Some Extensions and Analysis of Flux and Stress Theory

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Structures of the Mechanics of Complex Bodies
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It is a great pleasure and honor for me to invited to the
CITY OF PISA,
to the distinguished
SCUOLA NORMALE SUPERIORE,
to deliver these lectures at
the Center for Mathematical Research
in memory of the great mathematician
ENNIO DE GIORGI.

Many thanks to the organizers,
Reuven

Introduction

Objects of Interest

- Mathematical aspects of the theories of fluxes and stresses, particularly, *existence theory*.
- *Geometric aspects:* Formulations that do not use the traditional geometric and kinematic assumptions. For example, Euclidean structure of the physical space, mass conservation. Materials with micro-structure (sub-structure), growing bodies.
- *Analytic aspects:* Irregular bodies and flux fields. Fractal bodies.

Main Tool: *Various aspects of duality*

Topics

- Scalar-Valued Extensive Properties and Fluxes on Manifolds,
- Fluxes and Geometric Integration Theory: Fractal Bodies,
- The Material Structure Induced by an Extensive Property,
- Forces and Cauchy Stresses—Geometric Aspects,
- Variational Stresses,
- Stresses for Generalized Bodies,
- Stress Optimization, Stress Concentration, and Load Capacity.

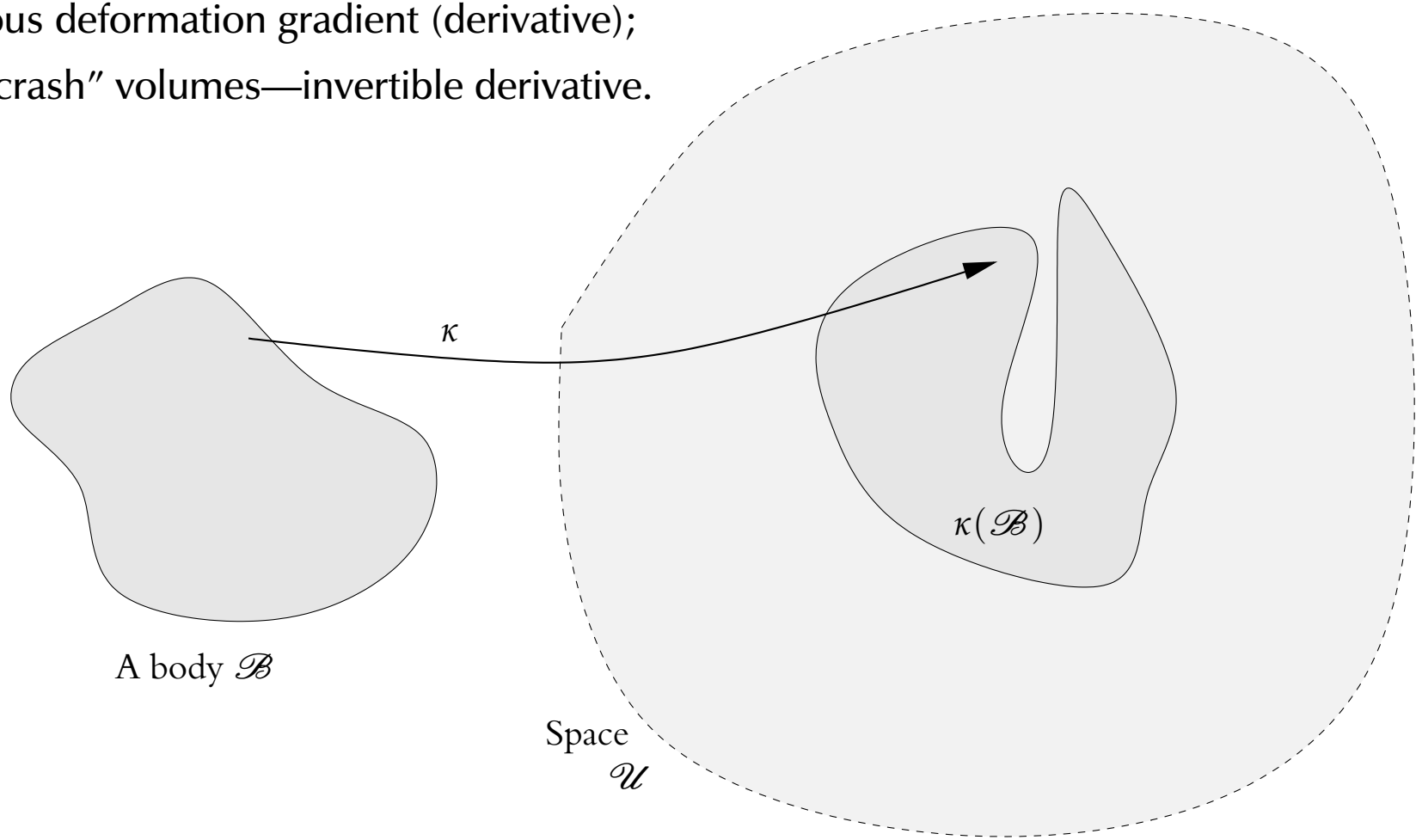
And maybe also

- The Global Point of View: C^1-Functionals,
- Locality and Continuity in Constitutive Theory.

Notation: Basic Variables of Continuum Mechanics

Kinematics

- A mapping of the body into space;
- material impenetrability—one-to-one;
- continuous deformation gradient (derivative);
- do not "crash" volumes—invertible derivative.



Fluxes: Traditional Approach

In terms of scalar extensive property p with density ρ in space, one assumes for every "control region" $\mathscr{B} \subset \mathscr{U} \cong \mathbb{R}^3$:

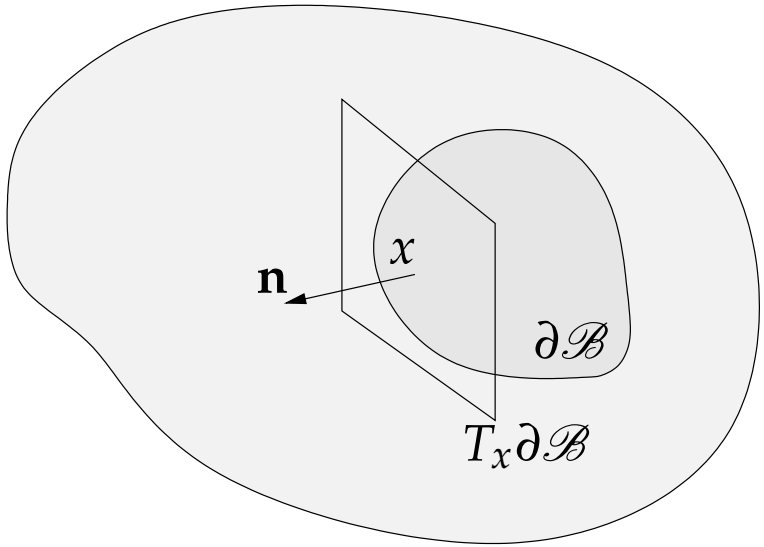
- Consider β, interpreted as the *time derivative* of the density ρ of the property, so for any control region $\mathscr{B}$ in space, $\int_{\mathscr{B}} \beta \mathrm{d}V$ is the rate of change of the property inside $\mathscr{B}$.
- For each control region $\mathscr{B}$ there is a *flux density* $\tau_{\mathscr{B}}$ such that $\int_{\partial\mathscr{B}} \tau_{\mathscr{B}} \mathrm{d}A$ is the total flux of the property out of $\mathscr{B}$.
- There is a positive m-form s on $\mathscr{U}$ such that for each region $\mathscr{B}$

$$\left| \int_{\mathscr{B}} \beta \, \mathrm{d}V + \int_{\partial\mathscr{B}} \tau_{\mathscr{B}} \, \mathrm{d}A \right| \leq \int_{\mathscr{B}} s \, \mathrm{d}V.$$

  Usually, equality is assume to hold (no absolute value) and s is interpreted as the *source density* of the property p (e.g., $s = 0$ for mass).

Fluxes: Traditional Cauchy Postulate and Theorem

Cauchy's postulate and theorem are concerned with the dependence of $\tau_{\mathscr{B}}$ on $\mathscr{B}$.



- It uses the metric properties of space.
- $\tau_{\mathscr{B}}(x)$ is assumed to depend on $\mathscr{B}$ only through the unit normal to the boundary at x. Generalize this to dependence on $T_x\partial\mathscr{B}$.
- The resulting Cauchy theorem asserts the existence of the flux vector $\boldsymbol{h}$ such that $\tau_{\mathscr{B}}(x) = \boldsymbol{h} \cdot \boldsymbol{n}$.

Cauchy's Theorem for Fluxes on Manifolds

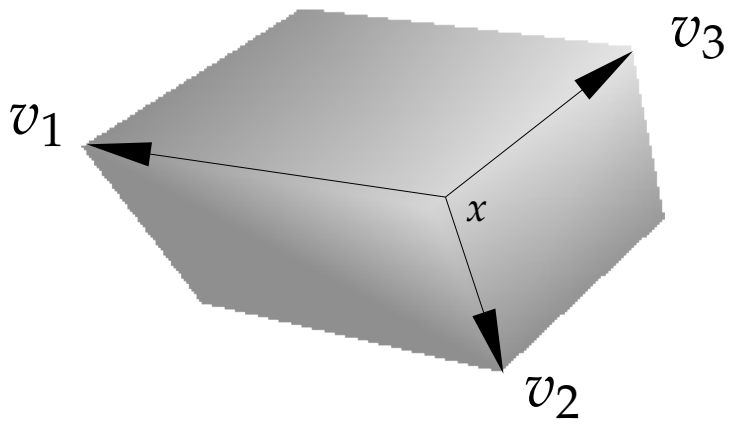
Scalar-Valued Extensive Properties

We will consider the generalization of the classical analysis to the geometry of differentiable manifolds where no particular metric is given. The concepts introduced will be useful later in the analytic generalizations.

Consider for example the heat flux field in a body. This will enable us to treat the *Cauchy heat flux* (defined relative to the current configuration of the body) and the *Piola-Kirchhoff heat flux* (defined relative to the reference configuration of the body) as two representations of a single mathematical entity. Clearly, a *vector field is not* the right mathematical object.

Integration: Volume Elements

An infinitesimal element defined by the tangent vectors $v_1, v_2, v_3 \in T_x\mathscr{U}$, $\mathscr{U}$—the space (3-dimensional) manifold.



- For a given property p, $\rho_x(v_1, v_2, v_3)$—the amount of the property in the element. $\rho_x : (T_x\mathscr{U})^3 \to \mathbb{R}$.
- ρ_x should be linear in each of the three vectors—ρ_x multi-linear.
- $\rho_x(v_1, v_2, v_3)$ should vanish if the three are not linearly independent (flat element). Hence, for example, since $\rho_x(v+u, v_2, v+u) = 0$

$$\begin{aligned} 0 &= \rho_x(v, v_2, v) + \rho_x(u, v_2, u) + \rho_x(v, v_2, u) + \rho_x(u, v_2, v) \\ &= \rho_x(v, v_2, u) + \rho_x(u, v_2, v). \end{aligned}$$

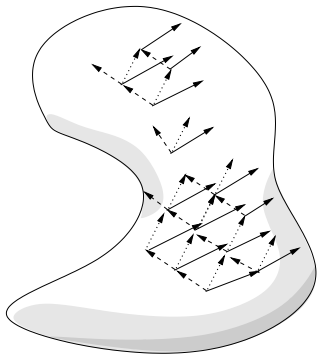
ρ_x is *anti-symmetric (alternating)*, i.e., $\rho_x(v, v_2, u) = -\rho_x(u, v_2, v)$!

Integration: Volume Elements and m-Forms

For a manifold $\mathscr{U}$ of dimension m integration for the total quantity of the property p is defined using alternating forms.

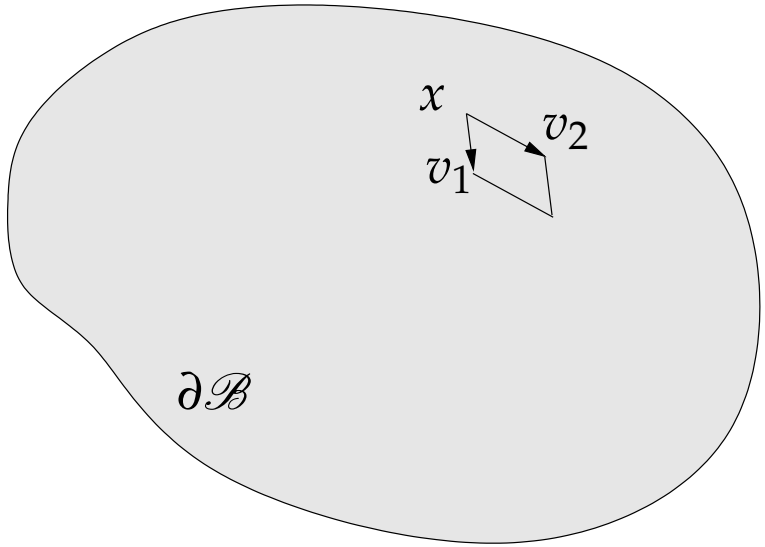
- $\bigwedge^m T_x^*\mathscr{U}$ is the collection of m-alternating multi-linear mappings on $T_x\mathscr{U}$. $\bigwedge^m(T^*\mathscr{U}) = \bigcup_{x\in\mathscr{U}} \bigwedge^m T_x^*\mathscr{U}$ is the bundle of m-multi-linear alternating forms on $\mathscr{U}$.
- An m-differential form $\rho\colon \mathscr{U} \to \bigwedge^m(T^*\mathscr{U})$, or a *volume element* (not the infinitesimal elements generated by the vectors), $\rho(x) \in \bigwedge^m T_x^*\mathscr{U}$ is integrated to give the sum of the contents of the extensive property in the various infinitesimal elements in any region $\mathscr{B} \subset \mathscr{U}$,

$$\int_{\mathscr{B}} \rho.$$



Integrating an $(m-1)$-Form over the Boundary: Flux Density

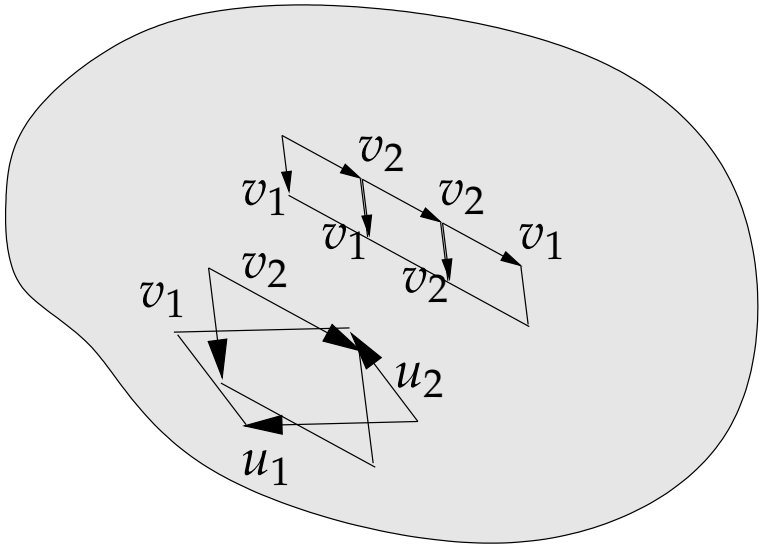
An *infinitesimal area element* is defined by the tangent vectors $v_1, v_2 \in T_x\partial\mathscr{B}$, $\partial\mathscr{B}$—the boundary (say 2-dimensional) of a control region $\mathscr{B}$.



- *For a given property p, we would like to integrate the flux density out of the boundary. Now* $\tau_x(v_1, v_2)$*—the flux through the the element.* $\tau_x\colon (T_x\partial\mathscr{B})^2 \to \mathbb{R}$.
- *Since* $\partial\mathscr{B}$ *is an* $(m-1)$*-dimensional manifold, the flux density is a mapping* $\tau\colon \partial\mathscr{B} \to \bigwedge^{m-1} T^*\partial\mathscr{B}$*, an* $(m-1)$*-form on* $\partial\mathscr{B}$.

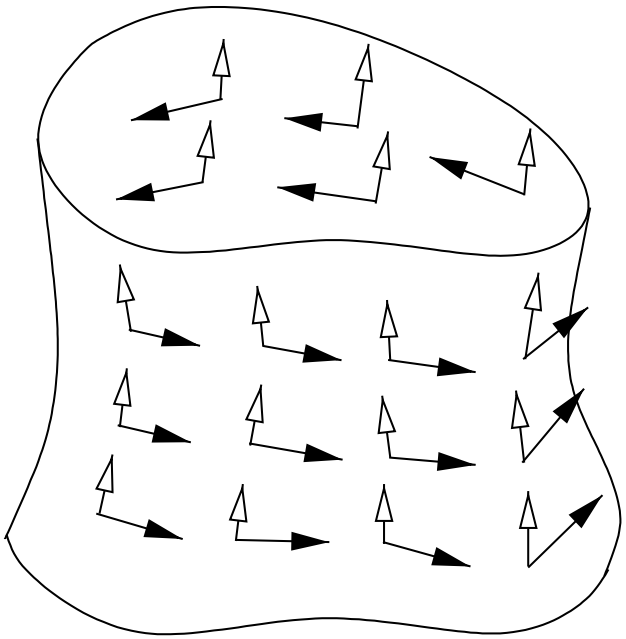
Orientation

The fact that the volume element is anti-symmetric causes a complication. The sign of the evaluation $\tau(v_1, v_2)$ (or $\rho(v_1, v_2, v_3)$) will change according to the way we order the vectors.

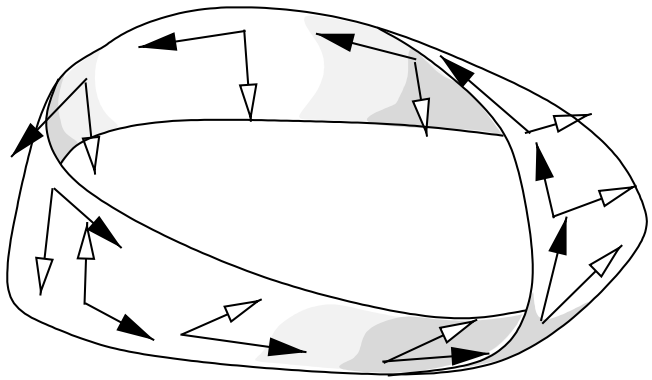


- *Orientability*—the ability to construct the various coordinate systems such that the Jacobian transformation matrix has a positive determinant.
- This is equivalent to the ability to construct *a volume element that does not vanish at any point on the manifold*.
- A choice of such a form, say θ, determines an *orientation* on the manifold. If $\theta(v_1, \ldots, v_m) > 0$, the vectors are *positively oriented*.

An orientable manifold



and a non-orientable manifold



The Balance of an Extensive Property

For an oriented manifold $\mathscr{U}$ of dimension m we consider *control regions*, m-dimensional compact submanifolds with boundary.

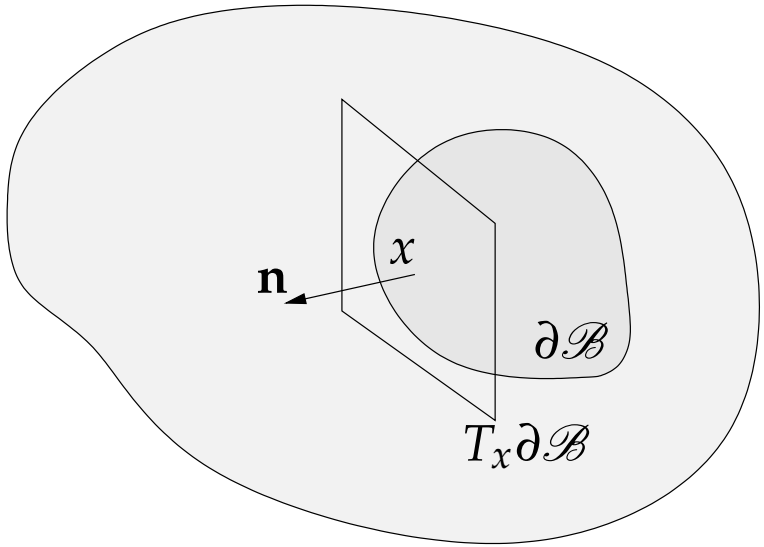
- ρ is time dependent with time-derivative β. For a fixed control region $\mathscr{B}$ in space $\int_{\mathscr{B}} \beta$ is the rate of change of the property inside $\mathscr{B}$.
- For each control region $\mathscr{B}$ there is a flux density $\tau_{\mathscr{B}}$ such that $\int_{\partial\mathscr{B}} \tau_{\mathscr{B}}$ is the total flux of the property out of $\mathscr{B}$.
- There is a positive m-form s on $\mathscr{U}$ such that for each region $\mathscr{B}$

$$\left| \int_{\mathscr{B}} \beta + \int_{\partial\mathscr{B}} \tau_{\mathscr{B}} \right| \leq \int_{\mathscr{B}} s.$$

Usually, equality is assume to hold (no absolute value) and s is interpreted as the *source density* of the property p.

Review of the Classical Cauchy Postulate and Theorem

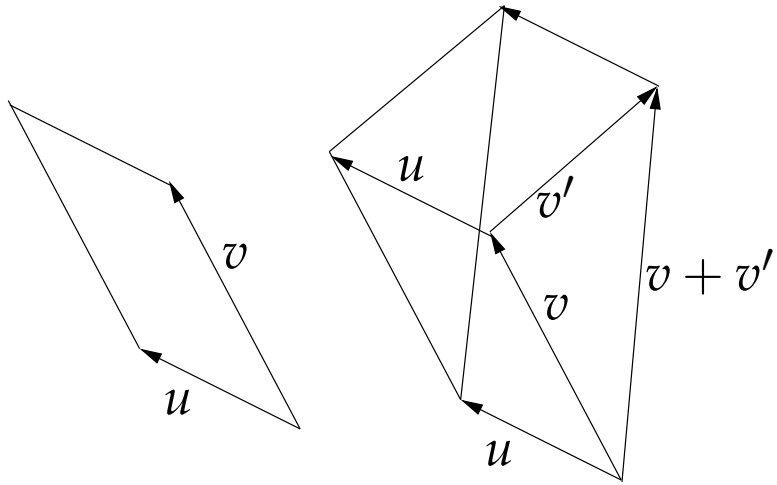
Cauchy's postulate and theorem are concerned with the dependence of $\tau_{\mathscr{B}}$ on $\mathscr{B}$.



- It uses the metric properties of space.
- $\tau_{\mathscr{B}}(x)$ is assumed to depend on $\mathscr{B}$ only through the unit normal to the boundary at x. Generalize this to dependence on $T_x\partial\mathscr{B}$.
- The resulting Cauchy theorem asserts the existence of the flux vector $\boldsymbol{h}$ such that $\tau_{\mathscr{B}}(x) = \boldsymbol{h} \cdot \mathbf{n}$.

The Generalization of Cauchy's Theorem
$(m-1)$-Forms on an m-Dimensional Manifold

For the 3-dimensional example, we want to measure the flux through any infinitesimal surface element (on the various planes through x), say the one generated by the vectors $v,\ u$.



Denote by $J(v,u)$ the flux through that infinitesimal element.

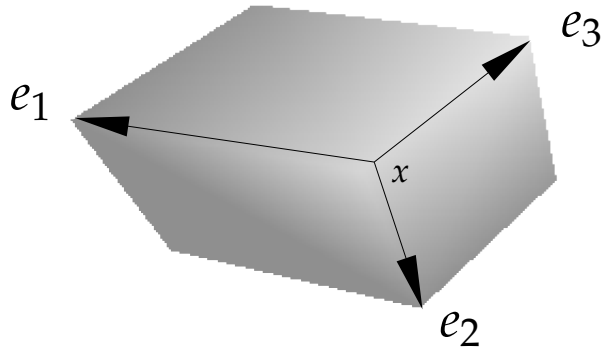
- $J(v,u)$ should be linear in both arguments—J is multilinear.
- $J(v,u)$ should vanish it they are not linearly independent—J is *alternating*.

Conclusion:

J should be a 2-form in a 3-dimensional space, or generally, an $(m-1)$-form on an m-dimensional manifold.

The Dimension of the Space of m-Forms

Say $\{e_1, e_2, e_3\}$ is a base of the tangent space at a fixed point x. The matrix of ρ is $\rho_{ijk} = \rho(e_i, e_j, e_k)$.



- However, because it is alternating, ρ has only one independent component, e.g., $\rho_{ijk} = 0$ if any two indices are equal.
- It is enough to know $\rho_{123} = \rho(e_1, e_2, e_3)$, the volume of the basic element, to know the amount of property in all other infinitesimal elements.

In general, the dimension of $\bigwedge^m(T^*_x\mathscr{U})$ *is* 1.

The Dimension of the Space of $(m-1)$-Forms

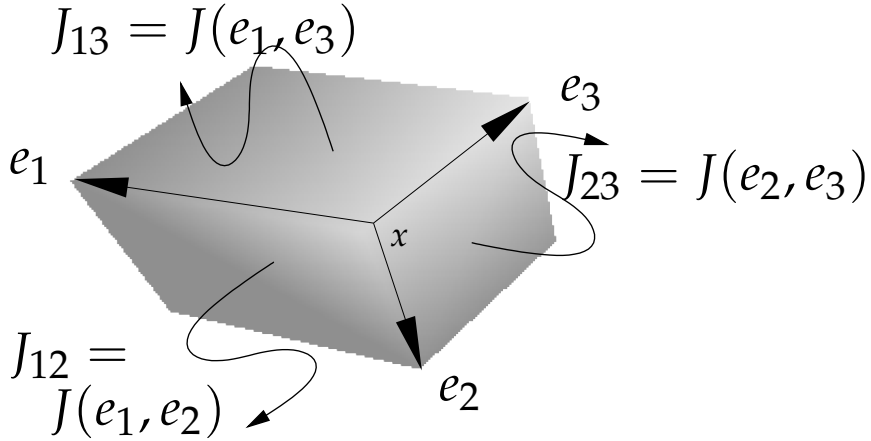
- Again, $\{e_1, e_2, e_3\}$ is a base of the tangent space at x. The matrix of the 2-form J is $J_{ij} = J(e_i, e_j)$.
- Now, as J is alternating there are 3 different independent components, namely, $J(e_2, e_3)$, $J(e_1, e_3)$, $J(e_1, e_2)$.

In general, the dimension of $\bigwedge^{m-1} T_x^* \mathscr{U}$ is m.

In other words, if we know the flux density through the three basic surface elements we know the flux through any other infinitesimal surface element.

$$J(u, v) = J_{ij} u_i v_j.$$

The three components of the flux 2-form are the generalizations of the three components of the flux vector field.

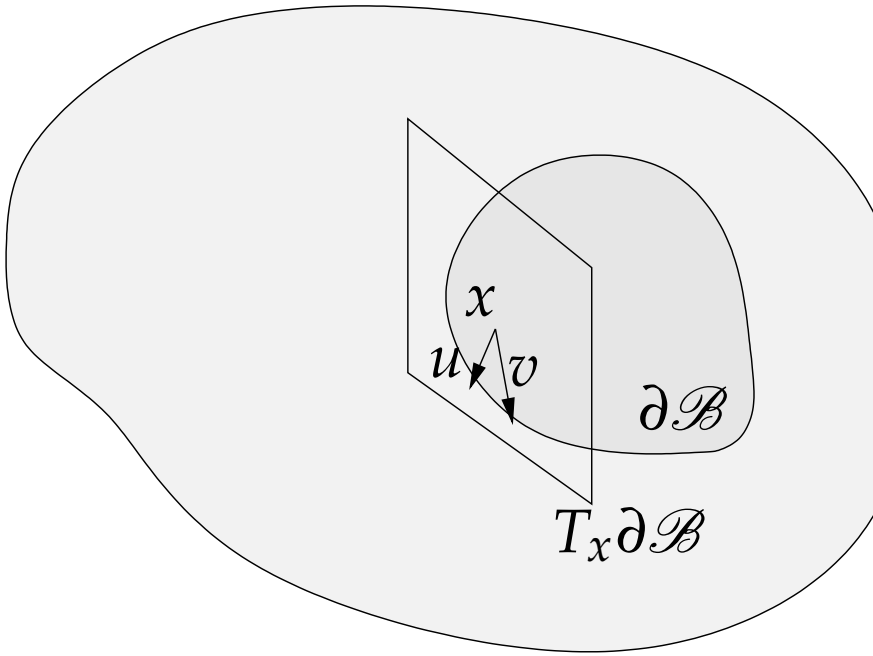


Cauchy's Formula and the Restriction of Forms

The $(m-1)$-form J on $\mathscr{U}$ (m components) induces by restriction an $(m-1)$-form τ on $\partial\mathscr{B}$.

- τ is given by

$$\tau(v,u) = J(v,u).$$



The induced form τ has a single component as it is an $(m-1)$-form on the $(m-1)$-dimensional manifold $\partial\mathscr{B}$. The mapping that assigns τ to J is the *restriction* and it is denoted as

$$\tau = \iota^*(J).$$

This equation is the required generalization of Cauchy's formula.

Inclusion and Restriction

The inclusion

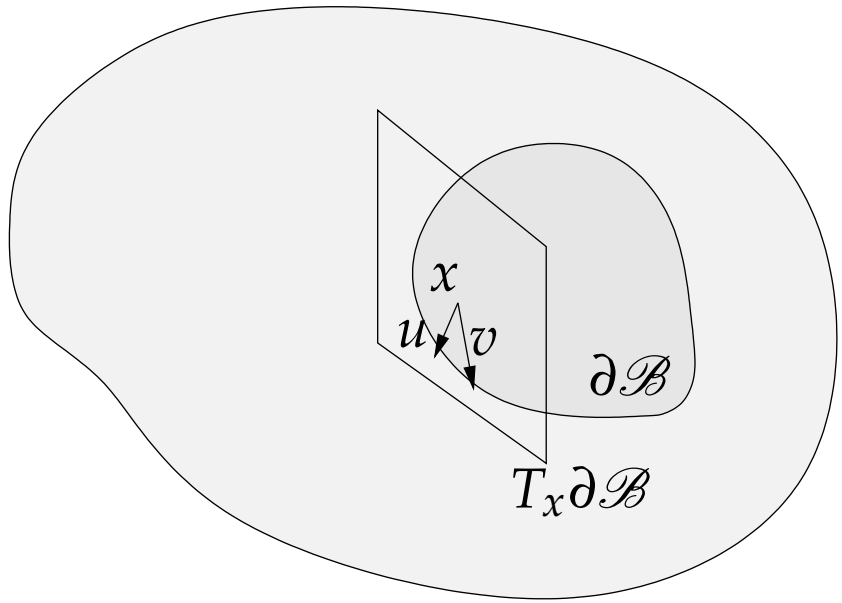
$$\iota : T_x\partial\mathscr{B} \times T_x\partial\mathscr{B} \to T_x\mathscr{B} \times T_x\mathscr{B}$$

induces the dual restriction mapping

$$\iota^* : (T_x\mathscr{B} \times T_x\mathscr{B})^* \to (T_x\partial\mathscr{B} \times T_x\partial\mathscr{B})^*,$$

which restricts to the mapping

$$\iota^* : \bigwedge^2 T_x^*\mathscr{B} \to \bigwedge^2 T_x\partial\mathscr{B}.$$



In the general m-dimensional case,

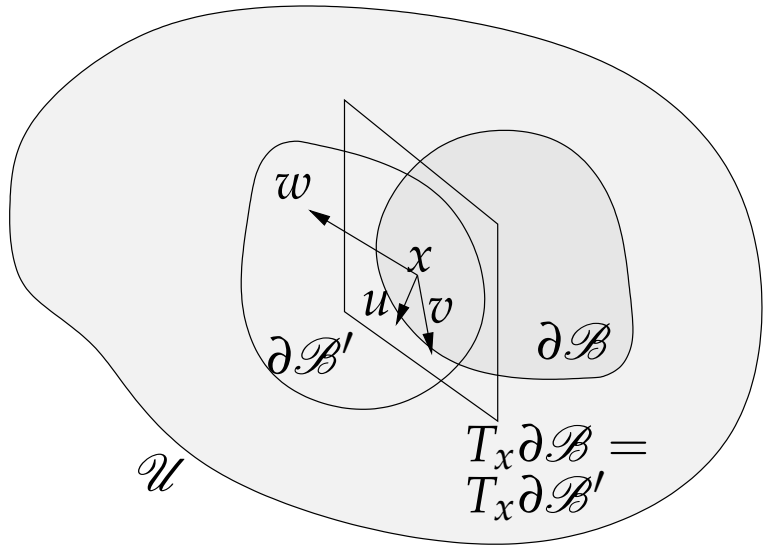
$$\iota^* : \bigwedge^{m-1} T_x^*\mathscr{B} \to \bigwedge^{m-1} T_x\partial\mathscr{B}$$

used in Cauchy's formula $\tau = \iota^*(J)$.

The Induced Orientation and Newton's Third Law

Now, $\mathscr{B}'$ has the same tangent space at x as $\mathscr{B}$. w is a vector pointing out of $\mathscr{B}$ (into $\mathscr{B}'$). The form $\iota^*(J)$ is one for both $\mathscr{B}$ and $\mathscr{B}'$.

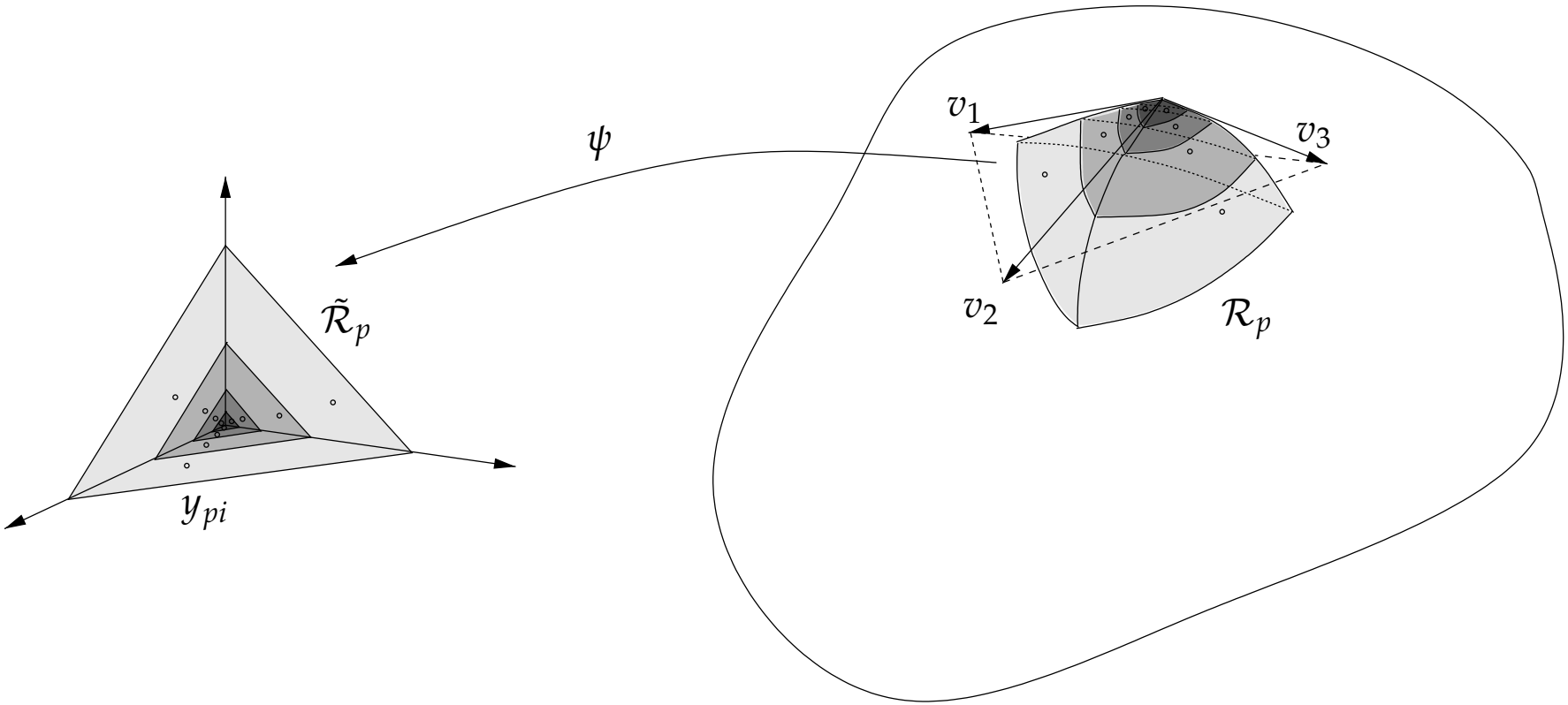
How do we distinguish the surface flux densities $\tau_{\mathscr{B}}$ and $\tau_{\mathscr{B}'}$?



- It was assumed that $\mathscr{U}$ was *oriented* so there is a way to tell whether any ordered triplet $\{u, v, w\}$ is positively or negatively oriented.
- This induces an *orientation on the boundary* of each region. At $x \in \partial\mathscr{B}$, take any outwards (relative to $\mathscr{B}$) pointing vector w and set $\{u, v\}$ to be positively oriented on $\partial\mathscr{B}$ if $\{w, u, v\}$ is positively oriented in $\mathscr{U}$.
- Hence, the orientation on $\partial\mathscr{B}'$ is opposite to that of $\partial\mathscr{B}$. Thus, if $J(u, v)$ is the flux out of the infinitesimal $\mathscr{B}$-*positively oriented* element $\{u, v\}$, the flux out of $\mathscr{B}'$ for the same geometric element is $J(v, u) = -J(u, v)$.

Notes on the Proof:

The proof is analogous to the proof of the classical version, using the image under a chart of a simplex.



Stokes' Theorem and the Differential Balance Law

- The boundary integral in the balance law

$$\int_{\mathscr{B}} \beta + \int_{\partial\mathscr{B}} \tau_{\mathscr{B}} = \int_{\mathscr{B}} s$$

of the property p assumes now the form

$$\int_{\partial\mathscr{B}} \tau_{\mathscr{B}} = \int_{\partial\mathscr{B}} \iota^*(J).$$

Stokes' theorem *(a generalization of the divergence theorem etc.): There is an m-form dJ (having a single component and calculated like the divergence of a vector field), such that*

$$\int_{\partial\mathscr{B}} \iota^*(J) = \int_{\mathscr{B}} dJ.$$

- Then, for each $\mathscr{B}$, the balance takes the form

$$\int_{\mathscr{B}} \beta + \int_{\mathscr{B}} dJ = \int_{\mathscr{B}} s, \quad \text{hence,} \quad \beta + dJ = s.$$