Hotelling-Downs Equilibria: Moving Beyond Plurality Variants*

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Abstract. Hotelling-Downs model is a classic model of political competition and strategizing candidates, almost always analyzed under plurality. Our paper presents a three-pronged development of the Hotelling-Downs model. First, we analyze competition under a variety of voting rules. Second, we consider not only a linear city model, but also a circular city model. Third, unlike most Hotelling-Downs papers, we solve the model under the winner-takes-all assumption, which saves many equilibria, and is more relevant to voting settings. In the case of three and four candidates we have found a measure of the set of equilibria.

Keywords: Spatial voting · Scoring methods · circular city.

1 Introduction

Since in many cases voters' behavior patterns are well known, we would expect candidates to react to each other's attempts to garner more support, until settling in a Nash equilibrium, in which none of them can profit by changing their location. This, of course, is not unique to political settings – candidates, being the set of options voters choose from, can be strategic in various settings. Restaurants try to calibrate their menus to the public's taste in their area, sometimes resulting in a sudden multiplication of a particular restaurant type when it becomes popular. A similar effect can be observed in many mass-produced items (e.g., clothing styles), in pricing of similar items, or in musical styles in major competitions (e.g. the Eurovision).

In the political domain, Downs [13] established the spatial model in which voters and candidates are all located in a metric space (with voters preference orders determined by their distance from each candidate), and used Hotelling's results [19] on facility location in metric spaces to show social choice results. Since that ground-breaking work much work has been done exploring the voting spatial model in general, and the Hotelling-Downs model in particular. Indeed,

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a lot of work specifically on candidate manipulation has been done extensively in this model. Despite this, we should stress that in many cases, candidates are located in particular location on the ideological spectrum due to other, exogenous concerns. Much of the existing work in this domain assumes the voting rules are plurality or variants of it, and usually assumes candidates have a utility they wish to maximize, rather than a winner-take-all approach, more common in the computational social choice framework.

In this work we expand results regarding the winner-take-all approach, extending the scope of strategic candidacy in Hotelling-Downs models beyond the limited voting rules discussed so far, to further our understanding of other voting rules, and in particular, to understand how scoring rule parameters affect the strategic space available to candidates. We also show the results regarding a circular model (which has been analyzed in economic Hotelling models, though not in political ones) – circular distribution of voters can be related with preferences over regional (geographical) development, such as a choice of airport location outside a compact city, or related with preferences over calendar events, such as holidays, etc.

In winner-takes-all approach multiplicity of equilibria is a common feature. In many cases candidates have no incentives to deviate from their sincere position. Moreover, in order to understand how likely are equilibria states to emerge as sincere positions points, we pioneer a measure probability – what is the likelihood that randomly placed candidates will be in a state of equilibrium. This helps to understand, for different voting rules (and equilibrium states thereof), how common is an equilibrium state – are there many of them, or only very few? Since candidates might be located in a particular area of the ideological spectrum due to other issues and may be constrained in leaving them (e.g., a political party usually has some spectrum of opinions within it, but one cannot usually extend this spectrum endlessly), seeing how common equilibria are helps to understand whether we are likely to reach them in an emergent, bottom up, candidate generation.

2 Related Work

There are many surveys of Hotelling-Downs models [21, 17], the most recent large-scale one is [29]. All attest to how common this model is in political science use when analyzing viewpoints and politics. We shall only mention results which are relevant to (or contrast with) our approach.

There are three main types of candidates objectives: win objective, rank objective, share/score/vote/support objective. "Economic" versions of the model with candidates wishing to maximize support/votes became popular as continuity of utility function allowed for more straightforward analysis. Surely in political competition discontinuity of payoff is a critical feature. The difference of 1% votes is insensible if it does not change the winner and it is crucial if it leads to a new winner.

If there is no profitable deviation in case of rank and support objectives, then there is no profitable deviation in case of win objective. If we have rank and support objectives Nash equilibrium, then we have also the a win objective Nash equilibrium.

For support objectives the model was solved for the run-off rule [18, 7, 31], probabalistic run-off (Apostolic Voting) [28], best-worst voting rules (mix of plurality and antiplurality) [9], some classes of scoring rules[10, 8] and for k-approval rules [32], though that result is problematic, as k can only be fixed (i.e., veto rule, which is $m-1$-approval, for m being the number of candidates, is not solved in that work, and we show the equilibria for this in this paper).

A more natural winner-takes-all assumption draws much less attention [16]. Chisik and Lemke [11] solved the model for three winning-motivated candidates case, uniform distribution of voters with plurality. In this case there are infinite number of equilibria, and the existence is shown for arbitrary number of candidates.

The "linear city" model was developed by Hotelling and Downs [19, 13], but a "circular city" model was initially developed by Salop [26]. This model is a basic industrial organizations model [30]. In non-Hotelling-Downs models, circular preferences have been used. Finite population circular domains, such as cyclic group domains [20], circular domains [27], top-circular domains [1], single-peaked on a circle domains [23] have been shown to be manipulable and lead to dictatorial social choice functions under various conditions. Infinite population models do not have these shortcomings. We believe Peeters et al [22] were the first who presented a Hotelling-Downs model of political competition on a circle. They claimed that far-left and far-right candidates are more similar to each other than candidates in political center.

While outside the scope of this paper, we note that there is some work on strategic candidates outside the Hotelling-Downs model, beginning with [14, 15] discussing strategic candidacy in tournaments, and recently further explored by [6, 24] and others. This line of work is mainly concerned with candidates deciding if to run or not (and the equilibria this may bring about). Another approach is viewing addition and removal of candidates as a form of control manipulation, studied by [3] (see summaries in [5, 25]).

3 Model

We describe the standard one-dimensional spatial model of voting. The policy space is assumed to be a closed $[0, 1]$ interval. Let a finite set $C = \{1, \ldots, m\}$ be the set of candidates. Each candidate chooses a point on $[0, 1]$ interval.

The set of voters is characterized by distribution of their ideal points $\mu : [0, 1] \to \mathbb{R}$ with $\int_0^1 d\mu = 1$ (it is assumed, that μ has support set $[0, 1]$ and it has no mass points). Each voter has complete transitive preferences over the set of candidates (linear order): let $\pi(C)$ be the set of all full orders over C. For each point $v \in [0, 1]$, the Euclidean distance from v defines a weak order of candidates (the closer the better). Two or more candidates may choose the same point. In this case we assume that all possible permutations of these candidates arise in voters preferences with equal probability. For each preference order $P \in \pi(C)$, we define an indicator function $I^P : [0,1] \to [0,1]$, such that $I^P(v) = 1/k$ if and only if there are k linear orders which are linearizations of the weak order of candidates defined by the Euclidean distance from v. Since we are dealing with a potentially infinite number of voters, a preference profile will be defined as a tuple $\mathcal{P} = (I, \mu)$, where I is a tuple of all indicator functions. Let $\mathcal{M}$ be the set of all preference profiles.

Let Q_α be the quantile of order α for voters' distribution μ, formally, we have $\int_0^{Q_\alpha} d\mu = \alpha$.

We consider the following game. m candidates independently and simultaneously choose their positions on a unit interval or on a unit circle. The winner is determined according to a voting rule. All candidates are motivated only by winning.The candidate's first best is to be a sole winner, the second best is to be a winner in the two-winners set, the third best is to be a winner in three-winners set, etc. All losing outcomes are worse and indistinguishable. We distinguish convergent Nash equilibrium (CNE), in which all agent have the same position, and non-convergent Nash equilibrium (NCNE), where some candidates have different position.

A voting rule is a function $F : \mathcal{M} \to 2^C \backslash \emptyset$ from the set of preference orders with their respective measures, obtained from the distribution of voters, to a set of candidates. As usually in Hotelling-Downs models, we consider equilibria in which voters vote sincerely.

We consider the following voting rules:

Definition 1. *A* ***Scoring rule*** *chooses a candidate with the highest sum of scores according to a score vector* $s = (s_1, s_2, \ldots, s_{m-1}, 0)$*, where for each* $i \in \{1, ..., m-1\}$ *we have* $s_i \geq s_{i+1}$*. The winner is*

$$\operatorname*{argmax}_{x \in C} \int_0^1 \sum_{P \in \pi(C)} I^P(v) Score(x, P)\, \mathrm{d}v$$

where function $Score(x, P) = s_i$ *if and only if candidate* x *is on position* i *at preference order* P*.*

We shall specifically mention these scoring rules:

Plurality The scoring vector $(1, 0, \ldots, 0)$.
Veto The scoring vector $(1, 1, \ldots, 1, 0)$.
2-approval The scoring vector $(1, 1, 0 \ldots, 0, 0)$.
Borda The scoring vector $(m-1, m-2, m-3, \ldots, 1, 0)$ (i.e., $s_i - s_{i+1} = 1$).

Definition 2. ***Scoring elimination rule*** *is an iterative rule that is based on a scoring rule. In the elimination rule, each iteration, a single candidate with the lowest scores (ties broken by some tie-breaking rule) is eliminated. The last candidate standing is the winner.*

For $m = 3$, the **run-off rule** is a special case of the scoring elimination rule with plurality scores $\alpha_1 = 1$, $s_2 = s_3 = 0$.

Definition 3. ***Kemeny rule*** *chooses a candidate at the top of a preference order with the lowest distance to all other preference orders in a profile according to swap distance metrics*

$$Topcandidate\left(\underset{K\in\pi(C)}{\operatorname{argmin}}\sum_{P\in\pi(C)}\int_0^1 I^P(v)Swaps(P,K)\,\mathrm{d}v\right),$$

where function $Swaps(P,K)$ is the swap distance between two preference orders (i.e., Kendall-Tau metric).

3.1 CNE. Known results

CNE with different objectives coincide.

Proposition 1 *(Cox, 1987)[12] For a given scoring rule s, there exists CNE, in which all candidates choose Q_α, if and only if $c(s,m)\le\alpha\le 1-c(s,m)$, where $c(s,m)=\frac{s_1-(1/m)\sum_{k=1}^m s_k}{s_1-s_m}$.*

Since median point satisfies condition from proposition1 it is a generalization of the median voter theorem. In particular CNE exists for the veto rule and the Borda rule.

Proposition 2 *(Cox, 1987)[12] For the Condorcet-consistent rules there is a CNE with all candidates located in the median voter position. There is no other equilibrium.*

3.2 NCNE. Known results

The literature of NCNE with support objectives for scoring rules [12, 10, 8, 9] consist of many existence/characterization results, in which candidates constitute a special structure, e.g. distribution of candidates with equal distance between neighbouring candidates.

A scoring rule s is convex if $s_1-s_2\ge s_2-s_3\ge s_{m-1}-s_m$.

Proposition 3 *(Cahan, Slinko 2017)[10] For a convex scoring rule s such that $s_n>s_{n+1}=\ldots=s_m$, where $1\le n\le m$, there is a NCNE with support objectives if and only if the subrule $s'=(s_1,\ldots,s_n,s_{n+1})$ is the Borda rule and $n+1\le\lfloor m/2\rfloor$.*

A scoring rule s is symmetric if $s_i-s_{i+1}=s_{m-i}-s_{m-i+1}$, for all $1\le i\le\lfloor m/2\rfloor$.

Proposition 4 *(Cahan, Slinko 2017)[10] For a symmetric scoring rule s, there is no NCNE with support objectives.*

A scoring rule s is weakly concave if $s_i-s_{i+1}\le s_{m-i}-s_{m-i+1}$.

Proposition 5 *(Cahan, Slinko 2018)[8] For a weakly concave scoring rule s has no NCNE with support objectives in which* $max(n_1, n_q) \leq \lfloor m/2 \rfloor + 1$*, where* n_1 *and* n_q *are numbers of candidates in the most left and the most right locations. If? in addition, s satisfies*

$$s_4 + s_{m-3} \geq \frac{1}{m-3}(\sum_{i=1}^{m-3} s_i + \sum_{i=4}^{m} s_i),$$

then no NCNE with support incentives exist.

These results are too special. We consider win objectives, which allow more equilibria.

4 Voters and Candidates Uniform on a Line

Firstly, we present results a benchmark case of voters uniformly distributed on a line.

Let us consider the scoring rule family with scores $(1, \alpha, 0)$, $\alpha < 1$ ($m = 3$ case). Chisik and Lemke [11] showed that only an extreme candidate wins in equilibrium under the plurality rule in three candidates elections. Lemma 1 generalizes this result for all scoring rules.

For the remainder of this subsection this section we consider a model with three candidates I, II, III located on $[0, 1]$ interval. Candidates I, II, III have positions x, y, z, correspondingly, where $x < y < z$.

Lemma 1. *For a scoring rule, there are no equilibria in which candidate II is a winner.*

Proof. The candidates' scores are shown in Table 1. The Table 1 summarizes partition of voters and scores from six possible linear orders (because of linear space we have only four linear orders with non-zero measure). Using definition 1 we find a winner.

Table 1. Candidates' scores for Lemma 1

	1 point	**α points**
Candidate I	Interval $[0, x + \frac{y-x}{2}]$, Length of interval $\frac{y+x}{2}$	Interval $[x + \frac{y-x}{2}, x + \frac{z-x}{2}]$, Length of interval $\frac{z-y}{2}$
Candidate II	Interval $[x + \frac{y-x}{2}, y + \frac{z-y}{2}]$, Length of interval $\frac{z-x}{2}$	Intervals $[0, x + \frac{y-x}{2}] \cup [z - \frac{z-y}{2}, 1]$, Length of intervals $1 - \frac{z-x}{2}$
Candidate III	Interval $[z - \frac{z-y}{2}, 1]$, Length of interval $1 - \frac{z+y}{2}$	Interval $[x + \frac{z-x}{2}, z - \frac{z-y}{2}]$, Length of interval $\frac{y-x}{2}$

If candidate II is a winner then

$$\frac{z-x}{2} + \alpha(1 - \frac{z-x}{2}) > \frac{x+y}{2} + \alpha\frac{z-y}{2},$$
$$\frac{z-x}{2} + \alpha(1 - \frac{z-x}{2}) > 1 - \frac{y+z}{2} + \alpha\frac{y-x}{2}.$$

Summing and simplifying these equations we get

$$z - x > \frac{2}{3} \cdot \frac{1-2\alpha}{1-\alpha}.$$

There should also be no incentive for candidate I to deviate to candidate II's location (y), so candidate III should win if that happens:

$$\begin{aligned}
\frac{1+\alpha}{2} \cdot \frac{z+y}{2} + \frac{\alpha}{2}(1 - \frac{z+y}{2}) &< 1 - \frac{z+y}{2}, \\
\frac{y+z}{4} + \frac{\alpha}{2} &< 1 - \frac{z+y}{2}, \\
\frac{3}{4}(y+z) &< 1 - \frac{\alpha}{2}.
\end{aligned}$$

Similarly, not allowing candidate III to deviate to y results in $-\frac{3}{4}(x+y) < -\frac{1+\alpha}{2}$. Combining these together we get $\frac{1}{2} - \alpha > \frac{3}{4}(z-x)$, which is simplified to $z - x < \frac{2}{3} - \frac{4}{3}\alpha$. Thus, we have

$$\frac{2}{3} - \frac{4}{3}\alpha > z - x > \frac{2}{3} \cdot \frac{1-2\alpha}{1-\alpha}.$$

Simplified, this results in $1 - 2\alpha > \frac{1-2\alpha}{1-\alpha}$ which cannot be true for any $\alpha < 1$.

We now begin to investigate how the α value changes the equilibria states.

Proposition 6 *For scoring rules with $\alpha \geq \frac{1}{2}$, there is no equilibrium with different locations of candidates.*

Proof. If there are more than two different locations for the candidates in equilibrium, candidate II should not want to deviate to x or to z (as we know from Lemma 1 that it is not the winner). Following the same calculation done in Lemma 1 for moving candidate I to y, this results in: $\frac{3}{4}(x+z) < 1 - \frac{\alpha}{2}$ (x has replaced y in the Lemma's expression). Similarly, candidate II cannot move to z and profit, thus (again, following Lemma 1), $-\frac{3}{4}(x+z) < -\frac{1+\alpha}{2}$. Summing these together, we have $0 < \frac{1}{2} - \alpha$ which is not true for $\alpha \geq \frac{1}{2}$.

Thus for $\alpha \geq \frac{1}{2}$, it is profitable for candidate II to deviate to one of the other candidates. But that means that there is a state where two candidates win, and hence the third would benefit from deviating to them. Thus, an equilibrium would contain a single location.

For a scoring rule with $\alpha < \frac{1}{2}$, equilibrium always exists with different locations of candidates.

4.1 Measure of the set of equilibria

Where we know there are equilibria, and yet do not have an easy and concise representation of it, we wish to understand how prevalent are equilibria states in the whole domain. In order to do this, we calculate the measure of the set of equilibria. 1 means that almost every distribution (except the set of measure zero) leads to an equilibrium. 0 means that it may exist, but it requires some special configurations of measure zero. Equilibrium is not a random event. Because of it we do not use term probability here.

Proposition 7 *For scoring rules with $\alpha < \frac{1}{2}$, the measure of the set of equilibria equals to $\frac{8}{27} \cdot \frac{(1-2\alpha)^3}{(1-\alpha)^2}$.*

Proof. Using the same values from Table 1, suppose candidate I is a winner:

$$\frac{x+y}{2} + \alpha\frac{z-y}{2} > \frac{z-x}{2} + \alpha(1 - \frac{z-x}{2}),$$
$$\frac{x+y}{2} + \alpha\frac{z-y}{2} > 1 - \frac{y+z}{2} + \alpha\frac{y-x}{2}.$$

Simplifying we have

$$x(1-\frac{\alpha}{2}) + y(\frac{1}{2} - \frac{\alpha}{2}) + z(\alpha - \frac{1}{2}) > \alpha,$$
$$x(\frac{1}{2} + \frac{\alpha}{2}) + y(1-\alpha) + z(\frac{1}{2} + \frac{\alpha}{2}) > 1.$$

There is no profitable deviation of candidate II with staying in the same point as the winner if:

$$\frac{1}{2} \cdot (x + \frac{z-x}{2} + \alpha) < \frac{z-x}{2} + 1 - z.$$

Which, simplified, is $z + x < \frac{4}{3} - \frac{2}{3}\alpha$.

Candidate II also should not be able to deviate profitably to $x < y' < z$, thus, it cannot be that:

$$\frac{z-x}{2} + \alpha(1 - \frac{z-x}{2}) > \frac{x+y'}{2} + \alpha\frac{z-y'}{2},$$
$$\frac{z-x}{2} + \alpha(1 - \frac{z-x}{2}) > 1 - \frac{y'+z}{2} + \alpha\frac{y'-x}{2}.$$

Summing those together, we get $z - x > \frac{2}{3}\frac{1-2\alpha}{1-\alpha}$, and since this should not happen, we will wish to maintain $z - x \leq \frac{2}{3} \cdot \frac{1-2\alpha}{1-\alpha}$. Candidate II should also not profit from moving to $y' < x$, thus it cannot be that:

$$\frac{x+y'}{2} + \alpha\frac{z-x}{2} > \frac{z-y'}{2} + \alpha(1 - \frac{z-y'}{2}),$$
$$\frac{x+y'}{2} + \alpha\frac{z-x}{2} > 1 - \frac{x+z}{2} + \alpha\frac{x-y'}{2}.$$

Summing together, and since the value for candidate II will increase the closest it gets to x, we get $x(3 - \frac{3}{2}\alpha) - \frac{3}{2}\alpha z > 1 + \alpha$, so for this not to be possible, $x(3 - \frac{3}{2}\alpha) - \frac{3}{2}\alpha z \leq 1 + \alpha$. Similarly, candidate II cannot become the winner due to moving to $y' > z$, so $z(3 - \frac{3}{2}\alpha) + \frac{3}{2}\alpha x \leq 2 - \alpha$.

A similar process for deviations of candidate III leads to additional constraints. However, the ones that are binding (many constraints are repetitive) are:

$$x(\frac{1}{2} + \frac{\alpha}{2}) + y(1-\alpha) + z(\frac{1}{2} + \frac{\alpha}{2}) > 1;$$
$$y < z; z + x < \frac{4}{3} - \frac{2}{3}\alpha; z - x > \frac{2}{3} \cdot \frac{1-2\alpha}{1-\alpha};$$
$$x(1.5 - \frac{\alpha}{2}) + z(0.5 + \frac{\alpha}{2}) \leq 1$$

The volume of this area equals to:

$$\begin{aligned}&\int_{(2\alpha-\alpha^2)/(1-\alpha)/3}^{(1-\alpha+\alpha^2)/(1-\alpha)/3}\int_{(2-x(1+\alpha))/(3-\alpha)}^{x+(2-\alpha 4)/(1-\alpha)/3}\int_{(2-x(1+\alpha)-z(1+\alpha))/(1-\alpha)/2}^{z} 1\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x\\ +&\int_{(1-\alpha+\alpha^2)/(1-\alpha)/3}^{\frac{1}{2}}\int_{(2-x(1+\alpha))/(3-\alpha)}^{(2-x(3-\alpha))/(1+\alpha)}\int_{(2-x(1+\alpha)-z(1+\alpha))/(1-\alpha)/2}^{z} 1\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x\\ =&\frac{2}{81}\frac{(1-2\alpha)^3}{(1-\alpha)^2}.\end{aligned}$$

Because there are six ways to rename candidates and two ways locate winner (near left end, or near right end of the interval) there are 12 equivalent cases. The measure of the set of equilibria equals

$$\frac{8}{27}\cdot\frac{(1-2\alpha)^3}{(1-\alpha)^2}.$$

Proposition 8 *For each scoring elimination rule, the measure of the set of equilibria equals to 0.*

Proof. For the third place candidate a strategy of staying in the same point with a winner guarantees a positive probability of winning the first round. On the second round the winner position guarantees positive probability of winning.

Proposition 9 *For each rule based on the majority relation, the measure of the set of equilibria equals to 0.*

Proof. Median candidate position deviation is profitable.

5 Voters and Candidates Uniform on a Circle

In this section we assume we have the unit circle, that is the point 0 and the point 1 are the same point. Note that thanks to this being a circle we can decide where the point 0 is on it.

5.1 Existence of Equilibrium

When analyzing single-peaked preferences on a circle, one of the important observations is that there is no median voter position. The equidistant distribution of candidates has direct counterpart in Salop circular city model [26]. It is quite straightforward to see that an equidistant distribution of candidates is an equilibrium under plurality rule and a run-off rule. The veto rule adds several equilibria – there is one where all candidates are at the same point, but also for an even m, candidates can be divided equally and located centrosymmetrically in two locations. There are also other equilibria. Furthermore, all rules based on the weighted tournament matrix have equilibria for each location of candidates.

For the plurality rule we completely solve the cases of $m = 3$ and $m = 4$.

5.2 Uniform on a circle, $m = 3$.

Since a circle has no fixed location anchor, let us mark a candidate I's position as 0. A direction towards candidate II is the direction of the axis. The position of candidate II is y and the position of candidate III is z.

Because of centrosymmetry the probability of each ranking equals to probability of reverse ranking $Pr(123) = Pr(321)$, $Pr(132) = Pr(231)$, $Pr(213) = Pr(312)$.

Proposition 10 *For each scoring rule, if $\alpha = \frac{1}{2}$, the measure of the set of equilibria equals to 1; if $\alpha < \frac{1}{2}$ it is $\frac{1}{6}$; and if $\alpha > \frac{1}{2}$ it is 0.*

Proof. The candidates' scores are shown in Table 2.

Table 2. Candidates' scores for Proposition 10

	1 point	**α points**
Candidate I	Intervals $[0, \frac{y}{2}] \cup [1 - \frac{1-z}{2}, 1]$, Length of intervals $\frac{y}{2} + \frac{1-z}{2}$	Intervals $[\frac{y}{2}, \frac{z}{2}] \cup [y + \frac{1-y}{2}, 1 - \frac{1-z}{2}]$, Length of intervals $z - y$
Candidate II	Interval $[\frac{y}{2}, y + \frac{z-y}{2}]$, Length of interval $\frac{z}{2}$	Intervals $[\max(0, z - 1 + \frac{1-z+y}{2}), \frac{y}{2}]$ $\cup[y + \frac{z-y}{2}, y + \frac{1-y}{2}]$ $\cup(\min(1, z + \frac{1-z+y}{2}), 1]$, Length of intervals $1 - z$
Candidate III	Interval $[z - \frac{z-y}{2}, 1 - \frac{1-z}{2}]$, Length of interval $\frac{1-y}{2}$	Intervals $[\frac{z}{2}, y + \frac{z-y}{2}]$ $\cup[1 - \frac{1-z}{2}, \min(z + \frac{1-z+y}{2}, 1)]$ $\cup[0, \max(0, z + \frac{1-z+y}{2} - 1)]$, Length of intervals y

If candidate I is a winner then

$$\begin{aligned} \frac{y}{2} + \frac{1-z}{2} + \alpha(z-y) >& \frac{z}{2} + \alpha(1-z), \\ \frac{y}{2} + \frac{1-z}{2} + \alpha(z-y) >& \frac{1-y}{2} + \alpha y \end{aligned}$$

.

Transforming, we have

$$\begin{aligned} y(\frac{1}{2} - \alpha) + z(-1 + 2\alpha) >& -\frac{1}{2} + \alpha, \\ y(1 - 2\alpha) + z(-\frac{1}{2} + \alpha) >& 0 \end{aligned}$$

.In case of $\alpha = \frac{1}{2}$ all candidates have the same scores in each realization of positions. The probability of equilibrium is equal to 1.

There is no profitable deviation of staying in the same point with the winner if $\frac{1}{4} + \frac{\alpha}{2} < \frac{1}{2}$.

In case of $\alpha > \frac{1}{2}$ there is no equilibrium. Let us consider the case of $\alpha < \frac{1}{2}$. There should be no profitable deviation of candidate II to $x < y' < z$, thus:

$$\frac{z}{2} + \alpha(1-z) < \frac{y'}{2} + \frac{1-z}{2} + \alpha(z-y'),$$
$$\frac{z}{2} + \alpha(1-z) < \frac{1-y'}{2} + \alpha y'.$$

Summing these up and simplifying, we get $z < \frac{2}{3}$. Similarly, there is no profitable deviation of candidate III to $y < z' < 1$, and similarly summing the inequalities will result in $y \geq \frac{1}{3}$. Thus, the measure of the set of equilibria equals to $\frac{1}{6}$.

Proposition 11 *For a scoring elimination rule with $\alpha \neq \frac{1}{2}$, the measure of the set of equilibria equals to 0.*

Proof. For the third ranked candidate deviating to the same point with a winner guarantees a positive probability of winning the first round. On the second round candidates would have exactly the same scores and probability of winning.

Proposition 12 *For the Kemeny rule, the measure of the set of equilibria equals to 0.*

Proof. Because of symmetry we have a pair of winning rankings with equal scores, and are opposite. The third place candidate has beneficial deviation to stay at the same point with a winner, as there is a positive probability of winning.

5.3 Uniform on a circle, $m = 4$.

In a quarter of cases, candidate I is a winner. Let us mark a candidate I's position as 0. We chose direction of the axis such that the position of candidate III is greater or equal than $\frac{1}{2}$. The position of candidate II is y, the position of candidate III is z ($z \geq \frac{1}{2}$), a position of candidate IV is t.

Proposition 13 *For the plurality rule, the measure of the set of equilibria equals to 0.*

Proof. The scores are:
Candidate I $\frac{y}{2} + \frac{1-t}{2}$ **Candidate III** $\frac{t-y}{2}$
Candidate II $\frac{z}{2}$ **Candidate IV** $\frac{1-z}{2}$

Candidate II obviously beats candidate IV (since $z \geq \frac{1}{2}$). Candidate II can deviate to the point $t - \frac{1}{2}$ which will make candidates I, III have equal score – $\frac{1}{4}$, which is below candidate II's score (since $z > \frac{1}{2}$). Thus, there is always a deviation for candidate II, i.e., there is no equilibrium.

Proposition 14 *For the run-off rule, the measure of the set of equilibria equals to 0.25.*

Proof. As shown in Proposition 13, candidate II can always make it to the top 2. Suppose candidates 1 and 2 are the first round winners, then we can write down the equations that maintain that candidates 3 and 4 score less, and thus we reach the constraints:

$$z \geq \frac{1}{2}; 0 \leq y \leq \frac{1}{2}; z \leq t; z \leq 1 - t + y; t \leq 1; t \leq \frac{1}{2} + y$$

The volume of this area equals to

$$\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^{\frac{1}{2}+y/2} \int_{\frac{1}{2}}^{t} 1 \, \mathrm{d}z \, \mathrm{d}t \, \mathrm{d}y + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}+y/2}^{\frac{1}{2}+y} \int_{\frac{1}{2}}^{1-t+y} 1 \, \mathrm{d}z \, \mathrm{d}t \, \mathrm{d}y = \frac{1}{96}.$$

Because there are 24 equivalent cases (possible renamings), the measure equals $\frac{1}{4}$.

The following propositions, for which only proof sketches are provided, hold for all m.

Proposition 15 *For the Borda rule, the measure of the set of equilibria equals to 1.*

Proof. Because of centrosymmetry the probability of each ranking equals to probability of reverse ranking. Scores of all candidates are equal due to the linear reduction in score in the vector. There are no incentives for deviation.

Proposition 16 *For each rule based on the majority relation, the measure of the set of equilibria equals to 1.*

Proof. We have ties for all paired comparisons.

Proposition 17 *For the veto rule, the measure of the set of equilibria equals to 0.*

Proof. Standing between the two consecutive candidates with the smallest distance is always profitable.

Proposition 18 *For the 2-approval, the measure of the set of equilibria equals to 1.*

Proof. Each candidate has two neighbour candidates and exactly one opposite candidate. For each pair (candidate-opposite candidate), there are two points in which distances from these candidates are equal. The distance between these points is 1/2. On each path between candidate and opposite candidate there is exactly one candidate. Thus each candidate receive exactly 1/2 approval votes.

6 Discussion

In this paper we have extended Hotelling-Downs settings in three directions: we extend the results on winner-take-all settings; we expand beyond plurality and its variants to many scoring rules, as well as a few non-scoring rules results; and we explore circular domains, and not just the line. Moreover, we have introduced the measure of the set of equilibria. In some sense it characterizes stability of of voting rule.

The Hotelling-Downs model results are of particular interest in today's research map – as there is a growing interest in political parties (e.g., gerrymandering [2] or primaries [4]), the shift of parties' candidates in intra-party competitions may be much more pronounced, as the party can choose its candidates to be anywhere on its spectrum of views. We hope this research will help contribute to this topic as well.

There are plenty of open problems still to explore – such as increasing the number of candidates; expanding to more domains beyond the unit circle; and integrating with party-based models. We also believe our metric can be useful in further domains which can benefit from understanding how likely an emergent state is to "accidentally" be an equilibrium.

References

1. Achuthankutty, G., Roy, S.: Dictatorship on top-circular domains. Theory and Decision **86**, 479–493 (2018)
2. Bachrach, Y., Lev, O., Lewenberg, Y., Zick, Y.: Misrepresentation in district voting. In: Proceedings of the 25th International Joint Conference on Artificial Intelligence (IJCAI). pp. 81–87. New York City, New York (July 2016)
3. Bartholdi III, J.J., Tovey, C.A., Trick, M.A.: How hard is it to control an election? Mathematical and Computer Modelling **16**(8-9), 27–40 (August-September 1992)
4. Borodin, A., Lev, O., Shah, N., Strangway, T.: Primarily about primaries. In: Proceedings of the 33rd Conference on Artificial Intelligence (AAAI). pp. 1804–1811. Honolulu, Hawaii (January-February 2019)
5. Brandt, F., Conitzer, V., Endriss, U., Lang, J., Procaccia, A.D. (eds.): Handbook of Computational Social Choice. Cambridge University Press (March 2016)
6. Brill, M., Conitzer, V.: Strategic voting and strategic candidacy. In: Proceedings of the 29th AAAI Conference on Artificial Intelligence (AAAI). pp. 819–826. Austin, Texas (January 2015)
7. Brusco, S., Dziubiński, M., Roy, J.: The hotelling–downs model with runoff voting. Games and Economic Behavior **74**, 447–469 (2012)
8. Cahan, D., McCabe-Dansted, J., Slinko, A.: Asymmetric equilibria in spatial competition under weakly concave scoring rules. Economics Letters **167**, 71–74 (2018)
9. Cahan, D., Slinko, A.: Electoral competition under best-worst voting rules. Social Choice and Welfare **51**, 259–279 (2018)
10. Cahan, D., Slinko, A.: Nonconvergent electoral equilibria under scoring rules: Beyond plurality. Journal of Public Economic Theory **19**(2), 245–460 (2018)
11. Chisik, R.A., Lemke, R.J.: When winning is the only thing: pure strategy nash equilibria in a three-candidate spatialvoting model. Social Choice and Welfare **26**(1), 209–215 (2006)

12. Cox, G.W.: Electoral equilibrium under alternative voting institutions. American Journal of Political Science **31**(1), 82–108 (1987)
13. Downs, A.: An Economic Theory of Democracy. New York: Harper and Row (1957)
14. Dutta, B., Jackson, M.O., Le Breton, M.: Strategic candidacy and voting procedures. Econometrica **69**(4), 1013–1037 (July 2001)
15. Dutta, B., Jackson, M.O., Le Breton, M.: Voting by successive elimination and strategic candidacy. Journal of Economic Theory **103**(1), 190–218 (March 2002)
16. Feldman, M., Fiat, A., Obraztsova, S.: Variations on the hotelling-downs model. In: Proceedings of the 30th Conference on Artificial Intelligence (AAAI). pp. 496–501. Phoenix, Arizona (February 2016)
17. Grofman, B.: Downs and two-party convergence. Annual Review of Political Science **7**, 24–46 (2004)
18. Haan, M., Volkerink, B.: A runoff system restores the principle of minimum differentiation. European Journal of Political Economy **17**, 157–162 (2001)
19. Hotelling, H.: Stability in competition. The Economic Journal **39**, 41–57 (1929)
20. Kim, K.H., Roush, F.W.: Special domains and nonmanipulability. Mathematical Social Sciences **1**, 85–92 (1980)
21. Osborne, M.J.: Spatial models of political competition under plurality rule: A survey of some explanations of the number of candidates and the positions they take. The Canadian Journal of Economics **28**(2), 261–301 (1995)
22. Peeters, R., Saran, R., Yüksel, A.M.: Strategic party formation on a circle and durverger's law. Social Choice and Welfare **47**, 729–759 (2016)
23. Peters, D., Lackner, M.: Preferences single-peaked on a circle. In: Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI) (2017)
24. Polukarov, M., Obraztsova, S., Rabinovich, Z., Kruglyi, A., Jennings, N.R.: Convergence to equilibria in strategic candidacy. In: Proceedings of the 24th International Conference on Artificial Intelligence (IJCAI). pp. 624–630. Buenos Aires, Argentina (July 2015)
25. Rothe, J. (ed.): Economics and Computation. Springer (2015)
26. Salop, S.C.: Monopolistic competition with outside goods. Bell Journal of Economics **10**, 141–156 (1979)
27. Sato, S.: Circular domains. Review of Economic Design **14**, 331–342 (2010)
28. Savčić, R., Xefteris, D.: Apostolic voting. Canadian Journal of Economics (2021)
29. Schofield, N.: The Spatial Model of Politics. No. 95 in Routledge Frontiers of Political Economy, Routledge (2008)
30. Tirole, J.: The Theory of Industrial Organization. MIT Press (1988)
31. Xefteris, D.: Mixed equilibria in runoff elections. Games and Economic Behavior **87**, 619–623 (2014)
32. Xefteris, D.: Stability in electoral competition: A case for multiple votes. Journal of Economic Theory **161**, 76–102 (2016)