating arrays $\{e^{\mathbf{i}}\}$, form a basis for the space of alternating arrays and that every alternating array of degree r may be written in the form

$$\omega = \omega_{(\mathbf{i})} e^{\mathbf{i}}. \tag{1.3.21}$$

The dual basis $\{e_{\mathbf{i}}\}$, of the space dual to the space of alternating arrays, satisfies

$$e_{\mathbf{j}}(e^{\mathbf{i}}) = \varepsilon^{(\mathbf{i})}_{(\mathbf{j})} \qquad e_{\mathbf{j}}(\omega) = \omega_{(\mathbf{j})}. \tag{1.3.22}$$

1.3.4. Exterior Product of Arrays. Let $\omega = (\omega_{i_1\dots i_r})$ and $\tau = (\tau_{j_1\dots j_p})$ be two alternating arrays of degrees r and p, respectively. We wish to define a product, the *exterior product*, of the two arrays that will give us an array $\omega\wedge\tau$ of degree $(r+p)$ by

$$(\omega\wedge\tau)_{k_1\dots k_{r+p}} = \operatorname{Alt}(\omega\otimes\tau)_{k_1\dots k_{r+p}} = \frac{1}{(r+p)!}\varepsilon^{i_1\dots i_r j_1\dots j_p}_{k_1\dots\dots k_{r+p}}\omega_{i_1\dots i_r}\tau_{j_1\dots j_p}, \tag{1.3.23}$$

or

$$(\omega\wedge\tau)_{\mathbf{k}} = \frac{1}{(r+p)!}\varepsilon^{\mathbf{ij}}_{\mathbf{k}}\omega_{\mathbf{i}}\tau_{\mathbf{j}}. \tag{1.3.24}$$

If we use only increasing sequences, we can use Equation (1.3.13) and write

$$\varepsilon^{\mathbf{ij}}_{\mathbf{k}}\omega_{\mathbf{i}}\tau_{\mathbf{j}} = \varepsilon^{\mathbf{ij}}_{\mathbf{k}}\varepsilon^{(\mathbf{l})}_{\mathbf{i}}\omega_{(\mathbf{l})}\varepsilon^{(\mathbf{m})}_{\mathbf{j}}\tau_{(\mathbf{m})} = r!\,p!\,\varepsilon^{(\mathbf{l})(\mathbf{m})}_{\mathbf{k}}\,\omega_{(\mathbf{l})}\tau_{(\mathbf{m})} \tag{1.3.25}$$

so the definition of the exterior product may be written in the form

$$(\omega\wedge\tau)_{\mathbf{k}} = \frac{r!p!}{(r+p)!}\varepsilon^{(\mathbf{l})(\mathbf{m})}_{\mathbf{k}}\omega_{(\mathbf{l})}\tau_{(\mathbf{m})}. \tag{1.3.26}$$

Example 1.3.2. For the particular case where ω is a 1-dimensional array, one has (suspending the summation convention in the third row)

$$\begin{aligned}
(\omega\wedge\tau)_{k_1\dots k_{p+1}} &= \frac{p!}{(p+1)!}\varepsilon^{j(m_1\dots m_p)}_{k_1\dots k_{p+1}}\,\omega_j\,\tau_{(m_1\dots m_p)},\\
&= \frac{1}{p+1}\varepsilon^{m_l(m_1\dots\widehat{m}_l\dots m_{p+1})}_{k_1\dots\dots\dots k_{p+1}}\,\omega_{m_l}\,\tau_{(m_1\dots\widehat{m}_l\dots m_{p+1})},\\
&= \frac{1}{p+1}\sum_{l=1}^{p+1}(-1)^{l-1}\varepsilon^{m_l(m_1\dots\widehat{m}_l\dots m_{p+1})}_{k_l(k_1\dots\widehat{k}_l\dots k_{p+1})}\,\omega_{m_l}\,\tau_{(m_1\dots\widehat{m}_l\dots m_{p+1})},\\
&= \frac{1}{p+1}\sum_{l=1}^{p+1}(-1)^{l-1}\,\omega_{k_l}\,\tau_{(k_1\dots\widehat{k}_l\dots k_{p+1})}.
\end{aligned} \tag{1.3.27}$$

It is noted that it is not necessary to use increasing sequences only in the equations above and one can write alternatively

$$\begin{aligned}(\omega\wedge\tau)_{k_1\dots k_{p+1}} &= \frac{1}{(p+1)!}\,\varepsilon^{jm_1\dots m_p}_{k_1\dots k_{p+1}}\,\omega_j\,\tau_{m_1\dots m_p},\\ &= \frac{1}{(p+1)!}\,\varepsilon^{m_l m_1\dots\widehat{m}_l\dots m_{p+1}}_{k_1\,\dots\,\dots\,\dots\,k_{p+1}}\,\omega_{m_l}\,\tau_{m_1\dots\widehat{m}_l\dots m_{p+1}}.\end{aligned} \tag{1.3.28}$$

Now in the sum over m_l it takes values from the fixed sequence $k_1,\dots,k_{p+1}$. When $m_l=k_l$, $\omega_{m_l}=\omega_{k_l}$ and

$$\varepsilon^{m_l m_1\dots\widehat{m}_l\dots m_{p+1}}_{k_1\,\dots\,\dots\,\dots\,k_{p+1}} = (-1)^l\,\varepsilon^{m_l m_1\dots\widehat{m}_l\dots m_{p+1}}_{k_l k_1\dots\widehat{k}_l\dots k_{p+1}} = (-1)^l\,\varepsilon^{m_1\dots\widehat{m}_l\dots m_{p+1}}_{k_1\dots\widehat{k}_l\dots k_{p+1}}. \tag{1.3.29}$$

Thus,

$$\begin{aligned}(\omega\wedge\tau)_{k_1\dots k_{p+1}} &= \frac{1}{(p+1)!}\sum_{l=1}^{p+1}(-1)^{l-1}\,\varepsilon^{m_1\dots\widehat{m}_l\dots m_{p+1}}_{k_1\dots\widehat{k}_l\dots k_{p+1}}\,\omega_{k_l}\,\tau_{m_1\dots\widehat{m}_l\dots m_{p+1}},\\ &= \frac{1}{(p+1)!}\sum_{l=1}^{p+1}(-1)^{l-1}\,\varepsilon^{m_1\dots\widehat{m}_l\dots m_{p+1}}_{k_1\dots\widehat{k}_l\dots k_{p+1}}\,\omega_{m_l}\,\tau_{m_1\dots\widehat{m}_l\dots m_{p+1}},\\ &= \frac{p!}{(p+1)!}\sum_{l=1}^{p+1}(-1)^{l-1}\,\omega_{k_l}\,\tau_{k_1\dots\widehat{k}_l\dots k_{p+1}},\\ &= \frac{1}{p+1}\sum_{l=1}^{p+1}(-1)^{l-1}\,\omega_{k_l}\,\tau_{k_1\dots\widehat{k}_l\dots k_{p+1}}.\end{aligned} \tag{1.3.30}$$

In case A and B are two arrays of degrees r and p respectively, one can set

$$A\wedge B = (\operatorname{Alt}A)\wedge(\operatorname{Alt}B). \tag{1.3.31}$$

It follows that

$$\begin{aligned}(A\wedge B)_{k_1\dots k_{r+p}} &= \frac{1}{(r+p)!}\,\varepsilon^{i_1\dots i_r j_1\dots j_p}_{k_1\dots\dots k_{r+p}}\,\frac{1}{r!}\,\varepsilon^{l_1\dots l_r}_{i_1\dots i_r}A_{l_1\dots l_r}\,\frac{1}{p!}\,\varepsilon^{m_1\dots m_p}_{j_1\dots j_p}B_{m_1\dots m_p},\\ &= \frac{1}{(r+p)!}\,\varepsilon^{l_1\dots l_r m_1\dots m_p}_{k_1\dots\dots k_{r+p}}A_{l_1\dots l_r}\,B_{m_1\dots m_p},\\ &= \operatorname{Alt}(A\otimes B)_{k_1\dots k_{r+p}},\end{aligned}$$

using the identity (1.3.5). Thus, one can define the exterior product of any two arrays by

$$A\wedge B = \operatorname{Alt}(A\otimes B) = (\operatorname{Alt}A)\wedge(\operatorname{Alt}B). \tag{1.3.32}$$

It is noted that the exterior product is associative. For arrays A, B, and C of degrees r, p, and s, respectively, we have

$$
\begin{aligned}
&((A\wedge B)\wedge C)_{k_1\dots k_{r+p+s}} = \\
&\quad = \frac{1}{(r+p+s)!}\,\varepsilon^{i_1\dots i_{r+p}j_1\dots j_s}_{k_1\dots\dots k_{r+p+s}}(A\wedge B)_{i_1\dots i_{r+s}}C_{j_1\dots j_s},\\
&\quad = \frac{1}{(r+p+s)!(r+p)!}\,\varepsilon^{i_1\dots i_{r+p}j_1\dots j_s}_{k_1\dots\dots k_{r+p+s}}\varepsilon^{l_1\dots l_r m_1\dots m_p}_{i_1\dots\dots\dots i_{r+p}}A_{l_1\dots l_p}B_{m_1\dots m_p}C_{j_1\dots j_s},\\
&\quad = \frac{1}{(r+p+s)!}\,\varepsilon^{l_1\dots l_r m_1\dots m_p j_1\dots j_s}_{k_1\dots\dots\dots\dots k_{r+p+s}}A_{l_1\dots l_p}B_{m_1\dots m_p}C_{j_1\dots j_s},\\
&\quad = (A\wedge(B\wedge C))_{k_1\dots k_{r+p+s}}
\end{aligned}
\tag{1.3.33}
$$

and one can write

$$((A\wedge B)\wedge C = (A\wedge(B\wedge C)) = A\wedge B\wedge C. \tag{1.3.34}$$

A number of authors use a somewhat alternative definition of th exterior product. See the discussion in [**War83**, pp. 59–60].

1.3.5. Inner Products. Let $\omega = (\omega_{i_1\dots i_r})$ be an alternating array and let $A = (A^{k_1\dots k_p})$ be any array with $p \leqslant r$. The inner product, denoted as $A\lrcorner\,\omega$ or $\mathbf{i}_A\omega$, is the alternating array of degree $r-p$ defined by

$$(A\lrcorner\,\omega)_{i_1\dots i_{r-p}} = \omega_{i_1\dots i_{r-p}k_1\dots k_p}A^{k_1\dots k_p}. \tag{1.3.35}$$

Clearly, $A\lrcorner\,\omega$ inherits the skew-symmetry property from ω. Since ω is alternating, one has by Equation (1.3.14)

$$(A\lrcorner\,\omega)_{i_1\dots i_{r-p}} = \frac{1}{r!}\,\varepsilon^{j_1\dots\dots j_r}_{i_1\dots i_{r-p}k_1\dots k_p}\omega_{j_1\dots j_r}A^{k_1\dots k_p}. \tag{1.3.36}$$

In addition, using Equation (1.3.14),

$$
\begin{aligned}
(A\lrcorner\,\omega)_{i_1\dots i_{r-p}} &= \frac{1}{r!}\,\varepsilon^{j_1\dots\dots j_r}_{i_1\dots i_{r-p}l_1\dots l_p}\,\frac{1}{p!}\,\varepsilon^{l_1\dots lp}_{k_1\dots k_p}\omega_{j_1\dots j_r}A^{k_1\dots k_p},\\
&= \frac{1}{r!}\,\varepsilon^{j_1\dots\dots j_r}_{i_1\dots i_{r-p}l_1\dots l_p}\omega_{j_1\dots j_r}(\operatorname{Alt}A)^{l_1\dots l_p},\\
&= \omega_{i_1\dots i_{r-p}l_1\dots l_p}(\operatorname{Alt}A)^{l_1\dots l_p},
\end{aligned}
\tag{1.3.37}
$$

and we conclude that

$$\operatorname{Alt}A\lrcorner\,\omega = A\lrcorner\,\omega. \tag{1.3.38}$$

We note that for an additional array B of degree q such that $r \geqslant p+q$, one has

$$B \lrcorner (A \lrcorner \omega)_{m_1 \dots m_{r-p-q}} = \omega_{m_1 \dots m_{r-p-q}\, i_1 \dots i_p\, k_1 \dots k_q} A^{i_1 \dots i_p} B^{k_1 \dots k_1}.$$

It follows that

$$B \lrcorner (A \lrcorner \omega) = (A \otimes B) \lrcorner \omega = (A \wedge B) \lrcorner \omega. \tag{1.3.39}$$

In the particular case of an alternating array ω and an array A both having the same degree r, we write

$$A \lrcorner \omega = \omega(A) = \omega \cdot A = \omega_{k_1 \dots k_p} A^{k_1 \dots k_p}. \tag{1.3.40}$$

For examle, for the r arrays of degree one $\alpha^1, \dots, \alpha^r$ and r arrays of degree one $v_1, \dots, v_r$,

$$\begin{aligned}(\alpha^1 \wedge \cdots \wedge \alpha^r)(v_1, \dots, v_r) &= (\alpha^1 \wedge \cdots \wedge \alpha^r)_{i_1 \dots i_r} v_1^{i_1} \cdots v_r^{i_r}, \\ &= \frac{1}{r!} \varepsilon^{k_1 \dots k_r}_{i_1 \dots i_r} \alpha^1_{k_1} \dots \alpha^r_{k_r} v_1^{i_1} \cdots v_r^{i_r}.\end{aligned} \tag{1.3.41}$$

For $r = 2$, we get

$$\begin{aligned}(\alpha \wedge \beta)(v, u) = (\alpha \wedge \beta)(v \wedge u) &= \tfrac{1}{2} \varepsilon^{ij}_{pq} \alpha_i \beta_j v^p u^q, \\ &= \tfrac{1}{2}(\alpha_i v^i \beta_j u^j - \alpha_i u^i \beta_j v^j), \\ &= \tfrac{1}{2}(\alpha(v)\beta(u) - \alpha(u)\beta(v)).\end{aligned} \tag{1.3.42}$$

Equation (1.3.41) may be presented in an alternative form. Using the $\varepsilon - \delta$-identity in Equation (1.3.11), we can write it as

$$(\alpha^1 \wedge \cdots \wedge \alpha^r)(v_1, \dots, v_r) = \frac{1}{r!} \sum_p \operatorname{sign}(p)\, \delta^{k_{p(1)}}_{i_1} \cdots \delta^{k_{p(r)}}_{i_r} \alpha^1_{k_1} \dots \alpha^r_{k_r} v_1^{i_1} \cdots v_r^{i_r}.$$

We note that for $l = 1, \dots, r$,

$$\delta^{k_{p(l)}}_{i_l} \alpha^{p(l)}_{k_{p(l)}} v_l^{i_l} = \alpha^{p(l)}_{i_l} v_l^{i_l} = \alpha^{p(l)}(v_l), \tag{1.3.43}$$

and setting $M^j_l = \alpha^j(v_l)$, we have

$$\begin{aligned}(\alpha^1 \wedge \cdots \wedge \alpha^r)(v_1, \dots, v_r) &= \frac{1}{r!} \sum_p \operatorname{sign}(p)\, M_1^{p(1)} M_2^{p(2)} \cdots M_r^{p(r)}, \\ &= \frac{1}{r!} \sum \varepsilon_{k_1 k_2 \dots k_r} M_1^{k_1} M_2^{k_2} \cdots M_r^{k_r}.\end{aligned} \tag{1.3.44}$$

Using the definition of the determinant (see Section 1.3.6) we may finally write

$$(\alpha^1 \wedge \cdots \wedge \alpha^r)(v_1, \dots, v_r) = \frac{1}{r!} \det[M] = \frac{1}{r!} \det[(\alpha^j(v_l))]. \tag{1.3.45}$$

In case the alternative definition of the exterior product is used as discussed in [**War83**, pp. 59–60], the $1/r!$ factor does not appear in the equation above and analog expressions proceeding it. We interpret the difference in the value as follows: There are $r!$ identical simplexes in a paralleliped. Thus, while in our definition $v_1 \wedge \ldots \wedge v_r$ is interpreted as the array associated with the oriented simples generated by these vectors, in the alternative definition, it is interpreted as that associated with the oriented parallelepiped.

1.3.5.1. *Duals.* We now consider the particular case in which the alternating symbol is used as the alternating array in a contraction operation.

Let $(v^1, \ldots, v^m) \in \mathbb{R}^m$ be a vector. There is an induced array, the *dual*

$$\widehat{v}_{j_1 \cdots j_{m-1}} = \varepsilon_{ij_1 \cdots j_{m-1}} v^i$$

which is clearly completely antisymmetric for transpositions of any pair of its indices. We note that in the expression for the definition of $\widehat{v}$ there is only one non-vanishing term in the sum and that the sequence $(j_1 \ldots j_{m-1})$ contains all the values $1, \ldots, m$ except for the value i for which $\widehat{v}_{j_1 \cdots j_{m-1}} = \pm v^i$. Thus, each sequence $(j_1, \ldots, j_{m-1})$ may be obtained from the sequence $(1, \ldots, \widehat{\imath}, \ldots, m)$, where the "hat" denotes an omitted item, by rearrangement. It follows from the antisymmetry of $\widehat{v}$ that there is only one independent component having a given collection of indices. The other terms can be obtained from it using the antisymmetry. We may choose this component to be the one with increasing indices determined by the missing $\widehat{\imath}$. Thus,

$$\widehat{v}_{j_1 \cdots j_{m-1}} = \varepsilon^{1 \ldots \widehat{\imath} \ldots m}_{j_1 \cdots j_{m-1}} \widehat{v}_{1 \ldots \widehat{\imath} \ldots m}, \quad \text{no sum over } \widehat{\imath}.$$

It follows from the definition of $\widehat{v}$ that

$$\widehat{v}_{1 \ldots \widehat{\imath} \ldots m} = (-1)^{i-1} v^i, \quad \text{no summation on } i.$$

Using the "hat" notation and renaming indices in the definition of the dual (i is renamed to j_p and for $l > p$, j_l is renamed to j_{p+1}), the definition of the alternating symbol implies that

$$\widehat{v}_{j_1 \cdots \widehat{j}_p \cdots j_m} = (-1)^{p-1} \varepsilon_{j_1 \cdots j_p \cdots j_m} v^{j_p}.$$

Given a completely antisymmetric array $\omega_{j_1 \cdots j_{m-1}}$, such that the indices $j_1, \ldots, j_{m-1} \in \{1, \ldots, m\}$, one may define a *dual vector* $\widehat{\omega} \in \mathbb{R}^m$

$$\widehat{\omega}^i = \frac{1}{(m-1)!} \varepsilon^{ij_1 \cdots j_{m-1}} \omega_{j_1 \cdots j_{m-1}}.$$

As

$$\omega_{j_1\cdots j_{m-1}} = \varepsilon^{1\ldots\hat{\imath}\ldots m}_{j_1\cdots j_{m-1}}\omega_{1\ldots\hat{\imath}\ldots m}, \quad \text{no sum on } \hat{\imath},$$

one has

$$\begin{aligned}\widehat{\omega}^i &= \frac{1}{(m-1)!}\,\varepsilon^{ij_1\cdots j_{m-1}}\varepsilon^{1\ldots\hat{\imath}\ldots m}_{j_1\cdots j_{m-1}}\omega_{1\ldots\hat{\imath}\ldots m}\\ &= \varepsilon^{i1\ldots\hat{\imath}\ldots m}\omega_{1\ldots\hat{\imath}\ldots m} \qquad \text{(no sum on the } i, \hat{\imath} \text{ indices)}\\ &= (-1)^{i-1}\omega_{1\ldots\hat{\imath}\ldots m}.\end{aligned}$$

It follows from the last equality that the two operations are inverses of one another, i.e., $\widehat{\widehat{v}}^i = v^i$.

1.3.6. Determinants and the alternating symbol. Let $[A] = (A^i_j)$, $i,j = 1,\ldots,m$, be a square matrix. The determinant of $[A]$ is given by

$$\det[A] = \varepsilon_{i_1\ldots i_m}A^{i_1}_1\cdots A^{i_m}_m.$$

We note that

$$\varepsilon_{i_1\ldots i_m}A^{i_1}_1\cdots A^{i_m}_m = \frac{1}{m!}\,\varepsilon^{j_1\cdots j_m}_{i_1\ldots i_m}A^{i_1}_{j_1}\cdots A^{i_m}_{j_m}.$$

This follows because each non-vanishing term in the sum on the j-indices may be transformed to the expression for the determinant by a rearrangement of the A-factors with a corresponding rearrangement of the i-sequence. Clearly, there are $m!$ terms in the sum and we conclude that

$$\det[A] = \frac{1}{m!}\,\varepsilon^{j_1\cdots j_m}_{i_1\ldots i_m}A^{i_1}_{j_1}\cdots A^{i_m}_{j_m}.$$

This expression makes it clearer that $\det[A]^T = \det[A]$.

Rewriting the expression for the determinant we obtain the following

$$\begin{aligned}\det[A] &= \varepsilon_{ij_1\cdots j_{m-1}}A^i_1A^{j_1}_2\cdots A^{j_{m-1}}_m\\ &= \widehat{A_1}_{j_1\cdots j_{m-1}}A^{j_1}_2\cdots A^{j_{m-1}}_m\end{aligned}$$

using the definition of the dual array. If we use

$$\widehat{A_1}_{j_1\cdots j_{m-1}} = \frac{1}{(m-1)!}\,\varepsilon^{i_1\cdots i_{m-1}}_{j_1\cdots j_{m-1}}\widehat{A_1}_{i_1\ldots i_{m-1}}$$

which follows from the antisymmetry of $\widehat{A_1}_{j_1\cdots j_{m-1}}$ we can continue the previous equalities to obtain

$$\det[A] = \frac{1}{(m-1)!}\,\widehat{A_1}_{i_1\ldots i_{m-1}}\varepsilon^{i_1\cdots i_{m-1}}_{j_1\cdots j_{m-1}}A^{j_1}_2\cdots A^{j_{m-1}}_m.$$

This expression is actually the rule for the expansion of the determinant by determinants $\varepsilon^{i_1\dots i_{m-1}}_{j_1\cdots j_{m-1}} A^{j_1}_2 \dots A^{j_{m-1}}_m$ of the matrices obtained by deleting the first row and the various columns of the matrix and multiplying them by the elements of the first row with the appropriate sign, $\widehat{A1}_{i_1\dots i_{m-1}}$ in our case.

We note that an antisymmetric array $A_{i_1\dots i_r}$ is completely determined by the values of its of elements for increasing sequence of indices, i.e., $i_1 < i_2 < \cdots i_r$ as the other values are determined from the antisymmetry. Thus, if we use only increasing sequences of indices in completely antisymmetric arrays (not including the alternating symbol), $\widehat{A1}_{i_1\dots i_{m-1}}$ in our case, then the division by $(m-1)!$ is not needed and we write

$$\det[A] = \widehat{A1}_{i_1\dots i_{m-1}} \varepsilon^{i_1\dots i_{m-1}}_{j_1\cdots j_{m-1}} A^{j_1}_2 \cdots A^{j_{m-1}}_m, \quad i_1 < i_2 < \cdots < i_{m-1}.$$

In the sequel we will sometimes use increasing sequences of indices and will indicate this in the notation.

Similarly, if we carry the analog calculation for $A^{j_r}_r$ for any value of r we have

$$\begin{aligned}
\det[A] &= \varepsilon_{j_1\cdots j_m} A^{j_1}_1 \cdots A^{j_m}_m \\
&= (-1)^{r-1} \varepsilon_{j_r j_1 \dots \widehat{j_r} \dots j_m} A^{j_r}_r A^{j_1}_1 \cdots \widehat{A^{j_r}_r} \cdots A^{j_m}_m \\
&= (-1)^{r-1} \widehat{(A_r)}_{j_1\dots\widehat{j_r}\dots j_m} A^{j_1}_1 \cdots \widehat{A^{j_r}_r} \cdots A^{j_m}_m \\
&= \frac{(-1)^{r-1}}{(m-1)!} \widehat{(A_r)}_{i_1\dots i_{m-1}} \varepsilon^{i_1 \dots\dots i_{m-1}}_{j_1\dots\widehat{j_r}\dots j_m} A^{j_1}_1 \cdots \widehat{A^{j_r}_r} \cdots A^{j_m}_m .
\end{aligned}$$

The last expression may be modified further by shuffling using the permutation mapping $p\colon \{1,\dots,m-1\} \to \{1,\dots,\widehat{r},\dots,m\}$ to obtain

$$\varepsilon^{i_1 \dots\dots i_{m-1}}_{j_1\dots\widehat{j_r}\dots j_m} A^{j_1}_1 \cdots \widehat{A^{j_r}_r} \cdots A^{j_m}_m = \varepsilon^{i_1 \dots\dots i_{m-1}}_{j_1\dots\widehat{j_r}\dots j_m} A^{j_{p(1)}}_{p(1)} \cdots A^{j_{p(m-1)}}_{p(m-1)}.$$

Since

$$\begin{aligned}
\varepsilon^{i_1 \dots\dots i_{m-1}}_{j_1\dots\widehat{j_r}\dots j_m} &= \varepsilon^{i_1 \dots\dots i_{m-1}}_{j_{p^{-1}\circ p(1)}\cdots j_{p^{-1}\circ p(m-1)}} \\
&= \operatorname{sign}(p^{-1}) \varepsilon^{i_1\cdots \quad \cdots i_{m-1}}_{j_{p(1)}\cdots j_{p(m-1)}} \\
&= \operatorname{sign}(p) \varepsilon^{i_1\cdots \quad \cdots i_{m-1}}_{j_{p(1)}\cdots j_{p(m-1)}},
\end{aligned}$$

this may be rewritten as

$$\begin{aligned}\varepsilon^{i_1\cdots\cdots i_{m-1}}_{j_1\cdots\widehat{j_r}\cdots j_m}A^{j_1}_1\cdots\widehat{A^{j_r}_r}\cdots A^{j_m}_m &= \operatorname{sign}(p))\varepsilon^{i_1\cdots\quad\cdots i_{m-1}}_{j_{p(1)}\cdots j_{p(m-1)}}A^{j_{p(1)}}_{p(1)}\cdots A^{j_{p(m-1)}}_{p(m-1)}\\ &= \frac{1}{(m-1)!}\varepsilon^{k_1\ldots k_{m-1}}\varepsilon^{i_1\ldots i_{m-1}}_{j_1\cdots j_{m-1}}A^{j_1}_{k_1}\cdots A^{j_{m-1}}_{k_{m-1}},\end{aligned}$$

where we renamed the sequence $p(1),\ldots,p(m-1)$ to $k_1,\ldots,k_{m-1}$ so $\operatorname{sign}(p)=\varepsilon^{k_1\ldots k_{m-1}}$, and renamed

$$j_{p(1)},\ldots,\widehat{j}_{p(r)},\ldots,j_{p(m-1)}$$

to $j_1,\ldots,j_{m-1}$. We conclude that

$$\det[A]=\frac{(-1)^{r-1}}{((m-1)!)^2}\widehat{(A_r)}_{i_1\ldots i_{m-1}}\varepsilon^{i_1\ldots i_{m-1}}_{j_1\cdots j_{m-1}}A^{j_1}_{k_1}\cdots A^{j_{m-1}}_{k_{m-1}}\varepsilon^{k_1\ldots k_{m-1}}$$

where the $((m-1)!)^2$ may be removed if we use inceasing sequences of indices only in the alternating tensors.

1.3.7. Differential Forms and Exterior Derivatives in $\mathbb{R}^m$. A smooth field over $\mathbb{R}^m$ of alternating array of degree r is a *differential r-form in $\mathbb{R}^m$*. All the operations defined above for alternating arrays may be applied pointwise to differential forms. For example, the exterior product of an r-form ω and a p-form τ is the $(r+p)$-form $\omega\wedge\tau$ defined by $(\omega\wedge\tau)(x)=\omega(x)\wedge\tau(x)$.

If $v_1,\ldots,v_r$ are linearly independent vectors in $\mathbb{R}^m$, it is natural to interpret $v_1\wedge\cdots\wedge v_r$ as the oriented r-oriented area of the simplex determined by these vectors. Thus, the area parallepiped determined by these vectors is $r!\,v_1\wedge\cdots\wedge v_r$ For an alternating array ω of degree r, the value $\omega(r!\,v_1\wedge\cdots\wedge v_r)$ is interpreted as a flux of a certain property across that parallelepiped. For an r-form ω in $\mathbb{R}^m$ this interpretation can still hold if we regard the vectors as infinitesimal vectors originating at some point in $\mathbb{R}^m$ and use the value of the form at a point within the parallelepiped determented by these vectors.

If we have $r+1$ vectors, they generate an $(r+1)$-dimensional parallelpiped to which we will refer as the box. The various faces of this box are obtained by omiting one vector at a time. Thus a face is determined by $\{v_1,\ldots,\widehat{v}_k,\ldots,v_{r+1}\}$. The positive orientation of the face determined by these vectors is defined to be $(-1)^{k-1}$ and so the oriented area of the face is determined by $(-1)^{k-1}r!\,v_1\wedge\cdots\wedge\widehat{v}_k\wedge\cdots\wedge v_{r+1}$ Evidently, there are two such faces seperated by the vector v_k. Thus, using $\nabla_v f=\nabla f(v)$ to denote the directional derivative of the function f in the direction of the vector v, the quantity

$$\nabla_{v_k}\left\{\omega[(-1)^{k-1}r!\,v_1\wedge\cdots\wedge\widehat{v}_k\wedge\cdots\wedge v_{r+1}]\right\} \quad \text{no sum on } k, \tag{1.3.46}$$

indicates the difference in the flux between these two faces. Adding up the differences on all pairs of faces, the total flux out of the box is

$$\Phi = \sum_{k=1}^{r+1}(-1)^{k-1}r!\,\nabla_{v_k}\{\omega[v_1\wedge\cdots\wedge\widehat{v}_k\wedge\cdots\wedge v_{r+1}]\}. \tag{1.3.47}$$

We have

$$\omega(v_1\wedge\cdots\wedge\widehat{v}_k\wedge\cdots\wedge v_{r+1}) = \omega_{i_1\ldots\widehat{i}_k\ldots i_{r+1}}v_1^{i_1}\cdots\widehat{v}_k^{i_k}\cdots v_{r+1}^{i_{r+1}} \tag{1.3.48}$$

and

$$\nabla_{v_k}[\omega(v_1\wedge\cdots\wedge\widehat{v}_k\wedge\cdots\wedge v_{r+1})] = \frac{\partial}{\partial x^{i_k}}\omega_{i_1\ldots\widehat{i}_k\ldots i_{r+1}}v_1^{i_1}\cdots v_k^{i_k}\cdots v_{r+1}^{i_{r+1}}. \tag{1.3.49}$$

Thus, the total flux out of the box is

$$\Phi = \sum_{k=1}^{r+1}(-1)^{k-1}r!\,\frac{\partial}{\partial x^{i_k}}\omega_{i_1\ldots\widehat{i}_k\ldots i_{r+1}}v_1^{i_1}\cdots v_k^{i_k}\cdots v_{r+1}^{i_{r+1}}. \tag{1.3.50}$$

Regarding $\partial/\partial x^{i_k}$ as the i_k-component ∇_{i_k} of the gradient array ∇, we may use Equation (1.3.30) to write

$$\begin{aligned}\sum_{k=1}^{r+1}(-1)^{k-1}\frac{\partial}{\partial x^{i_k}}\omega_{i_1\ldots\widehat{i}_k\ldots i_{r+1}} &= \sum_{k=1}^{r+1}(-1)^{k-1}\nabla_{i_k}\omega_{i_1\ldots\widehat{i}_k\ldots i_{r+1}},\\ &= (r+1)(\nabla\wedge\omega)_{i_1\ldots i_{r+1}}.\end{aligned} \tag{1.3.51}$$

The alternating array $\nabla\wedge\omega$ of degree $r+1$ is traditionally denoted by $d\omega$ and is referred to as the *exterior derivative* of the array ω. Thus,

$$\begin{aligned}d\omega_{i_1\ldots i_{r+1}} &= \frac{1}{r+1}\sum_{k=1}^{r+1}(-1)^{k-1}\frac{\partial}{\partial x^{i_k}}\omega_{i_1\ldots\widehat{i}_k\ldots i_{r+1}},\\ &= \frac{1}{(r+1)!}\varepsilon^{jm_1\ldots m_p}_{i_1\ldots i_{r+1}}\frac{\partial}{\partial x^j}\tau_{m_1\ldots m_p},\end{aligned} \tag{1.3.52}$$

where Example 1.3.2 was used in the second line.

We conclude that

$$\begin{aligned}\Phi &= (r+1)!\,d\omega(v_1,\ldots,v_{r+1}),\\ &== d\omega[(r+1)!v_1\wedge\ldots,\wedge v_{r+1}],\end{aligned} \tag{1.3.53}$$

and note that $(r+1)!v_1 \wedge \ldots, \wedge v_{r+1}$ is the oriented "volume" of the box.

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CHAPTER 7

Extensive Properties and Fluxes—Analytic Aspects

1. Multivectors and Differential Forms on Manifolds

Differential forms are roughly the variable counterparts of multilinear forms presented in the previous chapter. Thus, while alternating multilinear forms operate on multivectors (or sequences of vectors that induce them) to give the amount of a certain property they contain within their capacity, differential forms operate on infinitesimal capacities generated by (infinitesimal) tangent vectors to yield the infinitesimal amount of property they contain. These infinitesimal quantities can then be integrated to give the total amount of the property within finite regions as will be presented in the next section.

1.1. The bundles of multivectors and forms. The constructions of the previous chapter on spaces of alternating forms associated with a vector space $\mathbf{W}$ may be applied to the tangent space $T_x\mathcal{M}$ of any particular point. Thus one obtains the *space* $\bigwedge^r T_x\mathcal{M}$ *of* r*-multivectors at* x. The union of the various spaces of multivectors is the *bundle of* r*-multivectors* $\bigwedge^r TM$, i.e.,

$$\bigwedge^r TM = \bigcup_{x\in\mathcal{M}} \Big(\bigwedge^r T_x\mathcal{M}\Big).$$

Similarly, we have the *space* $\bigwedge^r(T_x^*\mathcal{M})$ *of* r*-forms at* x and the *bundle of* r*-forms*

$$\bigwedge^r T^*\mathcal{M} = \bigcup_{x\in\mathcal{M}} \Big(\bigwedge^r T_x^*\mathcal{M}\Big).$$

Clearly, for $r=1$, $\bigwedge^1 T_x^*\mathcal{M} = T^*\mathcal{M}$ and $\bigwedge^0 T_x^*\mathcal{M} = \mathbb{R}$.

The bundles of multivectors and the bundles of forms have natural projection mappings $\bigwedge^r T\mathcal{M} \to \mathcal{M}$ and $\bigwedge^r T^*\mathcal{M} \to \mathcal{M}$. These mappings assign to a multivector, or respectively an alternating mapping, the point x such that $T_x\mathcal{M}$ is the basic vector space for the construction of the multivector or form.

These spaces have natural manifold and bundle structures as follows. Let $(x^1, \ldots, x^m)$ be local coordinates in an open set $U \subset \mathcal{M}$. The coordinate system induces at any point in U, a base

$$\left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m} \right\}$$

of $T_x\mathcal{M}$ and a base $\{dx^1, \ldots, dx^m\}$ of $T^*_x\mathcal{M}$. Thus, these bases induce bases

$$\left\{ \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_r}} \right\}, \quad i_1 < \cdots < i_r,$$

of the spaces of multivectors, and bases

$$dx^{i_1} \wedge \cdots \wedge dx^{i_r}, \quad i_1 < \cdots < i_r,$$

for the spaces of forms at the various points. Thus, an element $\mathfrak{v}$ of $\bigwedge^r T_x\mathcal{M}$ for some $x \in U$ is of the form

$$\mathfrak{v} = \mathfrak{v}^{i_1 \ldots i_r} \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_r}}, \quad i_1 < \cdots < i_r,$$

where sum is implied over all increasing sequences of indices. Similarly, an element $\omega \in \bigwedge^r T^*_x\mathcal{M}$ is represented in the form

$$\omega = \omega_{i_1 \ldots i_r} \, dx^{i_1} \wedge \cdots \wedge dx^{i_r}, \quad i_1 < \cdots < i_r.$$

In case we do not restrict the sequences of indices to be increasing, the antisymmetry of both the arrays of components and the wedge products imply that

$$\mathfrak{v} = \frac{1}{r!} \, \mathfrak{v}^{i_1 \ldots i_r} \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_r}}, \quad \omega = \frac{1}{r!} \, \omega_{i_1 \ldots i_r} \, dx^{i_1} \wedge \cdots \wedge dx^{i_r}.$$

Thus, the local coordinate representation of $\mathfrak{v}$ is $(x^i, \mathfrak{v}^{i_1 \ldots i_r})$ and the natural projection is represented by $(x^i, \mathfrak{v}^{i_1 \ldots i_r}) \mapsto (x^i)$. The local representative of the alternating form ω is $(x^i, \omega_{i_1 \ldots i_r})$ and the natural projection is represented by $(x^i, \omega_{i_1 \ldots i_r}) \mapsto (x^i)$.

1.2. Multivector fields and differential forms. We recall that a section of a bundle $\pi\colon E \to \mathcal{M}$ is a mapping $\xi\colon \mathcal{M} \to E$ such that $\pi \circ \xi = 1_{\mathcal{M}}$—the identity on $\mathcal{M}$. Thus, sections of the bundles of multivectors and forms are mappings $\mathfrak{v}\colon \mathcal{M} \to \bigwedge^r T\mathcal{M}$ and $\omega\colon \mathcal{M} \to \bigwedge^r T^*\mathcal{M}$ such that $\mathfrak{v}(x) \in \bigwedge^r T_x\mathcal{M}$ and $\omega(x) \in \bigwedge^r T^*_x\mathcal{M}$. Note that for simplifying the notation we use for sections the same scheme of notation as for the objects in their co-domains. An *rmultivector field* is a smooth section of the bundle of r-multivectors, and an *r-differential form* is a smooth section of the bundle of forms.

Thus, locally, an r-differentiable form may be written as

$$\omega(x) = \omega_{i_1 \ldots i_r}(x^i) \, dx^{i_1} \wedge \cdots \wedge dx^{i_r},$$

where $\omega_{i_1\dots i_r}$, $i_1 < \cdots < i_r$, $1 \leq i_k \leq m$ are real valued functions on $\mathbb{R}^m$. Hence, the local representatives of the mapping ω are the mappings $(x^i) \mapsto (x^i, \omega_{i_1\dots i_r}(x^j))$. The collection of mappings $\omega_{i_1\dots i_r}$ will be referred to as the *principal* part of the local representative.

Various operations on alternating multi-linear forms are extended to differential forms by performing them on the values the differential forms assume i.e., performing the operations point-wise. For example, we have addition of forms $\omega_1+\omega_2$ and exterior products of forms $\omega_1\wedge\omega_2$ defined by

$$\big(\omega_1+\omega_2\big)(x) = \omega_1(x)+\omega_2(x) \quad \text{and} \quad \big(\omega_1\wedge\omega_2\big)(x) = \omega_1(x)\wedge\omega_2(x).$$

For an r-differential form ω and r vector fields $v_1,\dots,v_r$, $\omega(v_1,\dots,v_r)$ is the real valued function

$$\omega(v_1,\dots,v_r)(x) = \omega(x)\big(v_1(x),\dots,v_r(x)\big).$$

If we have a differential r-form ω and a vector field v we have the $(r-1)$-differential form $v\lrcorner\,\omega(x) = v(x)\lrcorner\,\omega(x)$. If $\kappa\colon \mathcal{M} \to \mathcal{N}$ is a smooth mapping and ω is a differential form on $\mathcal{N}$ then $\kappa^*(\omega)$ is a differential form on $\mathcal{M}$ defined by

$$\kappa^*(\omega)(v_1,\dots,v_r) = \omega\big(T\kappa(v_1),\dots,T\kappa(v_r)\big).$$

Obviously, similar operations may are defined for multivector fields.

***********THE FOLLOWING COULD APPEAR IN THE ALGEBRAIC CONTEXT EARLIER AND MAY BE REFERRED TO HERE ************** Let $\kappa\colon \mathcal{M} \to \mathcal{N}$ be a smooth mapping between two m-dimensional manifolds, ω an m-form on $\mathcal{N}$. Thus, for local coordinate systems x^i in $\mathcal{M}$ and $y^{j'}$ in $\mathcal{N}$, κ is represented by the m functions $\kappa^{j'}\colon \mathbb{R}^m \to \mathbb{R}$ such that $y^{j'} = \kappa(x)^{j'} = \kappa^{j'}(x^i)$. The differential form ω is represented as

$$\omega(y) = \omega_{1'\dots m'}(y^{i'})\, dy^{1'}\wedge\cdots\wedge dy^{m'},$$

for a single real valued function $\omega_{1\dots m}$ (as $1\dots m$ is the only increasing sequence of numbers in that range). We wish to find the local representation of $\kappa^*(\omega)$. Note that locally,

$$\kappa^*(\omega) = \kappa^*(\omega)_{i_1\dots i_m}\, dx^1\wedge\ldots\wedge dx^m,$$

where

$$\kappa^*(\omega)_{1\dots m} = \kappa^*(\omega)\left(\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^m}\right).$$

In order to determine $\kappa^*(\omega)_{i_1\dots i_m}$, we use the definition of $\kappa^*(\omega)$, where we observe that

$$T\kappa\left(\frac{\partial}{\partial x^i}\right) = \kappa^{j'}_{,i}\frac{\partial}{\partial y^{j'}},$$

to obtain

$$\begin{aligned}\kappa^*(\omega)_{1\ldots m} &= \omega\left(\kappa^{j'_1}_{,1}\frac{\partial}{\partial y^{j'_1}},\ldots,\kappa^{j'_m}_{,m}\frac{\partial}{\partial y^{j'_m}}\right)\\ &= \kappa^{j'_1}_{,1}\cdots\kappa^{j'_m}_{,m}\,\omega\left(\frac{\partial}{\partial y^{j'_1}},\ldots,\frac{\partial}{\partial y^{j'_m}}\right)\\ &= \varepsilon_{j'_1\ldots j'_m}\kappa^{j'_1}_{,1}\cdots\kappa^{j'_m}_{,m}\,\omega\left(\frac{\partial}{\partial y^{1'}},\ldots,\frac{\partial}{\partial y^{m'}}\right).\end{aligned}$$

Using the expression for the determinant of a matrix and the definition of $\omega_{1\ldots m}$ we finally get

$$\kappa^*(\omega)_{1\ldots m} = \det(\kappa^{j'}_{,i})\omega_{1'\ldots m'},$$

so

$$\kappa^*(\omega) = \det(\kappa^{j'}_{,i})\;(\omega_{1'\ldots m'}\circ\kappa)\,dx^1\wedge\cdots\wedge dx^m.$$

This result clearly holds in the particular case where $\mathcal{M}=\mathcal{N}=\mathbb{R}^m$. In addition, it applies in the case where $\kappa = 1_{\mathcal{M}}$, $\kappa^* = 1_{\bigwedge^m T\mathcal{M}}$, the identity mappings on a manifold and the bundle of forms. In this case, for two coordinate systems $(x^1,\ldots,x^m)$ and $(y^{1'},\ldots,y^{m'})$ with intersecting domains, the mappings κ^j represent the coordinate transformation on the intersection and $\kappa^j_{,i'}$ is the derivative—Jacobian matrix. The local representatives of an m-differential form ω will be of the form

$$\omega = \omega_{1\ldots m}\,dx^1\wedge\cdots\wedge dx^m = \omega_{1'\ldots m'}\,dy^{1'}\wedge\cdots\wedge dy^{m'}.$$

Hence, the last expression gives the transformation rule

$$\omega_{1'\ldots m'} = \det\left(\frac{\partial y^{j'}}{\partial x^i}\right)\omega_{1\ldots m}$$

for the of representative of the m-differential form.

If one performs the analogous calculation for an m-multivector field $\mathfrak{v}$ the resulting transformation represented locally as

$$\mathfrak{v} = \mathfrak{v}^{1\ldots m}\frac{\partial}{\partial x^1}\wedge\cdots\wedge\frac{\partial}{\partial x^m} = \mathfrak{v}^{1'\ldots m'}\frac{\partial}{\partial y^{1'}}\wedge\cdots\wedge\frac{\partial}{\partial y^{m'}},$$

the resulting transformation is

$$\mathfrak{v}^{1\ldots m} = \det\left(\frac{\partial y^{j'}}{\partial x^i}\right)\mathfrak{v}^{1'\ldots m'}.$$

2. Integration of Forms

Prerequisites: Simplexes (particularly, the r-vector of an oriented r-simplex in $\mathbb{R}^r$ and that of an r-chain in Whitney pp. 80–81), open neighborhoods, differentiability, orientation, affine space, homeomorphism, transformation of the representations of forms, compact support, manifold, manifold with a boundary, partition of unity, pullback of vector bundles (used in the section on restriction of forms), bases for tangent spaces and dual spaces, equivalence relations and classes, $(m-1)$-multivectors are simple,

References: Abraham, Marsden & Ratiu; Whitney; Spivak

Notes: See notation in the section on local representation of the flux form etc.

2.1. Overview. This section presents the main aspects of Riemannian integration theory on manifolds. The immediate application of integration we have in mind is clear. We want to calculate the total amount of some extensive property in a certain region in space. We have made a step towards integration theory in the last chapter. The multivector $v_1 \wedge \ldots \wedge v_r$ induced by the infinitesimal simplex constructed by the tangent vectors $(v_1, \ldots, v_r)$ at $T_x\mathcal{M}$ is conceived as the oriented capacity of the simplex to carry an extensive property. Then, the application of a form representing the property under consideration to the multivector yields the infinitesimal amount of property in that infinitesimal simplex. Roughly, in order to calculate the total amount of property in a region, one should subdivide the region into small simplexes and add up the amount of the property in the various simplexes. While this prescription is enough in order to follow the continuum mechanics track, a few complications to this naive description should be addressed.

Firstly, it is not clear what a small enough subdivision is because we do not have a metric that will enable one to measure the "size" of the simplexes. Secondly, one has to make precise what is meant by the subdivision and how it is being constructed. An important theorem, the triangulation theorem, asserts that any differentiable manifold may be divided into simplexes. On the other hand, the theory of integration on chains considers formal linear combinations of simplexes as domains of integration thus bypassing the problem. For domains of integration that are orientable manifolds with boundaries the basic tool is localization using a partition of unity (see 2.10 below).

2.2. Simplexes and Chains on Manifolds. Roughly, a simplex in a manifold is the image of a simplex in a vector space under a smooth mapping (see Figure 1).

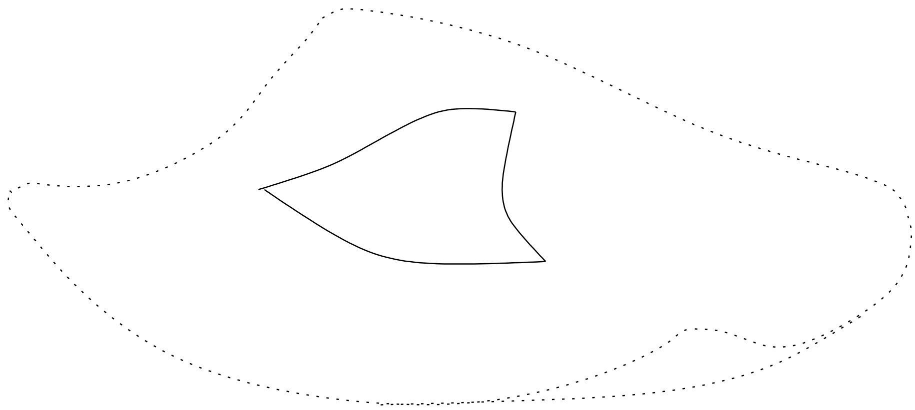

FIGURE 1. The image of a simplex on a manifold

More precisely, we first consider a model simplex, the *standard r-simplex*, as the set

$$\Delta_r = \left\{ (x^1, \dots, x^r) \in \mathbb{R}^m \;\middle|\; 0 \le x^i \le 1,\ \sum_{i=1}^{r} x^i \le 1 \right\}$$

(see Figure 2). The *vertices* of Δ_r are the $r+1$ points $q_0 = (0, \dots, 0)$, $q_1 = (1, 0, \dots, 0)$, $\dots$, $q_r = (0, \dots, 0, 1)$.

The faces of the simplex are numbered such that the i-th *face* is that opposing the vertex q_i. Again, the formal definition views the face as the mapping $k_i^{r-1} \colon \Delta_{r-1} \to \Delta_r$ of the standard $(r-1)$-simplex into the corresponding portion of the boundary, i.e., the 0-th face is given by

$$k_0^{r-1}(x^1, \dots, x^{r-1}) = \left(1 - \sum_{j=1}^{r-1} x^j, x^1, \dots, x^{r-1} \right),$$

and for $i \neq 0$

$$k_i^{r-1}(x^1, \dots, x^{r-1}) = (x^1, \dots, x^{i-1}, 0, x^i, \dots, x^{r-1}).$$

Formally, *singular r-simplex* on a differentiable manifold $\mathcal{M}$ is a smooth map $s \colon U \to \mathcal{M}$, where U is an open neighborhood of Δ_r in $\mathbb{R}^r$. We will often abuse the notation and write $s \colon \Delta_r \to \mathcal{M}$. The images under s of the vertices of the standard simplex are the *vertices*

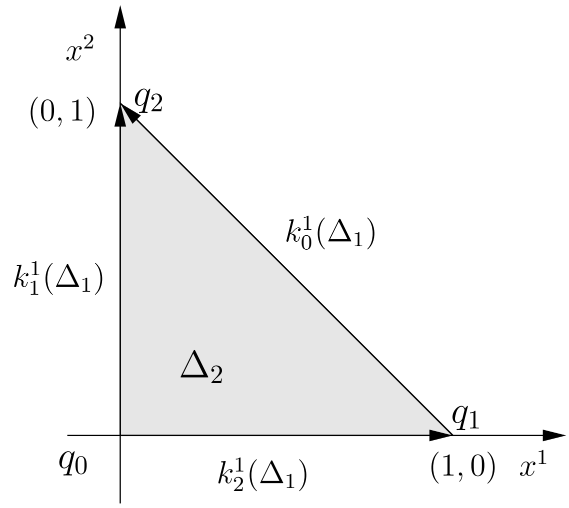

Figure 2. The standard 2-simplex

of the singular simplex. The faces of the simplex are defined as follows. One can extend the mappings k_i^{r-1} to an neighborhood of Δ_{r-1} in $\mathbb{R}^{r-1}$ using the same formula as above and define the i-th *face* of the simplex as the mapping $s \circ k_i^{r-1} \colon V \to \mathcal{M}$ (see Figure 3).

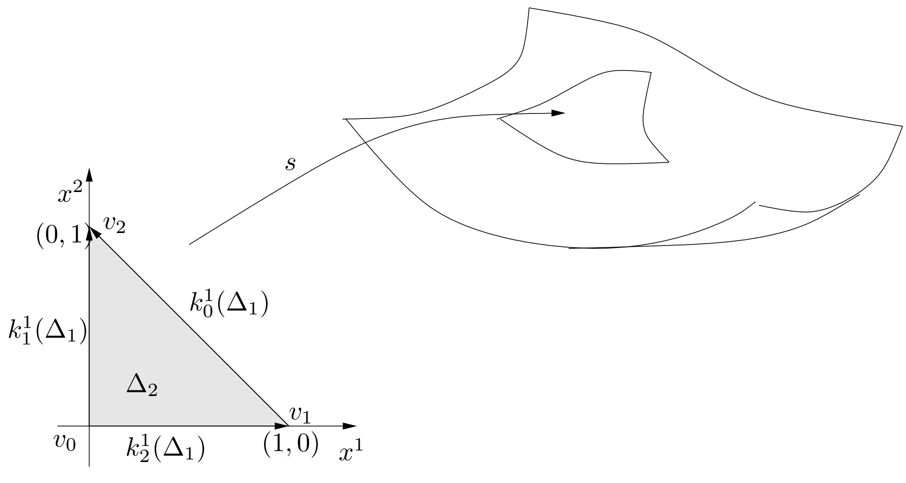

Figure 3. A simplex on a manifold

The standard orientation on $\mathbb{R}^r$ (using the standard basis as an oriented set of vectors) induces an orientation on the standard simplex and the mappings k_i^{r-1} induce orientations on its various faces. Similarly, the mappings s and $s \circ k_i^r$ induce orientations on their the images of the simplex and its faces. (see Figure 2), and s induces an orientation on its image. A formal linear combination $a^p s_p$ of simplexes on a manifold is a *chain*. This induces an infinite dimensional vector space structure on the space of chains on $\mathcal{M}$. The chain $1s$ is defined to be s and $(-1)s$ is interpreted as that simplex corresponding to s but with the reverse orientation. Thus, $s + (-1)s = 0$ has a simple interpretation. Naturally, if in a chain is given as $c = \sum_p s_p$ for r-simplexes s_p whose images do not intersect, then it is interpreted as being associated with the union of their images $\bigcup_p s_p(\Delta_p)$.

The *boundary* of an r-simplex s is defined as the $(r-1)$-chain

$$\partial s = \sum_{i=0}^{r-1} (-1)^i s \circ k_i^{r-1},$$

where we do not use the summation convention for the various faces. The boundary is extended to chains by linearity, i.e.,

$$\partial(a^p s_p) = \sum_{i=0}^{r} (-1)^i a^p s_p \circ k_i^r.$$

2.3. Linear and affine simplexes and chains. We now consider the situation where the ambient manifold $\mathcal{M}$ is replaced by a vector space $\mathbf{W}$ and the simplex mapping $s\colon \Delta_r \to \mathbf{W}$ is either a restriction of a linear mapping or an affine mapping $\mathbb{R}^r \to \mathbf{W}$. In the linear case, the simplex and its orientation are uniquely determined by the images $v_1, \dots, v_r$ of the standard basis elements $\mathbf{e}_1, \dots, \mathbf{e}_r$. In the affine case the simplex and its orientation is uniquely determined by the images $p_0, p_1, \dots, p_r$ of the vertices of the standard Δ_r. Alternatively, an affine simplex is determined by p_0 and the vectors $v_1, \dots, v_r$ where $v_i = p_i - p_0$ (see Figure 4). Clearly, $v_1, \dots, v_r$ are the images of the standard basis in $\mathbb{R}^r$ under the derivative of the simplex mapping—the linear part of the affine mapping. In the sequel we refer to these vectors as the *defining vectors* for the simplex. Clearly, the elements $\mathbf{e}_1, \dots, \mathbf{e}_r$ are the defining vectors for the standard simplex.

An affine r-simplex induces an r-multivector $\mathfrak{v}$ by

$$\mathfrak{v} = \frac{1}{r!} v_1 \wedge \cdots \wedge v_r,$$

where $v_1, \dots, v_r$ are the defining vectors. The $1/r!$ factor appears in the definition as it is the volume of the standard simplex.

An *affine chain* is a formal linear combination of affine simplexes. A linear r-chain $a^p s_p$ induces an r-multivector $\mathfrak{v} = a^p \mathfrak{v}_p$, where $\mathfrak{v}_p$ is the

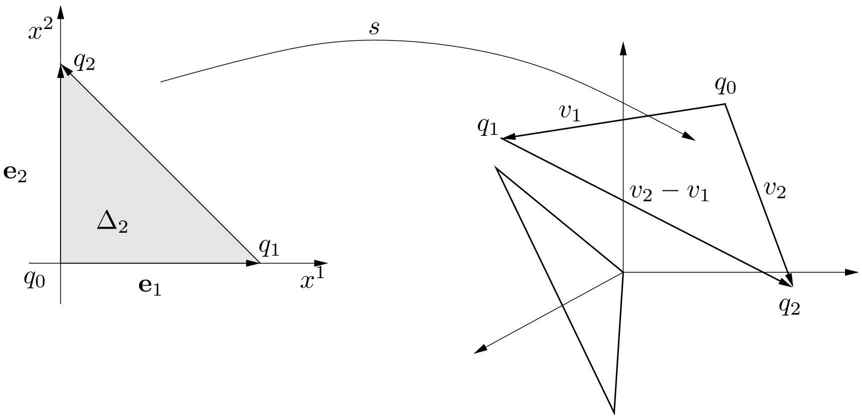

FIGURE 4. Linear and affine simplexes

multivector associated with the r-simplex s_p. Clearly, the multivector induced by the simplex can also be written as

$$\mathfrak{v} = v_1 \wedge (v_2 - v_1) \wedge \cdots \wedge (v_r - v_{r-1}).$$

Next, we give some useful examples of affine chains.

2.3.1. *The boundary of the standard simplex.* The defining vectors for the i-th face are $(k_i^{r-1})_*(\mathbf{e}_1), \dots, (k_i^{r-1})_*(\mathbf{e}_{r-1})$. By the definition of the k_i^{r-1}-mappings,

$$(k_i^{r-1})_*(\mathbf{e}_1), \dots, (k_i^{r-1})_*(\mathbf{e}_{r-1}) = \begin{cases} \mathbf{e}_2 - \mathbf{e}_1, \dots, \mathbf{e}_r - \mathbf{e}_1, & \text{for } i = 0, \\ \mathbf{e}_1, \dots, \widehat{\mathbf{e}}_i, \dots, \mathbf{e}_r, & \text{for } i > 0. \end{cases}$$

Hence, taking into account the orientation $(-1)^i$, the $(r-1)$-multivector $\mathfrak{v}_i$ defining the i-th face of the standard simples is given by

$$\mathfrak{v}_i = \begin{cases} \dfrac{1}{(r-1)!}(\mathbf{e}_2 - \mathbf{e}_1) \wedge \cdots \wedge (\mathbf{e}_r - \mathbf{e}_1) & \text{for } i = 0, \\ \dfrac{(-1)^i}{(r-1)!}\, \mathbf{e}_1 \wedge \cdots \wedge \widehat{\mathbf{e}}_i \wedge \cdots \wedge \mathbf{e}_r & \text{for } i > 0. \end{cases}$$

When we expand the the expression for $\mathfrak{v}_0$ and drop the vanishing exterior products where $\mathbf{e}_1$ appears more then once, we obtain

$$\mathfrak{v}_0 = \frac{1}{(r-1)!}\big[\mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_r - \mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \cdots \wedge \mathbf{e}_r \\ - \mathbf{e}_2 \wedge \mathbf{e}_1 \wedge \mathbf{e}_4 \cdots \wedge \mathbf{e}_r - \cdots - \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_{r-1} \wedge \mathbf{e}_1\big].$$

Now, skew-symmetry implies that these equation may be rewritten as

$$\mathfrak{v}_0 = -\frac{1}{(r-1)!}\sum_{i=1}^{r}(-1)^i\, \mathbf{e}_1 \wedge \cdots \wedge \widehat{\mathbf{e}}_i \wedge \ldots \wedge \mathbf{e}_r.$$

Comparing this last expression with those pertaining to $\mathfrak{v}_i$, for $i > 0$ we conclude that

$$\sum_{i=0}^{r} \mathfrak{v}_i = 0,$$

i.e., the multivector associated with the boundary of a standard simplex vanishes.

2.3.2. *The boundary of an affine simplex.* Let s be an affine r-simplex with defining vectors $v_1, \dots, v_r$ and consider its boundary ∂s. As $v_i = s_*(\mathbf{e}_i)$, $i = 1, \dots, r$ and the defining vectors for the j-th face, $j = 1, \dots, r-1$, are

$$(s \circ k_j^{r-1})_*(\mathbf{e}_1), \dots, (s \circ k_j^{r-1})_*(\mathbf{e}_{r-1}) = s_* \circ (k_j^{r-1})_*(\mathbf{e}_1), \dots, s_* \circ (k_j^{r-1})_*(\mathbf{e}_{r-1}),$$

the linearity of s_* and the previous example imply that the defining vectors of the j-th face are

$$v_2 - v_1, \dots, v_r - v_1, \quad \text{for } j = 0, \text{ and } \quad v_1, \dots, \widehat{v}_j, \dots, v_r, \text{ for } j > 0.$$

Thus, the $(r-1)$-multivectors associated with the faces are

$$\mathfrak{v}_j = \begin{cases} \dfrac{1}{(r-1)!}\,(v_2 - v_1) \wedge \cdots \wedge (v_r - v_1) & \text{for } j = 0 \\ \dfrac{(-1)^j}{(r-1)!}\, v_1 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_r & \text{for } j > 0. \end{cases}$$

Expanding the expression for $\mathfrak{v}_0$ as above we obtain again

$$\mathfrak{v}_0 = -\frac{1}{(r-1)!} \sum_{i=1}^{r} (-1)^i\, v_1 \wedge \cdots \wedge \widehat{v}_i \wedge \dots \wedge v_r,$$

and we immediately conclude that the multivector associated with the boundary of an affine simplex vanishes. Furthermore, from its definition, it follows that the multivector associated with the boundary of any affine chain vanishes.

2.3.3. *Prisms.* The standard r-prism is defined as

$$\Pi_r = [0,1] \times \Delta_{r-1} \subset \mathbb{R}^r.$$

The volume of the standard prism is $1/(r-1)!$ and since the volume of Δ_r is $1/r!$ we want to regard the prism as a chain made of r simplexes.

A chain structure (actually a complex) for the standard prism may be constructed as follows (see [**11**, pp. 365–366] for details). Let p_i, $i = 1, \dots, r$ be the points defining the standard simplex, i.e., p_0 is the origin and p_i, for $i > 0$ is on the the i-th axis. Then, denote the vertexes of the prism as follows: $q_i = (0, p_i)$ and $q_i' = (1, p_i)$. Now, if the simplex

τ_i is defined by its vertices $\tau_i = q_0 \cdots q_i q_i' \cdots q_r'$, then, $\Pi_r = \sum_i (-1)^i \tau_i$ (see Figure 5).

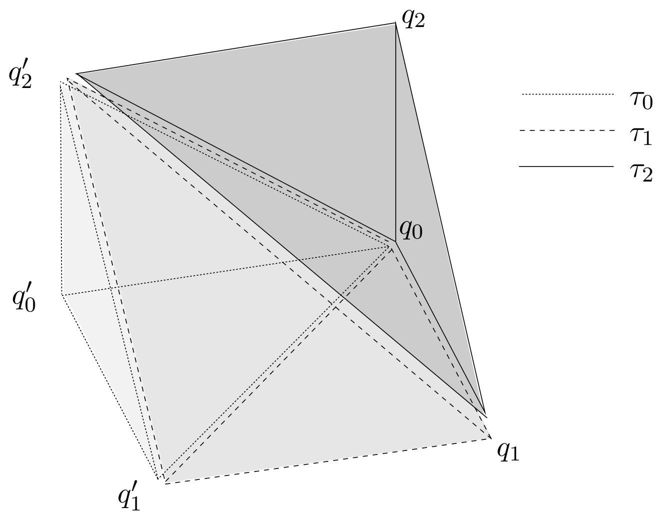

FIGURE 5. A prism complex

An *affine prism* is an affine mapping applied to the standard prism. Clearly, this construction holds for any affine prism.

2.4. Simplicial complexes and triangulations. Chains are rather general geometrical objects. In fact, every differentiable manifold may be regarded as a particular type of chain—a simplicial complex.

A simplicial complex in an affine space can be regarded roughly as a collection of simplexes that fit together. Specifically, *simplicial complex* K is a finite set of affine simplexes having the following properties. Each face of a simplex s in K is itself a simplex of K and whenever two simplexes intersect they do so on a common face.

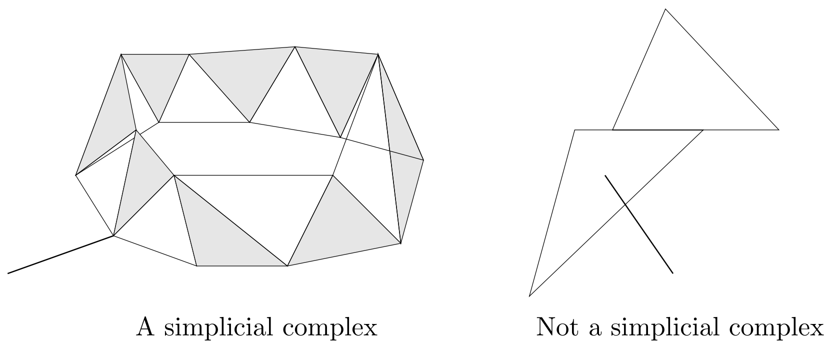

FIGURE 6

A *triangulation* of an m-dimensional manifold $\mathcal{M}$ consists of a simplicial complex K and a homeomorphism $\iota\colon K \to \mathcal{M}$ having the following property. For each m-simplex s of K, there is a chart (U, ϕ), defined in an open neighborhood U of s and $\phi \circ \iota$ is an affine in s (see Figure 7).

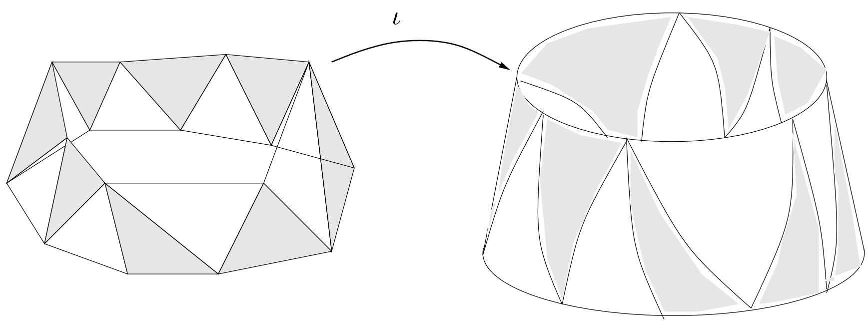

FIGURE 7. Triangulation of a manifold

The triangulation theorem due to Cairns (see Whitney [**11**, p. 124]) states that every differentiable manifold has a triangulation.

2.5. Integration of forms in $\mathbb{R}^m$. Let θ be an m-form defined on an open set $U \in \mathbb{R}^m$. Then, θ may be written uniquely as the product of a real valued function u on U and the standard m-form $dx^1\wedge\ldots\wedge dx^m$

$$\theta = u\, dx^1\wedge\ldots\wedge dx^m.$$

The integral of θ over a polyhedron $K \subset U$ is defined as

$$\int_K \theta = \int_K u\ dx^m.$$

It is noted that the sign of the function u is determined by the choice of the natural basis in $\mathbb{R}^n$ and the orientation it induces.

2.6. The transformation of variables formula. A stanrdard result of multivariable calculus in $\mathbb{R}^m$ (see Apostol [**2**, p. 421] or Sternberg [**9**, p. 381]) is the transformation of variables formula for the Riemann integral. It is concerned with a diffeomorphism $\psi\colon U \to \psi(U)$ of a bounded open set $U \in \mathbb{R}^m$. Using the notation $J = \det(D\psi)$, it asserts that for a continuous integrable function u defined on $\psi(U)$,

$$\int_{\psi(U)} u\ dy^m = \int_U |J| u \circ \psi\ dx^m.$$

The formula aslo holds if the domains U, $\psi(U)$ are replaced by polyhedrons $K \subset U$ and $\psi(K) \subset \psi(U)$.

The formula for the transformation of variables has a simple representation using the definition of the integral on an m-form in $\mathbb{R}^m$. Let $\psi\colon U \to \psi(U) \subset \mathbb{R}^m$ be a diffeomorphism. Then, for an m-form $\theta = u\, dy^1 \wedge \cdots \wedge dy^m$ in a neighborhood of $\psi(K)$, we showed in Section 1.2 that

$$\psi^*(\theta) = J(u \circ \psi)\, dx^1 \wedge \cdots \wedge dx^m.$$

Hence,

$$\begin{aligned} \int_{\psi(K)} \theta &= \int_{\psi(K)} u\; dy^m \\ &= \int_K |J| u \circ \psi \; dx^m \\ &= \pm \int_K J(u \circ \psi)\; dx^m \\ &= \pm \int_K \psi^*(\theta), \end{aligned}$$

where the sign of the integral is determined by the sign of J. In other words, if J is positive, then,

$$\int_{\psi(K)} \theta = \int_K \psi^*(\theta).$$

We will refer below (see Section 2.9) to a diffeomorphism ψ with positive Jacobian determinant as orientation preserving.

Intuitively, if $\psi^*(\theta)$ is interpreted as the form representing the density of a certain extensive property and if a small affine simplex s is represented by the multivector $\mathfrak{v} = v_1 \wedge \cdots \wedge v_m / m!$, then,

$$\int_s \psi^*(\theta)$$

may be approximated by $\psi^*(\theta)(\mathfrak{v}) = \theta\big(\psi_*(\mathfrak{v})\big)$.

For a general polyhedron K, we can use triangulation into a complex of "small" simplexes, approximation and additivity in order to arrive at the transformation formula (see Whitney [**11**, p. 87]). Thus, the transformation rule for forms implies the conservation of the property p under the diffeomorphism ψ.

2.7. Integration on simplexes and chains. Let s be an r-simplex on an m-dimensional manifold $\mathcal{M}$ and let ω be a continuous r-form defined in a neighborhood of $D = s(\Delta_r)$. The integral of ω over D is defined by

$$\int_s \omega = \int_D \omega = \int_{\Delta_r} s^*(\omega).$$

Here, $s^*(\omega)$ is the pullback of ω to a neighborhood of Δ_r using the simplex mapping s. Let $\psi\colon \Delta_r \to \Delta_r$ be a mapping that may be extended to a diffeomorphism in a neighborhood of Δ_r, and set $s' = s \circ \psi$. Then, using the transformation of variables formula we have

$$\begin{aligned}\int_{s'(\Delta_r)} \omega &= \int_{\Delta_r} s'^*(\omega) \\ &= \int_{\Delta_r} (s \circ \psi)^*(\omega) \\ &= \int_{\Delta_r} \psi^*\big(s^*(\omega)\big) \\ &= \int_{\Delta_r = \psi(\Delta_r)} s^*(\omega) \\ &= \int_{s(\Delta_r)} \omega,\end{aligned}$$

so the result is independent of the change of variables.

Let ω be an r-form on the manifold $\mathcal{M}$ having a compact support. Then, if c is an r-chain on $\mathcal{M}$, the integral of ω on $c = a^i s_i$ is defined by linearity as

$$\int_c \omega = \sum_i a^i \int_{D_i} \omega = \sum_i a^i \int_{s_i} \omega, \qquad D_i = s_i(\Delta_r).$$

We note that in particular this may be applied in the case where the manifold $\mathcal{M}$ is a vector space and the simplexes that make up the chain are affine.

2.8. The mean value theorem for integrals on simplexes. We recall (e.g., Apostol [**2**, p. 400]) that the mean value theorem of multi-variable calculus asserts that if a function u is continuous and bounded in a connected subset $D \subset \mathbb{R}^m$, then, there is a point $x_0 \in D$ such that

$$\int_D u\, dx^m = u(x_0) \int_D dx^m.$$

The mean value theorem for integrals has a particularly simple simple formulation for integration over a simplex s:

Let ω be an r-form on an r-simplex $s\colon \Delta_r \to \mathcal{M}$, then, there is a point $\mathbf{q} \in \Delta_r$, such that for $q = s(\mathbf{q})$,

$$\int_s \omega = \omega(q)\big(s_{*\mathbf{q}}(\mathfrak{e})\big),$$

where $\mathfrak{e}$ is the standard r-multivector in $\mathbb{R}^r$, i.e.,

$$\mathfrak{e} = \frac{1}{r!}\,\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_r$$

and $\mathbf{e}_i$ are the standard basis elements.

Let the r-form $s^*(\omega)$ in $\mathbb{R}^r$ be represented as

$$s^*(\omega) = u\,\mathbf{e}^1 \wedge \ldots \wedge \mathbf{e}^r$$

where u is a real valued function defined in a neighborhood of Δ_r. Then, the mean value theorem for integration in $\mathbb{R}^r$ asserts the existence of a point $\mathbf{q} \in \Delta_r$ such that

$$\begin{aligned}\int_s \omega &= \int_{\Delta_r} u\, dx^r \\ &= u(\mathbf{q}) \int_{\Delta_r} dx^r \\ &= \frac{u(\mathbf{q})}{r!}.\end{aligned}$$

However, as $\mathbf{e}^1 \wedge \ldots \wedge \mathbf{e}^r(\mathfrak{e}) = 1/r!$,

$$\begin{aligned}\frac{u(\mathbf{q})}{r!} &= u(\mathbf{q})\,\mathbf{e}^1 \wedge \ldots \wedge \mathbf{e}^r(\mathfrak{e}) \\ &= s^*_{\mathbf{q}}(\omega)\,(\mathfrak{e}) \\ &= \omega(q)\big(s_{*\mathbf{q}}(\mathfrak{e})\big).\end{aligned}$$

As $\mathfrak{e} = \mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_r/r!$, $s_{*\mathbf{q}}(\mathfrak{e}) = s_{*\mathbf{q}}(\mathbf{e}_1) \wedge \cdots \wedge s_{*\mathbf{q}}(\mathbf{e}_r)/r!$, the simplex generated by the images of the base vectors (see Figure 8).

This version of the mean value theorem supports the intuitive approach to integration. We triangulate the manifold into a fine complex and for each complex we simply evaluate the value of the form on the

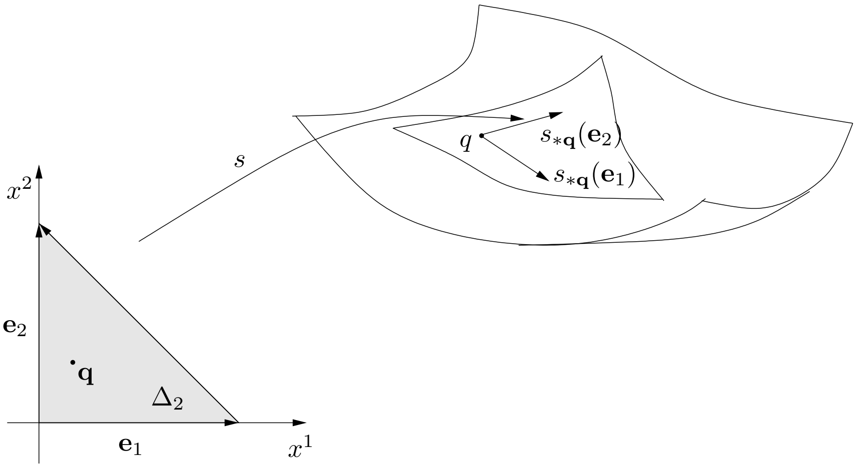

FIGURE 8. Notation for the mean value theorem

image of the defining vectors. The division is assumed to be fine enough so the value of s_* and ω at $\mathbf{q}$ are close to the respective values at the origin of $\mathbb{R}^r$.

2.9. Orientation. Let $\mathbf{W}$ be an m-dimensional vector space and consider an element $\mathfrak{v}_0 \neq 0$ in the 1-dimensional space of m-multivectors $\bigwedge^m \mathbf{W}$. Clearly, $\mathfrak{v}_0$ may serve a a basis and any other multivector $\mathfrak{v}$ may be written as $\mathfrak{v} = a\mathfrak{v}_0$ for a real a. If we ignore the zero element, this separates $\bigwedge^m \mathbf{W}$ into two separate components, namely, those multivectors for which the coordinate a is positive and those for which it is negative. We will refer to them as the *positively oriented* and *negatively oriented* multivectors relative to $\mathfrak{v}_0$, respectively. Clearly, any positively oriented multivector $\mathfrak{v}'_0$ will induce the same separation. Thus, we refer to such a choice of a base vector and the resulting separation as an *orientation* of $\mathbf{W}$. If one chooses a multivector $\mathfrak{v}'_0$, $\mathfrak{v}'_0 = a\mathfrak{v}_0$ with $a < 0$ as a basis it will reverse the separation. Hence, a vector space has two distinct orientations.

The orientation of $\mathbf{W}$ may be regarded as a separation of the nonzero m-simplexes in $\mathbf{W}$ into two separate collections. Thus, the simplex formed by the vectors $(v_1, \ldots, v_m)$ is positively oriented if $v_1 \wedge \cdots \wedge v_m$ is. Assume that the simplex generated by $(v_1, \ldots, v_m)$ is positively oriented and consider another simplex $(v_{1'}, \ldots, v_{m'})$. Since the vectors in each set are linearly independent, there is a nonsingular matrix $A^i_{i'}$

such that $v_{i'} = A^i_{i'} v_i$. Thus,

$$\begin{aligned} v_{1'} \wedge \cdots v_{m'} &= (A^{i_1}_{1'} v_{i_1}) \wedge \cdots \wedge (A^{i_m}_{m'} v_{i_m}) \\ &= A^{i_1}_{1'} \cdots A^{i_m}_{m'} v_{i_1} \wedge \cdots \wedge v_{i_m} \\ &= \varepsilon_{i_1 \dots i_m} A^{i_1}_{1'} \cdots A^{i_m}_{m'} v_1 \wedge \cdots \wedge v_m \\ &= \det(A^i_{i'})\, v_1 \wedge \cdots \wedge v_m. \end{aligned}$$

We conclude that the simplex generated by $(v_{1'}, \dots, v_{m'})$ is positive if and only if the determinant of the matrix $A^i_{i'}$ is positive. Clearly, the standard basis of $\mathbb{R}^m$ induce a natural orientation on it.

Alternatively, an orientation of $\mathbf{W}$ may be specified by an m-form θ. An m-multivector $\mathfrak{v}$ will be positively oriented if $\theta(\mathfrak{v}) > 0$. Clearly, this is equivalent to the other ways of specification of an orientation.

An orientation on a vector space is only a matter of choice, and hence the foregoing applies to the tangent space $T_x\mathcal{M}$ for any x in a manifold $\mathcal{M}$. However, an orientation on a manifold is the consistent assignments of orientations to the tangent spaces at the various points. An orientation is such a global sense does not necessarily exist on a general manifold. An orientation on a manifold is important when one wishes to fill the manifold with a certain external property that has a definite sign, either positive or negative. The amount of the property in a simplex may be calculated using the mean value theorem by applying the form ρ representing the density of the property to a multivector approximating a simplex on the manifold. However, since the form is alternating, the sign of the result depends on the orientation of simplex as reflected by the sign of the multivector. As we want to add up the amount of the property contained in distinct simplexes finite distant apart in a consistent manner we have to have a method for prescribing a "uniform" global orientation. Only this way the addition of the amount of property in two distinct simplexes is meaningful (see Figure 9).

From the discussion on orientation of vector spaces, it is natural to define an orientation on a manifold, if the manifold has one, as a smooth nowhere vanishing field of m-multivector. Alternatively, an orientation on a manifold is a smooth nowhere vanishing m-differential form. It is quite clear intuitively that in the case where the manifold $\mathcal{M}$ is not connected, the question whether the manifold is orientable or not may be applied to each connected component only. If we can define a nowhere vanishing multivector field on any connected component we can use these multivector fields to construct a nowhere vanishing field over the whole manifold.

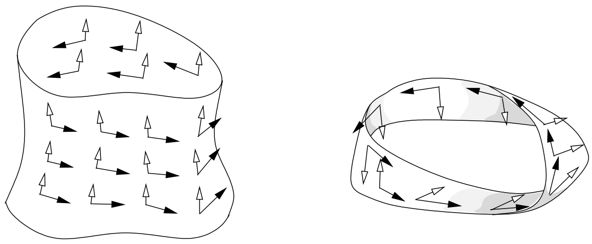

FIGURE 9. Orientation on a manifold

We show now how the notion of orientation is related to the transformation of variables formula. Let $\mathfrak{v}$ be a nowhere vanishing m-multivector field on $\mathcal{M}$ and let $(x^1, \dots, x^m)$, $(y^1, \dots, y^m)$ be two intersecting coordinate systems. Then, as in Section 1.2, for the two local representations

$$\mathfrak{v} = \mathfrak{v}_{1\dots m}\, dx^1 \wedge \cdot \wedge dx^m = \mathfrak{v}_{1'\dots m'}\, dy^{1'} \wedge \cdot \wedge dy^{m'},$$

we have,

$$\mathfrak{v}^{1\dots m} = \det\left(\frac{\partial y^{j'}}{\partial x^i}\right)\mathfrak{v}^{1'\dots m'}.$$

Clearly, the functions $\mathfrak{v}_{1\dots m}$ and $\mathfrak{v}_{1'\dots m'}$ are nowhere vanishing. The sign of each representing function may be inverted by inverting the sign of any coordinate function or by rearranging them. If $\mathfrak{v}$ defines the orientation of $\mathcal{M}$, one may always choose coordinate systems, *positive coordinates*, for which the representing functions are positive. As in the equation above, if both representatives are positive, it follows from the the transformation rule that for any two positive coordinate systems the Jacobian determinant is positive. Hence, for an orientable manifold there is always an atlas such that for any two charts $(x^1, \dots, x^m)$, $(y^1, \dots, y^m)$,

$$\det\left(\frac{\partial y^{j'}}{\partial x^i}\right) > 0.$$

Conversely, assume that for a manifold $\mathcal{M}$ there is an atlas such that the Jacobian determinant for the coordinate transformations is always positive. We will construct a nowhere vanishing m-multivector field on $\mathcal{M}$. The construction uses a partition of unity. Roughly, the idea is that if one has a property that is invariant under coordinate transformations, the sign of the Jacobian determinant in this case, one can use a partition of unity in order to construct a global form of the property using local representation. Specifically, this is done as follows.

Let $\{U_\alpha\}$ be a covering of $\mathcal{M}$ by domains of charts in such an atlas. Then, in each chart α one can define the local form $\omega_\alpha = dx^1 \wedge \cdots \wedge dx^m$, where $(x^1, \dots, x^m)$ are the local coordinates in this chart. In order to combine the various ω_α into a nowhere vanishing form on $\mathcal{M}$ we use the real valued functions $\{u_\alpha\}$ that make up a partition of unity subordinate to this atlas. In other words, the support of u_α is included in U_α, $0 \leqslant u_\alpha \leqslant 1$, and $\sum_\alpha u_\alpha = 1$. Thus, we may define the m-multivector field

$$\omega = \sum_\alpha u_\alpha \omega_\alpha,$$

that is clearly nowhere vanishing. (It is noted that while $u_\alpha \omega_\alpha$ is actually defined on U_α, it may be smoothly extended to $\mathcal{M}$. Strictly speaking, these extensions are added up.) We conclude that orientability is equivalent to the existence of an atlas for which the Jacobian determinants of the coordinate transformations are positive.

The foregoing analysis relates orientability to the transformation of variables formula of Section 2.6. Using an atlas with positive Jacobian determinants we may omit the absolute value function and the multiplicity of signs.

2.10. Integration on oriented manifolds. Integration theory on chains is rather general. For example, it allows for a covering of a manifold by a chain, where the various simplexes do not have the same orientation. It allows also integration on regions that are only piecewise smooth. Furthermore, it was not required earlier that the simplex mappings are diffeomorphisms and it is possible that the image of a singular simplex has a lower dimension than the standard simplex although this was not shown in the illustrations. In other words, integration on chains allows one to integrate r-forms on domains whose dimensions are smaller than r although the mean value theorem implies that the result will vanish.

We now develop the theory further and specialize it to the case where the domain of integration is an m-dimensional oriented manifold that may have a boundary. As expected the integrand is an m-form. Integration on manifolds can be developed on the basis of the theory of integration on chains using triangulations (see Whitney [**11**, p. 93]). However, following the classical presentations (e.g., [**9, 10**]), we use the approach that utilizes a partition of unity.

Simplexes enable the extension of integration of forms on $\mathbb{R}^n$ to chains. In integration theory of forms on oriented manifolds, this role is assumed by charts. Assume that (ψ, U) is a positively oriented chart in an oriented manifold $\mathcal{M}$. Then, the integral of an m-form ω over U

is defined by

$$\int_U \omega = \int_{\psi(U)} (\psi^{-1})^*(\omega),$$

where ψ^{-1}, the analog of the simplex mapping s, is used to pull the form ω to $\psi(U) \subset \mathbb{R}^n$. It follows from the transformation of variables formula 2.6 that any other chart defined on U will yield the same result. For the integration of a form defined on the manifold $\mathcal{M}$, a partition of unity is used in order to localize the form to the domains of charts. A more detailed description of the construction that is used in the proof of Stokes' theorem for integrals on manifolds with boundaries is outlined below.

We consider for an orientable m-dimensional manifold $\mathcal{M}$ equipped with a particular orientation, an m-dimensional submanifold $\mathcal{R} \subset \mathcal{M}$ that may have a boundary. The orientation of $\mathcal{M}$ induces a specific orientation on the boundary $\partial\mathcal{R}$ as follows. We first note that a a tangent vector to $\mathcal{R}$ at a point $x \in \partial\mathcal{R}$ may be inwards pointing, outwards pointing, or tangent to the boundary. These properties are invariant and do not depend on the chart. For example, if one chooses a 1-form ϕ that annihilates the tangent vectors to the boundary, the sign of $\phi(v)$ will determine whether v is outwards pointing. Thus, we may set the collection of vectors $v_1, \dots, v_{m-1}$ to be positively oriented on $\partial\mathcal{R}$ if the collection $v, v_1, \dots, v_{m-1}$ has positive orientation on $\mathcal{M}$.

A simplex s is *regular* if it extends to a diffeomorphism in a neighborhood of the standard simplex Δ_m and we will often refer to the extension as s also. Using the standard orientation on $\mathbb{R}^m$, one considers *oriented regular m-simplex*, i.e., a simplex for which the orientation it induces on its image conforms to that chosen on $\mathcal{M}$. In order to integrate a form ω having a compact support in $\mathcal{M}$ over $\mathcal{R}$, we consider a partition of unity subordinate to a special cover $U_1, \dots, U_k$ as follows. Each open set U in the cover, is contained in the interior of the image of an oriented regular simplex s (see Figure 10). The simplexes should serve to induce an atlas for the manifold with boundary $\mathcal{R}$ as follows. If s is such that $s(\Delta_m) \subset \mathrm{Int}\,(\mathcal{R})$, then the corresponding set U is contained in in interior of $s(\Delta_m)$. Otherwise, $s(\Delta_m) \subset \mathcal{R}$ such that $s(\Delta_m) \cap \partial\mathcal{R} = s \circ k_m^{m-1}$. In other words, the subset of the standard simplex whose image intersects $\partial\mathcal{R}$ is the m-th face. In this case U is chosen such that it is compatible with the submanifold with boundary structure of $\mathcal{R}$. So, there is an open set $U_0 \subset \mathbb{R}^m$ which is a neighborhood of a point on the m-th face of Δ_m such that U_0 intersects the boundary of Δ_m only on a subset of the m-th face and $U = s(U_0) \cap \mathcal{R} \subset s(\Delta_m)$. (In the last condition we used s to denote the extension of the simplex to a neighborhood of Δ_m.)

Assuming that ω has a compact support, one can cover $\mathcal{R} \cup \mathrm{Supp}(\omega)$ by a finite number of such open sets $U_1, \dots, U_k$ such that for each U_i there is a

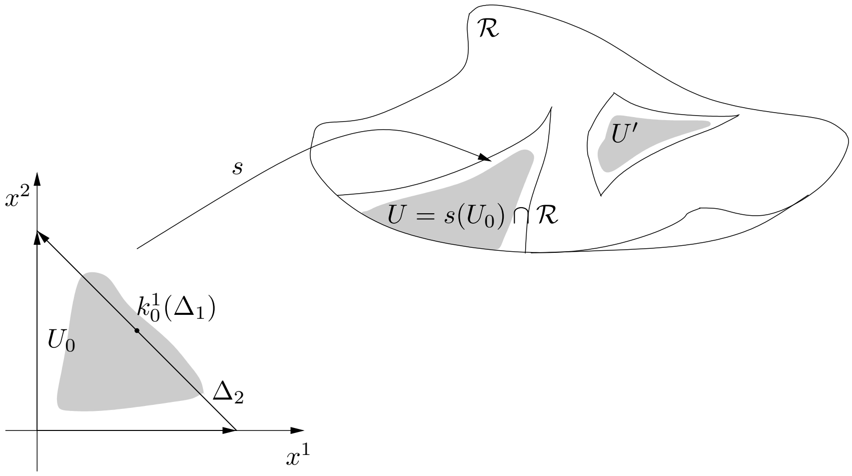

FIGURE 10. Simplexes and partition of unity for integration

corresponding oriented regular simplex s_i as above. Setting $V = \mathcal{M} - \mathcal{R} \cap \mathrm{Supp}\,(\omega)$, let $u, u_1, \ldots, u_k$ be a partition of unity subordinate to the cover $V, U_1, \ldots, U_k$ of $\mathcal{M}$. Finally the integral of ω over $\mathcal{R}$ is defined as

$$\int_{\mathcal{R}} \omega = \sum_{i=1}^{k} \int_{s_i} u_i \omega.$$

One can show that the result is independent of the cover and partition of unity.

2.11. Restriction of forms and integration on submanifolds. Among other differences between the theory of integration on chains and integration on oriented manifolds, it is noted that for an m-dimensional manifold $\mathcal{M}$, the former considers integration on any r-chain, $r \leq m$, while the latter considers integration of m-forms on m-dimensional submanifolds with boundary of $\mathcal{M}$. This difference is not substantial and is settled as follows.

Let $\mathcal{N}$ be an r-dimensional submanifold with boundary of $\mathcal{M}$. Then, we have the natural inclusion mapping

$$\iota \colon \mathcal{N} \to \mathcal{M}$$

with $\iota(x) = x$. The inclusion induces the tangent mapping

$$\iota_* \colon T\mathcal{N} \to T\mathcal{M},$$

and the conjugate mapping of q-forms, $q \leq r$,

$$\iota^* \colon \Big(\bigwedge^{q} T^*\mathcal{M}\Big)\Big|_{\mathcal{N}} \to \bigwedge^{q} T^*\mathcal{N}$$

$\iota^*(\omega)(v_1, \dots, v_r) = \omega(\iota_*(v_1), \dots, \iota_*(v_r))$. (Note that we may write $\iota^* \bigwedge^q T^*\mathcal{M}$ for $\left(\bigwedge^q T^*\mathcal{M}\right)\Big|_{\mathcal{N}}$ where here ι^* is the pullback of vector bundles.) We will refer to $\iota^*(\omega)$ as the *restriction* of ω to $\mathcal{N}$.

It follows that a q-form ω on $\mathcal{M}$, $q \leq r$, gives the q-form $\iota^*(\omega)$ on $\mathcal{N}$. In the particular case where $q = r$, the form $\iota^*(\omega)$ may be integrated on $\mathcal{N}$. We conclude that given an r-form ω on $\mathcal{M}$, the integral

$$\int_{\mathcal{N}} \iota^*(\omega)$$

is well defined for every r-dimensional submanifold with boundary $\mathcal{N}$ of $\mathcal{M}$,

3. Smooth Extensive Properties and Fluxes

One of the basic notions of continuum mechanics is that of an extensive property. The term extensive property is used to describe a property that may be assigned to subsets of a given universe. These include for example, the mass of the various parts of a material body, the electrical charge enclosed in a certain region in space, etc. Thus, an extensive property is a real valued set function p. Even in the most general treatments, it is usually assumed that the extensive propety is additive so that for disjoint regions $\mathcal{R}_1$ and $\mathcal{R}_2$,

$$p(\mathcal{R}_1 + \mathcal{R}_2) = p(\mathcal{R}_1) + p(\mathcal{R}_2).$$

With the proper regularity assumptions, additivity means that mathematically an extensive property is a measure either in space on on the material universe. Furthermore, it is assumed in most cases that the extensive property has a smooth density associated with it.

The balance of an extensive property is concerned with the rate of change of the property in the various regions. Of particular importance is the idea of flux of the property through the boundary of regions. The flux measures the rate of change of the property in any region as a result of interaction with other regions. Thus, it is assumed that exchange of the property between the various regions is done on mutual boundaries.

While the flux is a set function on the boundaries of the various regions, Cauchy's postulates and theorem reduce this complicated dependence to to a pointwise dependence on a global field.

3.1. Densities of extensive properties. The basic setting for the basic theory of extensive properties we present in this section is that of a fixed physical space modelled by an m-dimensional differentiable manifold $\mathcal{U}$. (This Greek view of the physical world implies that a point

x in space has an invariant meaning and it clearly contradicts Galilean invariance and relativity. Nevertheless, this restriction is removed if one considers balance principles in the setting of spacetime.) Alternatively, one may wish to interpret $\mathcal{U}$ as the material manifold so a point $x \in \mathcal{U}$ is a material point having an invariant meaning. Since we are going to use integration later on, we will assume that $\mathcal{U}$ is orientable and that a particular orientation was chosen.

Continuous extensive properties have densities associated with them. This implies that the property cannot be concentrated on subsets of dimensions lower than m. Thus, it is assumed that there is an m-form ρ defined on $\mathcal{U}$ that model the density of the property p. Using integration theory presented above, one can now calculate the total amount of the property

$$p(\mathcal{R}) = \int_{\mathcal{R}} \rho$$

in any "region" $\mathcal{R}$ for which the integral is defined.

3.2. Control regions and subbodies. We will refer to "regions" for which integration is defined as *control regions* when we interpret $\mathcal{U}$ as the space manifold and as *subbodies* when we interpret $\mathcal{U}$ as the material manifold. The term "region" will be used when the particular interpretation is immaterial. Thus, we may consider a restricted theory where the regions are compact m-dimensional submanifolds with boundary of $\mathcal{U}$ and a more general theory where the regions of integration are chains. (We used the double quotes because chains for which the real numbers multiplying the various simplexes in the formal linear combinations are different than 1 do not represent actual subsets of $\mathcal{U}$.)

More general integration theories (some of which will be described later) consider even more general "regions". In many cases, such theories start with simpler, restricted class of regions and complement them by adding the limits of some sequences to obtain a larger class.

3.3. The time axis and density rates. Continuing with our naive view of spacetime, we assume that any physical event can be assigned a specific time and will model the time manifold by $\mathbb{R}$. As customary, we will use t to denote the time variable. Our ability to assign a particular pair of time and place to any event implies that we have a particular global *frame* on spacetime. This means that in general, the density ρ of a property p should be time dependent although usually we do not exhibit this explicitly in the notation. Since the value $\rho(t, x) \in \bigwedge^m T^*_x\mathcal{U}$—a vector space, we may differentiate it with respect to the time variable and obtain the m-form

$$\beta = \frac{\partial \rho}{\partial t}$$

on $\mathcal{U}$.

Thus, for a fixed region $\mathcal{R}$

$$\frac{dp(\mathcal{R})}{dt} = \int_{\mathcal{R}} \beta$$

represents the rate of change of the amount of the property p inside $\mathcal{R}$.

3.4. Classical balance laws, flux densities and sources. In the classical setting of continuum mechanics it is assumed that the change of the amount of property within the region $\mathcal{R}$ is a result of two phenomena: the rate at which the property is produced inside $\mathcal{R}$ which increases the amount of p and the rate at which the property leaves $\mathcal{R}$ through its boundaries. This rate at which the property leaves $\mathcal{R}$ through its boundary is referred to as the *flux* of p. The equality between the rate of change of the property and the difference between the production the the property and the flux is the *balance equation* for p. Very roughly, the property p may be thought of as a product produced in a certain country, the region $\mathcal{R}$, with the rate at which the amount of p in the country increases due to production and decreases due to export through the borders. (This of course rules out export through airports inside the country and requires the production, storage and export to be distributed continuously.) Another example that one may think of is the balance of thermal energy due to heat production and heat flux through the boundaries.

As mentioned, the flux of the property is assumed to be distributed continuously on the boundary of $\mathcal{R}$. Hence, whether the admissible regions are compact submanifolds of $\mathcal{U}$ or chains, integration of $(m-1)$-forms on their boundaries is well defined. Thus, it is assumed that for each region $\mathcal{R}$, there is an $(m-1)$-form $\tau_{\mathcal{R}}$ called the *flux density* such that the flux of p is given as

$$\int_{\partial\mathcal{R}} \tau_{\mathcal{R}}.$$

In the sequel when no confusion can occur, we will omit the $\mathcal{R}$ subscript and use only τ.

The production rate of the property inside $\mathcal{R}$ is assumed to be represented by an m-form s, the *source density* which is a global form on $\mathcal{U}$ and independent of $\mathcal{R}$, as

$$\int_{\mathcal{R}} s.$$

Thus, the classical balance law assumes the form

$$\int_{\mathcal{R}} \beta = \int_{\mathcal{R}} s - \int_{\partial\mathcal{R}} \tau_{\mathcal{R}}.$$

In case the source term vanishes, $s = 0$, the property is *conserved* and the balance equation becomes the conservation equation.

For various results we present later, in particular Cauchy's theorem on the existence of flux forms, even a weaker form of the balance principle is sufficient. In the weaker form, the balance principle is regarded as a boundedness or regularity postulate on the fluxes for the various bodies. The boundedness postulate for the fluxes states that there is a positive m-form ς (relative to the given orientation) on $\mathcal{U}$ such that for any region $\mathcal{R}$

$$\left| \int_{\partial\mathcal{R}} \tau_{\mathcal{R}} \right| \leqslant \int_{\mathcal{R}} \varsigma.$$

Clearly, if the various flux densities satisfy a conservation equation, such a bounding form exists and the boundedness postulate is satisfied.

3.5. Flux forms and Cauchy's formula. Only 2 m-forms on $\mathcal{U}$, namely β and s, are required in order to specify the rate of change of the property p and its production in any region. On the other hand, in order to specify the flux for the various regions it seems that one has to specify $\tau_{\mathcal{R}}$ for any region $\mathcal{R}$. In other words, while the rate of change of the property and the production term are specified by functions whose domain is space, the flux term is specified by means of a set-function whose domain is the collection of all regions. It is customary to refer to the set function $\mathcal{R} \mapsto \tau_{\mathcal{R}}$ as a *system of flux densities*. To emphasize the dependence of the flux density on the region under consideration it is noted that for a fixed point $x \in \mathcal{U}$ that is on the boundary of two distinct regions $\mathcal{R}$ and $\mathcal{R}'$ the values fluxes densities $\tau_{\mathcal{R}}(x) \in \bigwedge^{m-1} T_x^*\mathcal{R}$ and $\tau_{\mathcal{R}'}(x) \in \bigwedge^{m-1} T_x^*\mathcal{R}'$ actually belong to different spaces and surely cannot be compared. From the physical point of view this is expected. For example, one expects that the number of fish that meet a particular point on a net would depend on the way the net is situated. See Fig. 11 where on the left the fish move towards the net and on the right the move along it.

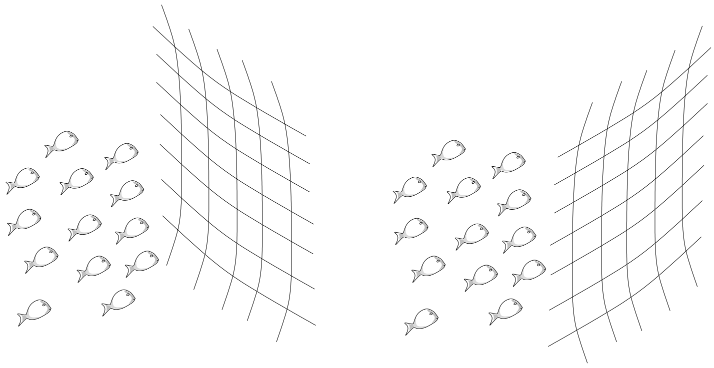

FIGURE 11. Fish, nets, and flux densities

Nevertheless, the integration theories presented above provide a simple means for specifying the flux densities for the various regions. Let J be an $(m-1)$-form on $\mathcal{U}$. Then, if we use chains as regions, J may be integrated on the chains that make up the boundaries of regions. If we use m-dimensional manifolds with boundary as regions, then, the boundaries are $(m-1)$-submanifolds (without boundaries) of $\mathcal{U}$. Hence, for every region $\mathcal{R}$, the inclusion $\iota_{\partial\mathcal{R}} \colon \partial\mathcal{R} \to \mathcal{U}$ induces the restriction $\iota^*_{\partial\mathcal{R}}(J)$ as in Subsection 2.11. Note that we add the subscript $\partial\mathcal{R}$ in order to specify the particular region under consideration. We will refer to such an $(m-1)$-form as a *flux form*.

Thus, a flux form J induce a collection of flux densities for the boundaries of the various subbodies by

$$\tau_{\mathcal{R}} = \iota^*_{\partial\mathcal{R}}(J).$$

The last equation will be referred to as the *Cauchy formula* and we will often omit the $\partial\mathcal{R}$-index if the particular region under consideration is clear from the context. The definition of the restriction of forms implies that for a point $x_0 \in \mathcal{U}$ and any region $\mathcal{R}$ such that $x_0 \in \partial\mathcal{R}$, we have for any collection $v_1, \dots, v_{m-1}$ of vectors in $T_{x_0}\partial\mathcal{R}$,

$$\begin{aligned} \tau_{\mathcal{R}}(v_1, \dots, v_{m-1}) &= J\big(\iota(v_1), \dots, \iota(v_{m-1})\big) \\ &= J(v_1, \dots, v_{m-1}). \end{aligned}$$

Alternatively, for the multivector $v_1 \wedge \cdots \wedge v_{m-1}$ induced by these tangent vectors to the boundary $\partial\mathcal{R}$ at x_0,

$$\tau_{\mathcal{R}}(v_1 \wedge \cdots \wedge v_{m-1}) = J(v_1 \wedge \cdots \wedge v_{m-1}),$$

where we arrive at the last equation using

$$\psi_*(v_1 \wedge \cdots \wedge v_r) = \psi(v_1) \wedge \cdots \wedge \psi(v_r)$$

In crude words this means that J "knows" how to calculate the flux through any infinitesimal $(m-1)$-simplex at x_0 and then in particular through the simplexes tangent to $\partial\mathcal{R}$.

for any linear linear mapping ψ and in particular for the inclusion ι.

It is one of the main results of continuum mechanics, namely Cauchy's theorem (see Section 5), that under rather general assumptions—Cauchy's postulates—every system of flux densities is induced by a unique flux form using Cauchy's formula. In other words, it will be shown that if the system of flux densities satisfies Cauchy's postulates then there is a unique $(m-1)$-flux form J such that the various flux densities are given by Cauchy's formula.

3.6. Extensive properties—local representation. We now present the coordinate description of the objects and relations given above. Let $x^1,\dots,x^m$ be a coordinate system in a neighborhood of a point x_0 and let

$$\left\{\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^m}\right\} \quad \text{and} \quad \{dx^1,\dots,dx^m\}$$

be the induces bases of the tangent and cotangent spaces. Then, as the space $\bigwedge^m T_x^*\mathcal{U}$ is one dimensional, the m-forms ρ and β are represented locally using the scalar functions $\rho_{1\dots m}(x^i)$ and $\beta_{1\dots m}(x^i)$ as

$$\rho(x) = \rho_{1\dots m}(x^i)\, dx^1\wedge\cdots\wedge dx^m$$

and

$$\beta(x) = \beta_{1\dots m}(x^i)\, dx^1\wedge\cdots\wedge dx^m,$$

respectively. We remark that the local representatives are indicated here by the inclusion of the indices only without any additional change in the notation.

The flux density $\tau_{\mathcal{R}}$ should be represented using a coordinate system on the $(m-1)$-dimensional manifold $\partial\mathcal{R}$, say $y^1,\dots,y^{m-1}$. Thus, in such a coordinate, $\tau_{\mathcal{R}}$ is represented on neighborhood of the boundary point y_0 using the scalar function $\tau_{\mathcal{R}1\dots(m-1)}$ in the form

$$\tau_{\mathcal{R}}(y) = \tau_{\mathcal{R}1\dots(m-1)}(y^j)\, dy^1\wedge\cdots\wedge dy^{m-1}.$$

The basic identity

$$dy^{\alpha_1}\wedge\cdots\wedge dy^{\alpha_r}\left(\frac{\partial}{\partial y^{\beta_1}},\dots,\frac{\partial}{\partial y^{\beta_r}}\right) = \varepsilon^{\alpha_1\dots\alpha_r}_{\beta_1\dots\beta_r}$$

implies that

$$\tau_{\mathcal{R}1\dots m-1} = \tau_{\mathcal{R}}\left(\frac{\partial}{\partial y^1},\dots,\frac{\partial}{\partial y^{m-1}}\right).$$

The value at $x_0\in\mathcal{U}$ of the flux form J is an element of $\bigwedge^{m-1} T_x^*\mathcal{U}$— an m-dimensional vector space. We recall that the natural basis of $\bigwedge^{m-1} T_x^*\mathcal{U}$ is

$$\left\{dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_{m-1}}\right\}, \quad i_1 < i_2 < \cdots < i_{m-1}, \quad 1\le i_k\le m.$$

Thus, we write locally

$$J(x) = J_{i_1 \dots i_{m-1}}(x)\, dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{m-1}}, \quad i_1 < i_2 < \dots < i_{m-1}.$$

Henceforth, unless it is explicitly indicated otherwise, whenever a form is written as above, we will omit the indication that the indices are increasing and this will be implied. Thus, the sum will be carried over only over increasing sequences of indices. The expression for the local components of the flux form is obtained by applying it to a typical collection of basis vectors (whose exterior product is a basis element of the space of multivectors) as

$$J_{i_1 \dots i_{m-1}} = J\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_{m-1}}}\right),$$

where we omitted the dependence on x in the notation.

Alternatively, an ordered set of $(m-1)$ indices $i_1, \dots, i_m$ out of $1, \dots, m$ is of the form $1, \dots, \widehat{k}, \dots, m$ where a superimposed hat indicates the omission of an item or a term. Hence, the natural basis of $\bigwedge^{m-1} T^*_x \mathcal{U}$ may be written as

$$\{dx^1 \wedge \dots \wedge \widehat{dx^k} \wedge \dots \wedge dx^m\}, \quad 1 \le k \le m.$$

It follows that the flux form is locally represented by m functions $J_{\hat{k}}$ in a neighborhood of x_0 as

$$J(x) = J_{\hat{k}}\, dx^1 \wedge \dots \wedge \widehat{dx^k} \wedge \dots \wedge dx^m,$$

where summation over the omitted repeated index is implied.

Locally, the inclusion $\partial\mathcal{R} \to \mathcal{U}$ is represented by m functions $x^i = x^i(y^\alpha)$ where Greek indices range up to $m-1$, i.e., $1 \le \alpha \le m-1$. Thus, using a comma to denote partial differentiation we have

$$\iota_*\left(\frac{\partial}{\partial y^\alpha}\right) = x^i_{,\alpha} \frac{\partial}{\partial x^i}$$

and for a vector $v \in T_{x_0}\partial\mathcal{R}$ represented locally by $v = v^\alpha\, \partial/\partial y^\alpha$ we have $\iota_*(v) = x^i_{,\alpha} v^\alpha\, \partial/\partial x^i$ which we may write with some abuse of notation as $v^i = x^i_{,\alpha} v^\alpha$.

The evaluation $\tau(v_1,\dots,v_{m-1})$ is represented as

$$\begin{aligned}
&\tau(v_1,\dots,v_{m-1})\\
&\quad= \tau_{1\dots m-1}\, dy^1\wedge\cdots\wedge dy^{m-1}\left(v_1^{\alpha_1}\frac{\partial}{\partial y^{\alpha_1}},\dots,v_{m-1}^{\alpha_{m-1}}\frac{\partial}{\partial y^{\alpha_{m-1}}}\right)\\
&\quad= \tau_{1\dots m-1} v_1^{\alpha_1}\cdots v_{m-1}^{\alpha_{m-1}}\, dy^1\wedge\cdots\wedge dy^{m-1}\left(\frac{\partial}{\partial y^{\alpha_1}},\dots,\frac{\partial}{\partial y^{\alpha_{m-1}}}\right)\\
&\quad= \tau_{1\dots m-1} v_1^{\alpha_1}\cdots v_{m-1}^{\alpha_{m-1}}\varepsilon_{\alpha_1\dots\alpha_{m-1}}\\
&\quad= \tau_{1\dots m-1}\det(v_{\beta}{}^{\alpha}),
\end{aligned}$$

where we used the antisymmetry to arrive at the 4-th line and the definition of the determinant in terms of the Levi-Civita epsilon to arrive at the last line ($v_{\beta}{}^{\alpha}$ is of course the matrix whose (β,α)-component is the α component of the vector v_β).

The evaluation of the flux form J is represented locally as

$$\begin{aligned}
&J(v_1,\dots,v_{m-1})\\
&\quad= J_{j_1\dots j_{m-1}}\, dx^{j_1}\wedge\cdots\wedge dx^{j_{m-1}}\left(v_1^{i_1}\frac{\partial}{\partial x^{i_1}},\dots,v_{m-1}^{i_{m-1}}\frac{\partial}{\partial x^{i_{m-1}}}\right)\\
&\quad= J_{j_1\dots j_{m-1}} v_1^{i_1}\cdots v_{m-1}^{i_{m-1}}\, dx^{j_1}\wedge\cdots\wedge dx^{j_{m-1}}\left(\frac{\partial}{\partial x^{i_1}},\dots,\frac{\partial}{\partial x^{i_{m-1}}}\right)\\
&\quad= J_{j_1\dots j_{m-1}}\, v_1^{i_1}\cdots v_{m-1}^{i_{m-1}}\,\varepsilon^{j_1\dots j_{m-1}}_{i_1\dots i_{m-1}}, \qquad j_1<j_2<\cdots<j_n\\
&\quad= J_{i_1\dots i_{m-1}}\, v_1^{i_1}\cdots v_{m-1}^{i_{m-1}}, \qquad J_{i_1\dots i_{m-1}}=\varepsilon^{j_1\dots j_{m-1}}_{i_1\dots i_{m-1}}J_{j_1\dots j_{m-1}},
\end{aligned}$$

so in the last line the $\mathbf{i}$ sequences are not increasing. (Note the difference in the ranges of the indices as implied by using roman letters to denote them.) Finally, comparing the third equality with the last expression for the determinant we obtained in Section 6.3, we conclude that $J(v_1,\dots,v_{m-1})$ is represented by the determinant of the matrix constructed by the components of J in the first column and the components of the vectors $v_1,\dots,v_{m-1}$ in the rest of the columns. (Clearly, we could replace "columns" by "rows" in the last sentence.)

We may denote this by

$$J(v_1,\dots,v_{m-1}) = \det(J_{j_1\dots j_{m-1}}; v_1{}^{i_1};\dots;v_{m-1}{}^{i_{m-1}}),$$

where the square matrix is constructed by inserting the vectors separated by semi-colons in the columns according to the free indices. Since the expression is invariant we may also write

$$J(v_1,\dots,v_{m-1}) = \det[J; v_1;\dots;v_{m-1}].$$

We can now combine the representations of the various variables and use them in Cauchy's formula

$$\tau(v_1,\dots,v_{m-1}) = J\big(\iota_*(v_1),\dots,\iota_*(v_{m-1})\big)$$

to obtain

$$\begin{aligned}
\tau_{1\dots m-1} &= \tau\left(\frac{\partial}{\partial y^1},\dots,\frac{\partial}{\partial y^{m-1}}\right)\\
&= J\left(\iota_*\left(\frac{\partial}{\partial y^1}\right),\dots,\iota_*\left(\frac{\partial}{\partial y^{m-1}}\right)\right)\\
&= J\left(x^{i_1}_{,1}\frac{\partial}{\partial x^{i_1}},\dots,x^{i_{m-1}}_{,m-1}\frac{\partial}{\partial x^{i_{m-1}}}\right),\\
&= x^{i_1}_{,1}\cdots x^{i_{m-1}}_{,m-1}\varepsilon^{j_1\dots j_{m-1}}_{i_1\dots i_{m-1}}J_{j_1\dots j_{m-1}}, \qquad j_1<j_2<\dots<j_{m-1}.
\end{aligned}$$

which is simply the determinant of the matrix $[J, Dx]$ whose first column consists of the components of $flow$ and the rest of the columns are occupied by the matrix of the derivative of the local representative of the inclusion mapping.

4. Cauchy's Theorem

This section considers the theory of existence of Cauchy fluxes. That is, we consider the conditions that the flux density fields $\{\tau_{\mathcal{R}}\}$ for the various regions are given using the Cauchy formula by a flux form. As mentioned earlier, if such a form exists, then it is unique. If indeed there is a flux-form J such that

$$\tau_{\mathcal{R}} = \iota^*_{\partial\mathcal{R}}(J),$$

we will say that the flux density system $\{\tau_{\mathcal{R}}\}$ is *consistent*.

4.1. Locality. The dependence of the flux density $\tau_{\mathcal{R}}$ on the region $\mathcal{R}$ is in general a set function—to each region $\mathcal{R}$ it assigns the differential form $\tau_{\mathcal{R}}$ on its boundary. Recalling that with an orientation on $\mathcal{U}$ a region defines a unique orientation on its boundary, it is natural to replace the dependence on the region $\mathcal{R}$ by dependence on the boundary $\partial\mathcal{R}$. This even makes the set function more symmetric, it assigns to any closed surface $\partial\mathcal{R}$ a flux density $(m-1)$-form on it. Clearly, it is difficult to specify such a set function in general. Locality assumptions simplify this dependence.

Consider a point $x \in \mathcal{U}$. Then, x may be on the boundary of various bodies and in general we want to find the dependence of the value of the flux density $\tau_{\mathcal{R}}(x)$ at x on $\partial\mathcal{R}$. The boundary $\partial\mathcal{R}$ is an $(m-1)$-chain and it suffices to know the values of the flux form only at the interior points of the simplexes that constitute it. Thus, it is enough to consider regions that contain x as an interior point on the simplexes that constitute the boundaries as shown in Figure 12.

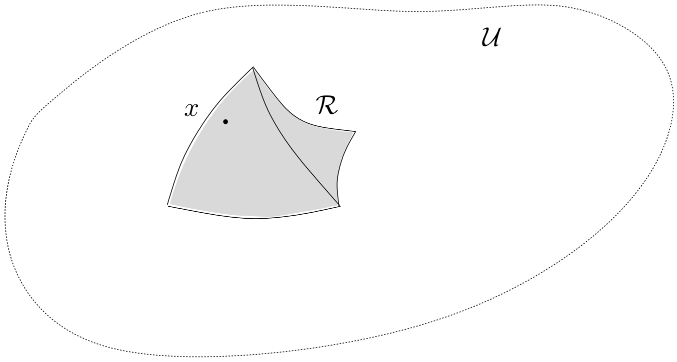

FIGURE 12. A point on the boundary of a region

Locality means that $\tau_{\mathcal{R}}(x)$ does not depend on the "shape" of $\partial\mathcal{R}$ away from x. In other words, if the two $(m-1)$-dimensional manifolds $\partial\mathcal{R}$ and $\partial\mathcal{R}'$ have the same "shape" in a neighborhood of x they will have identical flux density at x (see Figure 13).

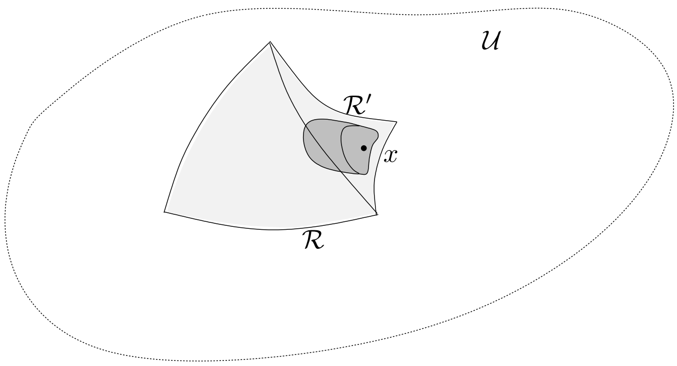

FIGURE 13. Locality

Specifically, the two boundaries have the "same shape" in a neighborhood of x if there is an $(m-1)$-dimensional manifold U containing x which is an open submanifold of both $\partial\mathcal{R}$ and $\partial\mathcal{R}'$. In such a case (no matter how small this neighborhood may be) we will say that the two boundaries have the same *germ* at x. This is clearly an equivalence relation and we will refer to a an equivalence class of boundaries as a *germ*. Thus, the locality requirement we described above means that the value of the flux density $\tau_{\mathcal{R}}(x)$ depends on $\mathcal{R}$ only through the germ of $\partial\mathcal{R}$ at x. We will refer to such form of locality as *germ locality*.

4.2. Tangent space locality and the Cauchy mapping. A stronger locality assumption may be postulated if we note that the small neighborhood of x in $\partial\mathcal{R}$ may be "approximated" by the tangent space $T_x\partial\mathcal{R}$ (see Figure 14). Thus, we will refer as *tangent space locality* to the assumption that the value of $\tau_{\mathcal{R}}(x)$ depends only on the tangent space to the boundary at x and its orientation. In this part of the book, only tangent space locality will be considered. The next section describes the way such a dependence is described mathematically.

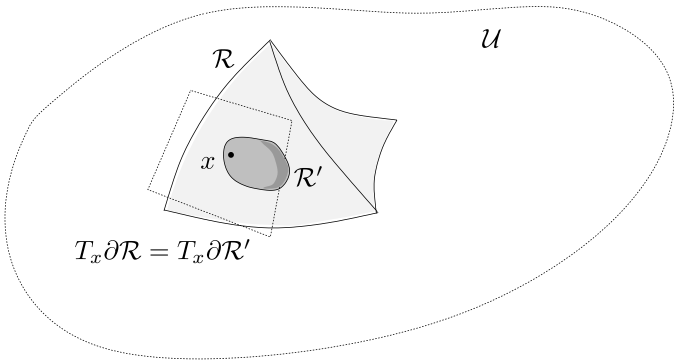

FIGURE 14. Tangent space locality

The tangent space $T_x\partial\mathcal{R}$ is an $(m-1)$-dimensional oriented subspace of $T_x\mathcal{U}$—an *oriented hyperplane*. Thus, tangent space locality implies that at x there is a mapping $\mathfrak{t}_x$ that assigns to any oriented hyperplane $h = T_x\partial\mathcal{R}$ the $(m-1)$-alternating mapping $\tau_{\mathcal{R}}(x) \in \bigwedge^{m-1} h^*$ defined on it. We will refer to $\mathfrak{t}_x$ as the *Cauchy mapping*. Thus, in general we write $\tau_{\mathcal{R}}(x) = \mathfrak{t}_x(h)$.

In the traditional formulations of continuum mechanics in which the the manifold $\mathcal{U}$ is a Euclidean space, the oriented tangent space is represented by the unit normal vector to the boundary at x and the flux density at a point is a given by a real number. Thus, in a Euclidean setting $\tau_{\mathcal{R}}(x)$ depends on the region through the normal $\mathbf{n}$ to the boundary of $\mathcal{R}$ at x. This is traditional written as $\tau_{\mathcal{R}} = \tau(x, \mathbf{n})$.

We recall that the sign of an integral of a form on an orientable manifold is meaningful only in relative to a given orientation. Thus, the Cauchy mapping is assumed to conform to the induced orientations on the various regions, i.e., the form $\mathfrak{t}_x(h) = \tau_{\mathcal{R}}(x)$ is indeed the flux density with respect to the natural orientation of $\partial\mathcal{R}$. Similarly, the sign of the value $\mathfrak{t}_x(h)(\mathfrak{v})$ is meaningful only if the multivector $\mathfrak{v}$ has the same orientation as h (and $\partial\mathcal{R}$).

4.3. Whitney's function. In order to consider properties of the Cauchy mapping such a continuity, the collection of oriented hyperplanes should be given a topological structure. While this may be done using Grassmann manifolds, we present here Whitney's function as the mathematical framework for the formulation of tangent space locality.

Following Whitney [**11**, p. 165] we note that with the tangent space locality assumption we can represent the Cauchy mapping $\mathfrak{t}_x$ with a real valued function $\mathfrak{t}'_x$ of $(m-1)$-multivectors as follows. Recalling that every $(m-1)$-multivector is simple, any multivector $\mathfrak{v} = v_1 \wedge \cdots \wedge v_{m-1}$ determines a unique oriented hyperplane h containing the vectors $v_1, \ldots, v_{m-1}$ and oriented accordingly. The flux density $\tau_{\mathcal{R}}(x) = \mathfrak{t}_x(h)$ may then be evaluated on $\mathfrak{v}$ to yield

$$\tau_{\mathcal{R}}(x)(\mathfrak{v}) = \mathfrak{t}_x(h)(\mathfrak{v}) \in \mathbb{R}.$$

Thus, we may set $\mathfrak{t}'_x(\mathfrak{v}) = \mathfrak{t}_x(h)(\mathfrak{v})$, where h is the oriented hyperplane determined by $\mathfrak{v}$. From its definition it is clear that for a positive $a \in \mathbb{R}$, $\mathfrak{t}'_x(a\mathfrak{v}) = \mathfrak{t}_x(h)(a\mathfrak{v}) = a\mathfrak{t}_x(h)(\mathfrak{v})$ as $\mathfrak{v}$ and $a\mathfrak{v}$ induce the same oriented hyperplane. It is noted that unlike the traditional Cauchy mapping, Whitney's function does not have the redundancy of specifying both the the oriented hyperplane and the multivector on it. A Whitney mapping $\mathfrak{t}'_x$ clearly defines a Cauchy mapping $\mathfrak{t}_x$ by the same relation. Thus, in the following we will use the same notation for both.

Since Whitney's mapping $\mathfrak{t}_x$ is defined on the vector space of $(m-1)$-multivectors at x one can vary x and consider the global Whitney mapping

$$\mathfrak{t} \colon \bigwedge^{m-1} T\mathcal{U} \to \mathbb{R}.$$

As the global Whitney mapping is defined on the bundle of multivectors continuity and differentiability requirements may be postulated for it. We will refer to the assumptions for existence of a Whitney mapping and its continuity as *Cauchy's postulate*.

4.4. The dependence on the orientation. The definition of the Whitney mapping implies that $\mathfrak{t}(\mathfrak{v}) = \tau_{\mathcal{R}}(x)(\mathfrak{v})$ only if $\mathfrak{v}$ is positively oriented with respect to to the orientation of $\partial\mathcal{R}$.h

Consider a point x on the boundary of a region $\mathcal{R}$ and an open neighborhood V of x in $\partial\mathcal{R}$ so that V is an $(m-1)$-dimensional submanifold of $\mathcal{U}$. Clearly, V is an open subset of the boundary of some manifold $\mathcal{R}'$ situated on the other side of $\partial\mathcal{R}$ (see Figure 15). The orientation of $\mathcal{U}$ induces an outwards-pointing orientation on $T_x\partial\mathcal{R} =$

$T_x\partial\mathcal{R}'$. As a vector pointing out of $\mathcal{R}$ points into $\mathcal{R}'$, the orientation on $T_x\partial\mathcal{R}'$ is opposite to that of $T_x\partial\mathcal{R}$.

Assume that an $(m-1)$-multivector $\mathfrak{v}$ at x is positively oriented with respect to the orientation of $T_x\partial\mathcal{R}$ so $\mathfrak{t}(\mathfrak{v}) = \tau_{\mathcal{R}}(x)(\mathfrak{v})$. Then, $-\mathfrak{v}$ has the opposite orientation, that corresponding to $T_x\partial\mathcal{R}'$, so $\mathfrak{t}(-\mathfrak{v}) = \tau_{\mathcal{R}'}(-\mathfrak{v})$. So far, we did not postulate or prove any relation between $\tau_{\mathcal{R}'}$ and $\tau_{\mathcal{R}}$ for a change in orientation only. Thus, we do not have a relation between $\mathfrak{t}(\mathfrak{v})$ and $\mathfrak{t}(-\mathfrak{v})$.

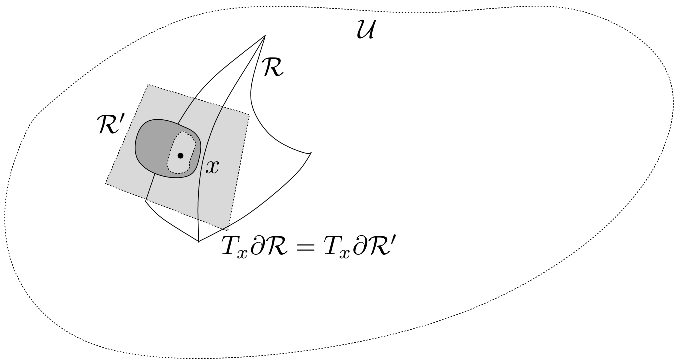

FIGURE 15

Since we interpret $\tau_{\mathcal{R}}(x)(\mathfrak{v})$ and $\tau_{\mathcal{R}'}(x)(-\mathfrak{v})$ as the flux densities out of an infinitesimal elements in $\partial\mathcal{R}$ and $\partial\mathcal{R}'$ respectively, we expect intuitively that

$$\mathfrak{t}(\mathfrak{v}) = -\mathfrak{t}(-\mathfrak{v}).$$

This indeed follows from the boundedness postulate for the fluxes of Section 3.4 and from continuity of Whitney's mapping as follows. Let ς be the positive bounding m-differential form such that

$$\left|\int_{\partial\mathcal{R}} \tau_{\mathcal{R}}\right| \leqslant \int_{\mathcal{R}} \varsigma,$$

and consider a region $\mathcal{R}$ and a point x_0 on its boundary. For a multivector $\mathfrak{v}$, let $v_1, \ldots, v_{m-1} \in T_{x_0}\partial\mathcal{R}$ be tangent vectors such that $\mathfrak{v} = v_1 \wedge \cdots \wedge v_{m-1}$. The definition of a manifold with a boundary implies that we can choose a chart (ψ, U), inducing the coordinate system $(x^1, \ldots, x^m)$ in a neighborhood $U \subset \mathcal{U}$ of x_0 such that $x^m = 0$ on $\partial\mathcal{R}$, $x^m \geqslant 0$ on $U \cap \mathcal{R}$ and $x^m < 0$ in $U - \mathcal{R}$. Without loss of generality we may assume that x_0 is represented locally by $(0, \ldots, 0)$ and that the chart is such that the vector v_i is represented by $\partial/\partial x^i$. We choose a positive $a_0 \leqslant 1$ such that $\psi(U)$ contains the cube $(-a_0, a_0)^m$ of side $2a_0$

centered at the origin or $\mathbb{R}^m$. Let $\tilde{\varsigma} = (\psi^{-1})^*(\varsigma)$ be the local representative of ς, $\tilde{\tau}_{\mathcal{R}} = (\psi^{-1})^*(\tau_{\mathcal{R}})$ the local representatives of the flux densities for the regions contained in U, and let $\tilde{\mathfrak{t}}$ be the local representative of $\mathfrak{t}$ so

$$\tilde{\mathfrak{t}}\big(\psi_*(v_1) \wedge \cdots \wedge \psi_*(v_{m-1})\big) = \mathfrak{t}(v_1 \wedge \cdots \wedge v_{m-1}).$$

For $p = 1, 2, \ldots$ set $a_p = 2^{-p} a_0$ and consider the boundedness postulate for the region $\mathcal{R}_p$ such that $\tilde{\mathcal{R}}_p = \psi(\mathcal{R}_p) = \Delta_p \times [-a_p^2, a_p^2]$, where Δ_p is the standard $(m-1)$-simplex multiplied by a_p, i.e., $\Delta_p = s_p(\Delta_{m-1})$, with $s_p(y) = a_p y$, $y \in \mathbb{R}^{m-1}$. Thus, the various $\tilde{\mathcal{R}}_p$ form a sequence of small prisms whose heights are an order of magnitude smaller than their bases (see Figure 16).

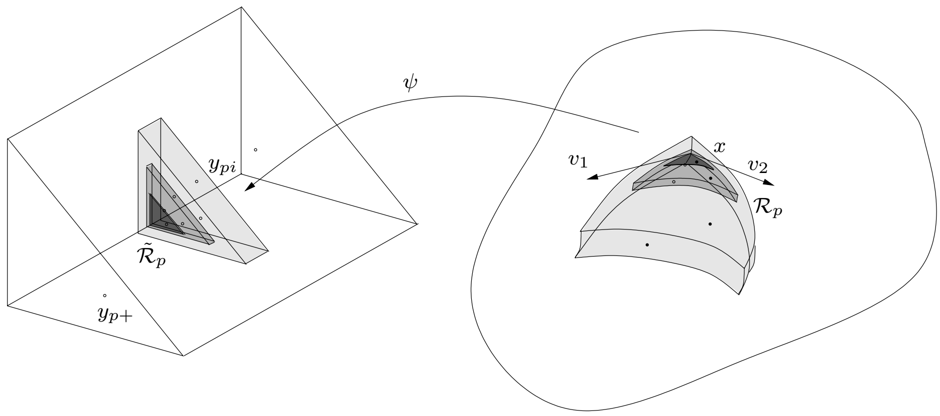

FIGURE 16

Evaluating the various integrals in $\psi(U)$ we obtain for $\mathcal{R}_p$

$$\left| \int_{\partial \tilde{\mathcal{R}}_p} \tilde{\tau}_{\mathcal{R}_p} \right| \leqslant \int_{\tilde{\mathcal{R}}_p} \tilde{\varsigma}.$$

It is noted that

$$\partial \tilde{\mathcal{R}}_p = \bigcup_{j=0}^{m-2} k_{p_j}^{m-2} \times [-a_p^2, a_p^2] \cup \Delta_p \times \{-a_p^2, a_p^2\}$$

where $k_{p_j}^{m-2} = s_p \circ k_j^{m-2}(\Delta_{m-2})$ is the j-th face of Δ_p. (the illustration does not depict the standard Δ_{m-2} and the various s_p but only the images.) The integral over this union of disjoint sets may be obtained by adding the individual integrals.

Consider the the integral over the face $k_{p_j}^{m-2} \times [-a_p^2, a_p^2]$. The mean value theorem implies that there is a point y_{pj} on the j-th face, $k_{p_j}^{m-2} \times$

$[-a_p^2, a_p^2]$, such that

$$\int\limits_{k_{pj}^{m-2}\times[-a_p^2,a_p^2]} \tilde{\tau}_{\mathcal{R}_p} = \tilde{\tau}_{\mathcal{R}_p}(y_{pj})\left((s_p \circ k_j^{m-2})_*(\mathfrak{e}_{m-2}) \wedge 2a_p^2 \frac{\partial}{\partial x^m}\right),$$

where $\mathfrak{e}_{m-2}$ is the standard $(m-2)$-vector in $\mathbb{R}^{m-2}$. We did not specify where the derivative is evaluated as it is constant.

Using the multi-linearity of the flux density and the definition of Whitney's mapping we have

$$\begin{aligned}
\int\limits_{k_{pj}^{m-2}\times[-a_p^2,a_p^2]} \tilde{\tau}_{\mathcal{R}_p} &= \tilde{\tau}_{\mathcal{R}_p}(y_{pj})\left((s_p \circ k_j^{m-2})_*(\mathfrak{e}_{m-2}) \wedge 2a_p^2 \frac{\partial}{\partial x^m}\right) \\
&= \tilde{\tau}_{\mathcal{R}_p}(y_{pj})\left(a_p^{m-2}(k_j^{m-2})_*(\mathfrak{e}_{m-2}) \wedge 2a_p^2 \frac{\partial}{\partial x^m}\right) \\
&= a_p^m\, \tilde{\tau}_{\mathcal{R}_p}(y_{pj})\left((k_j^{m-2})_*(\mathfrak{e}_{m-2}) \wedge \frac{\partial}{\partial x^m}\right) \\
&= a_p^m\, \tilde{\mathfrak{t}}(y_{pj})\left((k_j^{m-2})_*(\mathfrak{e}_{m-2}) \wedge \frac{\partial}{\partial x^m}\right).
\end{aligned}$$

We now consider the integrals over $\Delta_p^+ = \Delta_p \times \{a_p^2\}$ and $\Delta_p^- = \Delta_p \times \{-a_p^2\}$. In analogy with the foregoing notation we have

$$\begin{aligned}
\int\limits_{\Delta_p^{\pm}} \tilde{\tau}_{\mathcal{R}_p} &= \tilde{\tau}_{\mathcal{R}_p}(y_{p\pm})\big(\pm s_p(\mathfrak{e}_{m-1})\big) \\
&= a_p^{m-1}\tilde{\mathfrak{t}}_{\mathcal{R}_p}(y_{p\pm})(\pm\mathfrak{e}_{m-1}),
\end{aligned}$$

where, y_{p+} and y_{p-} are the integration points in Δ_p^+ and Δ_p^-, respectively, and it is noted that it is the simplex $-\mathfrak{e}$ that is positively oriented on Δ_p^-.

Similarly,

$$\begin{aligned}
\int\limits_{\tilde{\mathcal{R}}_p} \tilde{\varsigma} &= \tilde{\varsigma}(y_p)\left(s_p(\mathfrak{e}_{m-1}) \wedge 2a_p^2 \frac{\partial}{\partial x^m}\right) \\
&= 2a_p^{m+1}\tilde{\varsigma}(y_p)\left(\mathfrak{e}_{m-1} \wedge \frac{\partial}{\partial x^m}\right)
\end{aligned}$$

for some point $y_p \in \tilde{\mathcal{R}}_p$.

The boundedness assumption may be rewritten now as

$$\left|\int_{\Delta_p^+}\tilde{\tau}_{\mathcal{R}_p}+\int_{\Delta_p^-}\tilde{\tau}_{\mathcal{R}_p}+\sum_{j=0}^{m-2}\int_{k_{pj}^{m-2}\times[-a_p^2,a_p^2]}\tilde{\tau}_{\mathcal{R}_p}\right|\leqslant\int\limits_{\tilde{\mathcal{R}}_p}\tilde{\varsigma},$$

implying (as $|A+(-B)|\leqslant|A|+|B|$ etc.)

$$\left|\int_{\Delta_p^+}\tilde{\tau}_{\mathcal{R}_p}+\int_{\Delta_p^-}\tilde{\tau}_{\mathcal{R}_p}\right|\leqslant\sum_{j=0}^{m-2}\left|\int_{k_{pj}^{m-2}\times[-a_p^2,a_p^2]}\tilde{\tau}_{\mathcal{R}_p}\right|+\int\limits_{\tilde{\mathcal{R}}_p}\tilde{\varsigma}.$$

We may now substitute the expression we obtained using the mean value theorem to get for each p,

$$\begin{aligned}\left|a_p^{m-1}\tilde{\mathfrak{t}}_{\mathcal{R}_p}(y_{p+})(\mathfrak{e}_{m-1})+a_p^{m-1}\tilde{\mathfrak{t}}_{\mathcal{R}_p}(y_{p-})(-\mathfrak{e}_{m-1})\right|&\leqslant\\ \sum_{j=0}^{m-2}\left|a_p^m\tilde{\mathfrak{t}}(y_{pj})\left((k_j^{m-2})_*(\mathfrak{e}_{m-2})\wedge\frac{\partial}{\partial x^m}\right)\right|&\\ +2a_p^{m+1}\tilde{\varsigma}(y_p)\left(\mathfrak{e}_{m-1}\wedge\frac{\partial}{\partial x^m}\right)&.\end{aligned}$$

Dividing the inequality by a_p^{m-1} and considering the limit as $p\to\infty$ we observe that all points $y_{p+},\ y_{p-},\ y_{pj},\ y_p$ converge to $(0,\dots,0)$ (representing x_0) so the evaluation of the various forms are bounded. Hence, as $a_p\to 0$ we conclude that the limit of the right-hand side of the inequality vanishes. We conclude that

$$\tilde{\mathfrak{t}}_{\mathcal{R}_p}(0)(\mathfrak{e}_{m-1})+\tilde{\mathfrak{t}}_{\mathcal{R}_p}(0)(-\mathfrak{e}_{m-1})=0$$

which implies that $\mathfrak{t}(\mathfrak{v})=-\mathfrak{t}(-\mathfrak{v})$.

4.5. Relation to classical formulation. In this paragraph we discuss the differences and similarities between the classical formulation of the anti-symmetry of the Cauchy mapping and the one given above that uses differential forms. Consider a hyperplane H at a point $x\in\mathcal{U}$. As a result of anti-symmetry, the property $\mathfrak{t}_x(a\mathfrak{v})=a\mathfrak{t}_x(\mathfrak{v})$, $\mathfrak{v}\in\bigwedge^{m-1}H$, (see Section 4.3) now holds for both positive and negative real numbers a. Thus, the restriction $\mathfrak{t}_x|H$ is an $(m-1)$-form on H. Let $\mathcal{R}$ be a region such that $H=T_x\partial\mathcal{R}$. Then, for a multivector $\mathfrak{v}$, $\mathfrak{t}_x(\mathfrak{v})=\tau_{\mathcal{R}}(\mathfrak{v})$ only if $\mathfrak{v}$ is positively oriented relative to the orientation induced on $\partial\mathcal{R}$. Assume that $\mathfrak{v}$ is negatively oriented with respect to $\partial\mathcal{R}$. Then, if $\mathcal{R}'$ is on the other side of H, $-\mathfrak{v}$ is positively oriented with respect to the orientation of $\partial\mathcal{R}'$, hence,

$$\tau_{\mathcal{R}'}(-\mathfrak{v})=\mathfrak{t}_x(-\mathfrak{v})=-\mathfrak{t}_x(\mathfrak{v})=-\tau_{\mathcal{R}}(\mathfrak{v}).$$

Thus, as forms on H, $\tau_{\mathcal{R}'} = \tau_{\mathcal{R}}$.

The last equation seems to contradict the classical formulation where one usually writes for the situation under consideration $\mathbf{t}(\mathcal{R}) = -\mathbf{t}(\mathcal{R}')$, where $\mathbf{t}$ is the flux vector field. In the classical formulation the area elements do not have orientations and the sense in which the property flows through a surface element is included in the sense of the vector $\mathbf{t}$. For integration of forms, there is only one form $\tau_{\mathcal{R}} = \tau_{\mathcal{R}'}$. The difference in the senses by which the property flows relative to $\mathcal{R}$ and $\mathcal{R}'$ is accounted for by the convention that the flux out of a boundary of a region is calculated by applying the form to multivectors that are positively oriented relative to the orientation of $\partial\mathcal{R}$. (Equivalently, for integration of forms the chart used for the evaluation of the integral is positively oriented.) Thus, the sense of the flow of the property is accounted for by requiring a particular scheme of orientation which gives opposite results for $\mathcal{R}$ and $\mathcal{R}'$.

4.6. Cauchy's Theorem. In this section we prove that under the boundedness assumption and Cauchy's postulate there is a unique flux form J such that the Cauchy formula holds, i.e.,

$$\tau_{\mathcal{R}} = \iota^*_{\partial\mathcal{R}}(J).$$

If we consider the algebraic Cauchy theorem of the previous chapter, in order to prove the assertion of the Cauchy theorem it is enough to show that for any linear m-simplex s in the tangent space $T_x\mathcal{U}$, we have

$$\sum_{i=1}^{m} \mathfrak{t}_x(\mathfrak{v}_i) = 0,$$

where $\mathfrak{v}_i$ is the multivector associated with the i-th face of s.

As in the previous section we let ς be a positive bounding m-form and we consider an arbitrary linear simplex s in $T_x\mathcal{U}$ having $v_1, \cdots, v_m$ as defining vectors. Thus, as in 2.3.2

$$\mathfrak{v}_j = \begin{cases} -\dfrac{1}{(m-1)!} \displaystyle\sum_{i=1}^{m} (-1)^i \, v_1 \wedge \cdots \wedge \widehat{v}_i \wedge \ldots \wedge v_m & \text{for } j = 0, \\ \dfrac{(-1)^j}{(m-1)!} \, v_1 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_m & \text{for } j > 0, \end{cases}$$

and $\sum_i \mathfrak{v}_i = 0$. Using the same scheme of notation as in the previous section, we use for example $\tilde{\mathfrak{t}}$ for the local representative of $\mathfrak{t}$ in a coordinate system $(x^1, \ldots, x^m)$ in a chart (ψ, U) containing x, and we assume that the coordinates of x are $(0, \ldots, 0)$. Again, without loss of generality, we may assume that $\tilde{v}_i$, the local representative of v_i is parallel to $\mathbf{e}_i$. Choose a positive $a_0 \leqslant 1$ such that the linear simplex

$\tilde{s}_0$ induced by $a_0\tilde{v}_1, \ldots, a_0\tilde{v}_m$ in $\mathbb{R}^m$ is contained in the image of the coordinate neighborhood. For $p = 1, 2, \ldots$ we set $a_p = 2^{-p}a_0$ and consider the boundedness postulate for regions $\mathcal{R}_p$ such that $\tilde{\mathcal{R}}_p = \psi(\mathcal{R}_p)$ is the linear m-simplex $\tilde{s}_p$ generated by the vectors $a_p\tilde{v}_1, \ldots, a_p\tilde{v}_m$. In other words, the various simplexes $\tilde{s}_p$ form a sequence of decreasing linear simplexes $\tilde{s}_p = a_p\tilde{s}_0$ such that $\tilde{s}_0(\mathbf{e}_i) = a_0\tilde{v}_i$. The multivector $\tilde{\mathfrak{v}}_p$ associated with $\tilde{s}_p$ satisfies $\tilde{\mathfrak{v}}_p = s_{p*}(\mathfrak{e}_m)$. The local representatives $\tilde{\mathfrak{v}}_i$ of the multivectors $\mathfrak{v}_i$ associated with the faces of $\tilde{s}_0$ satisfy $\tilde{\mathfrak{v}}_i = (\tilde{s}_0 \circ k_i^{m-1})_*(\mathfrak{e}_{m-1})$ (see Figure 17 where only the images of the various s_p are shown on the left).

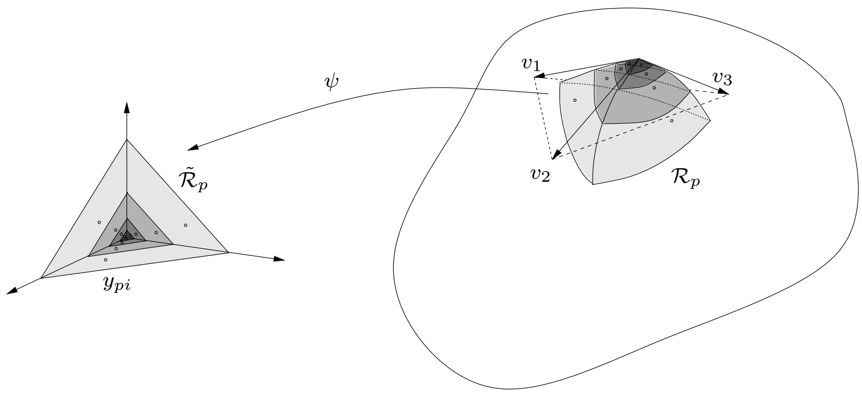

FIGURE 17

Evaluating the various integrals in $\psi(U)$ we have for $\mathcal{R}_p$

$$\left|\int_{\partial\tilde{\mathcal{R}}_p} \tilde{\tau}_{\mathcal{R}_p}\right| \leqslant \int_{\tilde{\mathcal{R}}_p} \tilde{\varsigma}.$$

The mean value theorem implies that there are points $y_{pi} \in \tilde{s}_p \circ k_i^{m-1}$, $i = 1, \ldots, m+1$, $p = 1, 2, \ldots$

$$\begin{aligned}\int_{\tilde{s}_p \circ k_i^{m-1}} \tilde{\tau}_{\mathcal{R}_p} &= \tilde{\tau}_{\mathcal{R}_p}(y_{pi})\big((\tilde{s}_p \circ k_i^{m-1})_*(\mathfrak{e}_{m-1})\big) \\ &= a_p^{m-1}\tilde{\tau}_{\mathcal{R}_p}(y_{pi})\big((\tilde{s}_0 \circ k_i^{m-1})_*(\mathfrak{e}_{m-1})\big) \\ &= (-1)^i a_p^{m-1}\tilde{\tau}_{\mathcal{R}_p}(y_{pi})(\tilde{\mathfrak{v}}_i) \\ &= (-1)^i a_p^{m-1}\tilde{\mathfrak{t}}_(y_{pi})(\tilde{\mathfrak{v}}_i).\end{aligned}$$

We did not indicate the point where the derivative of the mapping $\tilde{s}_0 \circ k_i^{m-1}$ is evaluated since it is affine. Similarly, there are points $y_p \in$

$\tilde{s}_p$ such that

$$\begin{aligned}\int_{\tilde{\mathcal{R}}_p} \tilde{\varsigma} &= \tilde{\varsigma}(y_p)\big(\tilde{s}_{p*}(\mathfrak{e}_m)\big)\\ &= a_p^m \tilde{\varsigma}(y_p)\big(\tilde{s}_{0*}(\mathfrak{e}_m)\big)\\ &= a_p^m \tilde{\varsigma}(y_p)\big(\tilde{\mathfrak{v}}\big).\end{aligned}$$

Since $\partial\tilde{\mathcal{R}}_p = \sum_i (-1)^i \tilde{s}_p \circ k_i^{m-1}$, the boundedness postulate becomes

$$\left|\sum_i \tilde{\mathfrak{t}}(y_{pi})(\tilde{\mathfrak{v}}_i)\right| \leqslant a_p \tilde{\varsigma}(y_p)\big(\tilde{\mathfrak{v}}\big).$$

Taking the limit as $p \to \infty$, we have $a_p \to 0$, $y_{pi} \to (0,\ldots,0) = \psi(x)$, hence, $\sum_i \tilde{\mathfrak{t}}(0,\ldots,0)(\tilde{\mathfrak{v}}_i) = 0$ and we conclude that

$$\sum_i \mathfrak{t}(\mathfrak{v}_i) = 0.$$

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