Empirical Aspects of Plurality Elections

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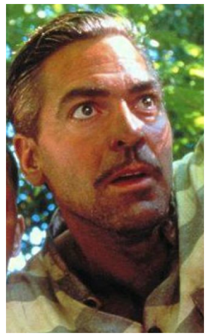
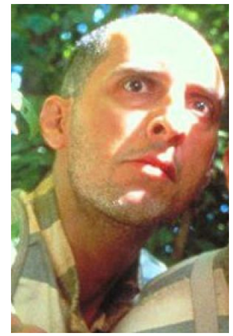
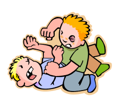
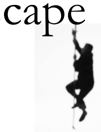
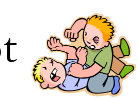
What is a (pure) Nash Equilibrium?

A **solution concept** involving games where all players know the strategies of all others. If there is a set of strategies with the property that no player can benefit by changing her strategy while the other players keep their strategies unchanged, then that set of strategies and the corresponding payoffs constitute the **Nash Equilibrium.**

Adapted from Roger McCain's *Game Theory: A Nontechnical Introduction to the Analysis of Strategy*

What is a Nash Equilibrium?

Example: voting prisoners' dilemma…

	 **Everett**	 **Pete**	 **Delmar**
1st preference	Escape 	Riot 	Stay in prison 
2nd preference	Riot 	Escape 	Escape 
3rd preference	Stay in prison 	Stay in prison 	Riot 

What is a Nash Equilibrium?

Example: voting prisoners' dilemma…

Everett

Pete

Delmar

Suppose tie is broken by deciding to stay in prison

	Everett	Pete	Delmar
1st preference	Escape	Riot	Stay in prison
2nd preference	Riot	Escape	Escape
3rd preference	Stay in prison	Stay in prison	Riot

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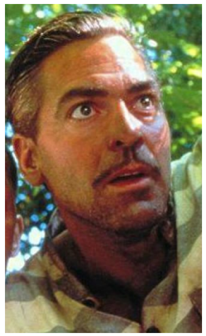
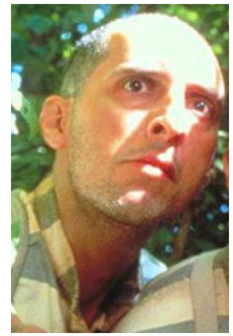
	 **Everett**	 **Pete**	 **Delmar**
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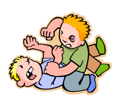
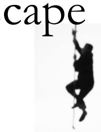
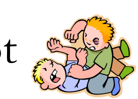
2nd preference	Riot 	Escape 	Escape 
3rd preference	Stay in prison 	Stay in prison 	Riot 

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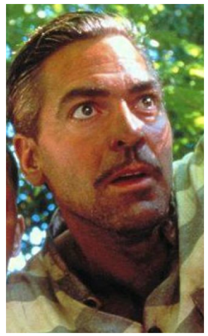
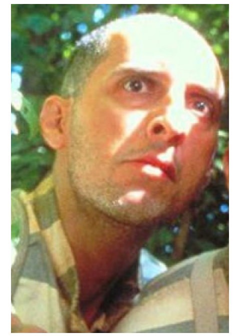
Example: voting prisoners' dilemma…

	 **Everett**	 **Pete**	 **Delmar**
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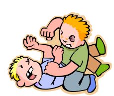
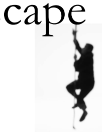
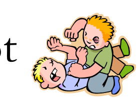
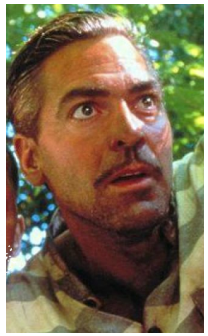
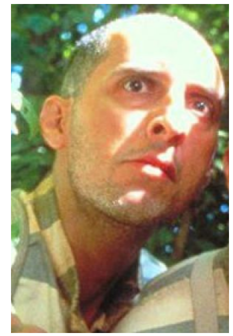
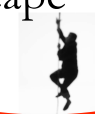
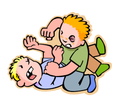
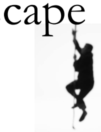
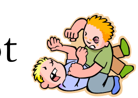
2nd preference	Riot 	Escape 	Escape 
3rd preference	Stay in prison 	Stay in prison 	Riot 

What is a Nash Equilibrium?

Example: voting prisoners' dilemma…

Condorcet winner

Nash Equilibrium

	 Everett	 Pete	 Delmar
1st preference	Escape 	Riot 	Stay in prison 
2nd preference	Riot 	Escape 	Escape 
3rd preference	Stay in prison 	Stay in prison 	Riot 

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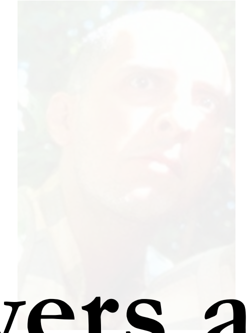


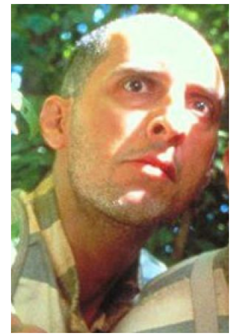
	Everett	Pete	Delmar
1st preference	Escape	Riot	Stay in prison
2nd preference	Riot	Escape	Escape
3rd preference	Stay in prison	Stay in prison	Riot

But if players are not truthful, weird things can happen…

What is a Nash Equilibrium?

Example: voting prisoners' dilemma...

Nash Equilibrium

	 **Everett**	 **Pete**	 **Delmar**
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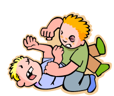
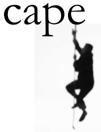
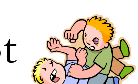
2nd preference	Riot 	Escape 	Escape 
3rd preference	Stay in prison 	Stay in prison 	Riot 

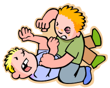
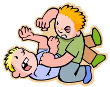
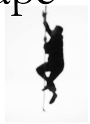
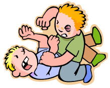
Problem 1:
Can we decrease the number of pure Nash equilibria?

(especially eliminating the senseless ones…)

The truthfulness incentive

Each player's utility is not just dependent on the end result, but players also **receive a small ε when voting truthfully**. The incentive is not large enough as to influence a voter's choice when it can affect the result.

The truthfulness incentive Example

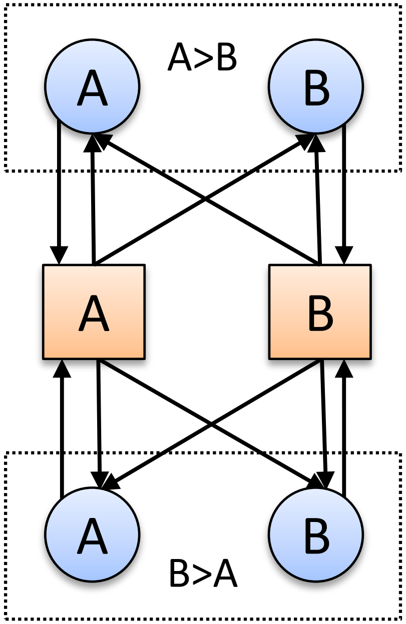
	 Everett	 Pete	 Delmar
1st preference	Escape 	Riot 	Stay in prison 
2nd preference	Riot 	Escape 	Escape 
3rd preference	Stay in prison 	Stay in prison 	Riot 

Problem 2:
How can we identify pure Nash equilibria?

Action Graph Games

A compact way to represent games with 2 properties:

Anonymity: payoff depends on own action and *number* of players for each action.



Context specific independence: payoff depends on easily calculable statistic summing other actions.

Calculating the equilibria using *Support Enumeration Method* (worst case exponential, but thanks to heuristics, not common).

Now we have a way to find pure equilibria, and a way to ignore absurd ones.

So?

The scenario

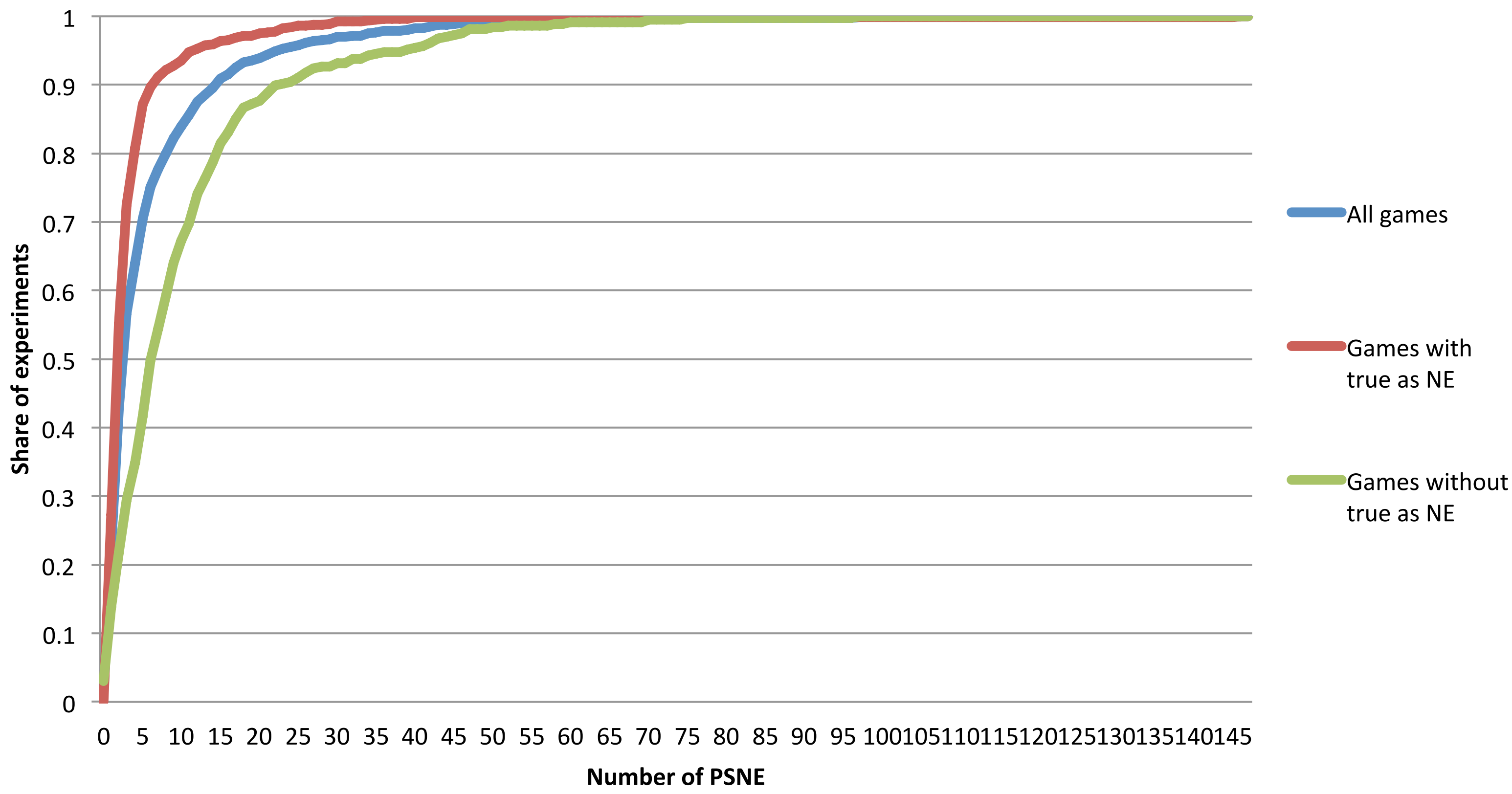
5 candidates & 10 voters.

Voters have Borda-like utility functions (gets 4 if favorite elected, 3 if 2nd best elected, etc.) with added *truthfulness incentive* of $\varepsilon = 10^{-6}$.

They are randomly assigned a preference order over the candidates.

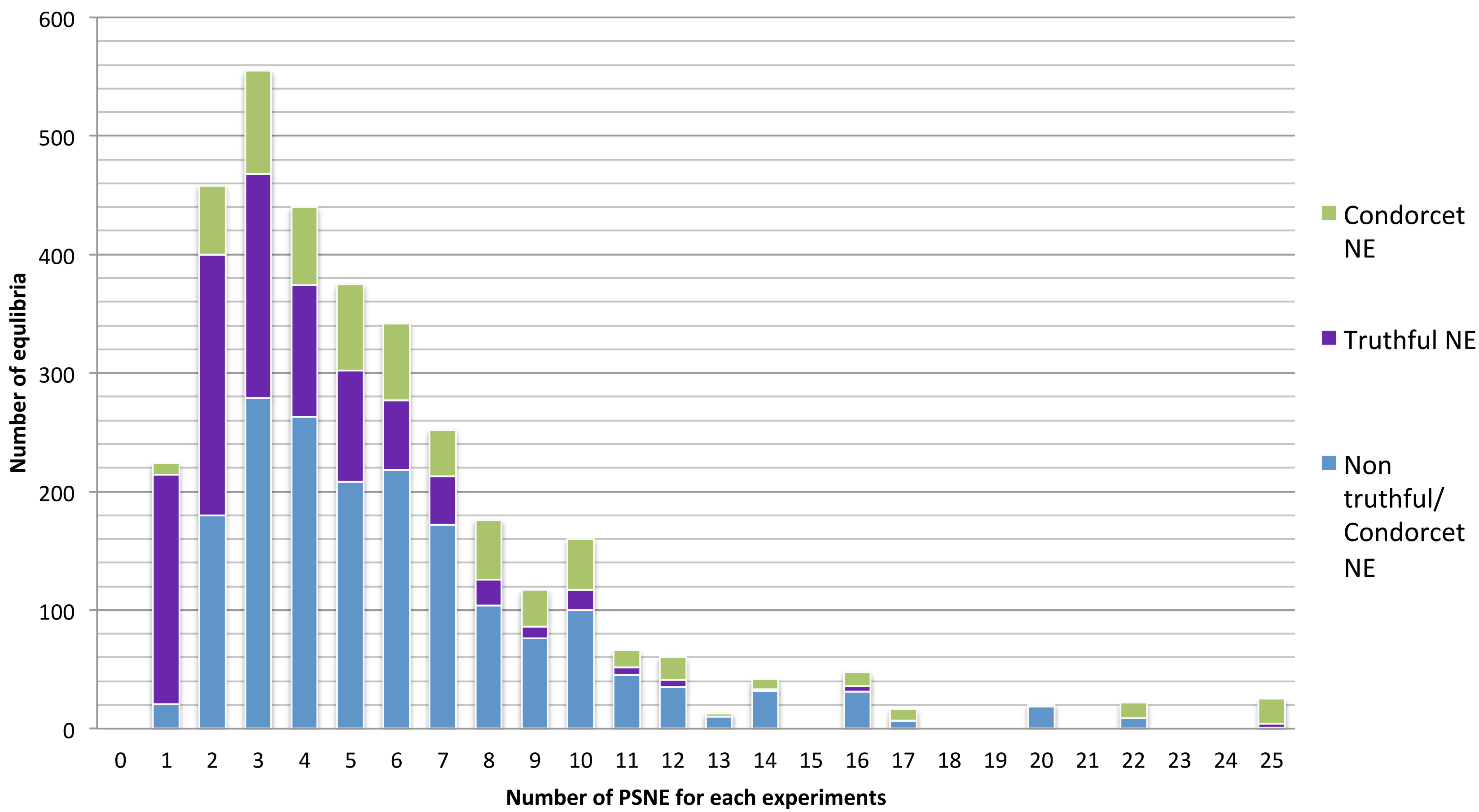
This was repeated 1,000 times.

Results: number of equilibria



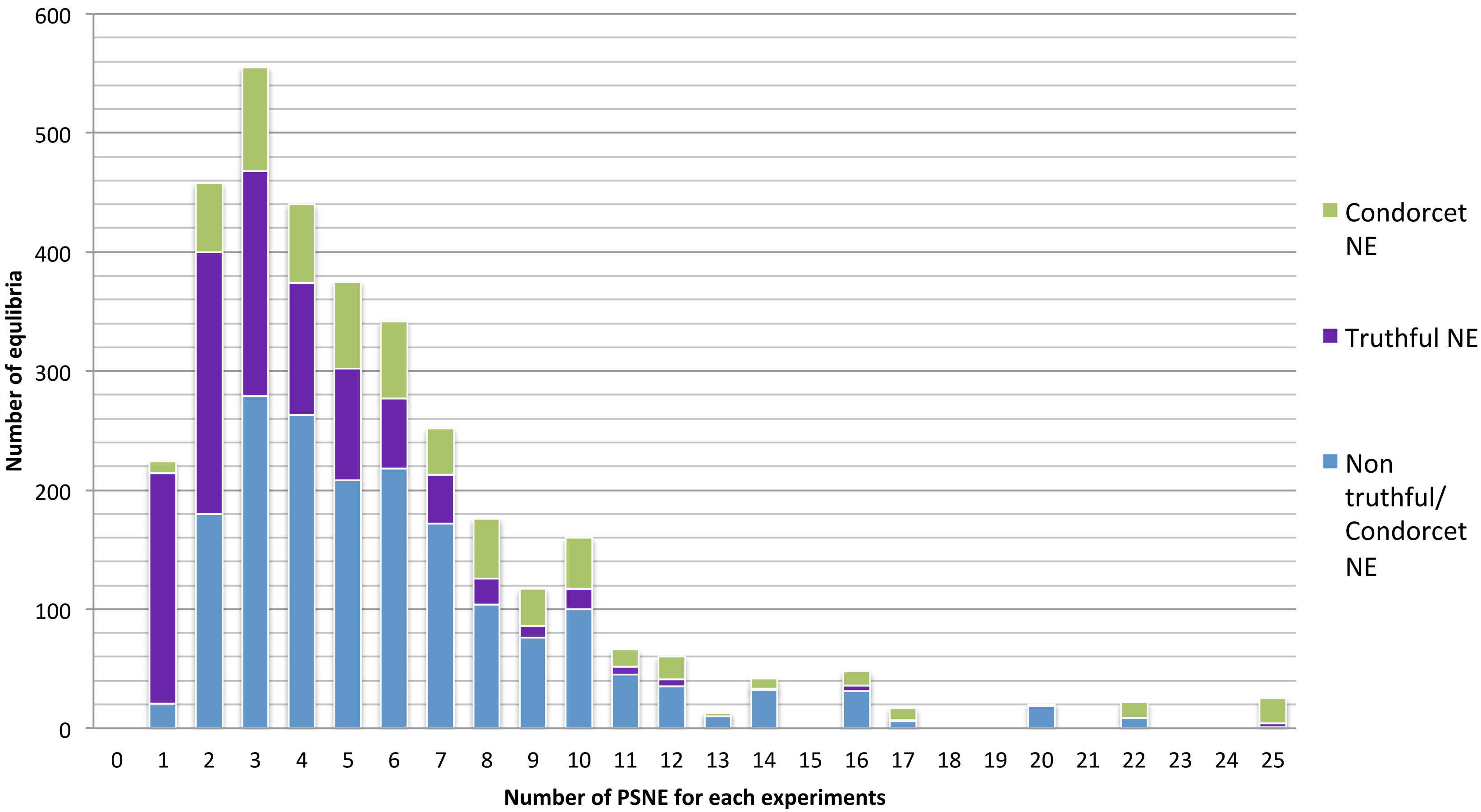
In 63.3% of games, voting truthfully was a Nash equilibrium.
96.2% have less than 10 pure equilibria (without permutations).
1.1% of games have **no pure Nash equilibrium** at all.

Results: type of equilibria truthful



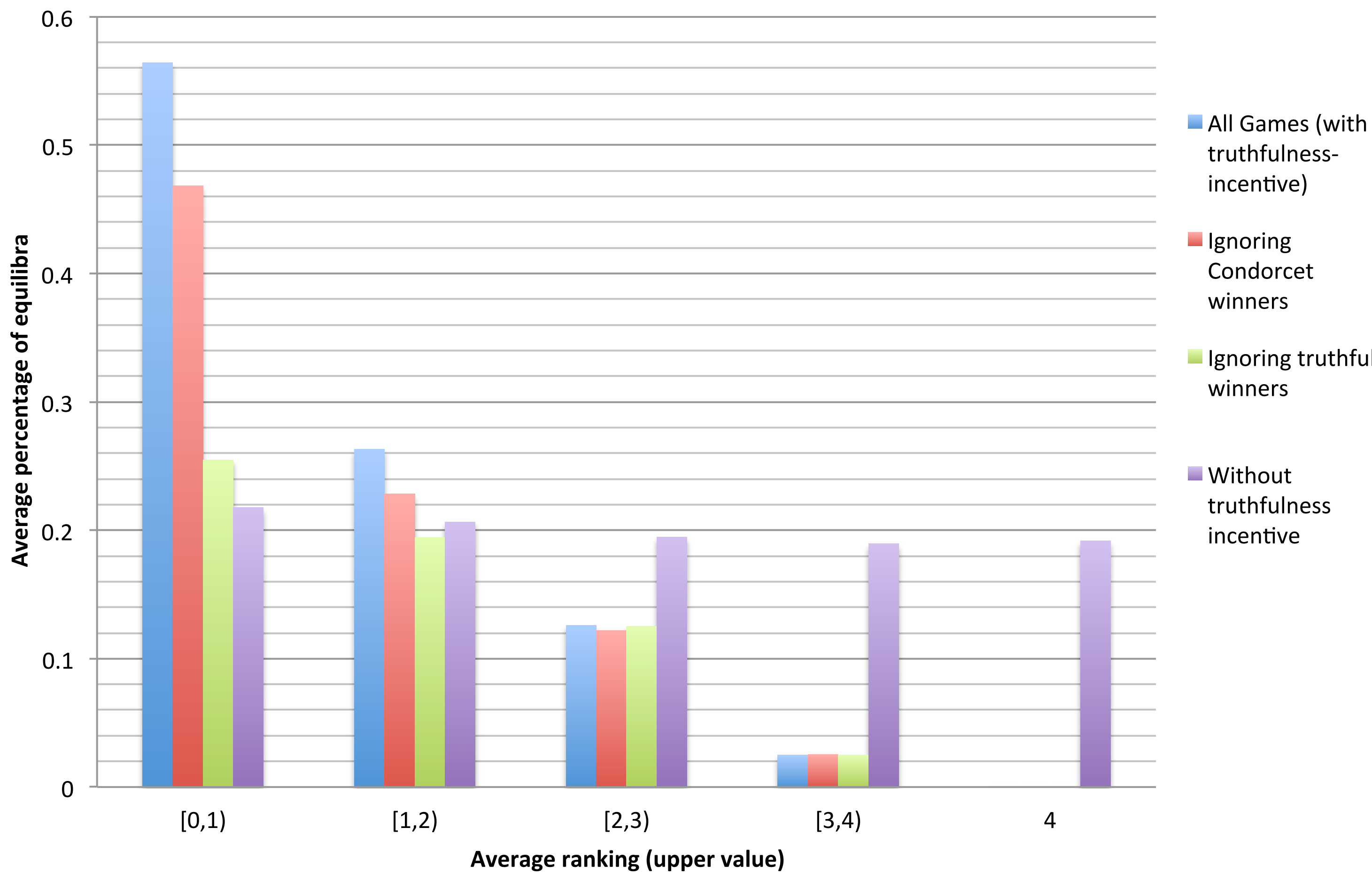
80.4% of games had at least one truthful equilibrium. Average share of truthful-outcome equilibria: 41.56% (without incentive – 21.77%).

Results: type of equilibria Condorcet



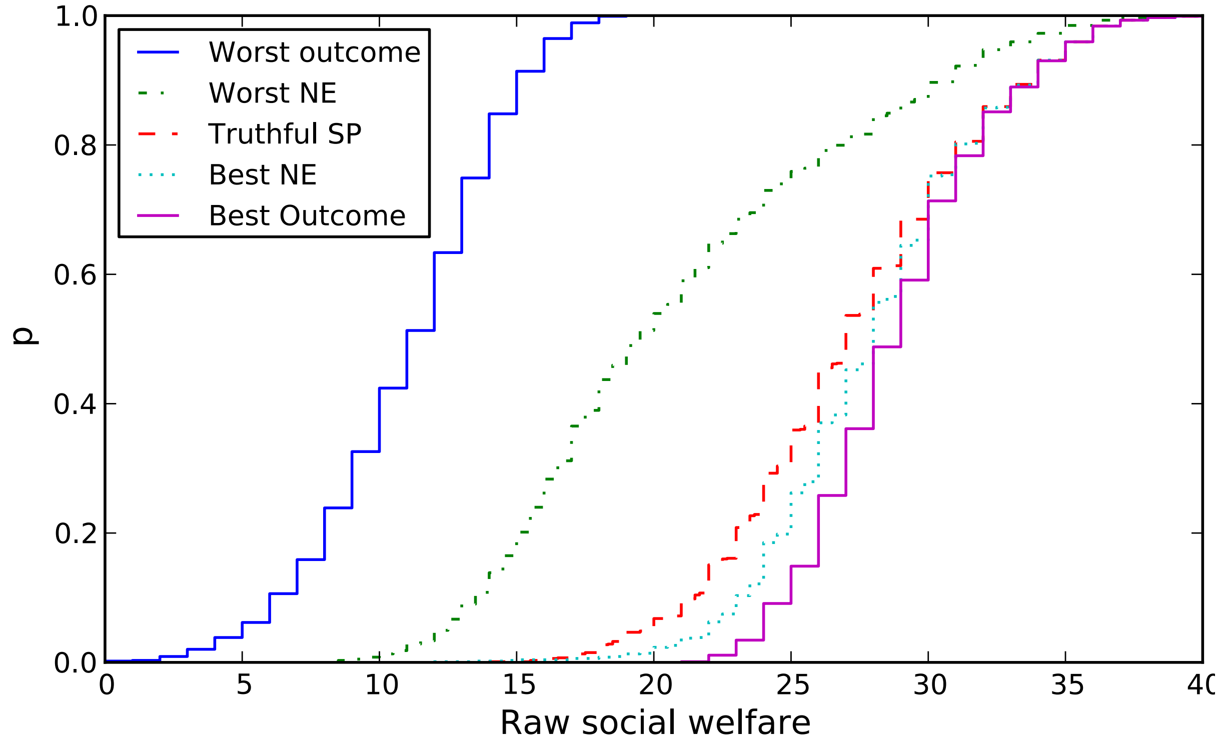
92.3% of games had at least one Condorcet equilibrium.
Average share of Condorcet equilibrium: 40.14%.

Results: social welfare average rank



71.65% of winners were, on average, above median.
52.3% of games had *all* equilibria above median.

Results: social welfare raw sum

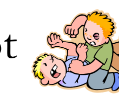
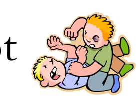
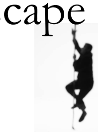
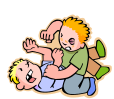
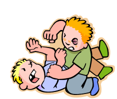


92.8% of games, there was no pure equilibrium with the worst result (only in 29.7% was best result not an equilibrium).
59% of games had truthful voting as best result (obviously dominated by best equilibrium).

But what about more common situations, when we don’t have full information?

Bayes-Nash equilibrium

Each player doesn't know what others prefer, but knows the **distribution according to which they are chosen**.
So, for example, *Everett* and *Pete* don't know what *Delmar* prefers, but they know that:

	50%	45%	5%	0%
1st preference	Stay in prison 	Escape 	Stay in prison 	Riot 
2nd preference	Escape 	Stay in prison 	Riot 	Escape 
3rd preference	Riot 	Riot 	Escape 	Stay in prison 

Bayes-Nash equilibrium scenario

5 candidates & 10 voters.

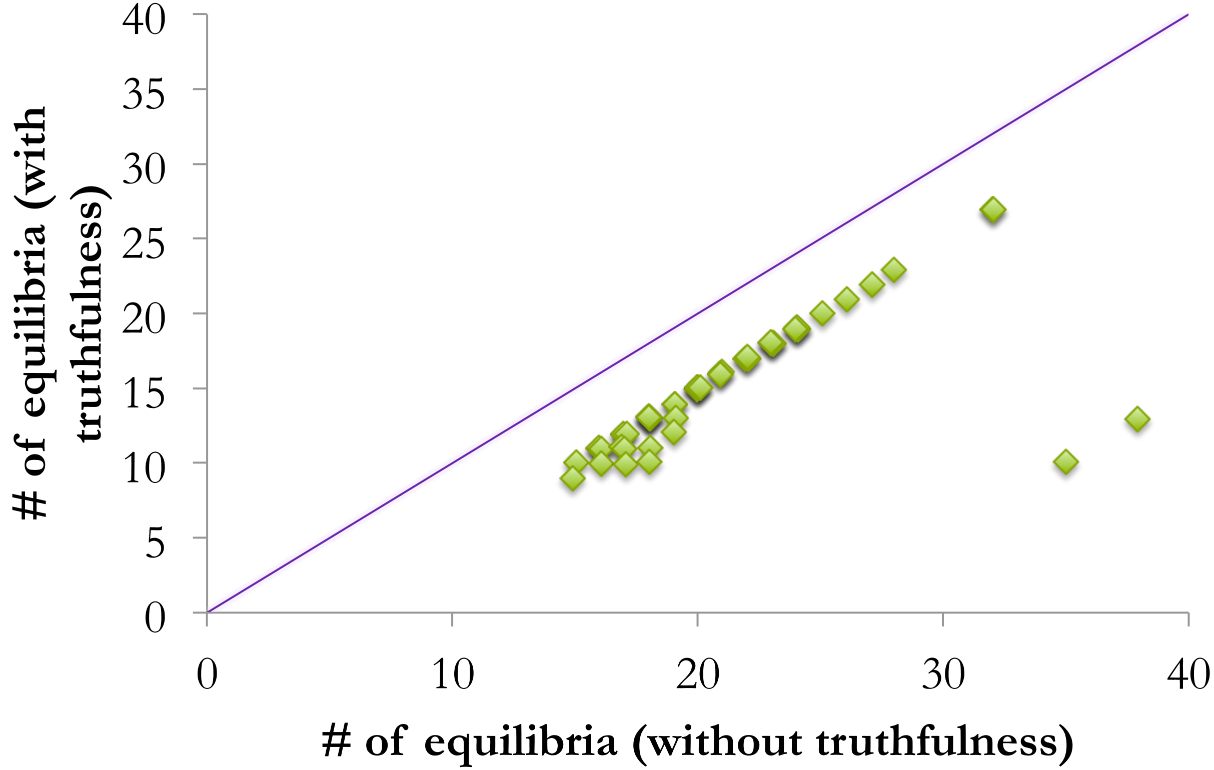
We choose a distribution: assign a probability to each preference order. To ease calculations – only 6 orders have non-zero probability.

We compute equilibria assuming voters are chosen i.i.d from this distribution.

All with Borda-like utility functions & *truthfulness incentive* of $\varepsilon=10^{-6}$.

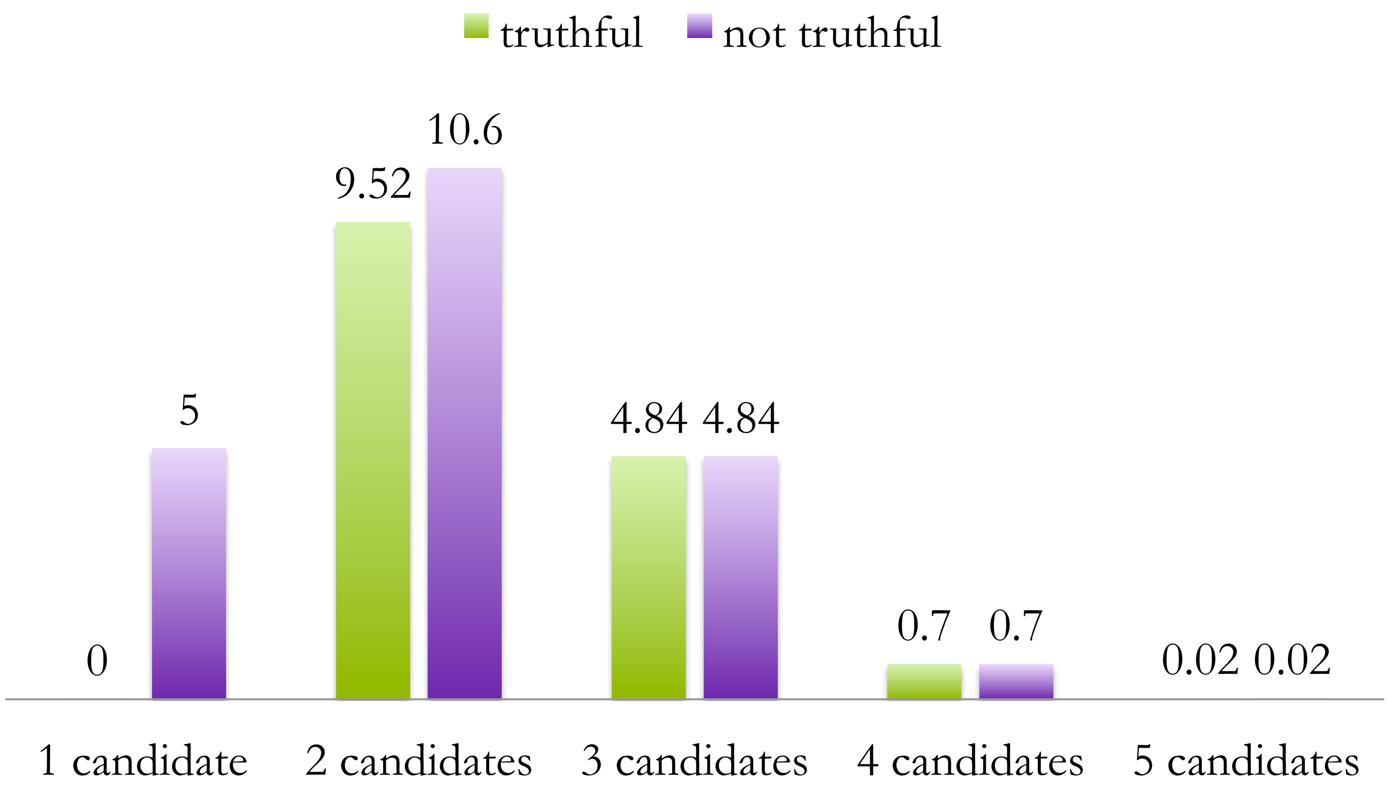
This was repeated 50 times.

Results: number of equilibria



Change (from incentive-less scenario) is less profound than in the Nash equilibrium case (76% had only 5 new equilibria).

Results: type of equilibria



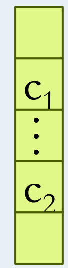
95.2% of equilibria had only 2 or 3 candidates involved in the equilibria. Leading to…

Results: proposition

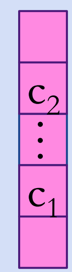
In a plurality election with a truthfulness incentive of ε, as long as ε is small enough, for every $c_1, c_2 \in C$ either c_1 Pareto dominates c_2 (i.e., all voters rank c_1 higher than c_2), or there exists a pure Bayes-Nash equilibrium in which each voter votes for his most preferred among these two candidates.

Proof sketch

Suppose I prefer c_1 to c_2.



If it isn't Pareto-dominated, there is a probability P that a voter would prefer c_2 over c_1,



and hence $P^{n/2}$ that my vote would be pivotal.

If ε is small enough, so one wouldn't be tempted to vote truthfully, each voter voting for preferred type of c_1 or c_2 is an equilibrium

What did we see?

Empirical work enables us to better analyze voting systems. E.g., potential tool enabling comparison according likelihood of truthful equilibria…

Truthfulness incentive induces, we believe, more realistic equilibria.

Clustering: in PSNE, clusters formed around the equilibria with "better" winners. In BNE, clusters formed around subsets of candidates.

Future directions

More cases – different number of voters and candidates.

More voting systems – go beyond plurality.

More distributions – not just random one.

More utilities – more intricate than Borda.

More empirical work – utilize this tool to analyze different complex voting problems, bringing about **More Nash equilibria…**

The End

(Yes, they escaped…)

Thanks for listening!