Some Extensions and Analysis of Flux and Stress Theory

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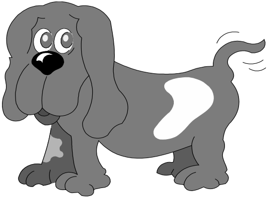
Generalized Bodies

The Material Structure Induced by an Extensive Property

Organisms

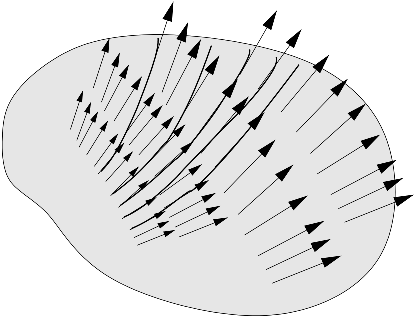
- *Material points, bodies and subbodies* are primitive concepts in continuum mechanics. These notions are somehow related to the conservation of mass.
- In growing bodies, material points are added and removed from the body.
- Examples: fingerprints, birthmarks are distinguishable.
- An *organism* has a body structure although mass is not preserved. Can formalize this idea?
- Assume we have an extensive property.



The Material Structure Induced by an Extensive Property

In the classical case we have the flux vector field $\boldsymbol{h}$. It can be integrated to give us a material structure.

A material point is identified with an integral line (a flow line). This procedure may induce material structure associated with any extensive property, e.g., color and energy.

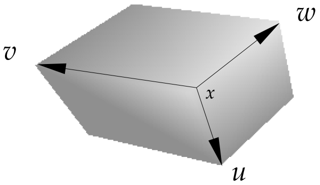


- $\dfrac{\boldsymbol{h}}{\rho}$ will be the velocity field of the material points.
- *Can we generalize the same idea for the general manifold case where the flow $(m-1)$-form replaces the vector field?*

The Case where a Volume Element is Specified

It is not necessary to have a metric structure in order that the flux form J be represented by a vector field.

Assume that you have a *volume element* θ (m-form) on $\mathscr{U}$. This may be thought of as the density of the property p if it is positive or another positive property, e.g., mass.



- *Given J and θ, find a vector v* such that for every pair of tangent vectors, u, w,

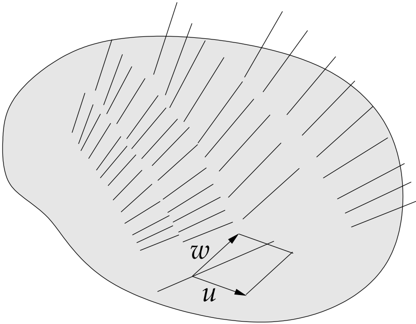
$$\theta(v, u, w) = J(u, w) \qquad \text{written as} \qquad J = v \lrcorner \theta.$$

- For a given θ there is a unique such vector v—*the kinematic flux*—a generalization of the velocity field.
- The vector field v depends linearly on the flux J.

The Flux Bundle

Let us examine how the kinematic flux v varies as we vary the volume element.

Since the space of m-forms at x is 1-dimensional, as we vary the volume element the resulting vectors v remain on a line (1-D subspace of the tangent space).

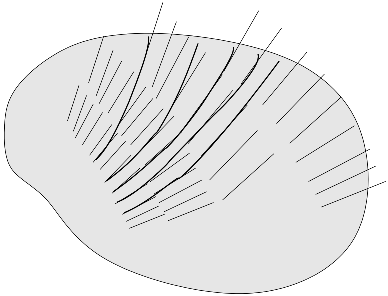


- Another characterization: *If a surface element (say the one defined by the vectors u, w) contains the line, the flux through it vanishes.*
- This is analogous to the situation with the velocity field.
- A collections of subspaces is referred to as a *distribution*. This distribution is the *flux bundle.*

Generalized Body Points

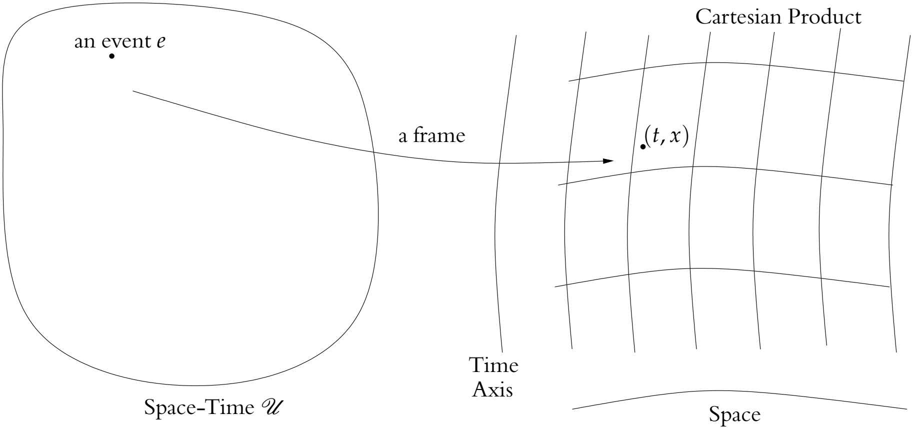
Integral manifolds of the distribution, the 1-dimensional flux bundle in this case, are submanifolds whose tangent space at a point is the corresponding line of the flux bundle at that point.

In general such integral manifolds need not exist (higher dimensions), however they always exist for 1-dimensional bundles as is the case here.



- Each integral line manifold is identified with a *body point*.
- Actual formulation is done on space-time manifold to allow time dependent fluxes. There β is included in τ and $dJ = s$.

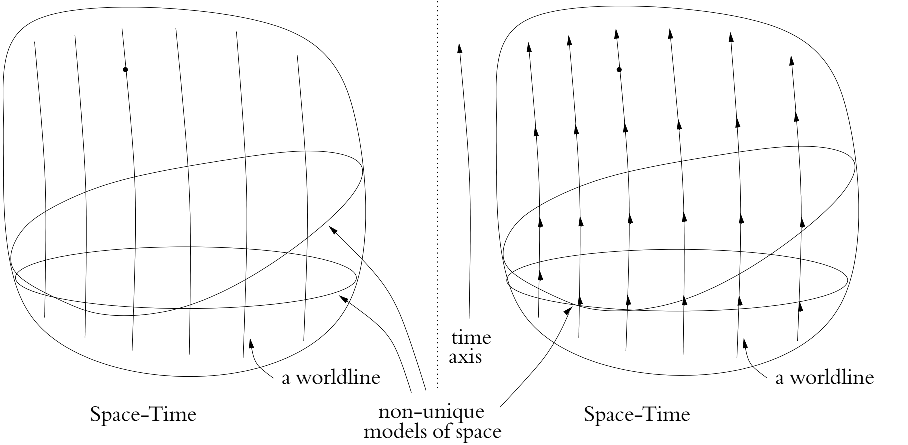
Frames in Space-Time



Property-Induced Fibration and Frame

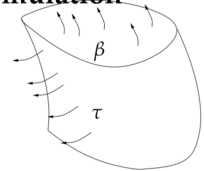
No volume element: Fibration
—no real valued time is assigned to events

A volume element: Integrable vector field
—real valued time is assigned to events



Space Formulation VS. Space-Time Formulation

	Space Formulation	**Space-Time Formulation**
	$\dim \mathscr{U} = 3$	$\dim \mathscr{E} = 4$
	$\dim \mathscr{B} = 3$	$\dim \mathfrak{R} = 4$
Balance	$\int_{\mathscr{B}} \beta + \int_{\partial\mathscr{B}} \tau = \int_{\mathscr{B}} s$	$\int_{\partial\mathfrak{R}} \mathfrak{t} = \int_{\mathfrak{R}} \mathfrak{s}$
surface term	2-form on a 3-D manifold	3-form on a 4-D manifold
source term	3-form on a 3-D manifold	4-form on a 4-D manifold
flux form	J—3 components	$\mathfrak{J}$—4 components
variables	—time dependent	—fixed values at events
field equation	$\beta + dJ = s$	$d\mathfrak{J} = \mathfrak{s}$



Flow Potentials

- Although we do not have vector velocity fields, we have material points.
- In addition, we have analogs for the flow potentials.
- In the case $s = 0$ we obtain (say the 4-D case) $dJ = 0$.
- Assume that A is any $(m-2)$-form on $\mathscr{U}$. Then, $J = dA$ satisfies the differential balance equation—A is a *flow potential*. Since in general,

$$\int_{\partial M} \iota^* \omega = \int_M d\omega,$$

for every control region $\mathscr{B}$

$$\int_{\mathscr{B}} dJ = \int_{\partial \mathscr{B}} \iota^*(J) = \int_{\partial \mathscr{B}} \iota^*(dA) = \int_{\partial(\partial \mathscr{B}) = \varnothing} \iota^*\left(\iota^*(A)\right) = 0.$$

Summary: The Structure on Space-Time manifold Associated with an Extensive Property

- Balance laws are formulated in terms of forms.
- The flux vector field is replaced by a flux $(m-1)$-form in the m-dimensional space.
- Flow lines still make sense using the flux bundle.
- Generalized body points may be associated with an arbitrary extensive property—*organisms*.
- A particularly compact formulation in space-time.
- A positive extensive property induces a material frame.

Stresses for Generalized Bodies

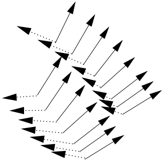
Forces for Generalized Bodies

- Force densities are linear mappings on the values of the generalized velocities.
- In the case where a material structure is induce by an extensive property and a volume element is given, the induced generalized velocity w depends linearly on the flux form J.
- It would be a natural generalization to replace generalized velocities by flux forms as fields on which forces operate to produce power.
- The physical dimension of forces will not be power per unit velocity but power per per unit flux of the property p.
- For the spacetime formulation $F_{\mathscr{B}}(J) = \int_{\partial\mathscr{B}} t_{\mathscr{B}}(J), \quad \mathscr{B} \subset \mathscr{E}$.
- $t_{\mathscr{B}}(e)\colon \bigwedge^{m-1} T_e^*\mathscr{E} \to \bigwedge^{m-1} T_e^*\partial\mathscr{B}$.

Stresses for Generalized Bodies

- Consider the energy extensive property. It has a flux density term $\int_{\partial\mathscr{B}} \tau^{(e)}$ and a corresponding flux form $J^{(e)}$ such that $\tau^{(e)} = \iota^* \circ J^{(e)}$.
- On the other hand the flux density of energy may be written in terms of the boundary force as $t_{\mathscr{B}}(J)$.
- Cauchy's theorem implies that $t_{\mathscr{B}} = \iota^* \circ \sigma$ so the energy flux density is $\tau^{(e)} = \iota^* \circ J^{(e)} = \iota^* \circ \sigma(J)$. Hence,

$$J^{(e)} = \sigma(J).$$



The Cauchy stress is the linear mapping that transforms the flux of the property p into the flux of energy.

- $\sigma_e \colon \bigwedge^{m-1} T_e^* \mathscr{E} \to \bigwedge^{m-1} T_e^* \mathscr{E}$. *The stress at a point (event) is a linear transformation on the space of $(m-1)$-forms.*
- May be applied to "resources" other then energy?

Local Representation of Stress-Tensors

- Denote by $\{\hat{e}^i\}$ the basis of the m-dimensional space of $(m-1)$-forms. Denote its dual basis by $\{\hat{e}_j\}$.
- Since the stress at a point is a linear transformation on the space of $(m-1)$-forms it may be represented in the form $\hat{\sigma}_i^{\,j}\,\hat{e}_j \otimes \hat{e}^i$.
- If we had a volume element θ we would have an isomorphism $\bigwedge^{m-1}(T^*\mathscr{U}) \leftrightarrow T\mathscr{U}$ of $(m-1)$-forms and vectors, such that $J \leftrightarrow v$ are given by $\theta(v,u,w) = J(u,w)$.
- Thus, with a volume element and due to the following structure,

$$\begin{array}{ccc} \bigwedge^{m-1}(T^*\mathscr{U}) & \xrightarrow{\ \sigma\ } & \bigwedge^{m-1}(T^*\mathscr{U}) \\ {\scriptstyle i_\theta^{-1}}\uparrow & & \downarrow{\scriptstyle i_\theta} \\ T\mathscr{U} & \xrightarrow{\ \tilde{\sigma}\ } & T\mathscr{U}, \end{array}$$

  one may represent a stress σ by a linear transformation $\tilde{\sigma}$ on $T\mathscr{U}$.
- *Surprisingly, $\tilde{\sigma}$ is independent of the volume element θ. In fact, you can construct a natural isomorphism $\sigma \leftrightarrow \tilde{\sigma}$ without a volume element.*

Maxwell Stress-Energy Tensor without a Metric

- Maxwell 2-form: $\mathfrak{g}$, a flow potential for J, i.e., $J = d\mathfrak{g}$.
- Faraday 2-form: $\mathfrak{f}$ such that $d\mathfrak{f} = 0$.
- Assume a volume element and set $w = i_\theta(J)$ to be the vector field representing the flux form.
- define the stress-energy tensor as the section σ of $L\big(\bigwedge^{m-1}(T^*\mathscr{U}), \bigwedge^{m-1}(T^*\mathscr{U})\big)$ by

$$\sigma(J) = \big(w \lrcorner\, \mathfrak{g}\big) \wedge \mathfrak{f} - \big(w \lrcorner\, \mathfrak{f}\big) \wedge \mathfrak{g}.$$

- The power is

$$d\sigma(J) = (w \lrcorner\, \mathfrak{f}) \wedge J + (\mathscr{L}_w \mathfrak{g}) \wedge \mathfrak{f} - (\mathscr{L}_w \mathfrak{f}) \wedge \mathfrak{g}.$$

—a generalization of the Lorentz force $(w \lrcorner\, \mathfrak{f}) \wedge J$. ($\mathscr{L}$ is the Lie derivative.) The two additional terms cancel in the traditional situation.