Spoiler Susceptibility in Party Elections

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Abstract

An electoral spoiler is usually defined as a losing candidate whose removal would affect the outcome by changing the winner. So far spoiler effects have been analyzed primarily for single-winner electoral systems. We consider this subject in the context of party elections, where there is no longer a sharp distinction between winners and losers. Hence, we propose a more general definition, under which a party is a spoiler if their elimination causes any other party's share in the outcome to decrease.
We characterize spoiler-proof electoral allocation rules for zero-sum voting methods. In particular, we prove that for seats-votes functions only identity is spoiler-proof. We also show that spoilers are ubiquitous under some of the most common electoral rules. However, their impact can vary depending on the rule. Hence, we introduce a measure of spoilership, which allows us to experimentally compare a number of multiwinner social choice rules according to their spoiler susceptibility. Since the probabilistic models used in COMSOC have been developed for non-party elections, we extend them to generate multi-district party elections.

1 Introduction

In the context of single–winner elections, a *spoiler* is usually defined as a losing candidate whose removal would affect the outcome by changing the winner [14, 34]. In this paper, we extend the definition of a spoiler to *party elections*, investigate spoiler-proofness of multiwinner electoral rules, and analyze their spoiler susceptibility.

We primarily seek to investigate spoiler effects in political elections to multi–member representative bodies. They are distinguished from other multiwinner elections by their character as *party elections*. By party election we mean such an election where the allocation of some payoff (e.g., *parliamentary seats*) among *parties* (rather than the identity of the winners) is the main outcome.[1] Thus, spoiler effects should be considered in terms of party seat allocations.

1.1 Contribution

The five principal contributions of the paper are:

First. We *generalize the definition of a spoiler* to party elections. A party is deemed a spoiler if there exists another party whose payoff decreases if the former is eliminated.

Second. We prove a *general impossibility theorem* regarding the existence of *spoiler-proof electoral rules* that can be described by *seats-votes functions* distinct from identity.

Third. We *characterize spoiler-proof electoral rules* for zero-sum voting methods like ordinal or cumulative voting.

Fourth. We *compare electoral rules* according to their expected spoiler susceptibility. We analyze seven rules based on ordinal voting: k–Borda, Chamberlin–Courant, Harmonic-Borda, Jefferson–D'Hondt, k–PAV, SNTV, and STV.

Fifth. Since party elections under ordinal voting have not been hitherto considered in COMSOC, we propose *probabilistic models of party elections* extending several common statistical cultures.

[1]This definition differs from that put forward by earlier authors to address party elections in COMSOC, such as Botan [11], for whom its defining characteristics was a block voting pattern. However, it generalizes the concept of apportionment methods, discussed in, e.g., [12] and [13].

1.2 Related Work

While spoiler effects have long been a familiar subject in the field of voting theory, there have been few attempts to formally define spoilers or to measure the immunity of electoral systems to spoilers. However, spoiler effects have been tangentially considered in classical social choice theory in the context of stronger postulates such as *independence of irrelevant alternatives* [1, 46, 10] or *candidate stability* [23, 24, 27, 48] or distinct though related postulates such as the *independence of clones* [55].

In computational social choice, spoilers have been addressed from the point of view of *electoral control problems* [40, 37, 17, 29, 42, 28], in particular of the problems of Constructive-Control-by-Adding-Candidates (CCAC) and Constructive-Control-by-Deleting-Candidates (CCDC), and their destructive control counterparts (DCAC and DCDC). While closely related to our subject, those problems are distinct, as they treat the election outcome in binary terms which are inapposite in party elections.

Kaminski [34] has been the first to propose a generalized definition of a spoiler applicable to party elections. He focused on distinguishing ways in which a potential spoiler can affect the seat payoff. Thus, apart from *classical spoilers* (who turn a majority winner into a majority loser, while making another player a majority winner), he distinguishes *kingmaker spoilers* (who turn a majority loser into a majority winner), *kingslayer spoilers* (who turn a majority winner into a majority loser), and *valuegobblers* (who affect the seat payoff of one player by an amount greater than their own payoff).

2 Introductory Example

While no formal definition of spoilers in party elections exists, psephologists have nevertheless regarded some parties as spoilers. For example, a small left-wing party *Razem* has been widely consider to be one in the Polish general election of 2015. We shall examine this example more closely to illustrate the intuition underlying our proposed Definition 3.

In Polish general elections, the Jefferson–D'Hondt rule is used to allocate a total of 460 seats in 41 districts with the district magnitude varying between 7 and 20. In addition, a statutory threshold is set at 5% of the total number of valid votes for parties and 8% for electoral coalitions. Only parties and coalitions whose vote shares exceed the appropriate threshold are eligible for seat allocation. In 2015, there have been eight major contenders, see Table 1, columns 1 to 3.

Let us consider what would happen if *Razem* had not participated in the election. Its votes would likely have been redistributed among other parties. Assume for the sake of argument that each party's share in those votes would be inversely proportional to its distance from *Razem* in the Chapel Hill Expert Survey dataset [2]. We thus obtain the outcome of a counterfactual election without *Razem*.

We could have expected that since every party gains votes when *Razem* is removed, its seats shares should not decrease. However, an inspection of the results reveals that four parties lose seats (including PiS, which no longer commands a majority). All in all, 9.5% of seats would change hands, even though *Razem* held none. Conversely, if we were instead to exclude PiS and redistribute its votes, everyone would benefit: no seats beyond PiS's share would be redistributed.

3 Seats-Votes Model

Let us now formalize the intuition described above. We start with introducing the *seats-votes model*, a highly simplified model of electoral systems that allows us to formulate and discuss the definition of electoral spoiler and all concepts requisite therefor with only the minimal formalism. We then establish that identity (i.e., seats-votes proportionality) is

We claim that the former effect, i.e., a party's removal causing another to lose seat shares, is the essence of spoilership. Hence, *Razem* was a spoiler in 2015, while PiS was not – in accord with the intuition that the election winner should not be regarded a spoiler.

name	actual		no Razem		no PiS	
	votes	seats	votes	seats	votes	seats
PiS	.376	.511	.380	.454	.000	.000
PO	.241	.300	.246	.278	.287	.332
Kukiz	.088	.091	.092	.076	.153	.156
Nowoczesna	.076	.061	.081	.062	.109	.099
Lewica (*)	.076	.000	.085	.067	.123	.116
PSL	.051	.035	.056	.029	.159	.164
Korwin	.048	.000	.051	.022	.091	.075
Razem	.036	.000	.000	.000	.069	.046

Table 1: Polish general election of 2015. Asterisk (*) marks coalitions.

the unique spoiler-proof electoral rule in the framework of this model. Our interest in the seats-votes model is not merely expository: it can be used to approximate a vast majority of electoral formulae used in real-life party elections.

We begin with some notation used throughout the paper:

$[t]$ For $t \in \mathbb{Z}_+$, let $[t]$ denote the set $\{1, \ldots, t\}$.

Δ Put $\Delta := \{\mathbf{x} \in \mathbb{R}_+^{\mathbb{N}} : \sum_{i \in \mathbb{N}} x_i = 1\}$.

Δ_S Let $\emptyset \neq S \subset \mathbb{N}$ be finite. Put $\Delta_S := \{\mathbf{x} \in \Delta : \sum_{i \in S} x_i = 1\}$. Note that the simplices Δ_K ($\emptyset \neq K \subset S$) are faces of Δ_S. Moreover, $\Delta := \bigcup_{S \in \mathcal{P}(\mathbb{N}) : S \neq \emptyset} \Delta_S$.

Δ_{-i} For $i \in S \subset \mathbb{N}$, put $\Delta_{-i} := \Delta_{S \setminus \{i\}}$ for short if S is given.

$\mathbf{x}_{-i}$ For $n \in \mathbb{N}$, $\mathbf{x} \in \mathbb{R}^n$, and $i \in [n]$, put $\mathbf{x}_{-i} := (x_j)_{j \in [n] \setminus \{i\}}$ for short if n is given.

Let $P \subset \mathbb{N}$ be a finite set of *parties*, and let $p := |P|$. Let us consider voting systems in which the profile can be fully represented by an actually p-dimensional vector of *vote shares*, $\mathbf{w} \in \Delta_P$. These include all single-district single- or multiple-choice voting systems like plurality or SNTV, as well as single-district scoring systems like k–Borda. Under such systems, the election and its outcome share a single domain, allowing us to model electoral rules by a particularly simple function.

Definition 1 (Seats-Votes Function)**.** A *seats-votes function* is a function $f : \Delta \to \Delta$ fulfilling the following properties:

symmetry $\sigma \circ f = f \circ \sigma$ for any σ that permutes the coordinates of a point in Δ,

weak monotonicity $x_i > y_i$ and $x_j \leq y_j$ for each $j \neq i$ implies that $f_i(\mathbf{x}) \geq f_i(\mathbf{y})$ for every $\mathbf{x}, \mathbf{y} \in \Delta$ and $i \in \mathbb{N}$,

negative unanimity $x_i = 0$ implies $f_i(\mathbf{x}) = 0$ for every $\mathbf{x} \in \Delta$ and $i \in \mathbb{N}$.

In other words, a seats-votes function maps a vector of vote shares to an election outcome (usually a vector of seat shares, but see the following remark). We require it to satisfy three natural axioms: the system should treat all parties in the same manner (*symmetry*), getting more votes should always be non-detrimental (*weak monotonicity*), and no party should be entitled to any seats without getting any votes (*negative unanimity*). Note that the former two conditions have natural Arrovian counterparts.

Remark 1. Despite its name, a seats-votes function not necessarily maps vote shares to seat shares. For example, a function mapping voting weights to *indices of voting power* [43, 4, 49] can also be (formally) consider as a seats-votes function under the above definition.

Apportionment methods, used to divide proportionally discrete goods (like parliamentary seats), are the most common example of real-life electoral systems that can be described by seats-votes functions. On the other hand, systems involving *multiple electoral districts* (like common multi-district plurality systems) are not seats-votes systems, since election results depend not only on the aggregate vote distribution, but also on district-level results. However, such systems, at least in some situations, may still be approximated by seats-votes functions under reasonable distributional assumptions.

Any discussion of spoiler effect in party elections presumes that a party is removed and its votes (or seats) are redistributed. Mathematically, such a process can be modeled by a *monotonic projection*:

Definition 2 (Monotonic Projection)**.** For any finite $P \in \mathcal{P}(\mathbb{N})$ and any $\emptyset \neq K \subseteq P$, a function $\pi : \Delta_P \to \Delta_K$ is a *monotonic projection* if $\pi^2 = \pi$ and $\pi_j(\mathbf{x}) \geq x_j$ for all $\mathbf{x} \in \Delta_P$ and $j \in K$. For $i \in P$, we denote the set of all monotonic projections $\Delta_P \to \Delta_{P\setminus\{i\}}$ by Π_{-i}.

We now have introduced all the concepts necessary for a definition of an electoral spoiler:

Definition 3 (Electoral Spoiler)**.** Let $\emptyset \neq P \subset \mathbb{N}$ be finite and let $\mathbf{w} \in \Delta_P$ be a vector of vote shares. Then the i-th party, $i \in P$, is an *electoral spoiler* under a seats-votes function f and a monotonic projection $\pi \in \Pi_{-i}$ if and only if there exists no monotonic projection $\rho \in \Pi_{-i}$ such that $f(\pi(\mathbf{w})) = \rho(f(\mathbf{w}))$.

Corollary 1. *If the i-th party is an electoral spoiler, then there exists a party $j \neq i$ such that $f_j(\pi(\mathbf{w})) \leq f_j(\mathbf{w}))$, i.e., j's share in the election outcome decreases when i is eliminated.*

From the definition of a spoiler, it is natural to proceed to the issue of spoiler-proofness:

Definition 4 (Spoiler-Proofness)**.** A seats-votes function f is *spoiler-proof* if for every finite $\emptyset \neq P \subset \mathbb{N}$, $\mathbf{x} \in \Delta_P$, $i \in P$, and $\pi \in \Pi_{-i}$ there exists $\rho \in \Pi_{-i}$ such that $f(\pi(\mathbf{x})) = \rho(f(\mathbf{x}))$.

Finally, our framework enables us not only to distinguish between spoilers and non-spoilers, but also to measure the magnitude of a spoiler's impact on election results (see Figure 1):

Definition 5 (Excess Electoral Impact)**.** Let a finite set of parties $\emptyset \neq P \subset \mathbb{N}$ and a vote share vector $\mathbf{w} \in \Delta_P$ be given. By $\lambda^f_{i,\pi}$, where $\pi \in \Pi_{-i}$, we denote the *excess electoral impact* of the i-th party under a seats-votes function f, i.e., the L_1 distance between $\pi(f(\mathbf{w}))$ and the *redistribution region* $\{\rho(f(\mathbf{w})) : \rho \in \Pi_{-i}\}$.

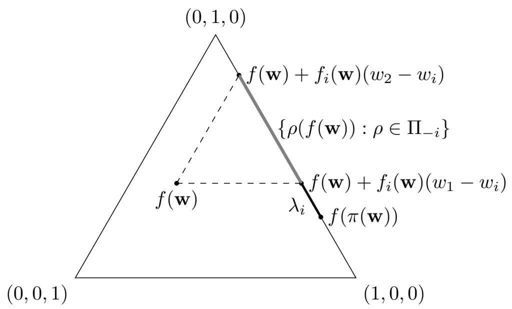

Figure 1: Geometric interpretation of Definitions 3 and 5.

Remark 2. Note that $\lambda^f_{i,\pi} > 0$ if and only if i is a spoiler under f and π.

3.1 Spoiler-Proofness

Our main theoretical result here is an impossibility theorem regarding the existence of spoiler-proof seats-votes functions that are distinct from identity (i.e., perfect proportionality between votes and seats for each party).

Theorem 1. *The identity function given by $I(\mathbf{x}) = \mathbf{x}$ for $\mathbf{x} \in \Delta$ is the unique spoiler-proof seats-votes function.*

Outline of Proof. (See Appendix on arXiv, [9], for the full proof). Let f be a spoiler-proof seats-votes function. Fix any $i \in \mathbb{N}$ and $v \in (0,1)$, and consider $\Lambda_i := \{\mathbf{x} \in \Delta : x_i = v\}$. We begin by demonstrating that $f_i \restriction_{\Lambda_i}$ achieves maxima in the vertices of Λ_i. It is enough to show that spoiler-proofness implies that for any face of Λ_i the function f_i is maximized at its facets. Then we establish that $f_i \restriction_{\Lambda_i}$ also achieves minima in the vertices of Λ_i, by adding a dummy party j and showing that every point in Λ_i can be obtained by a monotonic projection from a vertex where $x_i = v, x_j = 1 - v$.

Accordingly, we arrive at the conclusion that $f_i(\mathbf{x})$ depends only on x_i, i.e., that there exists some $\varphi : [0,1] \to [0,1]$ such that $f_j(\mathbf{x}) = \varphi(x_j)$ for each $j \in \mathbb{N}$. From weak monotonicity and negative unanimity of f we obtain a sequence of Schröder's functional equations $\varphi(px) = p\varphi(x)$ for any $x \in [0, 1/p]$. Applying Theorem A from [56], we then conclude that $\varphi(x) = x$ for all $x > 0$, which concludes the proof. □

3.2 Probability of the Occurrence of Spoilers

The natural next step is to consider whether spoilers, if unavoidable, are low-probability outliers or something that arises regularly. Accordingly, we investigate the probability of the occurrence of spoilers under seats-votes functions approximating two of the electoral rules that are most commonly used in real-life party elections: FPTP (plurality) and Jefferson–D'Hondt method of apportionment [33, 21, 22, 3, 44]. Because the proofs in this section are highly technical, they have been moved to the Appendix [9].

We assume that for any vote share vector $\mathbf{w} \in \Delta_P$ and every $i \in P$, the vote share under the restriction to $P \setminus \{i\}$ is given by the monotonic projection π such that $\pi_{-i}(\mathbf{w}) := \mathbf{w}_{-i} + w_i\mathbf{x}$, where $\mathbf{x} \sim \mathrm{Unif}(\Delta_{P\setminus\{i\}})$. We call π the *uniform projection*. We denote the set of spoilers by S.

3.2.1 Jefferson–D'Hondt

The following seats-votes function approximates the Jefferson–D'Hondt allocation rule under certain assumptions regarding vote distribution [31]. Let parties be ordered degressively by their vote shares, w_i, $i \in P$. Then

$$f_i(\mathbf{w}) = w_i \Big/ \left(\sum\nolimits_{j=1}^{r} w_j\right)\left(1 + \frac{r}{2k}\right) - \frac{1}{2k} \tag{1}$$

for $i \in [r]$, and $f_i(\mathbf{w}) = 0$ for $i > r$, where

$$r := \max\left\{ l \in P : \frac{w_l}{\sum_{j=1}^{l} w_j} > \frac{1}{2k + l} \right\}, \tag{2}$$

and $k \in \mathbb{R}_+$ represents the mean district magnitude.

Lemma 1. Let $\mathbf{W} \sim \mathrm{Unif}(\Delta_p)$ be a vote share vector, and f be a seats-votes function given by (1). The probability that S is empty, assuming the uniform projection, equals:

$$\Pr\big(\forall i \in P : f(\pi_i(\mathbf{W})) \in R_i\big) = \Psi^{\mathrm{JDH}}_{p,k} :=$$
$$(1-t)^{p-1}\Gamma(p)\Gamma(p-1)\left(\frac{1-t}{t}\right)^{p-2} \mathcal{L}^{-1}\left\{\left(\frac{1}{s}U\left(p-2, 0, \frac{s}{2k}\right)\right)^{p}\right\}\Bigg|_{s=1}, \tag{3}$$

where U is Tricomi's confluent hypergeometric function.

As far as we know, the inverse Laplace transform appearing in (3) needs to be evaluated numerically.

Lemma 2. Let $\mathbf{w} \in \Delta_p$ be a vote share vector, and f be given by (1). The probability that S is empty if $\pi_i \sim \text{Unif}(\Pi_{-i})$ for each $i \in P$ is bounded from above by:

$$\Psi^{\text{JDH}}_{p,k} \leq \tilde{\Psi}_{p,k} := (1-t)\left(\frac{2kp^2}{1+2kp^2}\right)^{p-2}. \tag{4}$$

Corollary 2. *Let p, k be a non-decreasing function of $x > 0$:*

- *if* $\lim\limits_{x\to\infty} \frac{\log p(x)}{\log k(x)} > \frac{1}{3}$, *then* $\lim\limits_{x\to\infty} \tilde{\Psi}_{p(x),k(x)} = 0$,
- *if* $\lim\limits_{x\to\infty} \frac{\log p(x)}{\log k(x)} < \frac{1}{3}$, *then* $\lim\limits_{x\to\infty} \tilde{\Psi}_{p(x),k(x)} = 1$,
- *if* $\lim\limits_{x\to\infty} \frac{\log p(x)}{\log k(x)} = \frac{1}{3}$, *then* $\lim\limits_{x\to\infty} \tilde{\Psi}_{p(x),k(x)} = 1/\sqrt{e}$.

Note in particular that under many formal models (e.g. [7]), as well as under common heuristics such as the Seat-Product Model [51], $p \geq k^{1/2}$ as $p, k \to \infty$. Thus, the probability that there are no spoilers converges to 0 (and is already negligible for commonly encountered values of those parameters).

3.2.2 FPTP (1–SNTV)

Per [52], the FPTP (1–SNTV) rule can be heuristically approximated by a seats-votes function given by:

$$f_i(\mathbf{w}) = w_i^\beta \Big/ \left(\sum\nolimits_{j=1}^{r} w_j^\beta\right). \tag{5}$$

Let the vote share vector $W \sim \text{Unif}(\Delta_p)$, and let f be given by (5). We have numerically evaluated probability that there are no spoilers for $\beta \in [1, 4]$ and $p \in \{3, \ldots, 8\}$ (see Fig. 2). For $\beta = 1$ the power-law formula (5) yields a perfectly proportional system which, by Theorem 1, is spoiler-proof.

We have thus seen that seats-votes functions approximating two of the most commonly used electoral rules not only fail to satisfy spoiler-proofness, but yield spoilers with a high probability. But perhaps this is an artifact of single-choice voting method underlying those rules? It has been claimed, although informally, that systems like STV, k-Borda, or approval voting are less susceptible to spoilers. To analyze such claims, we need a more general framework.

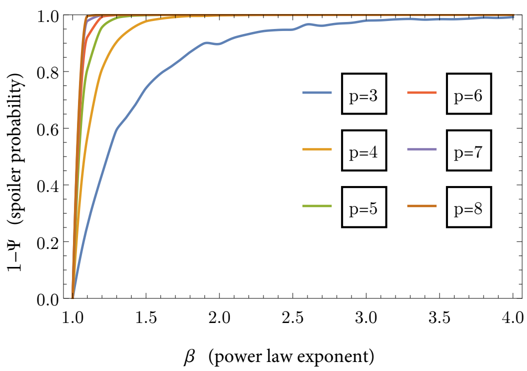

Figure 2: Spoiler occurrence probability, FPTP.

4 Generalized Model

The challenge in generalizing the model formulated in the preceding section lies in the fact that as we move beyond single-choice voting methods, there is no longer a natural definition of the number of votes cast for a given party, or at least that information is insufficient to obtain the election outcome. As we need to preserve potentially all the information contained in votes, we proceed by using a richer model of elections than in Section 3 and endowing it with a structure (order, restriction operation, etc.) that mirrors the structure of the unit simplex. While somewhat complex, this structure yields a powerful generalization of Theorem 1.

We start by introducing a framework model of *party elections*. Let C be a finite set of *candidates*, $m := |C|$, let $P := [p]$ be a set of *parties*, $p \in \mathbb{N}_+$, and let a be a function that assigns parties to candidates. On the other hand, the definition of a *vote* necessarily depends on the voting method, i.e., the structure of the ballot. For instance, a vote can be:

- an element of C (single-choice voting),
- a linear order on C (ordinal voting),
- a binary sequence $\{0,1\}^C$ (approval voting),
- a point in Δ_C (cumulative voting),
- a point in unit cube $[0,1]^C$ (cardinal voting).

Hence, we proceed by specifying the properties that the set of all possible votes must satisfy. Intuitively, we treat a vote as a function assigning to each candidate some 'judgment'. For example, for single-choice and approval voting the judgment is binary (with an additional restriction that only a single candidate can be approved in single-choice); for cumulative and cardinal voting it is an element of $[0,1]$ (again with a restriction on the sum of values for cumulative voting); and for ordinal voting it is the set of less favored candidates under the voter's underlying order.

Definition 6 (Domain of Admissible Votes)**.** Let $\mathcal{D}(C)$ be a set endowed with a *partial order* $\precsim_C$ such that every chain maximal by inclusion has a minimum, and with an order-preserving *inverse system* $r_{I,K} : \mathcal{D}(I) \to \mathcal{D}(K)$, $\varnothing \neq K \subseteq I \subseteq C$. A *domain of admissible votes* is a set $\mathcal{V}(C)$ of some maps from C to $\mathcal{D}(C)$. For any $v \in \mathcal{V}(C)$ we require a *restriction* of $r_{C,K} \circ v$ to a nonempty $K \subseteq C$ to be an element of $\mathcal{V}(K)$ if $v(x)$ is minimal in $(\mathcal{D}(C), \precsim_C)$ for each $x \in C \setminus K$.

For any $v \in \mathcal{V}(C)$ and $x \in C$ we denote $v(x)$ as v_x.

The order on $\mathcal{D}(C)$ corresponds to the intuition that some votes are less favorable for the candidate than others. For most of our examples, it is induced by the natural order on reals, except that for ordinal voting it is induced by the inclusion order on $\mathcal{P}(C)$. The reason for the requirement that each maximal chain have a minimum is to be able to define a counterpart to party getting no votes under the Sec. 3 model.

method	$\mathcal{D}(C)$	$\mathcal{V}(C)$	isomorphic to
single-choice	$\{0,1\}$	$\{\mathbf{e}_x : x \in C\}$	element of C
ordinal	$\mathcal{P}(C)$	$\mathcal{P}(C)^C$ (†)	linear order on C
approval	$\{0,1\}$	$\{0,1\}^C$	$\{0,1\}^C$
cumulative	$[0,1]$	Δ_C	Δ_C
cardinal	$[0,1]$	$[0,1]^C$	$[0,1]^C$

Table 2: Domains of admissible votes for common voting methods. † For ordinal voting, the image of C under each vote needs to be a permutation of some maximal chain in $\mathcal{P}(C)$ ordered by inclusion.

Definition 7. For each candidate $x \in C$, the order on $\mathcal{D}(C)$ induces a partial preorder $\precsim_{x,C}$ on $\mathcal{V}(C)$ such that $v \precsim_{x,C} w$ iff $v_x \precsim_c w_x$ for $v, w \in \mathcal{V}(C)$. Note that $\precsim_{x,C}$ inherits the property that each maximal chain has a minimum. To simplify notation if C is fixed we will use $\precsim_x$ to denote $\precsim_{x,C}$.

The restriction corresponds to a 'mechanical' removal of a subset of candidates. However, sometimes a candidate cannot be removed without the vote ceasing to be admissible. For instance, in single-choice voting we cannot remove x from a vote for x – otherwise, the vote would become empty. However, under any method we can remove a candidate who is irrelevant, i.e., for whom the vote is minimal under $\precsim_x$.

Definition 8. A vote $v \in \mathcal{V}(C)$ is *restrictible* to $\varnothing \neq K \subseteq C$ iff its restriction to K, $v_{C,K} := (r_{C,K} \circ v) \restriction_K \in \mathcal{V}(K)$.

As for the inverse system r, we need it to account for cases when $\mathcal{D}(C)$ is not constant. This is the case for ordinal voting, where $r_{C,K}(X) := X \cap K$, i.e., the set of less favored candidates X is restricted to a subset of K. The requirement that $r_{I,K}$ be order-preserving corresponds to the individual choice *independence of irrelevant alternatives* axiom [45], according to which removal of some candidates does not affect the judgment on others.

We can now define preference profiles, which correspond to *vote share vectors* from Sec. 3.

Definition 9 (Preference Profile)**.** For a set of candidates C and a domain of votes $\mathcal{V}(C)$, a *profile* V is a probability measure on $\mathcal{V}(C)$. We denote the set of all profiles as $\mathcal{M}(C)$.

Definition 10 (Marginal Profile)**.** For a profile $V \in \mathcal{M}(C)$ and a party $i \in P$, we denote the marginal measure of V on $\{(v_x)_{x \in a^{-1}(i)} : v \in \mathcal{V}(C)\}$, i.e., coordinates corresponding to the candidates of the i-th party, by V_i and call it a *marginal profile* of i. It corresponds to the i-th party's vote share.

Just as vote share vectors on Δ_p have been partially ordered for each party, we need a partial preorder on the set of profiles.

Definition 11. For each candidate $x \in C$, their preorder on votes $\preceq_x^\sim$ induces a partial preorder on profiles, $\preceq_x^*$, such that for $V, W \in \mathcal{M}(C)$ we have $V \preceq_x^* W$ if and only $V(U) \leq W(U)$ for each $U \subseteq \mathcal{V}(C)$ that is an upper set under $\preceq_x^\sim$. Note that $\preceq_x^*$ is a stochastic preorder on posets.

Definition 12. For each party $i \in P$, there exists a partial preorder on profiles, $\preceq_i$, such that for $V, W \in \mathcal{M}(C)$ we have $V \preceq_i W$ iff there exists such permutation of candidates of the i-th party, σ, that $V \preceq_x^* \sigma(W)$ for every $x \in a^{-1}(i)$. The permutation accounts for the fact that from a party's point of view it is irrelevant which candidate is first.

The intuition here is that a profile is 'better' for a candidate if some votes has been changed to be 'better' for him, and none have been change to be worse (up to a permutation). Similarly, it is better for a party if it is better for one of its candidates and not worse for every other (also up to a permutation). Note also that $\preceq_i$ corresponds to the partial order on the unit simplex induced by the i-th barycentric coordinate. In particular, a profile V being minimal under $\preceq_i$, $i \in [p]$, corresponds to the i-th party's vote share being equal to 0.

Two other operations for which we need counterparts are restriction of a vector and permutation of its coordinates:

Definition 13 (Restriction of a Profile)**.** A restriction of a profile V to a nonempty subset of candidates, $K \subset C$, is the pushforward under $v \mapsto v_{C,K}$ of the restriction of V to the set of votes restrictible to K. A restriction V_L of the profile V to a subset of parties, $L \subset P$, is its restriction to $a^{-1}(L) \subset C$, i.e., to the candidates affiliated with parties in L.

Note that if the profile V is concentrated on votes restrictible to $K \subseteq C$, then V_K is a probabilistic measure. In particular, if the profile is minimal under $\preceq_i$ for any party $i \in P$, it is always concentrated on votes restrictible to $P \setminus \{i\}$.

Definition 14 (Permutation of Parties)**.** A permutation of candidates σ is a *permutation of parties* if $a(x) = a(y)$ implies $a(\sigma(x)) = a(\sigma(y))$ for any two candidates $x, y \in C$.

Definition 15 (Party Election)**.** For sets of candidates C and parties P, an affiliation function $a : C \to P$, and a profile $V \in \mathcal{M}(C)$, we refer to $E := (C, P, a, V)$ as *party election.*

Allocation rules, introduced here, are the counterpart of seats-votes functions from Sec. 3:

Definition 16 (Allocation Rule)**.** An *allocation rule* is a function $f : \bigcup_{C \in \mathcal{P}(\mathbb{N})} \mathcal{M}(C) \to \bigcup_{C \in \mathcal{P}(\mathbb{N})} \Delta_{a(C)}$ that maps a preference profile into a unit simplex and satisfies the following axioms for any fixed C and each $V, W \in \mathcal{M}(C)$:

dimension preservation $f(V) \in \Delta_{a(C)}$,

symmetry $\sigma_\Delta \circ f = f \circ \sigma_\mathcal{M}$ for any permutation of parties σ, with σ_Δ and $\sigma_\mathcal{M}$ induced by σ on $\Delta_{a(C)}$ and $\mathcal{M}(C)$,

negative unanimity V being minimal under $\preceq_i$ implies $f_i(V) = 0$ for each party $i \in a(C)$,

consistency $f_j(V_{a(C)\setminus\{i\}}) = f_j(V)$, where V is minimal under $\preceq_i$, for any distinct parties $i, j \in a(C)$,

weak monotonicity $W \prec_i V$ and $V \preceq_j W$ for each $j \in a(C) \setminus \{i\}$ implies $f_i(W) \leq f_i(V)$ for every $i \in a(C)$.

Remark 3. Every multiwinner voting rule in the sense of [25] that satisfies neutrality, monotonicity, and never elects a candidate for whom the profile is minimal, naturally induces an allocation rule. It is obtained by normalizing, for each winning committee S, the counting measure of the intersection of S and each party's set of candidates, $a^{-1}(i)$, $i \in P$, and averaging over such committees.

Remark 4. Definition 16 can be naturally generalized to a system with multiple electoral districts (as in Section 5).

Definition 17 (Redistribution Function)**.** Fix any nonempty $L \subset P$. A *redistribution function* is any function $\pi_L : \mathcal{M}(C) \to \mathcal{M}(C)$ such that for each $V \in \mathcal{M}(C)$:

- $\pi_L^2 = \pi_L$,
- $\pi_L(V)$ is minimal under $\preceq_i$ for each $i \in P \setminus L$,
- $V \preceq_j \pi_L(V)$ for every $j \in L$,
- $V_L(X_L) \leq (\pi_{P,L}(V))_L(X_L)$ for all measurable $X \subseteq \{v \in \mathcal{V}(C) : v \text{ restrictible to } K\}$, where $K := a^{-1}(L)$ and X_L is the image of X under $v \mapsto v_{C,K}$.

For $i \in P$, we denote the set of all such $\pi_{P\setminus\{i\}}$ by R_{-i}.

Note that a redistribution function is a counterpart to a monotonic projection from Section 3. Condition 1 guarantees that it is a projection, condition 2 – that its codomain is embedded in an equivalent of a face of the simplex, and condition 3 – that it is monotonic in the sense of Definition 2. Unlike restriction, which is determined by the voting method, a redistribution function can be arbitrarily chosen as long as the axioms are satisfied. However, we only redistribute those votes that cannot be restricted (see condition 4 in Definition 17).

Definition 18 (Electoral Spoiler)**.** For a party election (C, P, a, V) the i-th party, $i \in P$, is an *electoral spoiler* under an allocation rule f and a redistribution function $\pi \in R_{-i}$ if and only if there exists no monotonic projection $\rho \in \Pi_{-i}$ such that $f(\pi(V)) = \rho(f(V))$. Note how this definition corresponds to Definition 3.

Definition 19 (Spoiler-Proofness)**.** An allocation rule f is *spoiler-proof* if for every profile $V \in \mathcal{C}$, party $i \in [p]$, and redistribution function $\rho \in R_{-i}$ there exists some monotonic projection $\pi \in \Pi_{-i}$ such that $f(\rho(V)) = \pi(f(V))$.

Definition of excess electoral impact mirrors Definition 5.

Definition 20 (Zero-Sum Voting Methods)**.** A voting method is *zero-sum* if, for any $x, y \in \mathcal{V}(C)$, $x_i \prec_\sim y_i$ for some $i \in C$ implies $x_j \succ_\sim y_j$ for some $j \in C$.

Examples of zero-sum voting methods include single-choice, cumulative, and ordinal voting.

Theorem 2. *For any zero-sum voting method an allocation rule f is spoiler-proof if and only if f_i is constant in every equivalence class in $\mathcal{M}(C)/\equiv_i$, i.e., the class of profiles equivalent under partial preorder $\preceq_i$, for each $i \in P$.*

The proof of the above theorem mirrors the first part of the proof of Theorem 1. A reader is referred to the Appendix [9].

Section 3	Section 4
unit simplex Δ_p	set of profiles $\mathcal{M}(C)$
vote share vector $\mathbf{w} \in \Delta_p$	profile $V \in \mathcal{M}(C)$
barycentric coordinate w_i	marginal profile V_i
order $\leq$ on each coordinate	preorder $\preceq_i$ on marginals
partial order $\leq_i$ on Δ_p	preorder $\preceq_i$ on profiles
section of Δ_p with w_i fixed	equivalence class in $\mathcal{M}(C)/\equiv_i$
$w_i = 0$	V minimal under $\preceq_i$
K-face of a unit simplex	profiles minimal under $\preceq_i$ for $i \in K$
restriction of a vector $\mathbf{w}_{-i}$	restriction of a profile $\mathbf{V}_{-i}$
monotonic projection	redistribution function
set of monotonic projections Π_{-i}	set of redistribution functions R_{-i}
seats-votes function	allocation rule

Table 3: Parallels between Sections 3 and 4.

5 Spoiler Susceptibility

In the preceding sections, we have demonstrated that spoiler-proofness is rare while spoilers are ubiquitous. But from a practical point of view the real issue is not whether spoilers exists, but what is the magnitude of their electoral impact.

Fix a family of redistribution functions, $(\pi_i)_{i \in P}$. We define spoiler susceptibility of an allocation rule f as the expected maximum excess electoral impact, where the maximum is takenover parties, and the expectation is over some probability distribution $\mathcal{D}$ on $\mathcal{M}(C)$:

$$\Phi(\mathcal{D}, f) := \mathbb{E} \max_{i \in P} \lambda^f_{i,\pi_i}(\mathbf{V}), \text{ where } \mathbf{V} \sim \mathcal{D}. \tag{6}$$

Spoiler susceptibility is difficult to model analytically, so we focus on experimental results.

5.1 Probabilistic Models

The probabilistic models commonly used in computational social choice (see generally [5, 50]) have been developed for single-district non-party elections. Thus, we need to appropriately extend those models, addressing two basic challenges: grouping of candidates into parties, and existence of multiple electoral districts.

Grouping of candidates into parties is based on a latent assumption of intra-party candidate clustering: candidates of the same party are assumed to be perceived by voters as, on average, more similar than candidates of different parties. Without that assumption, parties would tend to obtain equal seat shares, leading to an overall tie.

Existence of multiple electoral districts reflects another latent assumption – one of intra-district voter clustering. Voters within a single district are assumed to share preferences to a greater extent than an unbiased sample of the population. Otherwise the c-district allocation rule would (per the central limit theorem) converge to the expected value of the rule for the population (commingled) profile.

We consider four classes of probabilistic models:

Spatial Models. In a d-dimensional Euclidean model each party, voter, and candidate is assigned an ideal point in $\mathbb{R}^d$ [26, 41]. First, party ideal points are drawn from $\mathrm{Unif}((0, 1)^d)$, and then candidate ideal points for each district are drawn from the multivariate normal distribution with location at the party's ideal point and the correlation matrix $\mathbf{\Sigma} := \sigma I_d$, where I_d is a $d \times d$ identity matrix and $\sigma \in \mathbb{R}_+$. Voter ideal points are drawn independently from the uniform distribution on $(0, 1)^d$, then shifted in each district independently by a vector drawn from $\mathrm{Unif}(-1/4, 1/4)^d$ (to account for district clustering). A vote is obtained by sorting candidates according to the increasing L_2 distance from the voter's ideal point.

Single-Peaked Models. We consider two models for generating single-peaked profiles. In one [57] we are given an ordering on the set of candidates, and each vote is drawn from the uniform distribution on the set of all single-peaked votes consistent therewith. In the other one [18] the peak is drawn from a uniform distribution on candidates, and the remainder of the vote is obtained by a random walk. In both models, candidate ordering in each district is obtained in the same manner as in the 1-dimensional Euclidean model.

Mallows Model. The Mallows model [39, 19] is parametrized by a single parameter $\phi \in [0, 1]$, and a (central) vote $v_c \in \mathcal{D}(C)$. The probability of generating a vote v is proportional to $\phi^{f(v_c,v)}$, where $f(v_c, v)$ is the Kendall tau distance [35] between v_c and v. We first generate a central vote for each district, v_c^i with parameter ϕ_1 and a starting vote $v_c^0 := [m]$, then generate votes within each district with parameter ϕ_2 and a district–wide central vote v_c^i. Intra-party clustering is achieved by grouping party candidates together in the starting vote. On sampling from the Mallows model, see [38][2].

Impartial Culture (IC). Under IC, each vote is drawn randomly from the uniform distribution on linear orders on candidates [15]. This model does not account for intra-party and intra-district clustering, and is regarded as a poor approximation of real-life [47, 53].

5.2 Allocation Rules

We analyze seven well-known allocation rules (see the Appendix for formal definitions):

SNTV (plurality),
k–**Borda** [20, 30],
Chamberlin–Courant [16],
Harmonic Borda (HB) [30],
Proportional k–Approval (k–PAV) [36],
Single Transferable Vote (STV) [32, 54] with the fractional Droop quota,
Jefferson–D'Hondt (JDH), discussed in Section 3.2.

5.3 Experimental Results

To compare voting methods, we analyze their performance under 11 probabilistic models: four spatial models ($d = 1, 2$ and $\sigma = 0.05, 0.2$), four single-peaked models (Walsh and Conitzer, $\sigma = 0.05, 0.2$), two Mallows models ($\phi_1 = 0.75$, $\phi_2 = 0.25, 0.75$) and impartial culture. For every voting method, model, committee size $k \in \{1, 5, 15\}$, and number of parties $p \in \{3, \ldots, 10\}$ we run 600 simulations, with the number of districts $c = 100$, and the number of voters per district $n = 100$. We use exact results for SNTV, k-Borda, and STV, approximation (1) for JDH, and greedy approximations for CC, HB, and k-PAV.

Experimental results plotted on Figure 3 demonstrate several regularities. First, spoiler susceptibility depends strongly on the choice of the model. Under spatial models, which are most likely to approximate political elections, we can distinguish several classes of rules. Those least susceptible to spoilers are STV, Chamberlin–Courant, and Harmonic-Borda. The ordering on them depends on the number of parties – CC outperforms STV for large values of p. SNTV and Jefferson-D'Hondt are the middle performers, with JDH becoming the more resistant rule as the number of parties increases. On the other hand, k–Borda performs poorly against spoilers in models with high degree of party clustering ($\sigma = 0.05$), but is more spoiler–resistant than JDH and SNTV in models with greater candidate dispersion. In

[2] We use a Mallows model parameterization by Boehmer et al. [6], based on a normalized dispersion parameter norm-ϕ.

single-member districts, Borda rule is the most spoiler-susceptible one for three-party models, but for large values of p, it outperforms even STV. k–PAV in multi-member districts and FPTP in single-member districts are almost consistently the worst performers. Finally, switching from single–member to multi–member districts improves spoiler resistance.

The Conitzer model is generally most susceptible to spoilers, especially when districts are single-member. Again, STV and CC are the best performers in multi-member districts, followed by Jefferson–D'Hondt, k–Borda, and Harmonic-Borda. SNTV performs very poorly for $k = 5$, and better – but still worse than most alternatives – for $k = 15$. Finally, k–PAV is again most susceptible to spoilers.

The Mallows model exhibits high resistance to spoiler effects, especially for multi-member districts. In single-member districts, STV is the best performer, followed by Borda, and finally by FPTP. In multi-member districts, JDH and HB are the best performers, k–Borda is usually the worst, and results for other rules depend on k.

We treat Walsh and IC models as reference points, since they are considered unlikely to correspond to any real-life party elections. For both, spoiler susceptibility is an issue only for $k = 1$. Under Walsh, FPTP is more susceptible to spoilers than STV, while Borda depends on the parity of p. Under IC, Borda and STV perform best.

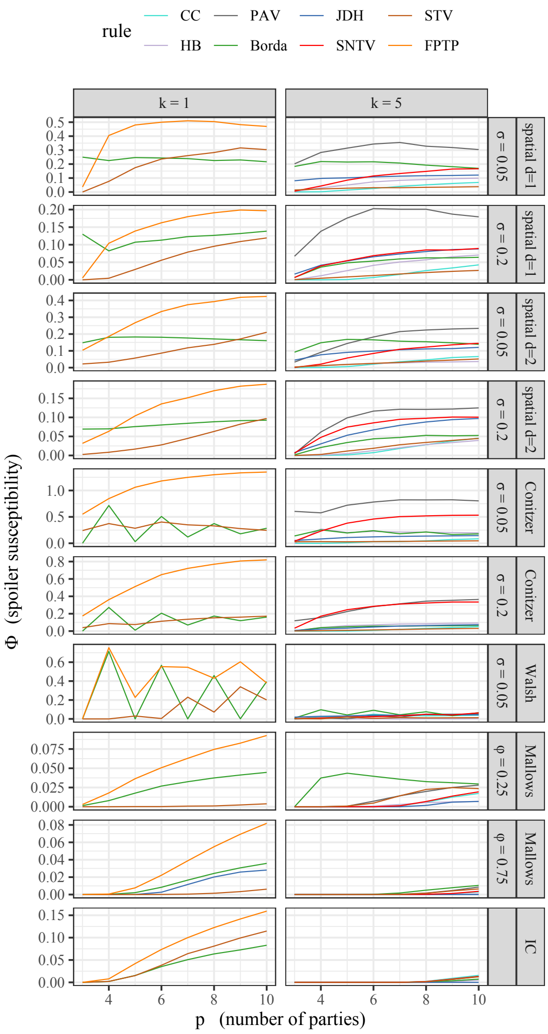

Figure 3: Spoiler susceptibility of electoral rules.

6 Summary

We have introduced a novel approach to defining spoilers in party elections, yielding both theoretical and experimental results. In particular we show that spoiler-proofness is a very strong postulate: the identity function, corresponding to proportional allocation of seats, is the unique spoiler-proof seats-votes function. Moreover, spoiler effects are ubiquitous at least under Jefferson–D'Hondt and FPTP.

The magnitude of spoiler impact varies with the allocation rule chosen. The results depend strongly on the distribution of preferences. An electoral system designer acting behind a veil of ignorance would do well to choose STV, CC, or HB. On the other hand, he should avoid k–PAV and FPTP. The performance of k–Borda depends on the degree of party clustering, while of that of Jefferson–D'Hondt – on single-peakedness of the model. Single-member districts are more susceptible to spoilers than multi-member ones.

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Appendix

A Proof of Theorem 1

Proof. Let f be a spoiler-proof seats-votes function. Fix any $i \in P$ and $v \in (0,1)$, and consider $\Lambda_i := \{\mathbf{x} \in \Delta_p : x_i = v\}$. We begin by demonstrating that $f_i \restriction_{\Lambda_i}$ achieves maxima in the vertices of Λ_i. It is enough to show that for any face of Λ_i the function f_i is maximized at its facets.

Each face of Λ_i has form of $L_K := \{\mathbf{x} \in \Lambda_i : \sum_{k \in K} x_k = 1 - v\}$ for some non-empty $K \subseteq P \setminus \{i\}$. Assume on the contrary that there exists some $\mathbf{y} \in \operatorname{int} L_K$ such that $f_i(\mathbf{y}) > \sup_{\mathbf{x} \in \partial L_K} f_i(\mathbf{x})$, where ∂L_K denotes the boundary of L_K, i.e., the union of its facets. But from the spoiler-proofness of f, it follows that $f_i(\pi(\mathbf{y})) \geq f_i(\mathbf{y})$ for every $j \in K$ and $\pi \in \Pi_{-j}$. As $\pi(\operatorname{int} L_K) \subset \partial L_K$, this is a contradiction. We can apply this reasoning inductively until we arrive at $\dim L_K = 0$. From the symmetry of f it follows that f_i is equal for every vertex of Λ_i.

On the other hand, we will now show that $f_i \restriction_{\Lambda_i}$ also achieves minima in the vertices of Λ_i. Let us first consider $\mathbf{z} \in \Delta_{p+1}$ such that $z_i = v$, $z_{p+1} = 1 - v$, and $z_j = 0$ for $j \in [p] \setminus \{i\}$. Note that for every $\mathbf{x} \in \Lambda_j$ there exists some projection $\pi \in \Pi_{-(p+1)}$ such that $\pi(\mathbf{z}) = \mathbf{x}$. From the spoiler-proofness of f it follows that $f_i(\mathbf{z}) \leq \inf_{\mathbf{x} \in \partial \Lambda_j} f_i(\mathbf{z})$. But from the symmetry of f we have $f_i(\mathbf{z}) = f_i(\mathbf{z}^j)$ for every $\mathbf{z}^j \in \Delta_{p+1}$, $j \in P \setminus \{i\}$, defined as $z^j_i := v$, $z^j_j := 1 - v$, and $z^j_k := 0$ for $k \in [p+1] \setminus \{i,j\}$. Finally, from the consistency of f it follows that for every $j \in [p] \setminus \{i\}$ we have $f_i(\mathbf{z}^j) = f_i(\mathbf{y}^j)$, where $\mathbf{y}^j \in \Delta_p$, $y^j_i := v$, $y^j_j := 1 - v$, and $y^j_k := 0$ for $k \in P \setminus \{i,j\}$. As the set $\{\mathbf{y}^j : j \in [p] \setminus \{i\}\}$ is the set of vertices of Λ_i, we conclude that $f_i \restriction_{\Lambda_i}$ achieves minima in those vertices. Thus, f_i is constant over Λ.

Accordingly, we arrive at the conclusion that $f_i(\mathbf{x})$ is independent on all coordinates of $\mathbf{x}$ save x_i. From the arbitrariness of the choice of j, the same holds for every other coordinate. Thus, by symmetry of f, there exists a function $\varphi : [0,1] \to [0,1]$ such that $f_j(\mathbf{x}) = \varphi(x_j)$ for each $j \in P$. From weak monotonicity of f it follows that φ is non-decreasing, while from negative unanimity it follows that $\varphi(0) = 0$ and $\varphi(1) = 1$. It also follows that for every $\mathbf{x} \in \Delta_k$, $k \in \mathbb{N}_+$, we have $\sum_{i=1}^{k} \varphi(x_i) = 1$. Hence, in particular, for any $n \in \mathbb{N}_+$ we have

$$n\,\varphi((n+1)^{-1} - \delta_n) = 1 - \varphi((n+1)^{-1} + n\delta_n),$$

for every

$$\delta_n \in \left[-(n(n+1))^{-1}, (n+1)^{-1}\right].$$

Substituting $\delta_n = (\delta_1 + 1/2 - (n+1)^{-1})/n$ in the n-th equation, $n > 1$, then subtracting both sides of every n-th equation from the respective sides of the first equation, and finally substituting $x = (1/2 - \delta_1)/n$, we obtain a sequence of Schröder's functional equations $\varphi(nx) = n\varphi(x)$ for any $x \in [0, 1/n]$. Applying Theorem A from [56], we then conclude that $\varphi(x) = \varphi(1)x = x$ for all $x > 0$. Thus, φ necessarily equals identity, as therefore does f. □

B Proof of Theorem 2

Definition 21. Fix some $i \in P$ and an equivalence class $\Lambda \in {}^{\mathcal{M}(C)}/_{\equiv_i}$, where $\equiv_i$ is induced by the preorder defined $\preceq_i$.

Let $\dim \Lambda := |\{j \in P \setminus \{i\} : \exists V, W \in \Lambda : V \prec_j W\}| - 1$.

A K-face of Λ, $K \subseteq P$, is any set of profiles $V \in \Lambda$ that V_j is minimal under $\preceq_j$ for any $j \in P \setminus K$.

A facet of Λ is any set of profiles $V \in \Lambda$ such that V is minimal under $\preceq_j$ for exactly two $j \in \{k \in P : \exists V, W \in \Lambda : V \prec_k W\}$.

Proof. Let f be a spoiler-proof allocation rule. Fix any $i \in [p]$ and any equivalence class $\Lambda_i \in {}^{\mathcal{M}(C)}/_{\equiv_i}$. We begin by demonstrating that $f_i \restriction_{\Lambda_i}$ achieves maxima in the vertices of Λ_i, i.e., such profiles that are minimal for all parties except i and at most one another. It is enough to show that for any face of Λ_i the function f_i is maximized at its facets.

Let L_K be a K-face of Λ_i for some non-empty $K \subseteq P \setminus \{i\}$. Assume on the contrary that there exists some $Y \in \operatorname{int} L_K$ such that $f_i(Y) > \sup_{X \in \partial L_K} f_i(X)$, where ∂L_K denotes the boundary of L_K, i.e., the union of its facets. But from the spoiler-proofness of f, it follows that $f_i(\pi(Y)) \geq f_i(Y)$ for every $j \in K$ and $\pi \in R_{-j}$. As $\pi(\operatorname{int} L_K) \subset \partial L_K$, this is a contradiction. We can apply this reasoning inductively until $\dim L_K = 0$. From the symmetry of f it follows that f_i is equal for every vertex of Λ_i.

On the other hand, we will now show that $f_i \restriction_{\Lambda_i}$ also achieves minima in the vertices of Λ_i. Let (C', P', a', Z) be a party election such that $C \subset C'$, $|P| = |P'| - 1$, $a = a' \restriction_C$, and $Z \in \mathcal{M}(C')$ is such that $Z_{-(p+1)} \equiv_i^P X$ for any $X \in \Lambda_i$ and Z is minimal for any $j \in [p] \setminus \{i\}$. Note that for every $X \in \Lambda_i$ there exists some vote redistribution function $\pi \in \Xi_{-(p+1)}$ such that $\pi(Z) = X$. From the spoiler-proofness of f it follows that $f_i(Z) \leq \inf_{X \in \partial \Lambda_j} f_i(X)$. But from the symmetry of f we have $f_i(Z) = f_i(Z^j)$ for every $Z^j \in \mathcal{M}(C \cup \{\dots\})$, where Z^j, $j \in P \setminus \{i\}$, are the vertices of $[Z]_{\equiv_i}$ other than Z. Finally, from the consistency of f it follows that for every $j \in [p] \setminus \{i\}$ we have $f_i(Z^j) = f_i(Z^j_{-(p+1)})$. As the set $\{Z^j_{-(p+1)} : j \in P \setminus \{i\}\}$ is the set of vertices of Λ_i, we conclude that $f_i \restriction_{\Lambda_i}$ achieves minima there. Thus, f_i is constant over Λ, as desired.

The proof in the other direction trivially follows from symmetry and weak monotonicity of f.

□

C Proofs of Theoretical Results for Jefferson–D'Hondt

For a vector $\mathbf{x} \in \mathbb{R}^n$, $n \in \mathbb{N}$, let $x_i^{\downarrow}$, $i \in [n]$, denote the i-th largest coordinate.

The j-th largest party is *relevant* iff

$$w_j^{\downarrow} \geq \frac{1}{2k + j} \sum_{i=1}^{j} w_i^{\downarrow}.$$

We denote the set of relevant parties as R. Let $n := |R|$.

If the i-th party is relevant, let

$$q_i := w_i \Big/ \left(\sum\nolimits_{j=1}^{n} w_j^{\downarrow} \right).$$

Otherwise, let $q_i := 0$.

Recall that under the uniform projection assumption, if the j-th party is eliminated, its votes are restributed as follows:

$$q_i' := q_i + \rho_i q_j$$

for $i \neq j$ and $(\rho_1, \dots, \rho_p)_{-j} \sim \operatorname{Unif}(\Delta_{p-1})$.

C.1 Probability that j-th Party is not a Spoiler

We begin with estimating the probability that a given party is not a spoiler for a given vote share vector.

Lemma 3. Let $\mathbf{w} \in \Delta_p$ be a vote share vector, and f be a seats-votes function given by (1). Then for any $i \in [p]$ the probability that i is not a spoiler under the uniform projection π equals:

$$\Pr\big(f(\pi(\mathbf{w})) \in R_i\big) = \left(\frac{1 - t/\hat{w}_i}{1 - t} \right)^{p-2}, \tag{7}$$

where $t = (2k+p)^{-1}$, if $\hat{w}_i > 0$, and 0, if $\hat{w}_i = 0$.

Note in particular that parties excluded from seat distribution by the iterative natural threshold given by (2) are almost surely spoilers.

Proof. There are two cases to consider:
Case 1: all parties are relevant.

$$\begin{aligned}
(q_i + \rho_i q_j)\frac{2k+n-1}{2m} - \frac{1}{2k} &\geq q_i \frac{2k+n}{2m} - \frac{1}{2k} \\
(q_i + \rho_i q_j)(2k+n-1) &\geq q_i (2k+n) \\
\rho_i q_j (2k+n-1) &\geq q_i \\
\rho_i &\geq \frac{q_i}{q_j}\frac{1}{2k+n-1} \\
\rho_i &\geq \sigma_i := \frac{q_i}{q_j}\frac{t}{1-t}
\end{aligned}$$

Probability of p being a no-spoiler:
Let $(\rho_1, \ldots, \rho_p) \sim \mathrm{Unif}(\Delta_{p-1})$, and assume (without the loss of generality) that $j = p$. Let Ψ_p be the probability that p is not a spoiler (i.e., that there are no spoilees). Relying on the fact that an integral of a product of functions over the unit simplex equals the value of their convolution at 1, we obtain:

$$\begin{aligned}
\Psi_p := \Pr\left(\forall_{i\in[n-1]}(\rho_i > \sigma_i)\right) &= \\
= \Gamma(n-1)\int_{\mathbf{x}\in\Delta_{p-1}} \prod_{i=1}^{n-1} u(x_i - \sigma_i)\, d\mathbf{v} &= \\
= \Gamma(n-1)\left(\bigotimes_{i=1}^{n-1} u(x_i - \sigma_i)\right)(1) &= \\
= \Gamma(n-1)\mathcal{L}^{-1}\left\{\prod_{i=1}^{n-1} \mathcal{L}\{u(v_i - \sigma_i)\}\right\}(1) &=
\end{aligned}$$

$$\begin{aligned}
= \Gamma(n-1)\mathcal{L}^{-1}\left\{\prod_{i=1}^{n-1} \frac{1}{s}\exp(-\sigma_i s)\right\}(1) &= \\
= \Gamma(n-1)\mathcal{L}^{-1}\left\{\frac{1}{s^{n-1}}\exp\left(-\left(\frac{1-q_n}{q_n}\frac{t}{1-t}\right)s\right)\right\}(1) &= \\
= \frac{\Gamma(n-1)}{\Gamma(n-1)}\left(1 - \frac{1-q_n}{q_n}\frac{t}{1-t}\right)^{n-2} &= \\
= \left(\frac{1-t/q_n}{1-t}\right)^{n-2} = \left(\frac{t}{1-t}\right)^{n-2}\left(\frac{2kw_n}{2kw_n+1}\right)^{n-2}
\end{aligned}$$

Case 2: j-th party is not relevant ($j \notin R$)
Let $r := w_j / \left(\sum_{k=1}^{n} w_k^{\downarrow}\right)$.

$$\begin{aligned}
s_i' &\geq s_i \\
\frac{q_i + \rho_i r}{1+r}\frac{2k+n}{2k} - \frac{1}{2k} &\geq q_i \frac{2k+n}{2k} - \frac{1}{2k} \\
q_i + \rho_i r &\geq q_i(1+r) \\
\rho_i &\geq q_i
\end{aligned}$$

Let $(\rho_1, \ldots, \rho_n) \sim \mathrm{Unif}\,(\Delta_{n-1})$:

$$\begin{aligned}
\Psi &:= \Pr\left(\forall_{i\in[p-1]}\left(\rho_i > \sigma_i\right)\right) = \\
&= \Gamma(p-1)\int_{\mathbf{v}\in\Delta_{p-1}} \prod\nolimits_{i=1}^{p-1} u\left(v_i - \rho_i\right)\, d\mathbf{v} = \\
&= \Gamma(p-1)\left(\bigotimes\nolimits_{i=1}^{p-1} u\left(v_i - \rho_i\right)\right)(1) = \\
&= \Gamma(p-1)\mathcal{L}^{-1}\left\{\prod\nolimits_{i=1}^{p-1} \mathcal{L}\left\{u\left(v_i - \rho_i\right)\right\}\right\}(1) = \\
&= \Gamma(p-1)\mathcal{L}^{-1}\left\{\frac{1}{s^{p-1}}e^{-s}\right\}(1) = 0
\end{aligned}$$

Intuition behind this result is as follows: if j-th party is not relevant, i-th party is not a spoilee only if it obtains at least proportional share of j's votes. But all parties can obtain at least proportional share only if all also obtain at most proportional share – and this result is non-generic. □

Probability of no party being a spoiler

Lemma 4. Let $\mathbf{W} \sim \mathrm{Unif}(\Delta_p)$ be a vote share vector, and f be a seats-votes function given by (1). The probability that S is empty, assuming the uniform projection, equals:

$$\begin{aligned}
&\Pr\big(\forall i \in [p] : f(\pi_i(\mathbf{W})) \in R_i\big) = \Psi^{\mathrm{JDH}}_{p,k} := \\
&\qquad (1-t)^{p-1}\Gamma(p)\Gamma(p-1)\left(\frac{1-t}{t}\right)^{p-2}\mathcal{L}^{-1}\left\{\left(\frac{1}{s}U\left(p-2, 0, \frac{s}{2k}\right)\right)^p\right\}(1), \qquad (8)
\end{aligned}$$

where U is Tricomi's confluent hypergeometric function.

Proof. Assume $(w_1, \ldots, w_p) \sim \mathrm{Unif}(\Delta_p)$ and $n = p$. Then, relying again on the fact that an integral of a product of functions over the unit simplex equals the value of their convolution at 1, we obtain:

$$\begin{aligned}
\Psi &= \Gamma(p)\left(\bigotimes\nolimits_{i=1}^{p}\left(\frac{t}{1-t}\frac{2kx}{2kx+1}\right)\right)(1) = \\
&= \Gamma(p)\mathcal{L}^{-1}\left\{\prod_{i=1}^{p}\mathcal{L}\left(\frac{t}{1-t}\frac{2kx}{2kx+1}\right)\right\}(1) = \\
&= \Gamma(p)\Gamma(p-1)\left(\frac{1-t}{t}\right)^p\mathcal{L}^{-1}\left\{\prod_{i=1}^{p}\mathcal{L}\left(\frac{2kx}{2kx+1}\right)\right\}(1) = \\
&= \Gamma(p)\Gamma(p-1)\left(\frac{1-t}{t}\right)^p\mathcal{L}^{-1}\left\{s^{-p}\,U\left(p-2, 0, \frac{s}{2k}\right)^p\right\}(1).
\end{aligned}$$

□

As far as we know, the inverse transform given above needs to be evaluated numerically.

Remark 5 (Technical Note)**.** Post's formula is quite inefficient for evaluating the above inverse transform, but we can numerically compute the Bromwich integral along the vertical line $\Re(x) = 1$. For common values of k and p it is well approximated by:

$$\mathcal{L}^{-1}\left\{\left(\frac{1}{s}U\left(p-2, 0, \frac{s}{2k}\right)\right)^p\right\}(1) \approx \frac{4}{5}\,U\left(p-2, 0, (2k)^{-1}\right)^p. \qquad (9)$$

Upper bound

We shall need the following proposition:

Proposition 1 (Order Statistics of the Uniform Distribution)**.** *Let* $\mathbf{W} \sim \mathrm{Unif}\,(\Delta_p)$*. Then the density of its* j*-th largest order statistic,* $W_j^{\downarrow}$*, is given by*

$$f_j^{\downarrow}(x) = p(p-1)\binom{p-1}{j-1}\sum_{l=j}^{\min(p,\lfloor 1/x\rfloor)}(-1)^{l-j}\binom{p-j}{l-j}(1-lx)^{p-2} \tag{10}$$

(see [8] for proof).

Our general idea is as follows: the probability that any party is a spoiler is bounded from below by the probability that the smallest one is a spoiler. By the Jensen inequality, that probability in turn is bounded by (7) evaluated at $W_p^{\downarrow}$:

$$\Psi(\mathbb{E}W_p^{\downarrow}) = \Psi\left(\frac{1}{p^2}\right) = \left(\frac{1}{1-t}\right)^{p-2}\left(\frac{2kp^2}{1+2kp^2}\right)^{p-2}. \tag{11}$$