Limited Voting for More Diversity?

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Abstract

Limited Voting (LV) is a method for approval based multi-winner elections where all ballots have the same fixed size. While it seems to be used as a standard way of voting in corporate governance and has some political applications, to the best of our knowledge, no formal analysis of the rule exists yet. We have been approached for advice on this voting rule by a company who uses it to elect its council. The core question of this paper is whether we can justify the use of LV in specific application domains. We focus on elections with organised parties, where parties broadcast voting instructions to their voters. We find that in such elections, LV can provide more diversity than basic approval voting.

1 Introduction

Limited Voting (LV) is a form of approval voting where ballots are limited: while voters can have general approval preferences, they can only submit a ballot of a given length l (where l is at most the desired committee size k). Although the rule is used in practice, it has not yet been analysed formally, as far as we know. We have been approached by a company who uses LV to elect its work council and wonders about the extent to which LV is the right rule for such purpose. In this paper we report on an initial analysis we made of LV in order to gain a better understanding of the behavior of the rule, in particular with respect to standard approval voting (AV). Since the main goal of a work council is to represent the employees of the company, we focus on representation and diversity in LV. Our intuition is that LV could provide more diversity than, for example, basic approval voting, at least in settings where voters are split up in different groups of voters with similar opinions. Since both in political applications and in corporate governance it is not unlikely that such a partition of voters exists, we consider so called party-list profiles, and especially the case where parties are organised and can broadcast 'voting instructions' to their members.We then use scores based on rules known for their diversity or proportionality to measure the diversity and proportionality of LV in comparison to AV.

Related work Although Limited Voting itself has, to the best of our knowledge, not yet been the object of any direct analysis, some very similar multi-winner voting rules are discussed in the literature. One is Bloc/Block Voting: in [9] (chapter 2) and [8], 'Bloc (Voting)' refers to the case where the ballot size l is exactly equal to the committee size k. In [5], ℓ-Bloc is used: every voter *approves* (and votes for) her ℓ most favorite candidates, assuming ordinal preferences, which differs from LV which makes limited votes a subset of approval ballots. Similarly, but for single-winner, [14] mentions k-approval voting, where a voter votes for the first k candidates in their ranking. In [10], 'Block Vote' allows voters to submit a ballot of length *at most* k (so the ballots of different voters can have different lengths), instead of insisting on length l. In [12], Block Voting, Limited Voting, and Single Non-Transferable Vote (SNTV) are mentioned, where Limited Voting refers to the case where voters may approve *at most* l candidates for some $l < k$, and SNTV is Limited Voting for $l = 1$. We study here the 'strict' case of limited voting where voters are requested to provide *exactly* l approved candidates.

With respect to the reasons for limiting ballot sizes, [13] observes that ballot length restrictions affect different voters to different extents, and therefore may be hard to justify

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
v_1	X	X				X				
v_2	X			X			X			
v_3	X	X						X		
v_4			X	X	X					
v_5			X	X	X					

Table 1: $l = 3$, $k = 5$ (the committee size). Approvals in grey, limited votes indicated by X. Some voters approve less that l candidates, then we let them fill their ballot lexicographically. The blue box shows the (unique) winning committee according to LV.

towards the voters. However, when voters have non dichotomous preferences, restricting ballots to the size of the committee can be justified in light of strategy-proofness. In contrast, in this paper, we do assume dichotomous preferences, and we explore the extent to which ballot size restrictions may still be justified, but now on diversity grounds.

The Chamberlin Courant (CC) rule (introduced for ordinal ballots by Thiele [19] and later by Chamberlin and Courant [7], and modified for approval ballots by e.g. Endriss [9]) can be used as a reference point for the diversity of committees. In approval-based elections, the rule chooses the committee W with the least amount of voters that are not represented at all in W. Underpinning this rule is the CC-score of a committee (the number of voters that are represented by at least one member of the committee), that can be used as a quantitative measure of a committee's diversity. We use this CC-score to compare the diversity of LV to that of approval voting without limit (AV). To have a lower bound on the diversity of a rule in any election, [11] use the CC-score to calculate the *CC-guarantee* of a multi-winner voting rule, which is the worst case proportion between the rule's least diverse outcome and the most diverse committee in the same election. We also study the CC-guarantee of LV in some restricted domains of elections.

Motivation At first sight, limited voting does not seem very appealing. Indeed, from the point of view of the voters, the rule is literally limiting them in what they are allowed to express in their vote. Some voters may have less than l approved candidates, and LV forces them to vote for candidates they do not approve. Other voters may have more than l approved candidates, and cannot vote for all candidates they approve. This makes the rule inefficient, as the following example shows:

Example 1. *LV does not satisfy Pareto efficiency. In the profile in Table 1, the committee* $W = \{c_1, c_2, c_3, c_4, c_5\}$ *wins according to LV: it has the highest number of votes. However, the committee* $W' = \{c_6, c_7, c_8, c_9, c_{10}\}$ *dominates it: no voter has less approved candidates in* W' *than in* W*, and* v_1 *and* v_3 *have more approved candidates in* W'*.*[1]

We see that when using LV instead of AV, we may loose information about the voters' preferences, which can result in worse outcomes. Given that this loss varies between elections, we are interested in finding out what is the worst possible loss of using LV instead of AV. Let us assume for a moment that the voters' welfare increases linearly with the number of approved candidates they have in the winning committee: for each candidate that they approve that is elected, they get one 'unit of welfare'. We can measure the welfare performance of a rule by looking at the sum of all voters' welfare, given the elected committee. With this measure—called the 'AV-score' of a committee—it is easy to construct examples where LV performs arbitrarily worse than AV.

Example 2. *In the election shown in Table 2, the committee* $W = \{c_1, c_2, c_3, c_4\}$ *is a winning committee according to LV (since all its members get one vote and no candidate*

[1] Approval voting does indeed satisfy Pareto efficiency for dichotomous preferences [12].

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
v_1	X								
v_2		X							
v_3			X						
v_4				X					
v_5					X				
v_6						X			

Table 2: $l = 1$, $k = 4$. Approvals in grey, votes indicated by X.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	b_1	b_2	b_3	c_1	c_2	c_3	c_4	c_5
v_1	X	X	X														
v_2	X	X	X														
v_3	X	X	X														
v_4	X	X	X														
v_5										X	X	X					
v_6										X	X	X					
v_7										X	X	X					
v_8													X	X	X		
v_9													X	X	X		

Table 3: Example: $l = 3$, $k = 8$. Approvals in grey, votes indicated by X. The red box shows a winning committee according to AV, the blue boxes a winning committee according to LV.

in the total election gets more than one vote), while any committee $W' \subseteq \{c_5, c_6, c_7, c_8, c_9\}$ (with $|W'| = k = 4$) wins according to AV. The AV-score of W is 4, since every elected candidate is only approved by one voter, while the AV-score of W' is 24, since every elected candidate is approved by all six voters. The structure of the example can be generalized to obtain arbitrarily large differences in AV-score, where any voter v_i approves $A_i = \{c_i \cup \{c_j : k+1 \leq j \leq m\}\}$ (with m the total number of candidates) and votes for c_i only. Then, $W = \{c_1, ..., c_k\}$ is a winning committee according to LV, since no candidate gets more than one vote and all candidates in W get one vote, but its AV-score is only k, while any committee $W' \subseteq \{c_j : k+1 \leq j \leq m\}$ (with $|W'| = k$) has an AV-score of $k \times n$ (with n the number of voters).

The worst case proportion between a rule's AV-score in an election and the maximum AV-score possible in that election is introduced formally as a rule's *AV-guarantee* in [11]. The above example shows that the AV-guarantee of LV tends to 0 as n grows large.

So, why is LV deployed in practice? Part of the motivation seems to be that LV intuitively restricts the influence of large majorities. Example 3 illustrates this intuition, and shows that, under some assumptions on the voters' ballots, LV gives a more diverse committee than AV.

Example 3. *Suppose we have three sets of candidates (parties) $C_a = \{a_1, a_2, ...\}$, $C_b = \{b_1, b_2, ...\}$, and $C_c = \{c_1, c_2, ...\}$ and three groups of voters V_a, V_b, V_c that approve the respective parties, and suppose the desired committee size $k = 8$ (see Table 3). Then if $|V_a| > |V_b| > |V_c|$, with AV, the winning committee will be completely filled with candidates from C_a. With limited voting however, with e.g. $l = 3$, if all voters can only vote for the first 3 candidates of their party, any winning committee will consist of 8 candidates from $\{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3\}$, which is clearly much more diverse.*

Contribution In this paper, we study the extent to which the intuition illustrated above can be used to 'rationalize' the use of LV in specific voting settings. As a measure of diversity, we use the CC-score, and we compare LV's diversity to that of AV to get the CC-gain in a certain election. We show that in elections where parties are able to tell their voters which candidates to include in their ballots (to which we refer as *broadcasted party-list elections*), LV does indeed provide higher diversity than AV, and we can determine the exact diversity difference given the number of parties, the committee size k and the ballot size l. We generalize the results slightly to *broadcasted laminar elections*, where the CC-gain is also non-negative. We then consider LV's proportionality and show that it does not satisfy many of the common proportionality axioms. To do this, we use a similar quantitative measure, the PAV-gain, to compare LV's proportionality to that of AV, and show that in broadcasted party-list elections, if the size of the parties is similar enough, the PAV-gain is also positive.

2 Preliminaries

An election $E = \langle N, C, k, l, A, L\rangle$ consists of a set of voters $N = \{v_1, ..., v_n\}$ and a set of candidates $C = \{c_1, ..., c_m\}$, a committee size $k \leq m$, a ballot limit $l \leq k$, an approval profile $A = \{A_i : i \in N\}$ where $A_i \subseteq C$ is the approval preference of voter i (the set of candidates that i approves), and a ballot profile $L = \{L_i : i \in N\}$, where $L_i \subseteq C$ with $|L_i| = l$ is the ballot of voter i (the set of candidates that i votes for). If it is not clear from the context, we sometimes use the notation A_E, k_E, etc. to refer to the A, k, etc. of election E. While in approval-based committee rules (ABC-rules, [12]), $A = L$, in our setting A and L may be different. Indeed, while we still assume that each voter has dichotomous preferences (their approval set), their ballot is in general not equal to their approval set. Still, we assume that the ballot of a voter i must remain consistent with their approval set, in the following sense: if $l \leq |A_i|$, then $L_i \subseteq A_i$, and if $l > |A_i|$, $A_i \subset L_i$.

Given such an election, Limited Voting selects k candidates with the highest number of votes. Let the LV-score of a candidate c be $s_{LV}(c) = |\{i \in N : c \in L_i\}|$. Then:

Definition 1 (Limited Voting (LV))**.** Given election $E = \langle N, C, k, l, A, L\rangle$, LV elects the committee W with size k that has the highest LV-score $s_{LV}(W) = \sum_{c\in W} s_{LV}(c)$.

We compare LV with plain Approval Voting (AV), which elects the k candidates that are approved by most voters. That is, the AV-score of a candidate c is $s_{AV}(c) = |\{i \in N : c \in A_i\}|$, and AV elects the committee W of size k with the highest AV-score $s_{AV}(W) = \sum_{c\in W} s_{AV}(c) = \sum_{c\in W} |\{i \in N : c \in A_i\}|$. We denote the set of outcomes of LV, respectively AV (note that neither of them is resolute), on an election E by $LV(E)$, respectively $AV(E)$.

In the context of corporate elections, it is not unusual for candidates to be divided over different parties, and for voters to vote either for a party as a whole or for different candidates that are members of the same party. In social choice theory, approval profiles where this is the case are called *party-list profiles* (for example in [16, 15, 4])[2]. Formally, a profile $A = \{A_1, \cdots, A_n\}$ is a party-list profile if for all $i, j \in N$, either $A_i = A_j$ or $A_i \cap A_j = \emptyset$. For notational reasons, we order the parties from most popular to least as $P_1, P_2, ...$ where $|P_i|$ denotes the number of voters in (that vote for) party P_i, so $|P_1| \geq |P_2| \geq ....$

A generalisation of elections with party-list profiles are *laminar profiles*, introduced in [16], where parties can consist of smaller subparties. Laminar elections can be defined recursively for LV elections as follows. An election $E = \langle N, C, k, l, A, L\rangle$ is laminar if either: (1) E is unanimous and $|C| \geq k$; (2) There is a candidate $c \in C$ such that $c \in A_i$ for all $i \in N$, the election $E_{-c} = \langle N, C\backslash\{c\}, k-1, l', A_{-c}, L'\rangle$ is not unanimous, and the election E_{-c} is laminar, where E_{-c} is E once we remove c, i.e., $A_{-c} = (A_1\backslash\{c\}, ..., A_n\backslash\{c\})$, l'

[2] [15] and [2] mention that proportional representation in party-list profiles is the same as the *apportionment problem.*

	c_1	c_2	c_3	c_4	c_5	c_6
v_1	X	X				
v_2	X	X				
v_3	X			X		
v_4	X			X		

	c_1	c_2	c_3	c_4	c_5	c_6
v_1	X		X			
v_2		X	X			
v_3					X	X
v_4	X			X		

Table 4: Example of an election E with $l = 2$ with a broadcasting order $c_1 \succ_E c_2 \succ_E c_3 \succ_E c_4 \succ_E c_5 \succ_E c_6$ (left), and the same approval profile with a non-broadcasted election (right). Approvals in grey, votes indicated by X.

and L' are any limit and ballot profile compatible with E_{-c}; or (3) There are two laminar elections $E_1 = \langle N_1, C_1, k_1, l_1, A_1, L_1 \rangle$ and $E_2 = \langle N_2, C_2, k_2, l_2, A_2, L_2 \rangle$ with $C_1 \cap C_2 = \emptyset$ and $|N_1| \cdot k_2 = |N_2| \cdot k_1$ such that $N = N_1 + N_2$ and $k = k_1 + k_2$. We refer to [16] for examples of laminar elections and intuitions motivating their definition.

In the above kinds of elections where voters and candidates are organized into distinct parties (or into a structure with parties and subparties), it is plausible to assume that parties are able to signal to their voters which candidates the party wishes to be elected. We call *broadcasting order* the order that the parties tell their voters to vote over the candidates. Notice that we assume parties to be able to signal to their base only one fixed order, and not different orders for different voters (hence the qualification 'broadcasted') as such level of fine-grained communication from parties to base appears unrealistic in the scenarios, such as corporate decision-making, that motivate our analysis.[3]

Definition 2 (Broadcasting order)**.** Given an election $E = \langle N, C, k, l, A, L \rangle$, a *broadcasting order* $\succ_E$ is a linear order over all candidates in C such that for any $c, c' \in C$: if $s_{AV}(c) > s_{AV}(c')$, then $c \succ_E c'$.

We call elections where the preferences are a party-list profile and the voters vote according to a broadcasting order *broadcasted party-list elections* and define *broadcasted laminar elections* similarly.

Definition 3 (Broadcasted party-list elections)**.** An election $E = \langle N, C, k, l, A, L \rangle$ is a *broadcasted party-list election* if A is a party-list profile and there is a broadcasting order $\succ_E$, such that for all voters i, for any $c, c' \in A_i$: if $c \in L_i$ and $c' \notin L_i$, $c \succ_E c'$; or, equivalently, for all $i, j \in N$, if $A_i = A_j$, then $L_i = L_j$.

Definition 4 (Broadcasted laminar elections)**.** An election $E = \langle N, C, k, l, A, L \rangle$ is a *broadcasted laminar election* if it is a laminar election and there is a broadcasting order $\succ_E$, such that for all voters i, for any $c, c' \in A_i$: if $c \in L_i$ and $c' \notin L_i$, $c \succ_E c'$.

See Table 4 for an example of a broadcasted laminar profile.

3 Measuring diversity: CC-score

The motivating example in the introduction points to the fact that in some situations, LV can enhance diversity of a committee compared to AV by hardcoding a limit on the size of the approval set. This raises the question of how to measure diversity and determine whether LV can increase it, and if so, under which circumstances. Intuitively, one way to increase diversity is to minimise the number of voters who are not satisfied at all, in the sense of not having any candidate selected. This is equivalent to maximising the number of

[3]It should be noted, however, that broadcasting unique orders to voters may not be optimal for a party as Example 6 in Appendix A witnesses.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
v_1	X	X	X					
v_2		X	X	X				
v_3			X	X	X			
v_4	X	X	X					
v_5	X	X		X				
v_6						X	X	X

Table 5: $l = 3$, $k = 4$. Approvals in grey, limited votes indicated by X. AV selects $\{c_2, c_3, c_4, c_5\}$ (red), LV selects $\{c_1, c_2, c_3, c_4\}$ (blue).

voters who have at least one candidate that they approve in the winning committee, that is, the number of voters who have at least one representative. This idea is captured by the Approval-Based Chamberlin-Courant (α-CC) rule (see e.g. [19, 7, 9]). The *CC-score* of a committee W given an approval profile A is defined by $s_{CC}(A, W) = |\{i \in N : W \cap A_i \neq \emptyset\}|$.

To compare the diversity of the outcome of LV to that of the outcome of AV, we define the *CC-gain* of LV with respect to AV as follows:

Definition 5 (CC-gain)**.** For an election $E = \langle N, C, k, l, A, L\rangle$, the gain in CC-score of LV with respect to AV is given by $Gain_{CC}(E) = \min_{W \in LV(E)} s_{CC}(A, W) - \max_{W \in AV(E)} s_{CC}(A, W)$.

We want to study under what conditions this gain is positive, and we start by showing that this is not the case in general:

Example 4. *Given the profile in Table 5, the outcome of AV is $W_{AV} = \{c_2, c_3, c_4, c_5\}$, with $s_{CC}(A, W_{AV}) = 6$, while the outcome of LV is $W_{LV} = \{c_1, c_2, c_3, c_4\}$ with $s_{CC}(A, W_{LV}) = 5$, since voter v_6 is left unrepresented. This gives a CC-gain of -1.*

3.1 CC-gain in broadcasted party-list elections

Our starting intuition was that LV would select more diverse committees when voters can be roughly divided into different cohesive groups. We hence start our analysis from party-list profiles. Note that in any party-list election, AV will first select all candidates from the most popular party, then from the second-most, etc., until the committee is filled. In LV we cannot directly say what the outcome will be, since it matters which candidates voters vote on from the candidates that they approve. In broadcasted party-list elections, however, we know voters will vote according to a specified order: LV will select l candidates from the most popular party, then l from the second-most popular, etc., until the committee is filled.

Theorem 1. *Let $E = \langle N, C, k, l, A, L\rangle$ be any broadcasted party-list election with parties $P_1, \ldots, P_g$. If the most popular party has at least k members, the CC-gain is*

$$Gain_{CC}(E) = \sum_{i=2}^{\min(\lceil \frac{k}{l} \rceil, g)} |P_i|,$$

which is strictly positive if $l < k$ and $g > 1$.

Proof. If the most popular party has at least k candidates, AV selects a committee of only candidates of the most popular party, so for any $W_{AV} \in AV(E)$, the CC-score will be $s_{CC}(A, W_{AV}) = |P_1|$, so $\max_{W \in AV(E)} s_{CC}(A, W) = |P_1|$. On the contrary, LV selects l members from P_1, l members from P_2, etc., up until the $\lceil \frac{k}{l} \rceil$th party (if $\lceil \frac{k}{l} \rceil \leq g$, otherwise it has to take arbitrary extra candidates that are not voted for), so for any $W_{LV} \in LV(E)$, $s_{CC}(A, W_{LV}) = \sum_{i=1}^{\min(\lceil \frac{k}{l} \rceil, g)} |P_i|$, so $\min_{W \in LV(E)} s_{CC}(A, W) = \sum_{i=1}^{\min(\lceil \frac{k}{l} \rceil, g)} |P_i|$. Subtracting the two scores gives the desired result. $\square$

However, if in a party-list profile the voters do not coordinate within their party on which candidates to vote, LV does not necessarily give a higher CC-score than AV. It can be the case that the voters from the most popular party spread their votes over more than l candidates, and in this way more than l from the first party can be chosen, or that the voters in the most popular party spread their votes too much, and do not get any candidate elected. Example 5 illustrates how the CC-gain of LV can be negative in such a situation.

Example 5. *Consider an election with $k = 4, l = 2$ and a party-list profile where the largest party has 5 voters and the second party has 4 voters. If the 5 voters in P_1 all spread their votes and all vote for two different candidates, while the 4 voters in P_2 pair up and two of them vote for candidates a and b, while two others vote for c and d, then a, b, c and d will get 2 votes each, while all candidates of party P_1 only get 1 vote. Hence, the winning committee of LV $\{a, b, c, d\}$ has a CC-score of $|P_2| = 4$, which is smaller than $|P_1| = 5$, which is the CC-score of the winning committee of AV: the CC-gain is -1.*

Even though LV may be more diverse than AV in broadcasted party-list elections, it should be stressed that LV does not maximize CC-score: its outcome is in general not the same as the outcome of α-CC. This rule will in such elections return a committee with at least one member of every party, if there are enough seats, and fill up the rest of the committee arbitrarily. Therefore, the score depends on the number of parties in the profile: if there are g parties, $s_{CC}(\alpha\text{-CC}) = \sum_{i=1}^{\min(k,g)} |P_i|$. The difference between the CC-score of an outcome of LV and the maximal CC-score is therefore $s_{CC}(\alpha\text{-CC}) - s_{CC}(A, W_{LV}) = \sum_{i=\lceil \frac{k}{l} \rceil+1}^{\min(k,g)} |P_i|$.

3.2 CC-guarantee

In [11], the CC-guarantee of a rule $\mathcal{R}$ is used as a quantitative measure of the rule's diversity, which is defined as follows: $\kappa_{cc}(k) = \inf_{A \in \mathcal{A}} \frac{\min_{W \in \mathcal{R}(A,k)} s_{CC}(A,W)}{\max_{W \in S_k(C)} s_{CC}(A,W)}$, where $\mathcal{A}$ is the set of all possible preference profiles and $S_k(C)$ is the set of subsets of C of size k.

Proposition 1. *The CC-guarantee of LV is 0.*

It is worth noticing that the CC-guarantee of even AV is better than that of LV: $\frac{1}{k}$ [11]. However, we can restrict the notion of CC-guarantee in order to focus on the performance of the rule on restricted domains of elections, rather than all of them, by simply restricting the set $\mathcal{A}$ to the domain that we are interested in. If D is a set of elections, $\kappa_{cc}(k)(D) = \inf_{E \in D | k_E = k} \frac{\min_{W \in \mathcal{R}(E)} s_{CC}(A_E,W)}{\max_{W \in S_k(C_E)} s_{CC}(A_E,W)}$ is the CC-guarantee restricted to elections of the domain D.

Proposition 2. *Let BP denote the set of all possible broadcasted party-list elections, then the CC-guarantee of LV restricted to BP is $\kappa_{cc}(k)(BP) = \frac{1}{k}$ in general, and $\kappa_{cc}(k)(BP) = \frac{\lceil \frac{k}{l} \rceil}{k}$ for all $E \in BP$ with $l_E = l$.*

The CC-guarantee of AV restricted to the domain of broadcasted party-list elections is $\frac{1}{k}$ as well.

3.3 CC-gain in broadcasted laminar elections

It seems that in more aligned elections (of which broadcasted party-list elections are an extreme case), the CC-gain is positive. This raises the question whether in slightly less aligned elections, such as laminar elections, this is still the case.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	c_{16}	c_{17}	c_{18}	c_{19}	c_{20}	c_{21}	c_{22}	c_{23}	c_{24}	c_{25}
v_1																X	X	X	X						
v_2																X	X	X	X						
v_3																					X	X	X		X
v_4																					X	X	X	X	
v_5																					X	X	X	X	
v_6	X	X	X	X																					
v_7					X	X	X	X																	
v_8									X	X	X	X													

Table 6: Laminar election with broadcasting order that does not prioritize the most popular candidates, $k = 8$ and $l = 4$. The blue boxed candidates are chosen by LV.

Theorem 2. *In any broadcasted laminar election E where AV and LV are resolute*[4]*, if for all $i \in N$, $|A_i| \geq l$, $Gain_{CC}(E) \geq 0$.*

Proof. Suppose towards a contradiction that there is such broadcased laminar election E where W_{AV} is the winning committee of AV and W_{LV} is the winning committee of LV, in which $s_{CC}(A, W_{AV}) > s_{CC}(A, W_{LV})$. Then there must be a voter v who has at least one of her approved candidates in W_{AV}, but none in W_{LV}. Therefore, even the first candidate (let's call it c_1) on which v will vote (the candidate in A_v that is ranked highest in $\succ_E$) is not included in W_{LV}. Since E is laminar, and $\succ_E$ is the same for all voters, and $l \geq 1$, all voters that approve c_1 must vote on it, so $s_{LV}(c_1) = s_{AV}(c_1)$. We also know that there is at least one $c' \in A_v \cap W_{AV}$. Since $c_1 \succ_E c'$, we know that $s_{AV}(c_1) \geq s_{AV}(c')$. If $s_{AV}(c_1) > s_{AV}(c')$ we can assume $c_1 \in W_{AV}$, so there exists a candidate $c'' \in A_v \cap W_{AV}$ with $s_{AV}(c'') = s_{AV}(c_1) = s_{LV}(c_1)$. However, since we have that for all $i \in N$, $|A_i| \geq l$, we know that for all candidates $c \in C, s_{AV}(c) \geq s_{LV}(c)$. Therefore, the threshold of approvals to be elected in W_{AV} must be at least as high as the threshold of votes to be elected in W_{LV}, and hence if the value of $s_{LV}(c_1)$ is not enough to be elected by LV, the value of $s_{AV}(c'') = s_{LV}(c_1)$ can also not be enough to be elected by AV. Hence, c'' cannot be elected by AV: a contradiction. $\square$

Note that if we let go of the assumption of a fixed voting order, the result does not hold, as shown by the counterexample for party-list profiles in Example 5. Moreover, if we do have a fixed order but one that does not start with the most popular candidates, the result does not hold either, as illustrated by the laminar profile in Table 6. With LV where candidates are voted according to lexicographic order, v_6, v_7, and v_8 will vote for c_1 to c_{12} instead of combining their forces on c_{13} to c_{15}. This makes the outcome of LV $\{c_{16}, ..., c_{19}, c_{21}, ..., c_{24}\}$ which leaves v_6, v_7, and v_8 without representative, while plain AV would elect at least one candidate from the approval set of every voter.

Finally, we show that the CC-guarantee of LV in broadcasted laminar elections is the same as in broadcasted party-list elections.

Proposition 3. *Let BL denote the set of all possible broadcasted laminar elections, then the CC-guarantee of LV restricted to BL is $\kappa_{cc}(k)(BL) = \frac{1}{k}$ in general, and $\kappa_{cc}(k)(BL) = \frac{\lceil \frac{k}{l} \rceil}{k}$ for all $E \in BL$ with $l_E = l$.*

Proof. Observe first that since broadcasted party-list elections are a subset of broadcasted laminar elections, the CC-guarantee of any voting rule can never be higher in broadcasted laminar elections than in broadcasted party-list elections. We show that for LV, the CC-guarantee in broadcasted laminar elections is also not lower than that in broadcasted party-list elections. Take any broadcasted laminar election E and let g be the number of super-parties in E, i.e. the lowest number g such that there exists a partition of V into g disjoint

[4]Example 7 in Appendix A shows why the condition that AV and LV are resolute is necessary.

sets of voters where every set has at least one candidate that all the voters in the set approve. Just as in party-list profiles, we order the superparties by number of voters: P_1 is the most popular, P_2 the second-most, etc. The maximum CC-score of any committee in E is $s_{CC}(\alpha\text{-CC}) = \sum_{i=1}^{\min(k,g)} |P_i|$, which we can obtain by taking one (in that party unanimously approved) candidate from every superparty, until all voters are satisfied or the committee is filled. The minimal CC-score of the outcome of LV over all broadcasted laminar profiles is the score of a party-list profile. To make the CC-score minimal, we want as few voters to be represented as possible. For the largest superparties (the ones that contain the most popular candidates), all voters will be satisfied anyway, since the most popular candidate is chosen first by LV. Hence, to minimise the CC-score, we need as many candidates as possible from the largest parties, since then we need less candidates that might represent other voters. Then, if for any superparty, a non-unanimous candidate of that party is elected by LV, that means that already the subparty that approves it has enough votes to elect a candidate. And that implies that if the unanimous candidates of the superparty were split into two candidates, one with the subparty as approvers and one with the rest of the voters of the superparty, at least as many candidates from the superparty could have been chosen, with at most as many voters being represented. But that implies that if the profile were party-list, at least as many candidates from the superparty could have been chosen, with at most as many voters being represented, so the CC-score in any broadcasted laminar profile is at least that of a party-list profile. Hence, the CC-guarantee of broadcasted laminar elections cannot be lower than that of broadcasted party-list elections. □

4 Measuring proportionality: axioms and PAV-score

After having focused on the diversity of LV, we now turn to analysing its proportionality. We first check whether it satisfies any of the most common proportionality axioms, and find that it does not. We then use the notion of PAV-gain to compare the proportionality of LV and AV, just as we used the CC-gain above to compare their diversity.

4.1 Proportionality axioms (or how LV fails them all)

Since LV is rather similar to AV, and AV is not known for its proportionality, it is unsurprising that also LV fails most of the axioms we study. In particular, there is a set of elections where AV and LV exactly overlap, namely when every voter approves exactly l candidates. Therefore, if a counterexample that proves that AV does not satisfy a given axiom comes from this set, it is also a counterexample for LV. For instance, such a counterexample can be used to show that LV does not satisfy the 'justified representation' axioms.

We briefly recall the definitions of some essential proportionality axioms: A voting rule $\mathcal{R}$ satisfies *justified representation (JR)* [1] if for each election $E = \langle N, C, k, l, A, L\rangle$, for each $W \in \mathcal{R}(E)$ and each 1-cohesive group of voters S, there is a voter $i \in S$ who is represented by at least one member of W, i.e. $|W \cap A_i| \geq 1$, where a group S is called 1-cohesive if $|S| \geq \frac{n}{k}$ and $|\cap_{i\in S} A_i| \geq 1$. The definition is extended to capture the idea that groups that agree on more candidates should be represented by more candidates: A rule satisfies *extended justified representation (EJR)* [1] if for each election $E = \langle N, C, k, l, A, L\rangle$, for each $W \in \mathcal{R}(E)$ and each ℓ-cohesive group of voters S (for $\ell \leq k$), there is a voter $i \in S$ who is represented by at least ℓ members of W, i.e. $|W \cap A_i| \geq \ell$, where a group S is ℓ-cohesive if $|S| \geq \ell\cdot \frac{n}{k}$ and $|\cap_{i\in S} A_i| \geq \ell$. The notion of JR is generalized to a weaker condition than EJR, *proportional justified representation (PJR)*, in [17]: A voting rule $\mathcal{R}$ satisfies PJR if for each election $E = \langle N, C, k, l, A, L\rangle$, for each $W \in \mathcal{R}(E)$ and each ℓ-cohesive group of voters S, the collective group (instead of just one member of the group) has at least ℓ representatives:

$|W \cap (\cup_{i \in S} A_i)| \geq \ell$. A rule $\mathcal{R}$ satisfies *laminar proportionality* [16] if, for every laminar election $E = \langle N, C, k, l, A, L \rangle$ it returns a committee that satisfies the following: (1) if E is unanimous, then $\mathcal{R}(E) \subseteq C$; (2) if there is a candidate $c \in C$ such that $c \in A_i$ for all $i \in N$ and the election E_{-c} is laminar, then $\mathcal{R}(E) = W' \cup \{c\}$, where W' is laminar proportional for E_{-c}; and (3) if E is the sum of E_1 and E_2, then $\mathcal{R}(E) = W_1 \cup W_2$, where W_1 is laminar proportional for E_1 and W_2 is laminar proportional for E_2. A rule $\mathcal{R}$ satisfies *priceability* [16] if for any election $E = \langle N, C, k, l, A, L \rangle$, $\mathcal{R}(E)$ is priceable: there exists a price system $\mathbf{ps} = (p, \{p_i\}_{i \in N}$ that supports it, where $p > 0$ is a price and p_i are payment functions $p_i : C \to [0, 1]$ such that: (1) if $p_i(c) > 0$, then $c \in A_i$; (2) $\sum_{c \in C} p_i(c) \leq 1$; (3) for all $c \in \mathcal{R}(E)$, $\sum_{i \in N} p_i(c) = p$; (4) for all $c \notin \mathcal{R}(E)$, $\sum_{i \in N} p_i(c) = 0$; and (5) for all $c \notin \mathcal{R}(E)$: $\sum_{i \in N: c \in A_i} \left(1 - \sum_{c \in \mathcal{R}(E)} p_i(c')\right) \leq p$.

Proposition 4. *LV fails JR, EJR, PJR, laminar proportionality, and priceability, even on broadcasted party-list elections.*

4.2 PAV-score

We saw above that LV does not seem to do well on proportionality axioms, even if we restrict ourselves to broadcasted party-list or broadcasted laminar elections. However, since those axioms are binary statements, and a rule can either fully satisfy them or not at all, it is interesting to move to a more quantitative approach to the proportionality of a rule instead. To measure how proportional a committee is, we introduce a measure similar to the CC-score from the previous section: the *PAV-score* [12]. The PAV-score of a committee W given a preference profile A is defined by $s_{PAV}(A, W) = \sum_{i \in N} \left(\sum_{j=1}^{|W \cap A_i|} \frac{1}{j}\right)$. Intuitively, this is higher when candidates that many people approve are in the winning committee, but gives less weight to voters who already have more approved candidates elected. Analogously to the CC-gain, we can define the *PAV-gain* as follows:

Definition 6 (PAV-gain)**.** For an election $E = \langle N, C, k, l, A, L \rangle$, the gain in PAV-score of LV with respect to AV is given by $Gain_{PAV}(E) = \min_{W \in LV(E)} s_{PAV}(A, W) - \max_{W \in AV(E)} s_{PAV}(A, W)$.

PAV-gain in general elections. We start by noting that the PAV-gain is not necessarily positive. Consider the situation where the limited votes are such that less than k candidates get any vote from the limited ballots. Then the rest of k has to be filled up with a tie-breaking rule, and it is therefore easy to construct an election where the PAV-score of AV is higher than that of LV. Consider for instance the very simple case where $l = 3, k = 4$, we have 6 candidates $c_1, ..., c_6$ and only one voter i, with $A_i = \{c_1, c_2, c_3, c_5, c_6\}$. Suppose the voter submits the first three of his approved candidates as his limited ballot, and that we use lexicographical tie-breaking for the remaining candidates. Then $\{c_1, c_2, c_3, c_4\}$ is elected by LV, while any committee that AV outputs consists only of candidates that i likes.

Let us turn to the slightly more interesting cases where no such extreme tie-breaking is needed. Assume that there are at least k candidates that get at least one vote. Despite this constraint, it is still possible for the PAV-gain to be negative. Consider for example the profile in Table 7a, where $l = 3$, $k = 4$. The outcome of AV is $W_{AV} = \{c_2, c_3, c_4, c_5\}$ with $s_{PAV}(A, W_{AV}) = 2(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}) + 2(1 + \frac{1}{2} + \frac{1}{3}) \approx 7.83$, while the outcome of LV is $W_{LV} = \{c_1, c_2, c_3, c_4\}$, with $s_{PAV}(A, W_{LV}) = 2(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}) + (1 + \frac{1}{2} + \frac{1}{3}) + (1 + \frac{1}{2}) = 7.5$. Hence, $s_{PAV}(A, W_{AV}) > s_{PAV}(A, W_{LV})$.

Importantly, the PAV-gain may also be positive. Table 7b provides an example where the approval ballots are more aligned than in the previous example (note that $|A_{v_4}| < l$, so v_4 has to vote for one candidate she does not approve of). Here, the outcome of AV is

	c_1	c_2	c_3	c_4	c_5	c_6
v_1	X	X	X			
v_2		X	X	X		
v_3			X	X	X	
v_4	X	X	X			

(a) AV (red) selects $\{c_2, c_3, c_4, c_5\}$ with $s_{PAV}(A, W_{AV}) \approx 7.83$, LV (blue) selects $\{c_1, c_2, c_3, c_4\}$ with $s_{PAV}(A, W_{LV}) = 7.5$.

	c_1	c_2	c_3	c_4	c_5	c_6
v_1	X	X	X			
v_2			X	X		X
v_3			X	X	X	
v_4			X		X	X

(b) AV (red) selects $\{c_1, c_2, c_3, c_4\}$ with $s_{PAV}(A, W_{AV}) = 6.25$, LV (blue) selects $\{c_3, c_4, c_5, c_6\}$, with $s_{PAV}(A, W_{LV}) \approx 6.67$.

Table 7: Two example elections with $l = 3$, $k = 4$. Approvals in grey, votes indicated by X.

$W_{AV} = \{c_1, c_2, c_3, c_4\}$ with $s_{PAV}(A, W_{AV}) = 3(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4})+0 = 6.25$, while the outcome of LV is $W_{LV} = \{c_3, c_4, c_5, c_6\}$, with $s_{PAV}(A, W_{LV}) = 2(1+\frac{1}{2}+\frac{1}{3})+2(1+\frac{1}{2}) \approx 6.67$. Hence, $s_{PAV}(A, W_{LV}) > s_{PAV}(A, W_{AV})$. Note that if we do not consider broadcasted elections, there can be a lot of variation in the outcome of LV depending on which of their approved candidates the voters choose to vote on. In this example however, the PAV-score of an LV-outcome can never be lower than that of the AV-outcome, since that has the lowest possible PAV-score for a 4-candidate committee anyways.

PAV-gain in broadcasted party-list elections. Since no clear conclusions can be drawn for general approval elections, and since it seems that more aligned ballots are better for LV's proportionality, we consider again the special case of broadcasted party-list elections. Under the assumption that all parties (or at least the most popular one) have more than k candidates, the PAV-score of any committee elected by AV in a broadcasted party-list election is $s_{PAV}(A, W_{AV}) = |P_1|(1 + \frac{1}{2} + \cdots + \frac{1}{k})$. The score of a committee elected by LV is $s_{PAV}(A, W_{LV}) = |P_1|(1 + \frac{1}{2} + \cdots + \frac{1}{l}) + |P_2|(1 + \frac{1}{2} + \cdots + \frac{1}{l}) + \cdots$, until W is filled. Usually, this will give a positive PAV-gain, but if P_1 is very big in comparison to the other parties, $|P_1| \sum_{j=l+1}^{k} \frac{1}{j}$ might be larger than $|P_2| \sum_{j=1}^{l} \frac{1}{j} + |P_3| \sum_{j=1}^{l} \frac{1}{j} + \cdots$. To give a more concrete example of what this may mean, we fix for the moment $l = \lceil \frac{k}{2} + 1 \rceil$, which is used for elections of the work council of at least one company that we know of. Then, if k is even, this gives $s_{PAV}(A, W_{LV}) = |P_1| \sum_{j=1}^{l} \frac{1}{j} + |P_2| \sum_{j=1}^{l-2} \frac{1}{j}$. This means that when $|P_1| \sum_{j=l+1}^{k} \frac{1}{j} > |P_2| \sum_{j=1}^{l-2} \frac{1}{j}$, the PAV-gain is negative.

So for broadcasted party-list elections, if the relative difference between the size of the first party and the other parties is large, the PAV-gain is positive. If, on the other hand, the parties are similar in size, then the PAV-gain will be negative. We can give a precise value to the PAV-gain on broadcasted party-list elections:

Proposition 5. *Let $E = \langle N, C, k, l, A, L \rangle$ be any broadcasted party-list election with parties $P_1, \ldots, P_g$. If the most popular party has at least k members, its PAV-gain is*

$$Gain_{PAV}(E) = \sum_{i=2}^{\min(\lfloor \frac{k}{l} \rfloor, g)} \left(|P_i| \cdot \sum_{j=1}^{l} \frac{1}{j} \right) + |P_{\lceil \frac{k}{l} \rceil}| \cdot \sum_{j=1}^{k \mod l} \frac{1}{j} - |P_1| \cdot \sum_{j=l+1}^{k} \frac{1}{j}.$$ [5]

Proof. This follows directly from the PAV-scores of winning committees by AV and LV in such elections: $\min_{W \in LV(E)} s_{PAV}(A, W) = \sum_{i=1}^{\min(\lfloor \frac{k}{l} \rfloor, g)} \left(|P_i| \sum_{j=1}^{l} \frac{1}{j} \right) + |P_{\lceil \frac{k}{l} \rceil}| \sum_{j=1}^{k \mod l} \frac{1}{j}$ and $\max_{W \in AV(E)} s_{PAV}(A, W) = |P_1| \sum_{j=1}^{k} \frac{1}{j}$. □

[5] Note that the middle term disappears when: i) $k \mod l \equiv 0$ since it takes the sum from $j = 1$ to $j = 0$: k is divisible by l and all candidates from $\frac{k}{l}$ parties are elected; or ii) $g < \lfloor \frac{k}{l} \rfloor$ since $P_{\lceil \frac{k}{l} \rceil}$ does not exist.

In the case where $l = \lceil \frac{k}{2} + 1 \rceil$, that is used in some companies' council elections, this gives us the following result:

Corollary 5.1 (PAV-gain of the special case where $l = \lceil \frac{k}{2} + 1 \rceil$)**.** *For any broadcasted party-list elections E where the most popular party has at least k members and $l = \lceil \frac{k}{2} + 1 \rceil$, $Gain_{PAV}(E) = |P_2| \sum_{j=1}^{l-2} \frac{1}{j} - |P_1| \sum_{j=l+1}^{k} \frac{1}{j}$.*

This means that, if the most popular party is much larger than the second party, the PAV-gain is negative, but if there is not a large difference in size between the first two parties, LV gives a more proportional committee than AV.

It would be interesting to compare the PAV-score of LV in broadcasted party-list elections to the maximal PAV-score (that of the winning committee according to PAV), but this is hard to express since the maximal PAV-score depends heavily on the relative differences in size of the parties. For example, if we fill the committee sequentially, it would be best to start with a candidate from P_1, but after that, if $|P_2| < \frac{|P_1|}{2}$, another candidate from P_1, but otherwise a candidate from P_2. After that, if $|P_3| < \frac{|P_2|}{2}$, another candidate from P_2, but otherwise a candidate from P_3, etc, until we are past all parties, then, after choosing another candidate from P_1, if $\frac{|P_2|}{2} < \frac{|P_1|}{3}$, choose another candidate from P_1, but otherwise a candidate from P_2, etc.

5 Conclusion

In general elections, we did not find any justification for the use of LV. Indeed, we have shown that restricting the ballot size in approval elections cannot increase the proportionality or diversity of the outcome. Moreover, LV may make voters feel restricted in expressing their preferences and have a lower efficiency when compared to other forms of approval voting. However, in broadcasted laminar and party-list elections, limited voting does significantly increase diversity and in some cases increase proportionality compared to unlimited approval voting. Therefore, in applications where elections have (or tend to have) such form, restricting approval ballots to a fixed size appears to be an effective 'hack' on simple approval voting in order to obtain a more diverse representation.

We conclude by mentioning a few directions for future work. First, in some applications of LV, voters do not have to vote on exactly l candidates, but can vote on *at most l* candidates. Most of the positive results in this paper will still hold for such elections, but some of the negative results may be mitigated by allowing for smaller votes since voters are not forced anymore to vote on candidates they do not approve. It would be good to extend our analysis also to this variant of LV. Second, we found that for more aligned elections, of which broadcasted party-list elections are an extreme case and broadcasted laminar elections a bit less extreme case, the CC-gain is positive. Since the property of being party-list or laminar is binary, it would be interesting to find a quantitative measure of alignment of elections using for example the insights from [3], and to see the extent to which it correlates with the CC-gain. Third, in Theorem 2, we gave a lower bound for the CC-gain in broadcasted laminar elections. To find an upper bound or an analytical expression for the exact CC-gain in such elections is left as an open question. Fourth, the PAV-gain is not the only quantitative measure for proportionality, it would be interesting to know how LV performs with other measures, e.g. the *proportionality degree* as defined in [18]. Finally, we made the common assumption that voters' preferences are binary and voters get equal satisfaction from all their approved candidates. A natural extension of this paper would be to analyse the diversity and proportionality of LV with other approval-based satisfaction functions, as is done in [6], or with cardinal or ordinal preferences.

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	c_1	c_2	c_3	c_4	c_5	c_6
v_1	X	X				
v_2				X	X	

Table 8: Example with AV and LV non-resolute and $Gain_{CC} = -1$ ($k = l = 2$).

A Additional results/observations

Example 6. *Take an election with $k = 6$, $l = 2$ with a party-list profile A such that there are two parties, a and b that both have 6 voters. Party a broadcasts to all its voters to vote on the same two candidates a_1 and a_2, party b is more coordinated and asks two voters to vote for b_1 and b_2, two voters to vote for b_3 and b_4, and two voters to vote for b_5 and b_6. In this way a gets only 2 seats in the winning committee, while b gets 4 seats, so clearly it is not the most strategic choice to ask all voters to vote for the same candidates.*

Example 7. *The condition that AV and LV are resolute in this election is necessary to prevent the possibility where $s_{LV}(c_1)$ was enough to be elected by LV, but just because of tie-breaking rules was not elected. We give an example of such election in Table 8: AV has a maximal CC-score of 2 with for example $\{c_3, c_4\}$, while LV has a minimal CC-score of 1 with for example $\{c_4, c_5\}$, so the CC-gain is -1.*

B Omitted proofs

Proposition 1. *The CC-guarantee of LV is 0.*

Proof. The idea of the proof is that a small group that coordinates well can overrule a large group that coordinates poorly. Let $\mathcal{E}$ be a class of elections of the following form: for any $E \in \mathcal{E}$ we have $E = \langle N, C, k, l, A, L\rangle$ where $k \geq 2$ is divisible by l[6], N consists of two groups $N = X \cup S$ with $X \cap S = \emptyset$, and $|S| = 2\frac{k}{l}$; for any $j \in S$, $A_j = W$ where $W \subseteq C$ with $|W| = k$ and for any $j \in X$, $A_j = Y$, where $Y \subseteq C$ with $|Y| \geq l|X|$ and $Y \cap W = \emptyset$; for any $j, j' \in X$, $L_j \cap L_{j'} = \emptyset$ (so all candidates in Y get at most one vote), for any $j \in S$ there is exactly one $j' \in S$ such that $L_j = L_{j'}$, and for all other j'', $L_j \cap L_{j''} = \emptyset$ (so all candidates in W get two votes). In any such election, W wins according to LV, and $s_{CC}(W) = |S| = 2\frac{k}{l}$. The most diverse committee W' is one that contains at least one candidate from W and at least one candidate from Y, with $s_{CC}(W') = 2\frac{k}{l} + |X|$. This gives us that for $E \in \mathcal{E}$, as $|X|$ becomes larger, $\frac{\min_{W \in LV(E)} s_{CC}(A,W)}{\max_{W \in S_k(C)} s_{CC}(A,W)} = \frac{2\frac{k}{l}}{2\frac{k}{l} + |X|}$ approximates 0, and therefore, the CC-guarantee of LV is $\kappa_{cc}(k) = \inf_{A \in \mathcal{A}} \frac{\min_{W \in LV(E)} s_{CC}(A,W)}{\max_{W \in S_k(C)} s_{CC}(A,W)} = 0$. □

Proposition 2
Let BP denote the set of all possible broadcasted party-list elections, then the CC-guarantee of LV restricted to BP is $\kappa_{cc}(k)(BP) = \frac{1}{k}$ in general, and $\kappa_{cc}(k)(BP) = \frac{\lceil \frac{k}{l} \rceil}{k}$ for all $E \in BP$ with $l_E = l$.

Proof. As we saw above, for any broadcasted party-list election $E = \langle N, C, k, l, A, L\rangle$ with g parties, the CC-score of any committee W chosen by LV is $s_{CC}(A, W_{LV}) = \sum_{i=1}^{\min(\lceil \frac{k}{l} \rceil, g)} |P_i|$, while the maximal CC-score of any committee $W \in C$ is $s_{CC}(\alpha\text{-CC}) = \sum_{i=1}^{\min(k,g)} |P_i|$. Hence, we need to find the election E for which $\frac{s_{CC}(A, W_{LV})}{s_{CC}(\alpha\text{-CC})}$ is lowest. All else being equal, if $g < k$, this fraction can never be smaller than if $g \geq k$, so for now we assume that $g \geq k$,

[6] This assumption is not necessary for the proof to work, but makes it easier to read.

which reduces the fraction that we have to minimise to $\frac{\sum_{i=1}^{\lceil \frac{k}{l} \rceil} |P_i|}{\sum_{i=1}^{k} |P_i|}$. If l is not given, $l = k$ gives the lowest value: $\frac{|P_1|}{\sum_{i=1}^{k} |P_i|}$, and to minimize this we should minimize $|P_1|$ (with the restriction that it has to be larger than the other parties). Let's say all parties except P_1 have x voters, and P_1 has $x+1$ voters. Then we have $\frac{|P_1|}{\sum_{i=1}^{k} |P_i|} = \frac{x+1}{x+1+(k-1)x}$, and if x goes to infinity, this will become $\frac{1}{k}$. If l is given, we have that the fraction $\frac{\sum_{i=1}^{\lceil \frac{k}{l} \rceil} |P_i|}{\sum_{i=1}^{k} |P_i|}$ is smallest when the last $\lceil \frac{k}{l} \rceil + 1$ to k parties are as large as possible in comparison with the rest. However, the first $\lceil \frac{k}{l} \rceil$ parties have to be at least 1 voter larger, in order to be ranked first. Hence, we minimise the fraction if the first $\lceil \frac{k}{l} \rceil$ parties all have 1 more voter: $\frac{\sum_{i=1}^{\lceil \frac{k}{l} \rceil} |P_i|}{\sum_{i=1}^{k} |P_i|} = \frac{(x+1)\lceil \frac{k}{l} \rceil}{(x+1)\lceil \frac{k}{l} \rceil + x(k - \lceil \frac{k}{l} \rceil)}$. This fraction is smallest when x becomes very large, and then its limit value is $\frac{\lceil \frac{k}{l} \rceil}{k}$. □

Proposition 4. *LV fails JR, EJR, PJR, laminar proportionality, and priceability, even on broadcasted party-list elections.*

Proof. We show that LV does not satisfy JR by constructing a counterexample. Take an election with $k = 4$ and $l = 3$, where we have 8 voters that vote as shown in Table 9 The

	a_1	a_2	a_3	w_1	w_2	w_3	w_4	w_5	w_6
v_1	X	X	X						
v_2	X	X	X						
v_3				X	X	X			
v_4				X	X	X			
v_5				X	X	X			
v_6							X	X	X
v_7							X	X	X
v_8							X	X	X

Table 9: $l = 3$, $k = 4$. Approvals in grey, Limited votes indicated by X.

group $V = \{v_1, v_2\}$ is 1-cohesive, since $|V| = 2 \geq \frac{n}{k} = \frac{8}{4}$, v_1 and v_2 agree on all their votes. However, the candidates they vote for all get two votes, while all other candidates $w_1, ..., w_6$ get three votes. Hence, there is no voter in V who is represented by at least one member in the winning committee. Note that in this example limited voting and approval voting are equivalent because all voters approve exactly 3 candidates, this is however not crucial for the counterexample to work.

Moreover, since EJR implies PJR and PJR implies JR[17], we can conclude that LV satisfies neither EJR nor PJR. Some voting rules that do not satisfy JR do satisfy PJR for $\ell \geq 2$, but this is not the case for LV.

We use another counterexample to show that LV does not satisfy laminar proportionality or priceability:

See the profile in Table 10. If $k = 6$ and $l = 4$, all voters vote exactly for the candidates they approve. Then c_1, c_2, c_3, c_4, c_5 all get two votes, and the other candidates get one vote. So the winning committee consists of $c_1, ..., c_5$, and one other candidate. An example of a winning committee is shown in grey in Table 10. The instance is laminar, but all outcomes are not laminar proportional, because laminar proportionality would mean that both groups $\{v_1, v_2\}$ and $\{v_3, v_4\}$ have three candidates that they approve in the winning committee, but here the first group has only two and the second group has 4.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}
v_1	X								X	X	X
v_2	X					X	X	X			
v_3		X	X	X	X						
v_4		X	X	X	X						

Table 10: Example to show that LV does not satisfy laminar proporitonality or priceability. $k = 6$ and $l = 4$, the committee in blue is a winning committee according to LV.

The same example shows that LV is not priceable: if the cost for all candidates is 1 and p is the budget per voter, v_3 and v_4 together have to pay for c_2, c_3, c_4, c_5, so $p \geq 2$. But then v_1 and v_2 together have a budget of 4 while they only paid 2, so at least one of them has at least 1 left, and could have bought another candidate. $\square$