Optimal Stress Fields and Load Capacity of Structures

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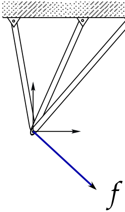
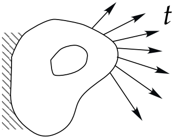
Stress Analysis

- For a given structure geometry Ω and an assumed *loading*, or a number of loading cases,
- Solve the equations of *equilibrium* with boundary conditions





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σ – stress field, b – volume force, t – boundary load, ν – unit normal

- *Problem: the system is under-determined (statically indeterminate):*
  6 independent stress components and 3 equations.
- *Solution:* Use constitutive relations (e.g., Hooke's Law) to relate the stress and the kinematics. One (hopefully) stress field σ_0 will solve the problem.
- Use a *failure criterion* $Y(\tau) \leqslant s_{\text{permitted}}$, where τ is a stress matrix.
- Find the maximal stress and check whether $\max_x Y(\sigma_0(x)) < s_{\text{permitted}}$.

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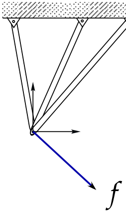

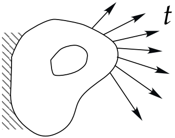

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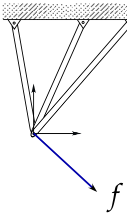

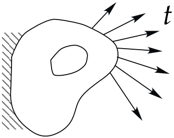

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Estimates for the Maximum of the Stress Field

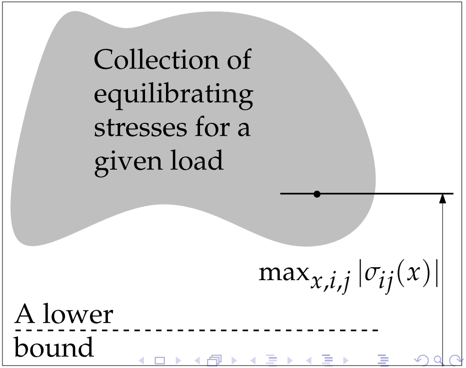
Question: For a given load on the structure,
what estimates or bounds apply to the stress field on the basis of equilibrium alone? (no reference to material properties!)

Signorini [1933], Grioli [1953], Truesdell & Toupin [1960], Day [1979]:
Lower bounds on the maximal stress in terms of the applied load only.

$$\max_{x,i,j}\left\{\left|\sigma_{ij}(x)\right|\right\} \geqslant \text{Bound}(t)$$

for all equilibrating stresses.

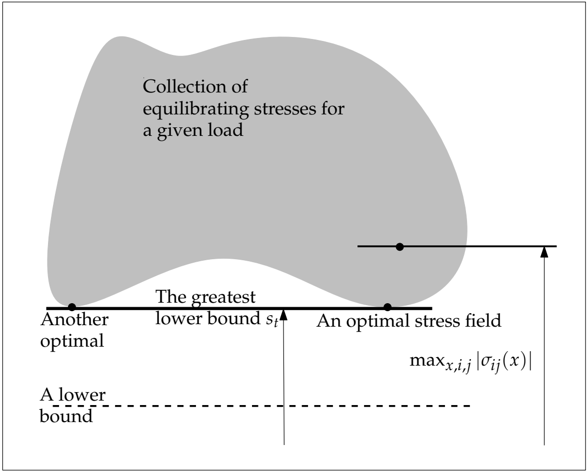
Note: the bounds are not exact!



Greatest Lower Bounds and Optimal Stresses

Notes:

- What is the greatest lower bound on the maximal stress components?
- Is the greatest lower bound attained for some stress field σ^{opt}?
- A stress field for which the bound it attained is *optimal* because it has the least maximum.



The Setting for the Problem

Definitions of the Main Variables

Ω – a given body (bounded), $\Gamma = \partial\Omega$ – its boundary,

Γ_0 – the part of the boundary where the body is fixed,

t – a surface traction field given on $\Gamma_t \subset \Gamma$,

ν – the unit normal to the boundary,

σ – a stress field that is in equilibrium with t,

$\sigma_{\max}$ – the maximal magnitude of the stress

$$\sigma_{\max} = \operatorname{ess\,sup}_{x\in\Omega} |\sigma(x)| = \|\sigma\|_\infty,$$

$|\tau| = Y(\tau)$ – a failure criterion function for the stress matrix τ, a norm.

Remark: The treatment may be generalized to include body forces.

- There is a class of stress fields that are in equilibrium with t.
- We denote this class of stress fields by Σ_t.

The Optimization Problem

- Find the least value s_t^{opt} of $\sigma_{\max}$, i.e.,

$$s_t^{\mathrm{opt}} = \inf_{\sigma \in \Sigma_t} \{\sigma_{\max}\} = \inf_{\sigma \in \Sigma_t} \{\|\sigma\|_\infty\}.$$

  - *Question*: Is there an optimal stress field σ_{opt} such that

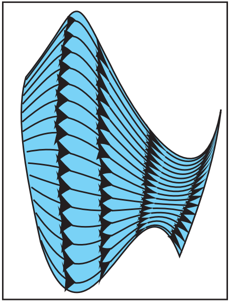
$$s_t^{\mathrm{opt}} = \|\sigma_{\mathrm{opt}}\|_\infty \,?$$

The Corresponding Scalar Problem: the Junction Problem

- Given the flux density ϕ on the boundary of Ω with $\int_{\partial\Omega} \phi \mathrm{d}A = 0$ (this constraint may be removed and we will get the optimal source distribution).
- Set $V_\phi = \{v \colon \Omega \to \mathbb{R}^3,\ v_{i,i} = 0 \text{ in } \Omega,\ ,v_i\nu_i = \phi \text{ on } \partial\Omega\}$
  —compatible velocity fields.
- For each $v \in V_\phi$, set $v_{\max} = \operatorname{ess\,sup}_{x\in\Omega} |v(x)|$.
- Find the least value v_ϕ^{opt} of $v_{\max}$, i.e.,

$$v_\phi^{\mathrm{opt}} = \inf_{v\in V_\phi} \{v_{\max}\}.$$

The optimal velocity field for the junction Ω.



The Result

Theorem

- *The optimal value* s_t^{opt} *is given by*

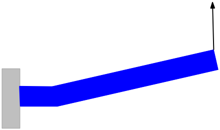
$$s_t^{\text{opt}} = \sup_{w \in C^\infty(\overline{\Omega}, \mathbb{R}^3)} \frac{\left|\int_{\partial\Omega} t \cdot w \, \mathrm{d}A\right|}{\int_\Omega |\varepsilon(w)| \, \mathrm{d}V} = \sup_{w \in C^\infty(\overline{\Omega}, \mathbb{R}^3)} \frac{|t(w)|}{\|\varepsilon(w)\|_1},$$

  $|\varepsilon(w)|$ *is the norm of the value of the stretching* $\varepsilon(w) = \frac{1}{2}(\nabla w + \nabla w^T)$.
- *The optimum is attained for some* $\sigma_t^{\text{opt}} \in \Sigma_t$.
- *Mathematically:*

$$s_t^{\text{opt}} = \|\textit{Force Functional}\|.$$

- *Motivation:* recall the principle of virtual work:

$$\int_\Omega \sigma_{ij} \varepsilon_{ij} \mathrm{d}V = \int_{\Gamma_t} t_i w_i \mathrm{d}A.$$



- *Note*: $s_{\lambda t}^{\text{opt}} = \lambda s_t^{\text{opt}}$, for all $\lambda > 0$.

Realization of an Optimal Stress Field

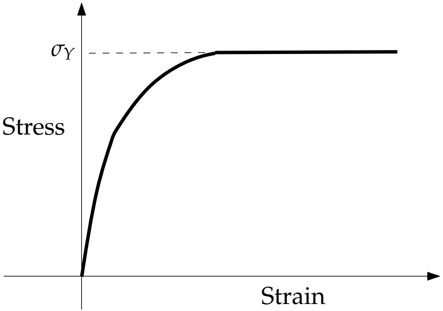
Question: Can the optimal stress field be realized?

- Introduce residual stresses in the structure (e.g., prestressed beams, tree trunks),
- Introduce additional external loading,
- Limit design for elastic perfectly plastic materials . . .

The Yield Condition and (Perfectly) Plastic Materials

Use the yield function as a norm for stress matrices.

- Hydrostatic pressure does not cause failure.
- $\tau = \tau^{\mathrm{H}} + \tau^{\mathrm{D}}$, where,
  $\tau^{\mathrm{H}} = \frac{1}{3}\mathrm{tr}(\tau)I$.
  τ^{D} – *deviatoric* component of the stress matrix.
- *Von Mises yield function:*
  $Y(\tau) = \left|\tau^{\mathrm{D}}\right| = \sqrt{\frac{3}{2}}\left|\tau^{\mathrm{D}}\right|_2,$
  $\left|\tau^{\mathrm{D}}\right|_2 = \sqrt{\tau_{ij}\tau_{ij}}$
  – the Euclidean norm.
- Yield condition:
  $Y(\tau) = \left|\tau^{\mathrm{D}}\right| = s_Y.$



A Perfectly Plastic Material

Yield Function and the (Semi-) Norm Induced

Deviatoric projection – $\pi_D(\tau) = \tau - \frac{1}{3}\tau_{ii}I$ for every matrix τ.
$\pi_D \colon \mathbb{R}^6 \longrightarrow D \subset \mathbb{R}^6$, the space of traceless matrices.

Yield function Y – a semi-norm on the space of matrices

$$Y(\tau) = \left|\tau - \tfrac{1}{3}\tau_{ii}I\right|, \quad |\cdot| \text{ is a norm on the space of matrices.}$$

Yield condition – $Y(\tau) = s_Y$.

Semi-norms – $\|\sigma\|^Y = \|Y \circ \sigma\|$, $\|\sigma\|^Y_\infty = \|Y \circ \sigma\|_\infty$
are norms on the subspaces of trace-less fields.

Thus, in the previous definitions of the optimal stress we have to use the semi-norms or restrict ourselves to the appropriate subspaces containing trace-less fields.

Limit Analysis of Plasticity Theory

- *The limit analysis problem:* Given t and s_Y, find the largest multiplier of the force for which collapse will not occur, i.e.,

$$\lambda_t^* = \sup \lambda, \quad \text{such that there exists } \sigma,\ \|\sigma\|_\infty^Y \leqslant s_Y,\ \sigma \in \Sigma_{\lambda t}.$$

Basic idea, the body can support any stress field σ as long as $\|\sigma\|_\infty^Y \leqslant s_Y$.

- Christiansen and Temam & Strang [1980's]:

$$\lambda_t^* = \sup_{\|\sigma\|_\infty^Y \leqslant s_Y} \inf_{t(w)=1} \int_\Omega \sigma_{ij}\varepsilon(w)_{ij} \mathrm{d}V$$
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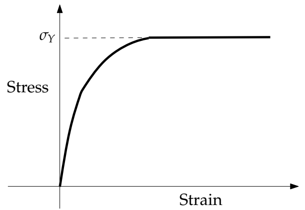

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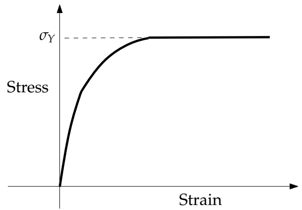

A Perfectly Plastic Material

Optimal Stresses and Limit Analysis

- *Task: Find the optimal stresses using the yield norm for stresses.* Result:

$$\text{Limit Design} \quad \Leftrightarrow \quad s_t^{\text{opt}} = s_Y, \quad \text{or,} \quad \frac{s_Y}{s_t^{\text{opt}}} = \lambda_t^*.$$

- The expression for s_t^{opt} is equivalent to the expression of Temam and Strang for the limit analysis factor.

Conclusions for Plasticity:

- Theoretically, optimal stress fields may be realized by choosing a perfectly plastic material for which the yield stress is equal to the optimal stress.
- *Perfectly plastic materials are optimal* in the following sense: If for a load t, the material satisfies $s_Y = s_t^{\text{opt}}$, the stress distribution will be automatically optimal. This holds for all loading distributions t satisfying this condition, independently of their distribution. (The optimality is not associated with a particular loading condition.)

Load Capacity Ratio

Notation

Ω – a given perfectly plastic body or a structure,

s_Y – the yield stress,

t – a loading traction field given on the boundary $\partial\Omega$,

$t_{\max}$ – the maximum of the external loading,

$$t_{\max} = \operatorname{ess\,sup}_{y\in\partial\Omega} |t(y)| = \|t\|_\infty$$

Result

There is a maximal number C such that the body will not collapse as long as

$$t_{\max} \leqslant C s_Y$$

independently of the distribution of the external traction t.

The Expression for the Load Capacity Ratio

The number C, a purely geometric property of the body Ω, is given by

$$\frac{1}{C} = \sup_{w} \frac{\int_{\Gamma_t} |w| \, \mathrm{d}A}{\int_{\Omega} |\varepsilon(w)| \, \mathrm{d}V} = \|\gamma_D\|,$$

where,

w – *an isochoric (incompressible,* $\operatorname{div} w = 0$*) vector field,*

$\varepsilon(w)$ – *the linear strain associated with w,*

$$\varepsilon(w)_{ij} = \tfrac{1}{2}(w_{i,j} + w_{j,i}).$$

Stress Concentration for Engineers

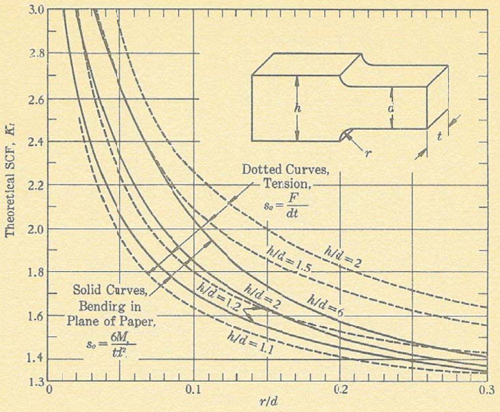

FIGURE AF 9 Flat Plate with Fillets. The tensile load is central. For $h/d = 1.1$, tensile and bending curves are very close together down to $r/d = 0.04$. (After R. E. Peterson)[4.21]

Generalized Stress Concentration Factors:

- Assume a body Ω is given (open, regular with smooth boundary).
- Assume a surface traction t is given and let σ be a stress field that is in equilibrium with t.
- The *stress concentration factor* associated with the pair t, σ is

$$K_{t,\sigma} = \frac{\operatorname{ess\,sup}_x \{|\sigma(x)|\}}{\operatorname{ess\,sup}_y \{|t(y)|\}}, \quad x \in \Omega, \quad y \in \partial\Omega.$$

Generalized Stress Concentration (Continued)

- Denote by Σ_t the collection of all possible stress fields that are in equilibrium with t. (There are many such stress fields because material properties are not specified.)
- The *optimal stress concentration factor* for the force t is defined by

$$K_t = \inf_{\sigma \in \Sigma_t} \{K_{t,\sigma}\}.$$

- The *generalized stress concentration factor* K—a purely geometric property of Ω—is defined by

$$K = \sup_t \{K_t\} = \sup_t \inf_{\sigma \in \Sigma_t} \left\{ \frac{\operatorname{ess\,sup}_x \{|\sigma(x)|\}}{\operatorname{ess\,sup}_y \{|t(y)|\}} \right\}.$$

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Concerning the Generalized Stress Concentration Factor

Theorem

- *Define the generalized stress concentration factor K by*

$$K = \sup_t \frac{s_t^{\mathrm{opt}}}{\operatorname{ess\,sup}_{y\in\partial\Omega} |t(y)|}\,.$$

- *Then,*

$$K = \|\gamma\| = \sup_{w\in C^\infty(\overline{\Omega},\mathbb{R}^3)_0} \frac{\int_{\Gamma_t} |w|\,\mathrm{d}A}{\int_\Omega |\varepsilon(w)|\,\mathrm{d}V}.$$

- *Motivation again:* recall the principle of virtual work:

$$\int_\Omega \sigma_{ij}\varepsilon_{ij}\mathrm{d}V = \int_{\Gamma_t} t_i w_i \mathrm{d}A.$$

- We have an analogous expression for cases where the allowed load is applied in an a-priori known part of the boundary (or body).

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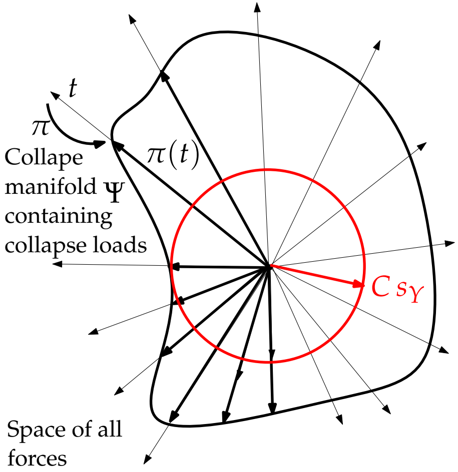
The Generalized Stress Concentration and the Load Capacity Ratio: Illustration



$$\Psi = \left\{ t \mid s_t^{\text{opt}} = s_Y \right\}, \quad \pi(t) = \frac{s_Y}{s_t^{\text{opt}}}\, t.$$

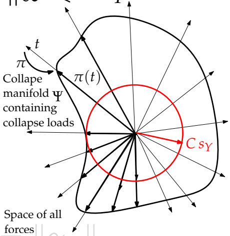
The Generalized Stress Concentration and the Load Capacity Ratio

- Given s_Y, consider the *collapse manifold*

$$\Psi = \left\{ t \mid s_t^{\text{opt}} = s_Y \right\}, \quad \text{with a projection } \pi(t) = \frac{s_Y}{s_t^{\text{opt}}}\, t$$

- Find the *load capacity ratio*

$$C = \frac{1}{s_Y} \inf_{t \in \Psi} \|t\|_\infty, \quad \Rightarrow \text{no collapse for } t \text{ with } \|t\|_\infty \leqslant C s_Y$$



- Easy to see that

$$C = \frac{1}{K}.$$

- The expression for K using the yield norms

$$K = \sup_{t \in L^\infty(\Gamma_t, \mathbb{R}^3)} s_t^{\text{opt}} = \sup_{w \text{ incomp}} \frac{\int_{\Gamma_t} |w| \, dA}{\int_\Omega |\varepsilon(w)| \, dV} = \|\gamma_D\|.$$

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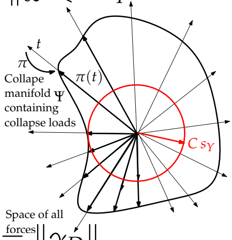

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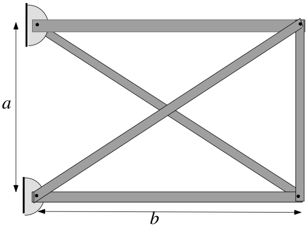
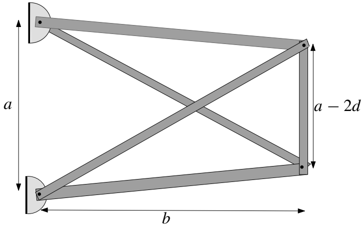
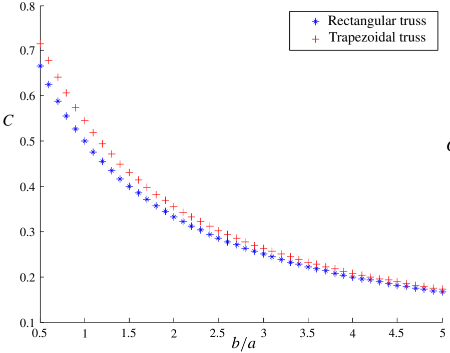
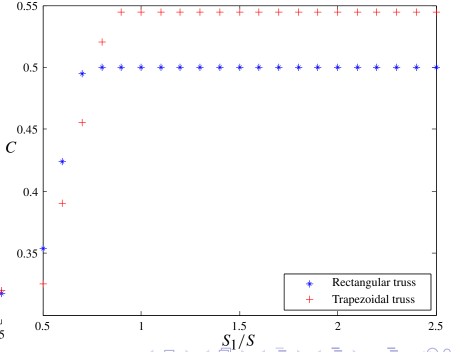
A Truss Example:

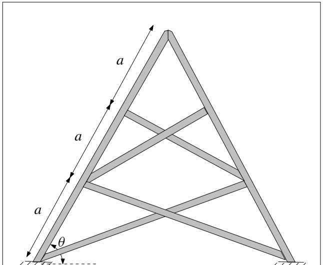








A Frame Example:



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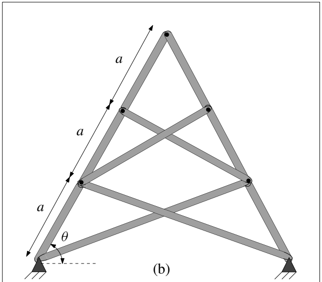

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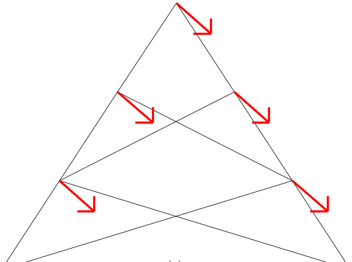
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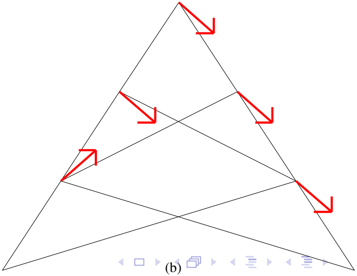
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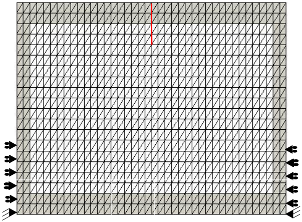
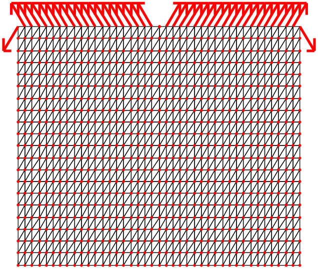
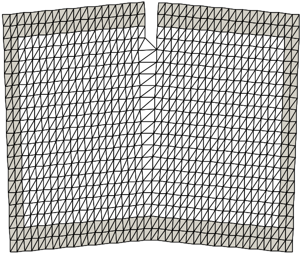
A Plane Stress Example:



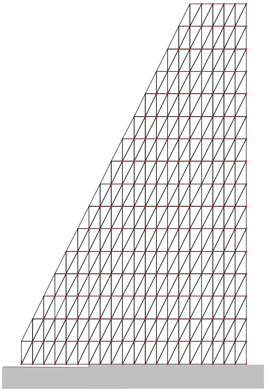


Distribution of a collapse load

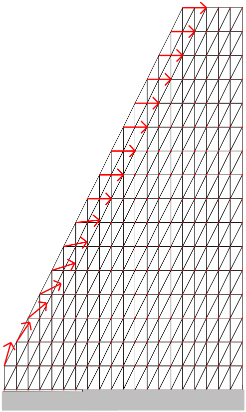


Maximizing virtual displacement

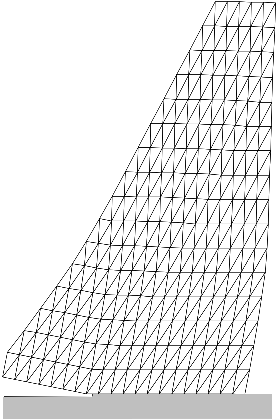
A Plane Strain Example:



A dam like structure



Collapse load



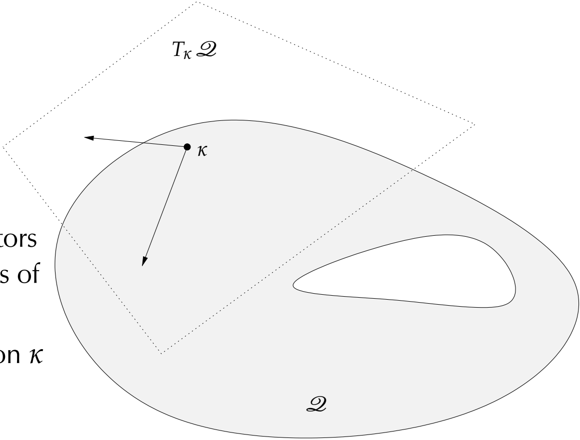
Maximizing virtual displacement

Tools Used in the Analysis

- Forces as linear functionals (all forces are generalized forces):
  - Representation theorems for forces
  - Equilibrium operator as a dual mapping
- Solutions of equilibrium equations using extensions of functionals Optimal extensions with Hahn-Banach Theorem.
- The right classes of functions: Sobolev spaces and LD-spaces
- Trace theorems

Force as Linear Functionals: Geometric Point of View

- The mechanical system is characterized by its configuration space—a manifold $\mathscr{Q}$.
- *Velocities* are tangent vectors to the manifold—elements of $T\mathscr{Q}$.
- A *Force* at the configuration κ is a continuous linear mapping $F\colon T_\kappa\mathscr{Q} \to \mathbb{R}$.



For a force F and a velocity w, the value $P = F(w)$ is interpreted as *mechanical power*.

Generalized Forces in Continuum Mechanics

- A *virtual velocity* w is a vector field on Ω.
- The tangent space at a point, the space of generalized velocities, is an infinite dimensional vector space $\mathscr{W}$. Forces are continuous linear functionals.
- One should specify the class of admissible vector fields and the norm used (or any other way to define the topology).
- Representation theorems will determine the nature of forces. Examples:
  - If we admit integrable vector fields with the L^1-norm, forces are represented by essentially bounded vector fields, $F(w) = \int_\Omega f \cdot w \, \mathrm{d}V$.
  - If we admit continuous vector fields with the supremum norm, forces are represented by measures.
- *The relevant class:* Vector fields whose components and corresponding linear strain (rate) are integrable over the body: LD-vector fields (a variation on the Sobolev spaces where all the derivatives are integrable).

$$\|w\| = \underbrace{\sum_i \int_\Omega |w_i| \, \mathrm{d}V}_{\text{omitted if rigidly supported}} + \sum_{i,j} \int_\Omega \left|\varepsilon(w)_{ij}\right| \, \mathrm{d}V.$$

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The Representation Theorem for the LD-Class:

- *The action of every force F may be represented by a tensor field σ in the form*

$$F(w) = \int_\Omega \sigma_{ij}\, \varepsilon(w)_{ij}\, \mathrm{d}V.$$

- *The tensor field σ representing F is not unique (unlike the previous examples). Implying: existence of stress and principle of virtual work.*
- *The norm of a linear operator such as F is defined as*

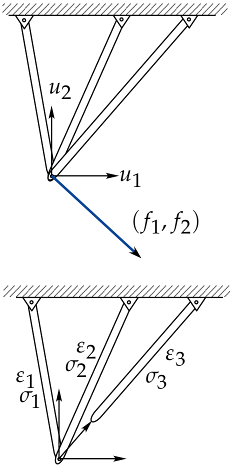
$$\|F\| = \sup_{w \in \mathscr{W}} \frac{|F(w)|}{\|w\|}.$$

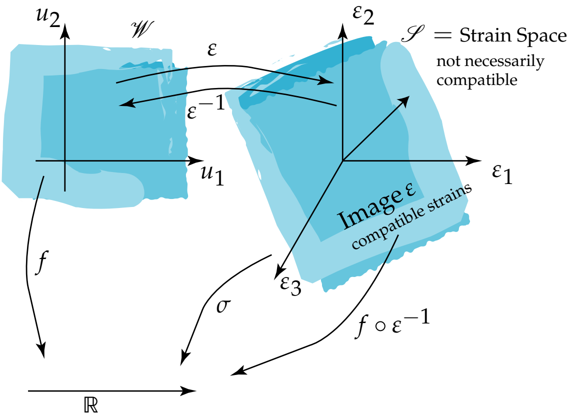
We have (using the Hahn-Banach Theorem),

$$\|F\| = \inf_\sigma \|\sigma\| = \inf_\sigma \{\operatorname{ess\,sup}_{x,i,j} |\sigma_{ij}(x)|\} = \|\sigma^{\mathrm{opt}}\|.$$

The inf *is taken over all tensor fields σ representing F.*

Extensions and Equilibrium





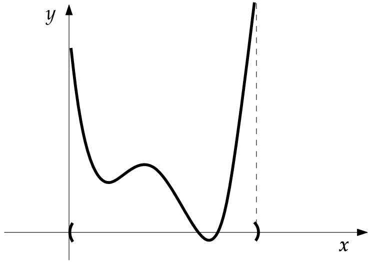
Extension of Functions and the Trace Mapping

How do you incorporate the boundary load?

- Differentiability of a function defined on an open set does not guarantee that it can be extended to the boundary.
- If the function has an integrable derivative, a Sobolev function, it may be extended to the boundary.
- For a vector field w, it is also sufficient that the corresponding linear strain $\varepsilon(w)$ has integrable components, an LD-vector field.
- The boundary values mapping

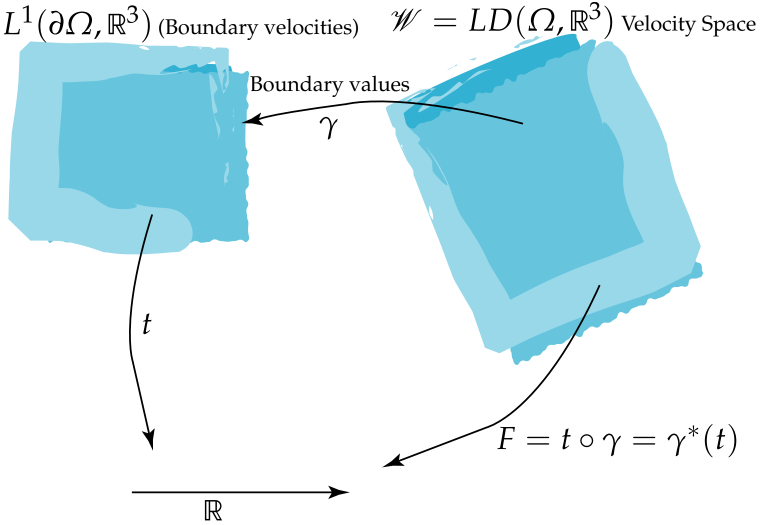
$$\gamma\colon LD(\Omega,\mathbb{R}^3) \longrightarrow L^1(\partial\Omega,\mathbb{R}^3)$$

is a well defined continuous, linear onto mapping.



Extension of a Funcion Defined on an Open Set

Forces on Ω Induced by Boundary Forces



Dual Mappings

$$x \in X \quad \xrightarrow{A} \quad Y \ni A(x)$$

$$A^*(g) \in X^* \xleftarrow{A^*} \quad Y^* \ni g$$

- Defined by $A^*(g)(x) = g(A(x)), \quad$ for all $\quad g \in Y^*, \ x \in X$.
- The condition $t(\gamma(w)) = F(w)$ may be written as $F = \gamma^*(t)$.
- Equilibrium, $\gamma^*(t)(w) = \sigma(\varepsilon(w))$, may be written as $\gamma^*(t)(w) = \varepsilon^*(\sigma)(w), \forall w$. Hence,

$$\text{Equilibrium} \quad \Longleftrightarrow \quad \gamma^*(t) = \varepsilon^*(\sigma).$$

- $\|A^*\| = \|A\|$,

$$\sup_{x \in X} \frac{\|A(x)\|}{\|x\|} = \sup_{g \in Y^*} \frac{\|A^*(g)\|}{\|g\|}.$$

Forces on Ω Induced by Boundary Forces

- The mapping $\gamma^* \colon (L^1(\partial\Omega, \mathbb{R}^3))^* \longrightarrow LD(\Omega)^*$ is injective. Thus, equilibrated surface forces t induce equilibrated forces $F = \gamma^*(t)$ on Ω, uniquely.
- We have

$$\|\gamma^*(t)\| = \sup_{w \in LD(\Omega)} \frac{|\gamma^*(t)(w)|}{\|w\|} = \sup_{w \in LD(\Omega)} \frac{|t(\gamma(w))|}{\|\varepsilon(w)\|_{L^1}}.$$

General Mathematical Structure

$$
\begin{array}{ccccc}
L^1(\Gamma_t, \mathbb{R}^3) & \xleftarrow{\;\gamma_0\;} & LD(\Omega)_0 & \xrightarrow{\;\varepsilon_0\;} & L^1(\Omega, \mathbb{R}^6) \\
\Big\| & & \Big\uparrow \iota & & \iota \Big\uparrow \Big\downarrow \pi_D^{\circ} \\
L^1(\Gamma_t, \mathbb{R}^3) & \xleftarrow{\;\gamma_D\;} & LD(\Omega)_D & \xrightarrow{\;\varepsilon_D\;} & L^1(\Omega, D)
\end{array}
$$

$$
\begin{array}{ccccc}
\text{boundary velocities} & \xleftarrow{\text{boundary value}} & \text{velocity fields} & \xrightarrow{\;\varepsilon_0\;} & \text{strain fields} \\
\Big\| & & \Big\uparrow \text{inclusion} & & \text{inclusion} \Big\uparrow \Big\downarrow \pi_D^{\circ} \\
L^1(\Gamma_t, \mathbb{R}^3) & \xleftarrow{\;\gamma_D\;} & \begin{array}{l}\text{incompressible} \\ \text{velocity fields}\end{array} & \xrightarrow{\;\varepsilon_D\;} & \begin{array}{l}\text{incompressible} \\ \text{strain fields}\end{array}
\end{array}
$$

General General Mathematical Structure - Continued

$$
\begin{array}{ccccc}
L^\infty(\Gamma_t, \mathbb{R}^3) & \xrightarrow{\gamma_0^*} & LD(\Omega)_0^* & \xleftarrow{\varepsilon_0^*} & L^\infty(\Omega, \mathbb{R}^6) \\
\Big\| & & \Big\downarrow \iota^* & & \iota^* \Big\downarrow \Big\uparrow \pi_D^{\circ *} \\
L^\infty(\Gamma_t, \mathbb{R}^3) & \xrightarrow{\gamma_D^*} & LD(\Omega)_D^* & \xleftarrow{\varepsilon_D^*} & L^\infty(\Omega, D).
\end{array}
$$

$$
\begin{array}{ccccc}
\text{boundary tractions} & \xrightarrow{\gamma_0^*} & \text{forces} & \xleftarrow{\varepsilon_0^*} & \text{stress fields} \\
\Big\| & & \Big\downarrow \text{inclusion}^* & & \text{restriction} \Big\downarrow \Big\uparrow \pi_D^{\circ *} \\
\text{boundary tractions} & \xrightarrow{\gamma_D^*} & \text{forces with deviatoric stresses} & \xleftarrow{\varepsilon_D^*} & \text{deviatoric stress fields}
\end{array}
$$

Properties of the Mappings

$$\begin{array}{ccccc}
\text{boundary velocities} & \xleftarrow{\gamma,\ \text{(b.v.)}} & \text{velocity fields} & \xrightarrow{\varepsilon_0} & \text{strain fields} \\
\Big\| & & \Big\uparrow \text{inclusion} & & \text{inclusion} \Big\uparrow\Big\downarrow \pi_D^\circ \\
L^1(\Gamma_t, \mathbb{R}^3) & \xleftarrow{\gamma_D} & \begin{array}{l}\text{incompressible}\\ \text{velocity fields}\end{array} & \xrightarrow{\varepsilon_D} & \begin{array}{l}\text{incompressible}\\ \text{strain fields}\end{array}
\end{array}$$

ε_0 – the strain mapping for velocity fields that satisfy the boundary conditions (zero on an open subset of the boundary).

Injective and norm preserving.

γ – the trace mapping. *Sujective.*

End

Appendix

Introducing $LD(\Omega)$ (Temam 85)

Recall: $\operatorname{ess\,sup}_x |\sigma(x)| = \|\sigma\|_\infty$ suggests:

$$\text{Stress Fields} = L^\infty(\Omega, \mathbb{R}^6) \quad \text{so} \quad \text{Stretching Fields} = L^1(\Omega, \mathbb{R}^6).$$

Conclusion:

$$\text{Body Velocities} = \left\{ w \colon \Omega \to \mathbb{R}^3 ; \varepsilon(w) \in L^1(\Omega, \mathbb{R}^6) \right\}.$$

Set

$$LD(\Omega) = \left\{ w \colon \Omega \to \mathbb{R}^3 ; w \in L^1(\Omega, \mathbb{R}^3), \varepsilon(w) \in L^1(\Omega, \mathbb{R}^6) \right\},$$

$$\|w\|_{LD} = \|w\|_1 + \|\varepsilon(w)\|_1.$$

Equivalent Norm for $LD(\Omega)$

- Let
$$\pi_{\mathscr{R}}\colon LD(\Omega) \longrightarrow \mathbb{R}^3 \times o(3)$$
be any projection on the space of rigid velocity fields on the body.
- An equivalent norm for $LD(\Omega)$:
$$\|w\|_{LD} = \|\pi_{\mathscr{R}}(w)\| + \|\varepsilon(w)\|_1.$$
- Displacement boundary conditions imply no rigid motion component:
$$\|w\| = \|\varepsilon(w)\|_1.$$
- $\varepsilon_0\colon LD(\Omega)_0 \longrightarrow L^1(\Omega, \mathbb{R}^6)$ *is norm preserving.*

Properties of $LD(\Omega)$

- *Approximations:* $C^\infty(\overline{\Omega}, \mathbb{R}^3)$ is dense in $LD(\Omega)$.
- *Traces:* There is a unique, continuous, linear trace mapping

$$\gamma \colon LD(\Omega) \longrightarrow L^1(\partial\Omega, \mathbb{R}^3)$$

such that $\gamma(u|_\Omega) = u|_{\partial\Omega}$, $u \in C(\overline{\Omega}, \mathbb{R}^3)$.

Proof of The Expression for the GSCF

We had

$$s_t^{\text{opt}} = \sup_{w \in LD(\Omega)_0} \frac{\left| \int_{\partial\Omega} t \cdot w \, \mathrm{d}A \right|}{\int_\Omega |\varepsilon(w)| \, \mathrm{d}V} = \sup_{w \in LD(\Omega)_0} \frac{|t(\gamma_0(w))|}{\|\varepsilon(w)\|_1},$$

$$= \sup_{w \in LD(\Omega)_0} \frac{|\gamma_0^*(t)(w)|}{\|w\|_{LD}} = \|\gamma_0^*(t)\|,$$

so,

$$K = \sup_{t \in L^\infty(\Gamma_t, \mathbb{R}^3)} \frac{s_t^{\text{opt}}}{\|t\|} = \sup_{t \in L^\infty(\Gamma_t, \mathbb{R}^3)} \left\{ \frac{\|\gamma_0^*(t)\|}{\|t\|} \right\} = \|\gamma_0^*\| = \|\gamma_0\|$$

where the last equality is the standard equality between the norm of a mapping and the norm of its dual.