Asymptotic Existence of Class Envy-free Matchings

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Abstract

We consider a one-sided matching problem where there are m items and n agents who are partitioned into disjoint classes. Each class must receive fair treatment in a desired matching. This model, recently proposed by Benabbou et al. [7] and Hosseini et al. [16], captures a wide range of real-life scenarios, such as the allocation of public housing and the allocation of medical resources across different ethnic groups and different age groups, respectively. The main interest has been to achieve class envy-free matchings, in which each class receives a total utility at least as large as the maximum value of a matching they would achieve from the items matched to another class. However, class envy-freeness is unattainable for worst case utilities without allowing some items to remain unused. To analyze the existence of a class envy-free matching in practice, we consider a distributional model, in which the agents' utilities for the items are drawn from a probability distribution. Our main result is the asymptotic existence of class envy-free and non-wasteful matchings when the number of agents approaches infinity. To this end, we propose a round-robin style algorithm and prove that it produces a desirable matching with high probability.

1 Introduction

One-sided matching is a fundamental problem both theoretically and practically. For instance, in the allocation of tasks to workers, or social housing to residents, ensuring fairness is of paramount importance. Consider the allocation of scarce medical resources to different regions in a country. To gain public acceptance, the social planner must ensure a fair distribution of the resources across various regional groups.

Recently, Benabbou et al. [7] and Hosseini et al. [16] proposed a model that captures such scenarios. They adapted fairness notions from fair division literature into the one-sided matching problem. In this model, there are m items and n agents forming k disjoint classes. Each class evaluates a matching through *assignment valuations*, representing the value of a class for the items allocated to another class. The optimum value of a matching between the members of the class and the items determines this value. A central concept of fairness is *class envy-freeness*, which requires that no class prefer being allocated to the items assigned to another class.

It is possible to achieve envy-freeness among classes if all items can be discarded, although such a matching would not be desirable at all. Unfortunately, a class envy-free matching is not guaranteed to exist without wasting any items. For example, consider a case with one item and two classes, each consisting of one agent who likes the single item. In any nonempty matching, one class receives nothing whereas the other class receives one item. This strong conflict between fairness and efficiency, however, does not necessarily imply that achieving a *non-wasteful*[1] class envy-free matching is impossible in practice. Experimental studies [7, 8] have demonstrated that an approximate notion of class envy-freeness can often be achieved with minimal wastage.[2]

[1]We provide a formal definition of non-wastefulness in Section 2.

[2]See Appendix C of Benabbou et al. [8] for experiments using real-life instances and Section 5 of Benabbou et al. [7] for experiments using random instances.

In the context of fair division, related studies have analyzed situations in which the agents' utilities for each item are drawn from a random distribution [12]. Although the non-existence of an envy-free allocation also holds in this setting, the distributional model allows us to analyze practical cases and ignore pathological inputs. Interestingly, a very simple algorithm is likely to output a complete envy-free allocation for agents with additive valuations. For instance, Manurangsi and Suksompong [29] demonstrated that the *round-robin algorithm*, which lets each agent choose the favorite item in the order, computes an envy-free allocation with high probability when agents have additive valuations.

Our contributions In this paper, we propose a distributional model in which each agent has utilities drawn from a continuous probability distribution. Our main result is the asymptotic existence of class envy-free and non-wasteful matchings, subject to mild assumptions on the number and the sizes of classes. To prove this, we adopt the round-robin algorithm from fair division into our setting. Specifically, in each round, each class selects an item with highest marginal utility among the remaining items to create a maximum weight matching with one more edge. We demonstrate that this algorithm produces an envy-free non-wasteful matching when the number of agents goes to infinity.

Although our algorithm behaves in a very similar manner to the round-robin algorithm for the fair division problem with additive valuations, the non-additivity of classes' valuations presents significant differences in our analysis compared with that for the additive setting [29] and poses technical challenges. To be more concrete, for additive valuations, the envy between any pair of agents based on the output of the round-robin algorithm can be decomposed into *per-round envies*, which represent the differences between the values of each agent i for the item they received and that by another agent in each round. However, obtaining such a decomposition is challenging for assignment valuations, owing to the combinatorial structure of the matchings.

To address these challenges, our proof crucially utilizes the techniques from the random assignment theory [2, 9, 14, 37, 40]. The random assignment theory employs a bipartite graph with random edge weights, primarily focusing on analyzing the expected value of a minimum weight perfect matching, essentially equivalent to a maximum weight perfect matching. In Section 3.1, rather than examining an individual pair of items allocated to two classes p and q in each round, we present the investigation of the marginal utility that each class p enjoys in each round. This approach allows us to analyze the expected value of the items allocated to class p. In addition, we evaluate the expected value of the items allocated to another class q from the viewpoint of p using a randomly selected bundle of the class p's size. In Section 3.2 discusses the concentration of probability around these expected values. Finally, in Section 3.3, we combine these results to establish our main theorem. See Appendix A for a more extensive discussion about the further related work.

2 Model

In this paper, we use $[k]$ to denote the set $\{1, 2, ..., k\}$. Let $I = [m]$ be the set of items and $N = [n]$ be the set of agents. We call a subset of I a *bundle*. The set of agents N is partitioned into k *classes*, $N_1, N_2, ..., N_k$, where N_p denotes the set of agents with class p and $n_p = |N_p|$ denotes the number of agents in the class p. We arrange k classes so that $n_1 \leq n_2 \leq \cdots \leq n_k$. We refer to N_p as class p. Each agent $i \in N$ possesses utility $u_i(j) \in [0, 1]$ for every item $j \in I$, where we assume that $u_i(j)$ is drawn from distribution $\mathcal{D}$ on $[0, 1]$, with some assumptions of the distribution presented in Section 2.1.

Consider a bipartite graph $G = (N \cup I, E)$, where $\{i, j\} \in E$ if and only if $u_i(j) > 0$ for every pair of $i \in N$ and $j \in I$. A *matching* M of G is a set of edges such that each vertex appears in at most one edge of M. Given a matching M, we denote by M^p the submatching of M, the edges of which have endpoints in N_p. For $S \subseteq N$, let $M(S)$ be the set of items assigned to S by matching M. We write $(M(N_1), M(N_2), ..., M(N_k))$ as the allocation to k classes induced by matching M, where $M(N_p)$ is a bundle allocated to class $p \in [k]$. Let $M(N) \subseteq I$ denote the set of items allocated to some agent and $I_0 = I \setminus M(N)$ denote the set of unallocated items.

We assume that each class evaluates a bundle allocated to another class with an assignment valuation [7]. Its definition is as follows:

Definition 1 (Assignment valuation)**.** The *assignment valuation* $v_p(I')$ for class p to a bundle $I' \subseteq I$ is the maximum total weight of matching of items in I' to the agents in N_p. Namely, $v_p(I') = \max_{M \in \mathcal{M}(N_p, I')} \sum_{\{i,j\} \in M} u_i(j)$, where $\mathcal{M}(N_p, I')$ denotes the whole set of matchings between N_p and I' in G.

It is worth noting that the value $v_p(I')$ for bundle $I' \subseteq I$ is at most the size n_p of each class $p \in [k]$, since each utility $u_i(j)$ is at most 1 for all $i \in N$ and $j \in I$. Each value $v_p(I')$ can be computed in polynomial time since, given a bipartite graph and edge weights, a maximum weight matching can be computed in polynomial time [26].

Next, we define the notion of class envy-freeness, which demands the total utility each class achieves to be higher or equal to the class p's assignment valuation for the items allocated to every other class q.

Definition 2 (Class envy-freeness)**.** Let M be a matching in the bipartite graph $G = (N \cup I, E)$. We say that class p *envies* class q if $\sum_{\{i,j\} \in M^p} u_i(j) < v_p(M(N_q))$. A matching M is *class envy-free* if no class envies another class, i.e., $\sum_{\{i,j\} \in M^p} u_i(j) \geq v_p(M(N_q))$ for every pair $p, q \in [k]$ of distinct classes.

If we allow each class to optimally shuffle items within the members of the class, then the class would choose a maximum weight matching between the members of the class and their bundle. In the round-robin algorithm which we will consider in Section 3, we allow shuffling and therefore the class envy-freeness requirement is equivalent to the definition above where we replace the left-hand side $\sum_{\{i,j\} \in M^p} u_i(j)$ with $v_p(M(N_p))$.

We define below non-wastefulness, which requires that there is no waste of an item j such that (a) j is an unallocated item that can increase the total utility of some class p or (b) j can be reallocated from class q to class p to increase the total utility of class p without hurting class q. For a bundle $I' \subseteq I$, class $p \in [k]$, and an item $j \in I$, we define the *marginal utility* $\Delta_p(I'; j)$ as $\Delta_p(I'; j) = v_p(I' \cup \{j\}) - v_p(I')$ if $j \in I \setminus I'$ and $\Delta_p(I'; j) = v_p(I') - v_p(I' \setminus \{j\})$ if $j \in I'$.

Definition 3 (Non-wastefulness)**.** An item $j \in I$ is said to be *wasted* for a matching M if either (a) $j \notin M(N)$ such that $\Delta_p(M(N_p); j) > 0$ for some class p, or (b) there exists class q such that $j \in M(N_q)$ and $\Delta_q(M(N_q); j) = 0$ but $\Delta_p(M(N_p); j) > 0$ for some class p. A matching is *non-wasteful* if no item is wasted.

The above definition of non-wastefulness is the same as the definition of non-wastefulness by Benabbou et al. [7] and stronger than that by Hosseini et al. [16], who define a non-wasteful matching to be a maximal matching.

As is observed in Hosseini et al. [16], unfortunately, a class envy-free matching that is non-wasteful may not exist. Note that the asymptotic existence of a "complete" class envy-free matching where each agent is matched to exactly one item readily follows from the result in the house allocation problem [15, 29], though such a matching may be wasteful. We discuss details in Appendix B.1. In Section 3, we demonstrate that under mild conditions, we can in fact obtain the asymptotic existence of a matching that satisfies both class envy-freeness and non-wastefulness.

2.1 Distributions

For each agent $i \in N$ and item $j \in I$, we assume that the utility $u_i(j)$ is independently drawn from a given distribution $\mathcal{D}$ supported on $[0,1]$. We assume that all the agents have the same distribution. We denote the density function of $\mathcal{D}$ by $f_{\mathcal{D}}$ and the cumulative distribution function of $\mathcal{D}$ by $F_{\mathcal{D}}$. A distribution is said to be *non-atomic* if it does not put positive probability on any single point. Throughout this paper, we assume that a distribution $\mathcal{D}$ is non-atomic. We define that a distribution $\mathcal{D}$ with density function $f_{\mathcal{D}}$ is (α, β)*-PDF-bounded* for constants $0 < \alpha \leq \beta$, if $\alpha \leq f_{\mathcal{D}}(x) \leq \beta$ for all $x \in [0,1]$. We say that $\mathcal{D}$ is *PDF-bounded* if it is (α, β)-PDF-bounded for some $\alpha, \beta > 0$. If $\alpha = \beta = 1$, then $\mathcal{D}$ is the uniform distribution over $[0,1]$ since $f_{\mathcal{D}}(x) = 1$ for all $x \in [0,1]$. The PDF-boundedness assumption was originally introduced by Manurangsi and Suksompong [29] in the context of fair division as a natural class of distributions, such as a truncated normal distribution is PDF-bounded. Bai and Gölz [5] also assumed this condition to prove the asymptotic existence of an envy-free allocation in the asymmetric model. We denote by $\mathrm{Unif}(0,1)$ (resp., $\mathrm{Exp}(\lambda)$ and $\mathrm{Bin}(n,p)$) the uniform distribution on $[0,1]$ (resp., the exponential distribution with rate λ on $[0,\infty)$ and the binomial distribution with parameter n and p). We say that an event occurs *almost surely* if the event occurs with probability 1. We here introduce the Talagrand's inequality, which will be used in the proof of Theorem 1.[3]

Lemma 1 (Talagrand's inequality)**.** *Let $\mathcal{F}$ be a family of d-tuples of non-negative real numbers $\omega = (\omega_1, \omega_2, \ldots, \omega_d)$ and $X_1, X_2, \ldots, X_d$ be an independent sequence of random variables on $[0,1]$. In addition, let $Z = \min_{\omega \in \mathcal{F}} \sum_{i=1}^{d} \omega_i X_i$, $\sigma = \max_{\omega \in \mathcal{F}} \|\omega\|_2$ and $\varepsilon > 0$ be a constant, then we obtain*

$$\mathbf{Pr}[|Z - \mathbb{E}[Z]| \geq \varepsilon] \leq 4 \exp\left(-\frac{\varepsilon^2}{4\sigma^2}\right).$$

3 Asymptotic Existence of Non-wasteful Class Envy-free Matchings

In this section, we show the asymptotic existence of class envy-free and non-wasteful matchings between agents and items. We say that an event happens *with high probability* if the event's probability converges to 1 as $n \to \infty$. Our main result is as follows:

Theorem 1. *Suppose that the distribution $\mathcal{D}$ is (α, β)-PDF-bounded and that the following conditions hold:*

(a) *the number k of classes is a constant satisfying $k > \frac{\beta}{\alpha^2}$,*

(b) *the number m of items is sufficiently large such that $k \cdot \max_{p \in [k]}(n_p + 1) \leq m$,*

(c) *the sizes of the classes are almost proportional to the total population; more precisely, there exists a constant $c > 0$ such that $n \leq c \cdot (\min_p n_p)^{3/2} \cdot (\log \min_p n_p)^{-5/2}$.*

Then, the round-robin algorithm, presented as Algorithm 1, produces a class envy-free and non-wasteful matching exists with high probability.

[3] The original Talagrand's inequality [37] is proven as the stochastic concentration around the median of Z, where the *median* ν of a random variable X distributed according to a density function f is a real number satisfying $\int_{(-\infty,\nu]} f(x)\mathrm{d}x \geq 1/2$ and $\int_{[\nu,\infty)} f(x)\mathrm{d}x \geq 1/2$. Nevertheless, it is known that the concentration inequality around the median with Gaussian upper bound implies that for the mean and vice versa. See [11] or Exercise 2.14 in [38] for details.

Note that the condition $k > \frac{\beta}{\alpha^2}$ in (a) is very mild for some distributions, e.g. for uniform distributions, it is equivalent to $k > 1$ since $\alpha = \beta = 1$. A perhaps more intuitive but stronger condition of (c) is the case where the total number of agents is within a constant factor of the size of every class, i.e., $n = C \cdot \min_p n_p$ where $C \geq 1$ is a constant. This is relevant in scenarios when the sizes of the classes under consideration are proportional to the total population size, e.g. classes may correspond to gender groups, ethnic groups, and groups of people with the same political interest.

Here, we refer to each iteration of the **while** loop (Lines 2–12) in Algorithm 1 as a *round*. As explained in Section 1, in each round r, each class selects its most preferred item j_r^p, which has the highest marginal utility to the current bundle $M(N_p)$ (Line 5) and updates its matching to create a new maximum weight matching M_r^p with one more edge (Line 6). In our model, the edge weights are drawn from a non-atomic distribution independently. So, no pair of distinct matchings with size r have the same total weight almost surely.[4] Hence, the item j_r^p as well as the maximum weight matching M_r^p for each class p can be uniquely determined in every round r.

In the sequel, let M be the matching produced by Algorithm 1 and M_r be the matching at the end of round r in the algorithm. We denote the set of remaining items just before class p selects an item in round r by $I_r^p = I \setminus (M_r(N_1 \cup \cdots \cup N_{p-1}) \cup M_{r-1}(N_p \cup \cdots \cup N_k))$. Let $m_r^p = |I_r^p|$.

Algorithm 1 Round-robin algorithm for classes with assignment valuations

Input: $(N = N_p \cup \cdots \cup N_k, I, \{u_i(j)\}_{i \in N, j \in I})$
Output: M

1: Set $M \leftarrow \emptyset$, $I_0 \leftarrow I$ and $r \leftarrow 1$.
2: **while** there is a remaining item for which some class p has positive marginal utility, i.e., $I_0 \neq \emptyset$ and $\exists j \in I_0\ \exists p \in [k]\ \Delta_p(M(N_p); j) > 0$ **do**
3:   **for** $p = 1, 2, ..., k$ **do**
4:     **if** there is $j \in I_0$ such that $\Delta_p(M(N_p); j) > 0$ **then**
5:       $j_r^p \leftarrow \text{argmax}_{j \in I_0} \Delta_p(M(N_p); j)$.
6:       $M_r^p \leftarrow$ the maximum weight matching between N_p and $M(N_p) \cup \{j_r^p\}$.
7:       $M \leftarrow (M \setminus M^p) \cup M_r^p$.
8:       $I_0 \leftarrow I_0 \setminus \{j_r^p\}$.
9:     **end if**
10:   **end for**
11:   $r \leftarrow r + 1$.
12: **end while**
13: **return** M.

As mentioned before, we allow each class to optimally shuffle items within the members of the class in each round and so the class selects a maximum weight matching between the members of the class and the items allocated to them. Thus, the total utility $\sum_{\{i,j\} \in M^p} u_i(j)$ that class p receives under Algorithm 1 is $v_p(M(N_p))$.

To begin with, we show that the output of Algorithm 1 is non-wasteful.

Lemma 2. *The resulting matching M of Algorithm 1 is non-wasteful.*

Proof. It is immediate that the resulting matching satisfies the condition (a) in Definition 3 by construction of the algorithm. To show that M satisfies the condition (b) in Definition 3, we prove a stronger statement that no class is allocated to an item whose removal does not change their total utility. We use the fact that for assignment valuations, the greedy

[4] This is from a similar proof of the isolation lemma [31]. See Appendix D for details.

algorithm that sequentially adds an item with the highest marginal utility and constructs a size-r subset is guaranteed to compute an optimal set of items that maximizes the valuation across all size-r subsets (See Theorem 17.2 in [33]). This implies that for each round r and each class p,

$$M_r^p(N_p) \in \arg\max\{\, v_p(I') \mid |I'| = r \ \wedge \ I' \subseteq M_{r-1}^p(N_p) \cup I_r^p\}. \tag{1}$$

Suppose towards a contradiction that there exists class p and item $j \in M(N_p)$ such that $\Delta_p(M(N_p); j) = 0$. Thus, $v_p(M(N_p) \setminus \{j\}) = v_p(M(N_p))$. Since each item j_r^p added to the bundle of class p has a positive contribution, we have

$$v_p(M(N_p)) > v_p(M_{r^*-1}(N_p)),$$

where r^* denotes the last round when class p obtains some item. However, this implies $v_p(M(N_p) \setminus \{j\}) > v_p(M_{r^*-1}(N_p))$, which contradicts (1) as $M(N_p) \setminus \{j\}$ has the same cardinality as $M_{r^*-1}(N_p)$ and yields a higher value for class p. □

It thus suffices to prove that a matching produced by Algorithm 1 is class envy-free with high probability. Throughout this section, we fix any two classes p and q.

In Section 3.1, we show Lemma 3 which presents a lower bound on the expected difference value between $v_p(M(N_p))$ and $v_p(M(N_q))$. Lemma 3 is proved from Claims 1 and 2. To this end, we use techniques from the random assignment theory, which analyzes the expected value of a maximum weight perfect matching given random edge weights. We evaluate the expected total weight of a maximum weight perfect matching by using a technique developed in [13, 14, 40]. Specifically, we add a special vertex whose incident edges are weighted by an exponential distribution. By adding a special vertex, we can estimate the difference between the expected values each class achieves in two consecutive rounds.

In Section 3.2, to achieve a probabilistic concentration around the expected value of the maximum weight matching for class p, we demonstrate that both the weights of the edges in the matching M^p and the weights of the edges in the maximum weight matching between N_p and $M(N_q)$ are sufficiently heavy. This is formalized in Lemma 4 and follows from the fact that a bipartite graph with only heavy edges satisfies the property of the expanding bipartite graph.

In Section 3.3, we combine Lemmas 3 and 4, we prove that the probability that class p itself envies another class q is bounded from above by using Talagrand's inequality. Finally, we show that the probability that at least one class has envy converges to 0 as the number of agents n approaches infinity, thereby establishing the asymptotic existence of a class envy-free and non-wasteful matching.

It is worth noting that the deterministic behavior of the round-robin algorithm exhibits some important differences between the additive and our settings. For additive valuations, if an agent i is ahead of another agent i' in the round-robin order, i does not envy i' in the resulting outcome. However, a similar property does not hold in our setting with assignment valuations, since the greedy strategy does not necessarily produce an optimal matching. Due to this, unlike the additive setting, our round-robin algorithm may not produce a matching that satisfies EF1, an approximate notion of envy-freeness [10]. See Appendix C for details.

Notations and definitions Here we introduce additional notations and definitions. An *alternating* path P (respectively, a cycle C) of a matching M' in a bipartite graph H is a path (respectively, a cycle) in H where for every pair of consecutive edges on P, one of them is in M' and another one is not in M'. An *augmenting* path P of a matching M' in a bipartite graph H is an alternating path where no edge from M' is incident to the first or the last vertex of P. We say that a vertex is saturated by a matching if it is an endpoint of one of the edges in the matching. We call a matching *saturated* if all the vertices on one side are saturated. For a bipartite graph H and each vertex i that appears in a matching M' of H (namely $\{i, j\} \in M'$ for some j), we denote by $M'(i)$ the vertex matched to i under M'.

3.1 Difference between two expected values

To establish the asymptotic existence of a class envy-free and non-wasteful matching, it is necessary to understand how likely each class may envy another class in expectation. Lemma 3 provides a lower bound on the expected envy under matching M, expressed as the ratio between the class sizes. In this section, we present the proof of Lemma 3.

Lemma 3. *Suppose that $\mathcal{D}$ is (α, β)-PDF-bounded and $k \cdot \max_{p\in[k]}(n_p + 1) \leq m$. Then we have*

$$\mathbb{E}[v_p(M(N_p))] - \mathbb{E}[v_p(M(N_q))] \geq n_p - \min(n_p, n_q) + \left(\frac{\alpha}{\beta} - \frac{1}{\alpha k}\right) \frac{\min(n_p, n_q)}{n_q} - o(1).$$

Let us explain Lemma 3. Here, from the condition (a) in Theorem 1: $k > \frac{\beta}{\alpha^2}$, we obtain $\alpha/\beta - 1/\alpha k > 0$. Therefore, when $n_p \geq n_q$, Lemma 3 implies that there exists a positive constant $c > 0$ such that $\mathbb{E}[v_p(M(N_p))] - \mathbb{E}[v_p(M(N_q))] \geq c - o(1)$. Note that when $n_p > n_q$, then we obtain an even stronger lower bound $c \geq 1$ since $n_p - \min(n_p, n_q) \geq 1$. On the other hand, when $n_p < n_q$, according to Lemma 3, we have $\mathbb{E}[v_p(M(N_p))] - \mathbb{E}[v_p(M(N_q))] \geq (\alpha/\beta - 1/\alpha k)\, n_p/n_q - o(1)$ where the lower bound depends on n_p and n_q.

To prove Lemma 3, we analyze the expected values of $v_p(M(N_p))$ and $v_p(M(N_q))$ separately. Let us note here the difficulty in calculating the exact expectations of $v_p(M(N_p))$ or $v_p(M(N_q))$ exactly. The exact calculations of the expected minimum total weight of matchings have been studied in the random assignment theory. Typically, these models assume that the weight of each edge is drawn from the exponential distribution. Assuming the exponential distribution, we can compute the expected minimum total weight exactly, as the distribution is *memoryless*, that is, the past process does not influence the future behavior. However, in this study, we assume the PDF-bounded distributions that do not have the memorylessness property. Nevertheless, we can show that, just like the fact shown by Aldous [2] that the uniform and the exponential distributions asymptotically behave in the same way, the PDF-bounded distributions also asymptotically exhibit these same behaviors. Therefore, we can learn about the asymptotic behavior of the difference between the expected values of $v_p(M(N_p))$ and $v_p(M(N_q))$ and derive the inequality in Lemma 3.

Lower bound on $v_p(M(N_p))$: Our goal here is to show the lower bound on the expected value of $v_p(M(N_p))$, as described in Claim 1.

Claim 1. *Under the same conditions as in Lemma 3, we have*

$$\mathbb{E}[v_p(M(N_p))] \geq n_p - \frac{1}{\alpha} \sum_{r=1}^{n_p} \frac{1}{r} \sum_{r'=1}^{r} \frac{1}{m^p_{r'} + 1}.$$

The formal proof of Claim 1 is deferred to Appendix E.1. Here we outline a brief sketch of the proof. Recall that, in each round r, class p selects an item $j^p_r = \mathrm{argmax}_{j\in I^p_r} v_p\left(M_{r-1}(N_p) \cup \{j\}\right)$ and updates their matching from M^p_{r-1} to M^p_r, which is the maximum weight matching between N_p and $M_{r-1}(N_p) \cup \{j^p_r\}$. Moreover, since all edge weights are positive almost surely and we assume $m \geq n$, the size of the matching M^p is n_p, and the size of the bundle $M(N_p)$ is also n_p. By linearity of expectation, we can decompose $\mathbb{E}[v_p(M(N_p))]$ into the expected difference piled in each round, i.e., $\mathbb{E}[v_p(M_r(N_p))] - \mathbb{E}[v_p(M_{r-1}(N_p))]$.

To analyze the expected difference between $v_p(M_r(N_p))$ and $v_p(M_{r-1}(N_p))$, we analyze an augmenting path that has been used to update matchings. Our approach is inspired by an idea of Wästlund [40]. Specifically, we add a special vertex $\hat{\jmath}$ to I and edges from $\hat{\jmath}$ to all n_p vertices in N_p. We assume that these edge weights are drawn independently from the distribution over $(-\infty, 1]$ whose density function is given by $f(x) = \lambda e^{-\lambda(1-x)}$. This probability distribution is the reversed exponential distribution, which is a reflection of the exponential distribution over the line $x = 1/2$. Also, we add edges from $\hat{\jmath}$ to vertices in $N \setminus N_p$ whose weights are $-\infty$. As $\lambda \to 0$, the maximum weight matching in this bipartite graph converges to the maximum weight matching in the original bipartite graph since $f(x) = \lambda e^{-\lambda(1-x)} \to -\infty$ as $\lambda \to 0$ and any edges connected to $\hat{\jmath}$ does not participate in the maximum weight matching. Let $\hat{M}$ denote the matching which is produced by the round-robin algorithm when the input items are $I \cup \{\hat{\jmath}\}$. Let $\hat{M}_r$ denote the matching, computed by the algorithm, at the time of the end of the round r for the modified instance. Also, for the modified instance, let $\hat{I}_r^p$ denote the remaining items just before class p selects a new item in round r and let $\hat{\jmath}_r^p$ denote an item selected by class p in round r.

For the proof of Claim 1, we consider the probability that the added item $\hat{\jmath}$ is selected by class p until round r ends. We consider the following two steps.

In the first step, we show that, under the same conditions as in Lemma 3, the expected difference is equivalent to one minus the probability that class p has selected item $\hat{\jmath}$ until round r ends, namely, $\hat{\jmath}$ belongs to the bundle $\hat{M}_r(N_p)$. This is formalized as follows:

$$\mathbb{E}[v_p(M_r(N_p))] - \mathbb{E}[v_p(M_{r-1}(N_p))] = 1 - \frac{1}{r} \lim_{\lambda \to 0} \frac{1}{\lambda} \mathbf{Pr}[\hat{\jmath} \in \hat{M}_r(N_p)]. \tag{2}$$

The proof of the equation (1) above is similar to the proof given by Wästlund [40].

In the second step, we show the upper bound on the probability that $\hat{\jmath}$ is selected by class p until round r ends. That is, for $r = 1, 2, ..., n_p$,

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \mathbf{Pr}[\hat{\jmath} \in \hat{M}_r(N_p)] \leq \sum_{r'=1}^{r} \frac{1}{\alpha} \frac{1}{m_{r'}^p + 1}. \tag{3}$$

We have $m_{r'}^p + 1 = |\hat{I}_{r'}^p|$. To prove the inequality (2), we consider an augmenting path of the matching $\hat{M}_{r-1}$ on the modified bipartite graph between N_p and $\hat{M}_{r-1}(N_p) \cup \hat{I}_r^p$. More precisely, we condition $\hat{\jmath} \notin \hat{M}_{r-1}(N_p)$ and consider an augmenting path of the matching $\hat{M}_{r-1}$: $P = (i_r, ..., i_a, \hat{\jmath}_r^p)$ as Figure 1. By considering the conditioned distribution and the nesting lemma by Buck et al. [9], we obtain a bound on the probability that $\hat{\jmath}$ is the item selected by class p in round r (that is, $\hat{\jmath}_r^p = \hat{\jmath}$).

Finally, combining (1) and (2), we obtain an inequality about the increment of the expected valuation. Summing the inequalities from $r = 1$ to $r = n_p$, we establish Claim 1.

Upper bound on $v_p(M(N_q))$: Next, we examine the upper bound on the expected value $v_p(M(N_q))$ of class p for q's bundle under matching M. Note that unlike the additive setting, class p may envy class q even when class p selects an item before q in every round; see an example in Appendix C.1.

A natural way to obtain the upper bound is to estimate the expected value $\mathbb{E}[v_p(M(N_q))]$ directly. However, it is difficult to do this since the bundle $M(N_q)$ is determined after the round-robin algorithm terminates and the edge weights on the bipartite graph between N_p and $M(N_q)$ are conditioned. Instead, we analyze $v_p(I_u)$ where I_u is a bundle of size n_q, which is selected uniformly at random from I; recall that all edges weights between N_p and I_u are drawn independently from the distribution $\mathcal{D}$. In Appendix E.2, we prove $\mathbb{E}[v_p(M(N_q))] \leq \mathbb{E}[v_p(I_u)]$, which yields Claim 2 below.

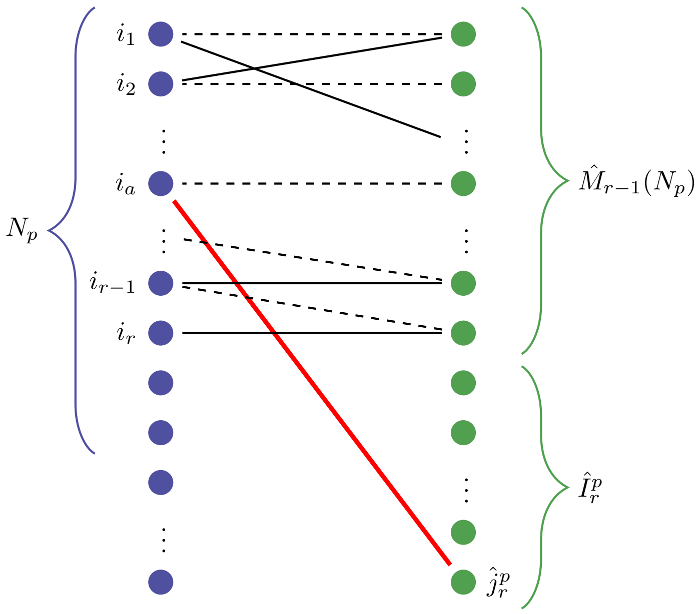

Figure 1: A new edge which is selected by an augmenting path.

Claim 2. *Under the same conditions in Lemma 3, we have*

$$\mathbb{E}[v_p(M(N_q))] \leq \min(n_p, n_q) - \sum_{r=1}^{\min(n_p,n_q)} \frac{\alpha}{\beta}(1-o(1))\frac{1}{r}\sum_{r'=1}^{r}\frac{1}{n_q - r' + 1}.$$

The formal proof of Claim 2 is given in Appendix E.2. The upper bound on $\mathbb{E}[v_p(I_u)]$ is calculated in a similar manner to the technique which we used to show the lower bound on $\mathbb{E}[v_p(M(N_p))]$. Finally, from Claims 1 and 2, we can prove Lemma 3 shown in Appendix E.3.

3.2 No light edges in maximum weight matchings

To prove Theorem 1, we need not only to investigate the expected envies among classes but also to show that the variance around the expected values of $v_p(N_p)$ and $v_p(N_q)$ is small. As defined in Section 3.1, we define I_u to be a bundle of size n_q, which is selected uniformly at random from I. Our goal in this section is to prove Lemma 4 described below. In (a) of Lemma 4, we claim that the weights of edges in the matching M^p are sufficiently heavy, which will be used in Section 3.3 to obtain a stochastic concentration for $v_p(M(N_p))$. In (b) and (c) of Lemma 4, we provide lower bounds on the weights of edges in the maximum weight matchings between N_p and $M(N_q)$, and between N_p and I_u.

Lemma 4 (No light edge lemma)**.** *Suppose that distributions are (α, β)-PDF-bounded. With high probability, we have the followings:*

(a) *No edge of weight at most $1 - c_p\frac{(\log n_p)^2}{n_p}$ appears in the matching M^p. Here $c_p > 0$ is a constant.*

(b) *No edge of weight at most $1 - c_q\frac{(\log \min(n_p,n_q))^2}{\min(n_p,n_q)}$ appears in a maximum weight matching in the bipartite subgraph of G induced by N_p and $M(N_q)$ with edge weights $u_i(j)$ for $i \in N_p$ and $j \in M(N_q)$. Here $c_q > 0$ are sufficiently large constant.*

(c) *No edge of weight at most* $1-c_u\frac{(\log\min(n_p,n_q))^2}{\min(n_p,n_q)}$ *appears in a maximum weight matching in the complete bipartite graph between* N_p *and* I_u *with edge weights drawn from* $\mathcal{D}$ *independently. Here* $c_u > 0$ *is a constant.*

To show Lemma 4, we employ an approach inspired by a technique for expanding bipartite graphs in the random assignment theory [13, 14, 37]. Especially, the process to obtain a stochastic concentration resembles the argument by Frieze and Johansson [14] and Talagrand [37]. Specifically, we consider a bipartite graph restricted to the "heavy" edges and analyze the edge weights of a maximum weight matching in the original bipartite graph by considering an alternating cycle of a maximum weight matching in the bipartite graph restricted to the heavy edges and bounding its length. Now, we define the notions of a θ-*expanding* bipartite graph and a *heavy* θ-*expanding* bipartite graph. The definition of an expanding bipartite graph is similar to the definition given by Talagrand [37].

Let us explain some additional notations which we use in this section. Consider a bipartite graph $H = (L \cup R, E_H)$ with left vertices L and right vertices R. For a bipartite graph H and $S \subseteq L$, we denote the neighborhood of S in H by $\Gamma_H(S) \subseteq R$, i.e., $\Gamma_H(S) = \{j \in R \mid \{i,j\} \in E_H \text{ for some } i \in S\}$.

Definition 4 (θ-expanding bipartite graph)**.** Consider a bipartite graph $H = (L \cup R, E_H)$ with left vertices L and right vertices R. Suppose that $|L| \geq |R|$. A bipartite graph H is θ-*expanding* for a constant $\theta > 1$ if

- for every $S \subseteq L$ such that $|S| \leq \frac{|R|}{2}$, $|\Gamma_H(S)| \geq \min\left(\theta|S|, \frac{|R|}{2}\right)$,
- for every $S \subseteq L$ such that $\frac{|R|}{2} < |S|$, $|R| - |\Gamma_H(S)| \leq \frac{1}{\theta}(|R| - |S|)$.

It is known that a θ-expanding bipartite graph admits an alternating cycle of bounded length [37] in the balanced case ($|L| = |R|$). Note that we define expanding bipartite graph even in the unbalanced case ($|L| > |R|$). This is described in Lemma 5 whose proof is presented in Appendix F.1. Here, we denote by O_a big O notation with respect to the parameter a.

Lemma 5. *Consider the complete bipartite graph* H^* *with left vertices* L *and right vertices* R *with* $|L| \geq |R| = a$ *and its saturated matching* M'*. Suppose that* $H = (L \cup R, E_H)$ *is a* θ*-expanding subgraph of* H^* *for some constant* $\theta > 1$*. Then, for any vertex* $i \in L$ *that participates in* M'*, there exists an alternating cycle* C *of length* $\ell = O_a(\log a)$ *in* H^* *such that* C *includes* i*, uses an edge in* H *when it goes from a left vertex to a right vertex, and uses an edge in* M' *when it goes from a right vertex to a left vertex.*

Next, we turn our attention to the edge weights in a bipartite graph. We denote the weight of edge $\{i,j\}$ by $w(i,j)$ here. Consider a bipartite graph H^* with left vertices L and right vertices R with $|L| \geq |R| = a$ and with edge weights w where $w(i,j) \in (0,1]$ for every edge $\{i,j\}$. We say that H^* is *heavy* θ-*expanding* if the bipartite subgraph $H = (L \cup R, E_H)$ where $\{i,j\} \in E_H$ if and only if $w(i,j) = 1 - O_a(a^{-1}\log a)$ is θ-expanding.

Then, from Lemma 5, we can obtain a bound on the weights of edges in a maximum weight matching of a heavy θ-expanding bipartite graph as the following lemma:

Lemma 6. *Let* H^* *be a heavy* θ*-expanding complete bipartite graph with left vertices* L *and right vertices* R *with* $|L| \geq |R| = a$ *and with edge weights* w *where* $w(i,j) \in (0,1]$ *for every edge* $\{i,j\}$*. Then, the weights of all edges in a maximum weight matching in* H^* *are* $1 - O_a\left(\frac{(\log a)^2}{a}\right)$.

From Lemma 6, to prove Lemma 4, it suffices to show that the bipartite graphs between N_p and $M(N_p)$, between N_p and $M(N_q)$, and between N_p and I_u are heavy 2-expanding with high probability. We provide the full proof of Lemma 4 in Appendix F.3.

3.3 Proof of Theorem 1

In this section, we show Theorem 1. We bound from above the probability that class p envies class q. From the results in Section 3.2, we can show the the stochastic concentration for the value of maximum weight matchings around its expectation from Talagrand's inequality. This technique is similar to the one in Frieze and Johansson [14].

First, we consider the stochastic concentration for the value of the maximum weight matching M^p. On the complete bipartite graph between agents in N_p and items in $M(N_p)$, there are $n_p!$ perfect matchings. Given $\omega \in \{0,1\}^{n_p^2}$, we denote by M_ω the set of edges between N_p and $M(N_p)$ where $\{i,j\} \in M_\omega$ if and only if $\omega_{i,j} = 1$. We define $\mathcal{F} = \{\omega \in \{0,1\}^{n_p^2} \mid M_\omega$ is a perfect matching between N_p and $M(N_p)\}$. Let a random variable $X_{i,j} = 1 - u_i(j)$ and thus we have

$$v_p(M(N_p)) = \max_{M \in \mathcal{M}(N_p, M(N_p))} \sum_{(i,j)\in M} u_i(j) = \max_{\omega\in\mathcal{F}} \sum_{(i,j)\in M_\omega} u_i(j) = n_p - \min_{\omega\in\mathcal{F}} \sum_{(i,j)\in M_\omega} X_{i,j}.$$

(Recall that $\mathcal{M}(N_p, M(N_p))$ corresponds to the set of perfect matchings between N_p and $M(N_p)$.) Let $\varepsilon > 0$ be a constant. Then, we obtain

$$\mathbf{Pr}\left[|v_p(M(N_p)) - \mathbb{E}[v_p(M(N_p))]| \geq \varepsilon\right] = \mathbf{Pr}\left[|\mathcal{X} - \mathbb{E}\left[\mathcal{X}\right]| \geq \varepsilon\right], \tag{4}$$

where $\mathcal{X} = \min_{\omega\in\mathcal{F}} \sum_{\{i,j\}\in M_\omega} X_{i,j}$. By (a) in Lemma 4, if $\{i,j\} \in M^p = \mathrm{argmax}_{M'\in\mathcal{M}(N_p,M(N_p))} \sum_{\{i,j\}\in M'} u_i(j)$, then $u_i(j) \geq 1 - c_p \cdot (\log n_p)^2/n_p$ with high probability. So, with high probability, we have $X_{i,j} = 1 - u_i(j) \leq c_p \frac{(\log n_p)^2}{n_p}$.

Let $Y_{i,j} = X_{i,j} \cdot \frac{n_p}{c_p(\log n_p)^2}$. Then we get $Y_{i,j} \leq 1$ with high probability. Let $\mathcal{F}'$ be the set of vectors obtained by multiplying the vectors in $\mathcal{F}$ with $c_p \frac{(\log n_p)^2}{n_p}$. Then, we get

$$\min_{\omega\in\mathcal{F}} \sum_{(i,j)\in M_\omega} X_{i,j} = \min_{\omega\in\mathcal{F}} \sum_{(i,j)\in M_\omega} c_p \frac{(\log n_p)^2}{n_p} Y_{i,j} = \min_{\omega'\in\mathcal{F}'} \sum_{(i,j)\in M_{\omega'}} Y_{i,j}$$

where $M_{\omega'}$ is a matching whose edges are given by $\{i,j\}$ with $\omega'_{ij} > 0$. So, we can set σ^2 in the Talagrand's inequality as $\sigma^2 = \max_{\omega'\in\mathcal{F}'} \|\omega'\|_2^2 = \max_{\omega'\in\mathcal{F}'} \sum_{i=1}^{n_p^2} \omega_i'^2 = \max_{\omega'\in\mathcal{F}} n_p \left(c_p \frac{(\log n_p)^2}{n_p}\right)^2 = c_p^2 \frac{(\log n_p)^4}{n_p}$. Note that $Y_{i,j}$ ($i \in N_p, j \in M(N_p)$) are not independent but conditional independent with conditioning $M(N_p)$, since the distributions of the weights of edges between N and M are all independent. Thus, from Lemma 1, we achieve

$$\mathbf{Pr}\left[|\mathcal{Y} - \mathbb{E}\left[\mathcal{Y}\right]| \geq \varepsilon \mid M(N_p)\right] \leq 4\exp\left(-\frac{\varepsilon^2 n_p}{4c_p^2(\log n_p)^4}\right),$$

where $\mathcal{Y} = \min_{\omega'\in\mathcal{F}'} \sum_{\{i,j\}\in M_{\omega'}} Y_{i,j}$. By taking the expected values of the left-hand and right-hand sides in terms of the set $M(N_p)$ of items matched to class p, we obtain

$$\mathbf{Pr}\left[|\mathcal{Y} - \mathbb{E}\left[\mathcal{Y}\right]| \geq \varepsilon\right] = \mathbb{E}_{M(N_p)}\left[\mathbf{Pr}\left[|\mathcal{Y} - \mathbb{E}\left[\mathcal{Y}\right]| \geq \varepsilon \mid M(N_p)\right]\right] \quad \leq 4\exp\left(-\frac{\varepsilon^2 n_p}{4c_p^2(\log n_p)^4}\right)$$

and, from (4), we have the following inequality:

$$\mathbf{Pr}\left[|v_p(M(N_p)) - \mathbb{E}[v_p(M(N_p))]| \geq \varepsilon\right] \leq 4\exp\left(-\frac{\varepsilon^2 n_p}{4c_p^2(\log n_p)^4}\right). \tag{5}$$

Second, we consider the stochastic concentration for the value of the maximum weight matching between N_p and $M(N_q)$. Given $\omega_q \in \{0,1\}^{n_p \cdot n_q}$, we denote by M_{ω_q} the set of edges between N_p and $M(N_q)$ where $\{i,j\} \in M_{\omega_q}$ if and only if ω_q takes value 1 at $\{i,j\}$. Similar to $\mathcal{F}$, we define $\mathcal{F}_q$ to be the set of vectors $\omega_q \in \{0,1\}^{n_p \cdot n_q}$ where M_{ω_q} is a saturated matching between N_p and $M(N_q)$. Then, similarly to the discussion on $v_p(M(N_p))$, we have

$$\mathbf{Pr}\left[|v_p(M(N_q)) - \mathbb{E}[v_p(M(N_q))]| \geq \varepsilon\right] = \mathbf{Pr}\left[|\mathcal{X}_q - \mathbb{E}\left[\mathcal{X}_q\right]| \geq \varepsilon\right],$$

where $\mathcal{X}_q = \min_{\omega_q \in \mathcal{F}_q} \sum_{\{i,j\} \in M_{\omega_q}} X_{i,j}$. Here, ε is a constant. From (b) in Lemma 4, for an edge $\{i,j\}$ which appears in the maximum weight matching between N_p and $M(N_q)$, we have $u_i(j) \geq 1 - c_q \frac{(\log \min(n_p,n_q))^2}{\min(n_p,n_q)}$ with high probability. By the same argument as above, we obtain the following inequality:

$$\mathbf{Pr}\left[|v_p(M(N_q)) - \mathbb{E}[v_p(M(N_q))]| \geq \varepsilon\right] \leq 4\exp\left(-\frac{\varepsilon^2 \min(n_p,n_q)}{4c_p^2(\log \min(n_p,n_q))^4}\right). \tag{6}$$

Finally, we bound the probability that class p envies class q. Let $D_{p,q} = n_p - \min(n_p,n_q) + \left(\frac{\alpha}{\beta} - \frac{1}{\alpha k}\right)\frac{\min(n_p,n_q)}{n_q} - o(1)$. Then, from Lemma 3, $\mathbb{E}[v_p(M(N_p))] - \mathbb{E}[v_p(M(N_q))] \geq D_{p,q}$. Also, if class p envies class q, then $v_p(M(N_p)) < \mathbb{E}[v_p(M(N_p))] - \frac{1}{2}D_{p,q}$, or $v_p(M(N_q)) > \mathbb{E}[v_p(M(N_q))] + \frac{1}{2}D_{p,q}$. Hence, from (5) and (6), we obtain

$$\begin{aligned}
&\mathbf{Pr}\left[v_p(M(N_p)) < v_p(M(N_q))\right] \\
&\leq \mathbf{Pr}\left[v_p(M(N_p)) < \mathbb{E}[v_p(M(N_p))] - \frac{1}{2}D_{p,q}\right] + \mathbf{Pr}\left[v_p(M(N_q)) > \mathbb{E}[v_p(M(N_q))] + \frac{1}{2}D_{p,q}\right] \\
&\leq 4\exp\left(-\frac{D_{p,q}^2 n_p}{16c_p(\log n_p)^4}\right) + 4\exp\left(-\frac{D_{p,q}^2 \min(n_p,n_q)}{16c_q(\log \min(n_p,n_q))^4}\right),
\end{aligned}$$

Here, we summarize the constant part using constants c'_p and c'_q. In the case of $n_p \leq n_q$, from $D_{p,q} = (\alpha/\beta - 1/\alpha k)n_p/n_q - o(1)$, we have

$$\mathbf{Pr}\left[\text{Class } p \text{ envies class } q\right] \leq 4\exp\left(-c'_p \frac{n_p^3}{n_q^2(\log n_p)^4}\right) + 4\exp\left(-c'_q \frac{n_p^3}{n_q^2(\log n_p)^4}\right).$$

From the condition (c) in Theorem 1, the following statement (c′) holds:

$$(\text{c}')\ \max(n_p,n_q) = n_q \leq c \cdot n_p^{3/2}(\log n_p)^{-5/2}$$

for some constant $c > 0$. See Appendix G.1 for the details. Then we have

$$\mathbf{Pr}\left[\text{Class } p \text{ envies class } q\right] \leq 8\exp\left(-\frac{\min(c'_p,c'_q)}{c^2}\log n_p\right)$$

and the probability that class p envies class q converges to 0 as $n_p \to \infty$. In the case of $n_p > n_q$, from $D_{p,q} = 1 - o(1)$, we have

$$\mathbf{Pr}\left[\text{Class } p \text{ envies class } q\right] \leq 4\exp\left(-c'_p \frac{n_p}{(\log n_p)^4}\right) + 4\exp\left(-c'_q \frac{n_q}{(\log n_q)^4}\right),$$

which approaches 0 as $n_p \to \infty$ and $n_q \to \infty$. From the condition (c) in Theorem 1, the following statement (d) holds (See Appendix G.1 for the details):

$$(\text{d})\ n_p \to \infty \text{ and } n_q \to \infty \text{ as } n \to \infty.$$

Therefore, we obtain that the probability that class p envies class q approaches 0 as $n \to \infty$ for all $p,q \in [k]$. Moreover, since k is just a constant, we can show that the probability that the matching M is not class envy-free approaches 0 as $n \to \infty$ using the union bound over ${}_kC_2$ pairs of classes. This completes the proof of Theorem 1.

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A Further Related Work

Fair division Our work is closely related to the growing literature on asymptotic fair division [3, 5, 6, 12, 19, 27, 28, 29, 36]. Dickerson et al. [12] initiated the study of asymptotic fair division. They showed that a welfare maximizing algorithm produces an envy-free allocation with high probability when the number of agents n and the number of items m satisfy $m = \Omega(n \log n)$. They assumed that valuations are additive and drawn from a distribution with positive variances. Following their work, Manurangsi and Suksompong [28] proved that, with high probability, an envy-free allocation exists among agents with additive valuations, assuming that utilities are drawn from a polynomial-bounded distribution when m is divisible by n. However, they also showed that there is no envy-free allocation with high probability when $m = O(n \log n / \log \log n)$ under a polynomial-bounded distribution. Moreover, Manurangsi and Suksompong [29] showed that under the assumption that distributions are PDF-bounded, an envy-free allocation exists with high probability when agents have additive valuations and $m = \Omega(n \log n / \log \log n)$. Here, a PDF-bounded distribution is polynomial-bounded, and a polynomial-bounded distribution with a positive variance. Several papers have studied the asymptotic existence of allocations that satisfy other fairness notions, such as proportionality [36] or a maximin share guarantee [3, 19].

Manurangsi and Suksompong [27] studied the asymptotic existence of an allocation that satisfies a group fairness criterion. The main difference between their setting and ours is that in their model, the agents within each group *share* the items allocated to that group, i.e., the utility of an agent is determined by the sum of the total utilities of the items allocated to their group. A group fairness notion requires that each agent within a group finds the bundle allocated to their group at least as valuable as a bundle allocated to every other group.

In the one-to-one house assignment problem, Gan et al. [15] studied the asymptotic existence of an envy-free assignment. They demonstrated that an envy-free assignment exists with high probability when the agents' preferences are drawn uniformly at random and $m = \Omega(n \log n)$. Furthermore, Manurangsi and Suksompong [29] showed that, if $m/n \geq \mathrm{e}+\varepsilon$ where e is Napier's constant and $\varepsilon > 0$ is any constant, an envy-free assignment can be found with high probability. Conversely, if $m/n \leq \mathrm{e} - \varepsilon$, there is no envy-free assignment with high probability. In the house assignment problem, the cardinal model is equivalent to the ordinal setting because agents must compare only individual items. However, in our setting, classes must compare bundles of items.

Our problem can be formulated as a fair division problem with *assignment* valuations, which may violate the additive assumption prevalent in literature. Specifically, each class is represented by a "meta" agent, and the meta agent's valuation for each set of items is determined by the maximum total weight of an optimal matching between the class members and the items in a given bipartite graph. An assignment valuation is also known as an OXS valuation which is an important subclass of gross substitutes valuations [22, 23] and submodular valuations. A *binary* assignment valuation, the underlying bipartite graph of which has binary edge weights, is a matroid rank function of a transversal matroid. An assignment valuation is a generalization of additive valuations. That is, given an arbitrary additive valuation v for m items, an equivalent assignment valuation can be created by representing each agent i with a class i with m copies of agents, with every copy having the same edge weight $v_i(j)$ towards each item j. However, the same construction cannot be used to represent the distributional model for additive valuations using assignment valuations. It is known that when agents have binary assignment valuations and more generally submodular valuations with dichotomous marginals, an allocation satisfying the approximate fairness notion of envy-freeness, known as EF1, and non-wastefulness exists and can be computed in polynomial time [4, 7, 8]. In fact, EF1 is compatible with a stronger efficiency notion of utilitarian-optimality in the binary assignment setting. However, it remains an open question whether the same compatibility between EF1 and non-wastefulness holds for general assignment valuations. For further discussion, see Section 5 of Benabbou et al. [8]. There are several group fairness notion other than assignment valuation [20, 27, 30, 34]. Almost all of these study the case in which each group has an additive valuation.

Compared with previous studies on asymptotic fair division that assume additive valuations, our setting is inherently non-additive, rendering previous results and techniques inapplicable.

Random assignment We briefly review the literature on the theory of random assignments [1, 2, 13, 14, 24, 32, 39, 41, 42]. The theory of random assignments deals with a bipartite graph with random edge weights, and the main focus has been on the analyzing the expected minimum total weight of a matching. More specifically, let $C_{n,m,r}$ be the minimum total weight of the matching with r edges in a bipartite graph with n and m vertices on each side when edge weights are independently assigned from the exponential distribution with rate 1. Here, we assume that $n \leq m$. When $n = m = r$, Walkup [39] showed that the expected value is bounded above when n goes to infinity. Following numerous papers on experimental results and improved bounds (see the introduction in [41] for details), Karp [18] improved the upper bound and showed that the expected value is smaller than 2 for any n. Aldous [1, 2] showed that $\mathbb{E}[C_{n,n,n}]$ converges to $\frac{\pi^2}{6}$ as n goes to infinity. For a more general combination of n, m, and r, Linusson and Wästlund [24] and Nair et al. [32] obtained a concrete formula for the expected minimum total weight of a matching given by $\mathbb{E}[C_{n,m,r}] = \sum_{i=1}^{r} \frac{1}{n} \sum_{j=0}^{i-1} \frac{1}{m-j}$. Wästlund [41] provided a concise and elegant proof for this result, by analyzing the expected difference between the minimum weight of matching with r edges and that with $r-1$ edges and showing that $\mathbb{E}[C_{n,m,r}] - \mathbb{E}[C_{n,m,r-1}] = \frac{1}{n} \sum_{j=0}^{r-1} \frac{1}{m-j}$. Frieze and Johansson [14] and Frieze [13] explored a similar approach for random bipartite graphs and non-bipartite graphs, and Wästlund [42] and Larsson [21] extended the result in [41] to more general distributions in some pseudo-dimension.

Throughout the paper, we show that the techniques from the random assignment theory can be usefully exported to our problem.

B Omitted Proofs from Section 2

B.1 Asymptotic existence of complete class envy-free matchings

In Section 2, we have mentioned that one can establish the asymptotic existence of a "complete" class envy-free matching, by using the result from the house allocation problem [29]. Here, a matching is defined to be *complete* if each agent is matched to exactly one item. The formal statement is as follows:

Proposition 1. *Suppose that each agent's utility is given by a distribution $\mathcal{D}$. Let $\varepsilon > 0$ be any constant. If $m \geq (\mathrm{e} + \varepsilon) \cdot \sum_{p \in [k]} n_p$, then there exists a complete class envy-free matching with high probability.*

Proof. We say that a matching M is *envy-free* if no agent envies the others, i.e., $u_i(j) \geq u_i(j')$ for every pair of agents $i, i' \in N$ where j (resp. j') is an item matched to i (resp. i') under M. The proof of Theorem 6.1 in [29] shows that, when $m \geq (\mathrm{e} + \varepsilon) \cdot n = (\mathrm{e} + \varepsilon) \cdot \sum_{p \in [k]} n_p$, Algorithm 5 in [29] computes a complete envy-free matching M with high probability. Observe that since M is envy-free, we have that for every pair $p, q \in [k]$

of classes and every matching $M' \in \mathcal{M}(N_p, M(N_q))$,

$$v_p(M(N_p)) = \sum_{\{i,j\} \in M^p} u_i(j) \geq \sum_{\{i,j\} \in M'} u_i(j).$$

Thus, $v_p(M(N_p)) \geq v_p(M(N_q))$ for every pair $p, q \in [k]$ of classes. This means that there exists a complete class envy-free matching with high probability. □

We note that Algorithm 5 in [29] may produce a wasteful matching that violates (a). Further, another algorithm (Algorithm 1 in [15]), which computes a complete envy-free matching with high probability in the context of the house allocation problem, may not satisfy the condition (a) in the non-wastefulness definition. We provide such an example below.

Consider four agents, two classes and five items in Table 1 below. Item j_1 is the most preferred by agents i_1 and i_3. So, both algorithms remove item j_1 to get an envy-free matching. Then these algorithms find a complete envy-free matching $M = \{(i_1, j_2), (i_2, j_5), (i_3, j_4), (i_4, j_3)\}$. Here item j_1 is wasted under this matching and therefore this matching does not satisfy (a).

Table 1: An example showing that both Algorithm 1 in [15] and Algorithm 5 in [29] return a class envy-free and wasteful matching.

		j_1	j_2	j_3	j_4	j_5
N_1	i_1	5	4	3	2	1
	i_2	1	2	3	4	5
N_2	i_3	5	2	3	4	1
	i_4	1	4	5	2	3

On the hand, in the same example as in Table 1, our round-robin algorithm (Algorithm 1 in Section 3) produces a class envy-free matching that is non-wasteful. In the first round in the algorithm, agent $i_1 \in N_1$ is matched to item j_1 and agent $i_4 \in N_2$ is matched to item j_3. In the second round, agent $i_2 \in N_1$ is matched to item j_5 and agent $i_3 \in N_2$ is matched to item j_4. Finally we get the matching $M = \{\{i_1, j_1\}, \{i_2, j_5\}, \{i_3, j_4\}, \{i_4, j_3\}\}$. Class 1's utility is $\sum_{\{i,j\} \in M^1} u_i(j) = u_{i_1}(j_1) + u_{i_2}(j_5) = 10$ and class 2's utility is $\sum_{\{i,j\} \in M^2} u_i(j) = u_{i_3}(j_4) + u_{i_4}(j_3) = 9$. This matching is class envy-free and non-wasteful.

C The Round-robin Algorithm for Deterministic Utilities

There are some interesting differences between the round-robin algorithm for the fair division problem with additive valuations and the round-robin algorithm for our problem. First, for the additive setting, an agent who chooses an item earlier does not envy another agent who picks up an item later. However, in our setting, the former class may envy the latter class under Algorithm 1. We show this through an example in Appendix C.1. Moreover, for deterministic utilities, the round-robin algorithm may not produce a class EF1 matching. We describe details in Appendix C.2 by using an example in Hosseini et al. [16] with binary utilities and another example with non-binary utilities.

C.1 An example where earlier agents envy later agents under Algorithm 1.

Consider an example in Table 2. Consider four agents i_1, i_2, i_3, i_4, two classes $N_1 = \{i_1, i_2\}$ and $N_2 = \{i_3, i_4\}$, and four items j_1, j_2, j_3, j_4. Each agent has the utilities as described in Table 2. We assume that class 1 selects an item before class 2. In the first round of Algorithm 1, class 1 chooses item j_1 and class 2 chooses item j_3. The matching at the end of the first round is $M_1 = \{\{i_1, j_1\}, \{i_3, j_3\}\}$. In the second round, class 1 selects item j_2 and class 2 selects item j_4. Thus, the final output of Algorithm 1 is the matching $M = \{\{i_1, j_1\}, \{i_2, j_2\}, \{i_3, j_3\}, \{i_4, j_4\}\}$. Observe that class 1 envies class 2 under M. Indeed, we have $v_1(M(N_1)) = u_{i_1}(j_1) + u_{i_2}(j_2) = 4 + 1 = 5$, while $v_1(M(N_2)) = u_{i_1}(j_4) + u_{i_2}(j_3) = 3 + 3 = 6 > 5 = v_1(M(N_1))$.

Table 2: An example which illustrates the former class may envy the latter class.

		j_1	j_2	j_3	j_4
N_1	i_1	4	0.1	0.1	3
	i_2	0.1	1	3	0.1
N_2	i_3	4	0.1	4	0.1
	i_4	0.1	3	0.1	1

C.2 An example where Algorithm 1 may not produce a CEF1 and non-wasteful matching.

In this section, we consider an approximate fairness notion, class envy-freeness up to one item [16]. A matching M is *class envy-free up to one item (CEF1)* if, for every pair of classes $p, q \in [k]$, we have $\sum_{\{i,j\} \in M} u_i(j) \geq v_p(M(N_q))$, or there exists an item $j \in M(N_q)$ such that $\sum_{\{i,j\} \in M} u_i(j) \geq v_p(M(N_q) \setminus \{j\})$.

We show that Algorithm 1 may not produce a CEF1 matching, by using Example 1 of Hosseini et al. [16] who showed that no online deterministic algorithm computes a CEF1 and non-wasteful matching.

Consider an instance as described in Table 3. In the first round of Algorithm 1, class 1 selects j_2 and matches it to $i_2 \in N_1$. Class 2 selects j_1 and matches it to $i_4 \in N_2$. In the second round, class 1 chooses j_3 and assigns it to i_3. At this point, no item can be allocated to class 2 because the last remaining item j_4 has zero marginal contribution to the class 2's current bundle. In the third round, class 1 selects j_4. The resulting matching is $M = \{\{i_1, j_4\}, \{i_2, j_2\}, \{i_3, j_3\}, \{i_4, j_1\}\}$, which violates CEF1 since class 2 envies class 1 by more than one item.

Table 3: An example where the round-robin algorithm may not produce a CEF1 and non-wasteful matching for deterministic and binary utilities.

		j_1	j_2	j_3	j_4
N_1	i_1	1	0	0	1
	i_2	0	1	0	0
	i_3	0	0	1	0
N_2	i_4	1	0	0	1
	i_5	0	1	0	0
	i_6	0	0	1	0

In the example in Table 3, the proof tactically uses the property that some agents have zero utilities, which prevents some class to receive a new item under Algorithm 1. We can show, even if all the utilities are positive, the round-robin algorithm may not produce a CEF1 matching.

To this end, consider an instance described in Table 4. There are three rounds. In the first round of the algorithm, class 1 selects an item j_1 and matches it to agent i_1 and class 2 selects item j_2 which is matched to agent i_4. In the second round, class 1 selects an item j_4 and matches it to agent i_2 and class 2 selects item j_3 which is matched to agent i_5. In the third round, class 1 selects an item j_5 and matches it to agent i_3 and class 2 selects item j_6 which is matched to agent i_6. Thus, we obtain a matching $M = \{\{i_1, j_1\}, \{i_2, j_4\}, \{i_3, j_5\}, \{i_4, j_2\}, \{i_5, j_3\}, \{i_6, j_6\}\}$. Note that in this example, the ordering of items to be picked is uniquely determined, i.e., there is no tie when each class selects an item. Observe that class 2 gets the value $v_2(M(N_2)) = v_2(\{j_2, j_3, j_6\}) = 5.1$ and envies class 1 as $v_2(M(N_1)) = v_2(\{j_1, j_4, j_5\}) = 13.1$. Moreover, we have $v_2(M(N_1)\setminus\{j_1\}) = 6 > v_2(M(N_2))$ and hence the matching M does not satisfy CEF1.

Table 4: An example where the round-robin algorithm may not produce a CEF1 and non-wasteful matching for deterministic utilities.

		j_1	j_2	j_3	j_4	j_5	j_6
N_1	i_1	10	0.1	0.1	0.1	0.1	0.1
	i_2	0.1	4	0.1	4	0.1	0.1
	i_3	0.1	0.1	3	0.1	1	0.1
N_2	i_4	10	4	0.1	0.1	3	0.1
	i_5	0.1	0.1	1	3	0.1	0.1
	i_6	0.1	0.1	0.1	0.1	0.1	0.1

Note that if we do not impose non-wastefulness, one can always construct a CEE1 matching, using the envy-graph algorithm by Lipton et al. [25]. As we explained in Section A, our problem can be considered as a fair division problem with "meta" agents having assignment valuations. Since the envy-cycle algorithm is guaranteed to produce a complete and EF1 allocation for monotone valuations, one can apply the algorithm to this fair division instance, create an EF1 allocation $(A_1, A_2, \ldots, A_k)$ of the items, and construct a matching by taking a union of maximum matchings between N_p and A_p in G for $p \in [k]$. However, the resulting matching may not satisfy non-wastefulness as pointed out in [7].

D Uniqueness of a Maximum Weight Matching

In Section 3, we mentioned that, in each round r, the maximum weight matching of size r in the bipartite graph between N_p and $M^p_{r-1}(N_p) \cup I^p_r$ is unique for all class p. Here, we show it for completeness from a similar proof of the isolation lemma as that given by Spencer [35]. See also Lemma 11.5 in [17].

For proof, we consider a general bipartite graph. Let $H = (L \cup R, E)$ be a bipartite graph of the set of vertices $L \cup R$ and the set of edges E. Here, we assume $|L| \geq |R|$. The weights of edges are drawn independently from a non-atomic distribution over $[0, 1]$. For an edge $e \in E$, we denote the weight of edge e by $w(e) \in [0, 1]$. Here, let $\mathcal{M}$ be the set of weighted matchings which cover R in H. For an edge e in E, we define $s(e)$ by

$$s(e) = \max_{M' \in \mathcal{M}: e \notin M'} w(M') - \max_{M' \in \mathcal{M}: e \in M'} w(M' \setminus \{e\}),$$

where $w(M')$ denotes the weight of the matching M', i.e., $w(M') = \sum_{e \in M'} w(e)$. This value of $s(e)$ does not depend on that of $w(e)$. So, since the distribution, which generates $w(e)$, is non-atomic, we have

$$\Pr[w(e) = s(e)] = 0,$$

and

$$\Pr[w(e) = s(e) \text{ for some } e \text{ in } E] \leq \sum_{e \in E} \Pr[w(e) = s(e)] = 0.$$

Let M'_1 and M'_2 be two maximum weight matchings which cover R in H. Consider an edge $e \in M'_2 \setminus M'_1$. Now, we have

$$w(M'_1) = \max_{M' \in \mathcal{M}: e \notin M'} w(M')$$

and

$$w(M'_2) - u_i(j) = \max_{M' \in \mathcal{M}: e \in M'} w(M' \setminus \{e\}).$$

So, if $w(M'_1) = w(M'_2)$, then $w(e) = s(e)$. Let the event that there exist two maximum weight matchings which cover R in H be $\mathfrak{E}$. Thus, we get

$$\Pr[\mathfrak{E}] \leq \Pr[w(e) = s(e) \text{ for some } e \text{ in } E] = 0.$$

Therefore, we show that the maximum weight matching which cover R in H is unique almost surely.

Similarly, we show, for all class p and for any round r, the maximum weight matching of size r in the bipartite graph between N_p and $M^p_{r-1}(N_p) \cup I^p_r$ is unique almost surely. Then, item j^p_r also uniquely determined almost surely.

E Omitted Proofs from Section 3.1

E.1 Proof of Claim 1

Claim 1. *Under the same conditions as in Lemma 3, we have*

$$\mathbb{E}[v_p(M(N_p))] \geq n_p - \frac{1}{\alpha} \sum_{r=1}^{n_p} \frac{1}{r} \sum_{r'=1}^{r} \frac{1}{m^p_{r'} + 1}.$$

Proof of Claim 1. We consider the following two steps. In the first step, we show the equation (1) in Section 3.1, which is written again below:

$$\mathbb{E}[v_p(M_r(N_p))] - \mathbb{E}[v_p(M_{r-1}(N_p))] = 1 - \frac{1}{r} \lim_{\lambda \to 0} \frac{1}{\lambda} \mathbf{Pr}[\hat{j} \in \hat{M}_r(N_p)]. \tag{1}$$

This describes that the expected difference can be written by the probability that the bundle $\hat{M}_r(N_p)$ includes $\hat{j}$. To show the equation (1), we consider the maximum weight matching version of Lemma 1.3 in Wästlund [41].

We denote the set of agents who join in the matching $\hat{M}_r$ by $\{i_1, i_2, ..., i_r\} \subseteq N_p$. Now we fix an agent $i \in N_p$ which is chosen uniformly at random from $\{i_1, i_2, ..., i_r\}$. We condition that $\hat{j}$ does not participate in $\hat{M}_{r-1}$. Then, we consider the probability that added item $\hat{j}$ is chosen by class p in round r. Note that the probability that any other class selects item $\hat{j}$ is very small, so we can ignore it.

Let us write $v_p(\hat{M}_r(N_p))$ by X and $v_p(\hat{M}_r(N_p))$ with conditioning $\{i, \hat{j}\} \in \hat{M}_r$ by Y_i. We have, as $\lambda \to 0$, X converges to $v_p(M_r(N_p))$ and $\mathbb{E}[Y_i]$ converges to $\mathbb{E}[v_p(M_r(N_p))]$. Now we proceed to show the equation (1). If the edges $\{i, \hat{j}\}$ participate in the maximum weight matching with r edges $\hat{M}_r$, then we have $u_i(\hat{j}) + Y_i > X$. This is because Y_i is the weight of the maximum weight matching with $r-1$ edges and $u_i(\hat{j}) + Y_i$ is the weight of the maximum weight matching with r edges, then $u_i(\hat{j}) + Y_i$ is larger than X. From this discussion, we get

$$\mathbf{Pr}[(i, \hat{j}) \in \hat{M}_r] \leq \mathbf{Pr}[u_i(\hat{j}) + Y_i > X].$$

Conversely, if $u_i(\hat{j}) > X - Y_i$ and no other edge from $\hat{j}$ other than i has weight larger than $X - Y_i$, then the edge $(i, \hat{j})$ must participate in the maximum weight matching $\hat{M}_r$. Let $\mathcal{E}$ denote an event that more than two edges adjacent to $\hat{j}$ has edge weight more than $X - Y_i$. So, we have

$$\mathbf{Pr}[\{u_i(\hat{j}) > X - Y_i\} \cap \mathcal{E}] \leq \mathbf{Pr}[(i, \hat{j}) \in \hat{M}_r].$$

Recall that we assumed that edge weights of $\{i, \hat{j}\}$ are drawn independently from the distribution over $(-\infty, 1]$ whose density function is $f(x) = \lambda e^{-\lambda(1-x)}$ for all $i \in N_p$. From this, the probability that the event $\mathcal{E}$ happens is smaller as λ is smaller as follows:

$$\mathbf{Pr}[\mathcal{E}] \leq n_p^2 \mathbb{E}_{X,Y_i}[(1 - \mathrm{e}^{-\lambda(1-X+Y_i)})^2] = o(\lambda).$$

So, the probability that $\mathcal{E}$ happens is $O(\lambda^2)$. From this, we get an equation:

$$\mathbf{Pr}[u_i(\hat{j}) > X - Y_i] = \mathbf{Pr}[(i, \hat{j}) \in \hat{M}_r] + o(\lambda). \tag{7}$$

Next we calculate the probability $\mathbf{Pr}[u_i(\hat{j}) + Y_i > X]$ by the expectations of X and Y_i. We denote the distribution function whose density function is $f(x) = \lambda e^{-\lambda(1-x)}$ by $F_{\mathcal{D}_{1-\mathrm{Exp}}}$. Here we have $F_{\mathcal{D}_{1-\mathrm{Exp}}}(x) = \mathrm{e}^{-\lambda(1-x)}$. With conditioning X and Y_i, we get

$$\begin{aligned}\mathbf{Pr}[u_i(\hat{j}) + Y_i > X] &= \mathbb{E}_{X,Y_i}\left[1 - F_{\mathcal{D}_{1-\mathrm{Exp}}}(X - Y_i)\right] \\ &= \mathbb{E}_{X,Y_i}\left[1 - \mathrm{e}^{-\lambda(1-X+Y_i)}\right].\end{aligned}$$

From this, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbf{Pr}[u_i(\hat{j}) + Y_i > X] = \mathbb{E}_{X,Y_i}[(1 - X + Y_i)\mathrm{e}^{-\lambda(1-X+Y_i)}]. \tag{8}$$

Moreover, since it holds that $\lim_{\lambda \to 0} \mathbb{E}[X] = \mathbb{E}\left[v_p(M_{r-1}(N_p))\right]$ and $\lim_{\lambda \to 0} \mathbb{E}[Y_i] = \mathbb{E}\left[v_p(M_{r-1}(N_p))\right]$, we consider the limit of the right hand side of the equation above as

$$\lim_{\lambda \to 0} \mathbb{E}_{X,Y_i}[(1 - X + Y_i)\mathrm{e}^{-\lambda(1-X+Y_i)}] = 1 - \mathbb{E}\left[v_p(M_{r-1}(N_p))\right] + \mathbb{E}\left[v_p(M_{r-1}(N_p))\right]. \tag{9}$$

Finally, from (7), (8) and (9), we get the equation (1) as follows:

$$
\begin{aligned}
&1 - \frac{1}{r}\lim_{\lambda\to 0}\frac{1}{\lambda}\mathbf{Pr}[\hat{\jmath} \in \hat{M}_r(N_p)] \\
&= 1 - \frac{1}{r}\lim_{\lambda\to 0}\frac{1}{\lambda}\sum_{i=i_1^*}^{i_r^*}\mathbf{Pr}\left[(i,\hat{\jmath}) \in \hat{M}_r\right] \\
&= 1 - \frac{1}{r}\lim_{\lambda\to 0}\frac{1}{\lambda}\sum_{i=i_1^*}^{i_r^*}\Big(\mathbf{Pr}[u_i(\hat{\jmath}) + Y_i > X] - o(\lambda)\Big) \\
&= 1 - \frac{1}{r}\sum_{i=i_1^*}^{i_r^*}\lim_{\lambda\to 0}\frac{1}{\lambda}\mathbf{Pr}[u_i(\hat{\jmath}) + Y_i > X] \\
&= \frac{1}{r}\sum_{i=i_1^*}^{i_r^*}\left(\mathbb{E}\left[v_p(M_r(N_p))\right] - \mathbb{E}\left[v_p(M_{r-1}(N_p))\right]\right) \\
&= \mathbb{E}\left[v_p(M_r(N_p))\right] - \mathbb{E}\left[v_p(M_{r-1}(N_p))\right].
\end{aligned}
$$

This completes the first step.

In the second step, we show the inequality (2) in Section 3.1. We rewrite the inequality, that is, for $r = 1, 2, ..., n_p$,

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\mathbf{Pr}[\hat{\jmath} \in \hat{M}_r(N_p)] \leq \sum_{r'=1}^{r}\frac{1}{\alpha}\frac{1}{m_{r'}^p + 1}. \tag{2}$$

From here, we prove this inequality. Our proof strategy to get this inequality is similar to the proof of Wästlund [40] but not same since we are now considering (α, β)-PDF-bounded distributions. We first consider the probability that the added item $\hat{\jmath}$ participates in the matching $\hat{M}_r$ with conditioning $\hat{\jmath}$ does not participate in the matching $\hat{M}_{r-1}$. This probability can be written by $\mathbf{Pr}[\hat{\jmath} \in \hat{M}_r(N_p) \mid \hat{\jmath} \notin \hat{M}_{r-1}(N_p)]$ and is equal to $\mathbf{Pr}[\hat{\jmath}_r^p = \hat{\jmath}]$. Note that any other class q does not select the item $\hat{\jmath}$ since $u_i(\hat{\jmath}) = -\infty$ for all $i \in N_q$ and all $q \in [k] \setminus \{p\}$. In round r, class p selects an item from $\hat{I}_r^p$, whose size is $|\hat{I}_r^p| = |I_r^p| + 1 = m_r^p + 1$.

Assume that $\hat{\jmath}$ does not participate in the matching $\hat{M}_{r-1}$ and $(i_1, ..., i_{r-1})$ are agents in N_p that participate in $\hat{M}_{r-1}$. In round r, class p selects an item through making the maximum weight matching with r edges between N_p and $\hat{M}_{r-1}(N_p) \cup \hat{I}_r^p$. Note that $(i_1, ..., i_{r-1})$ are also included in $\hat{M}_r$ since the uniqueness of the maximum weight matching. This fact is called the nesting lemma which is proved in Lemma 3 in Buck et al. [9]. Then, class p finds some augmenting path. Here, for the augmenting path for the matching $\hat{M}_{r-1}$, we write the new vertex in left side as i_r and the new vertex in right side is $\hat{\jmath}_r^p$ by its definition. Let i_a be the vertex in $\{i_1, ..., i_{r-1}\}$ that is connected to $\hat{\jmath}$ in the augmenting path. Then we can write the augmenting path by $P = (i_r, ..., i_a, \hat{\jmath}_r^p)$. See Figure 1 in Section 3.1.

Now, we consider the probability that $\hat{\jmath}_r^p$ just equals to $\hat{\jmath}$. We consider conditioning on (i) i_a, (ii) the weights of all edges other than the edges $\{i_a, j\}$ $(j \in \hat{I}_r^p)$ and (iii) $t_r = \max\{u_{i_a}(j) \mid j \in \hat{I}_r^p\}$. From Lemmas 2 and 5 in Buck et al. [9], with these conditions, $\hat{M}_{r-1}^p$ is fixed and so is the path $(i_r, ..., i_a)$. Then, the event that $\hat{\jmath}_r^p$ just equals to $\hat{\jmath}$ is equivalent to the event that i_a chooses $\hat{\jmath}$ among $\hat{I}_r^p$, which also means that $u_{i_a}(\hat{\jmath})$ is a maximizer of $\{u_{i_a}(j) \mid j \in \hat{I}_r^p\}$. Here $u_{i_a}(j)$ $(j \in \hat{I}_r^p)$ are given from some conditioned distributions. Note that other class does not affect this probability since they are independent. Then, we consider the probability of $u_{i_a}(\hat{\jmath}) = \max\{u_{i_a}(j) \mid j \in \hat{I}_r^p\}$. Here, for each $j \in \hat{I}_r^p \setminus \{\hat{\jmath}\}$, $u_{i_a}(j)$ is drawn from the conditioned $\mathcal{D}$ independently and $u_{i_a}(\hat{\jmath})$ is drawn from the reversed exponential distribution. Note that, when $u_{i_a}(\hat{j}) < 0$, the probability that $\hat{j}$ are selected in round r is zero. Finally, we have

$$
\begin{aligned}
&\mathbf{Pr}\left[\hat{j} \in \hat{M}_r^p(N_p) \mid \hat{j} \notin \hat{M}_{r-1}^p(N_p) \text{ and } \max\{u_{i_a}(j) \mid j \in \hat{I}_r^p\} = t_r\right] \\
&= \mathbf{Pr}\left[u_{i_a}(\hat{j}) = \max\{u_{i_a}(j) \mid j \in \hat{I}_r^p\} \mid \max\{u_{i_a}(j) \mid j \in \hat{I}_r^p\} = t_r\right] \\
&= \frac{\mathbf{Pr}\left[u_{i_a}(\hat{j}) = \max\{u_{i_a}(j) \mid j \in \hat{I}_r^p\} \bigwedge \max\{u_{i_a}(j) \mid j \in \hat{I}_r^p\} = t_r\right]}{\mathbf{Pr}\left[\max\{u_{i_a}(j) \mid j \in \hat{I}_r^p\} = t_r\right]} \\
&= \frac{\int_0^{t_r} \lambda \mathrm{e}^{-\lambda(1-x)} \prod_{j \in \hat{I}_r^p \setminus \{\hat{j}\}} \mathbf{Pr}[u_{i_a}(j) \le x] \, \mathrm{d}x}{\int_{-\infty}^{t_r} \lambda \mathrm{e}^{-\lambda(1-x)} \mathrm{d}x \cdot \prod_{j \in \hat{I}_r^p \setminus \{\hat{j}\}} \mathbf{Pr}[u_{i_a}(j) \le t_r]} \\
&= \frac{\int_0^{t_r} \lambda \mathrm{e}^{-\lambda(1-x)} F_{\mathcal{D}}(x)^{m_r^p} \mathrm{d}x}{\mathrm{e}^{-\lambda(1-t_r)} \cdot F_{\mathcal{D}}(t_r)^{m_r^p}} \\
&= \frac{\int_0^{t_r} \lambda \mathrm{e}^{-\lambda(1-x)} F_{\mathcal{D}}(x)^{m_r^p} \mathrm{d}x}{\mathrm{e}^{-\lambda(1-t_r)} \cdot F_{\mathcal{D}}(t_r)^{m_r^p}} \\
&= \lambda \mathrm{e}^{\lambda(1-t_r)} \int_0^{t_r} \mathrm{e}^{-\lambda(1-x)} \left(\frac{F_{\mathcal{D}}(x)}{F_{\mathcal{D}}(t_r)}\right)^{m_r^p} \mathrm{d}x.
\end{aligned}
$$

From this, we can get

$$
\begin{aligned}
&\mathbf{Pr}\left[\hat{j} \in \hat{M}_r^p(N_p) \mid \max\{u_{i_a}(j) \mid j \in \hat{I}_r^p\} = t_r\right] \\
&= 1 - \mathbf{Pr}\left[\hat{j} \notin \hat{M}_r^p(N_p) \mid \max\{u_{i_a}(j) \mid j \in \hat{I}_r^p\} = t_r\right] \\
&= 1 - \prod_{r'=1}^{r} \mathbf{Pr}\left[\hat{j} \notin \hat{M}_{r'}^p(N_p) \mid \hat{j} \notin \hat{M}_{r'-1}^p(N_p) \text{ and } \max\{u_{i_a}(j) \mid j \in \hat{I}_r^p\} = t_{r'}\right] \\
&= 1 - \prod_{r'=1}^{r} \left(1 - \mathbf{Pr}\left[\hat{j} \in \hat{M}_{r'}^p(N_p) \mid \hat{j} \notin \hat{M}_{r'-1}^p(N_p) \text{ and } \max\{u_{i_a}(j) \mid j \in \hat{I}_r^p\} = t_{r'}\right]\right) \\
&= 1 - \prod_{r'=1}^{r} \left(1 - \lambda \mathrm{e}^{\lambda(1-t_{r'})} \int_0^{t_{r'}} \mathrm{e}^{-\lambda(1-x)} \left(\frac{F_{\mathcal{D}}(x)}{F_{\mathcal{D}}(t_{r'})}\right)^{m_{r'}^p} \mathrm{d}x\right) \\
&= \sum_{r'=1}^{r} \lambda \mathrm{e}^{\lambda(1-t_{r'})} \int_0^{t_{r'}} \mathrm{e}^{-\lambda(1-x)} \left(\frac{F_{\mathcal{D}}(x)}{F_{\mathcal{D}}(t_{r'})}\right)^{m_{r'}^p} \mathrm{d}x + o(\lambda).
\end{aligned}
$$

Moreover, from uniform convergence, we can get

$$
\begin{aligned}
&\lim_{\lambda \to 0} \frac{1}{\lambda} \mathbf{Pr}\left[\hat{j} \in \hat{M}_r^p(N_p) \mid \max\{u_{i_a}(j) \mid j \in \hat{I}_r^p\} = t_r\right] \\
&= \lim_{\lambda \to 0} \sum_{r'=1}^{r} \mathrm{e}^{\lambda(1-t_{r'})} \int_0^{t_{r'}} \mathrm{e}^{-\lambda(1-x)} \left(\frac{F_{\mathcal{D}}(x)}{F_{\mathcal{D}}(t_{r'})}\right)^{m_{r'}^p} \mathrm{d}x \\
&= \sum_{r'=1}^{r} \int_0^{t_{r'}} \left(\frac{F_{\mathcal{D}}(x)}{F_{\mathcal{D}}(t_{r'})}\right)^{m_{r'}^p} \mathrm{d}x.
\end{aligned}
$$

Let $y = \frac{F_{\mathcal{D}}(x)}{F_{\mathcal{D}}(t_{r'})}$. Here $\mathrm{d}y = \frac{F'_{\mathcal{D}}(x)}{F_{\mathcal{D}}(t_{r'})}\mathrm{d}x$, where $F'_{\mathcal{D}}$ denotes the $F_{\mathcal{D}}$'s derivative. Since $\mathcal{D}$ is (α, β)-PDF-bounded, we have $F'_{\mathcal{D}}(x) = f_{\mathcal{D}}(x) \ge \alpha$. Also we have $F_{\mathcal{D}}(x) \le 1$ for all

$0 \le x \le 1$. Then we can get

$$\begin{aligned}\int_0^{t_{r'}} \left(\frac{F_{\mathcal{D}}(x)}{F_{\mathcal{D}}(t_{r'})}\right)^{m^p_{r'}} \mathrm{d}x &= \int_0^1 y^{m^p_{r'}} \cdot \frac{F_{\mathcal{D}}(t_{r'})}{F'_{\mathcal{D}}(x)} \mathrm{d}y \\ &\le \frac{1}{\alpha}\int_0^1 y^{m^p_{r'}} \mathrm{d}y \\ &= \frac{1}{\alpha} \cdot \frac{1}{m^p_{r'}+1}.\end{aligned}$$

Therefore, we can remove the conditioning about t_r and then

$$\lim_{\lambda \to 0} \frac{1}{\lambda}\mathbf{Pr}[\hat{\jmath} \in \hat{M}^p_r(N_p)] \le \sum_{r'=1}^{r} \frac{1}{\alpha} \cdot \frac{1}{m^p_{r'}+1}.$$

This completes the second step.

Finally, from the first step and the second step, we can show Claim 1 as follows:

$$\begin{aligned}\mathbb{E}[v_p(M(N_p))] &= \sum_{r=1}^{n_p} \left(\mathbb{E}[v_p(M_r(N_p))] - \mathbb{E}[v_p(M_{r-1}(N_p))]\right) \\ &= n_p - \sum_{r=1}^{n_p} \frac{1}{r} \lim_{\lambda \to 0} \frac{1}{\lambda}\mathbf{Pr}[\hat{\jmath} \in \hat{M}_r(N_p)] \\ &\ge n_p - \sum_{r=1}^{n_p} \frac{1}{r} \sum_{r'=1}^{r} \frac{1}{\alpha}\frac{1}{m^p_{r'}+1}.\end{aligned}$$

□

E.2 Proof of Claim 2

Claim 2. *Under the same conditions in Lemma 3, we have*

$$\mathbb{E}[v_p(M(N_q))] \le \min(n_p, n_q) - \sum_{r=1}^{\min(n_p,n_q)} \frac{\alpha}{\beta}(1 - o(1)) \frac{1}{r} \sum_{r'=1}^{r} \frac{1}{n_q - r' + 1}.$$

Proof of Claim 2. As defined in Section 3.2, we define I_u to be a bundle whose size is n_q which is selected uniformly at random from I. Here we analyze $v_p(I_u)$ instead of the expected value of $v_p(M(N_q))$. All edges weights between N_p and I_u are drawn independently from the distribution $\mathcal{D}$. On the other hand, since items in $M(N_q)$ are the ones which were not selected by class p, all edges weights between N_p and $M(N_q)$ are drawn independently from the distribution $\mathcal{D}$ conditioned from above. So, we have $\mathbb{E}[v_p(M(N_q))] \le \mathbb{E}[v_p(I_u)]$ for all class p.

Here the size of the maximum weight matching between N_p and I_u is $\min(n_p, n_q)$. For $r = 1, 2, ..., \min(n_p, n_q)$, let $I_{u,r}$ be the set of items which participate in the maximum weight matching with size r on the bipartite graph between N_p and I_u. Let $M^u_r(N_p, I_u)$ denote the maximum weight matching with r edges on the complete bipartite graph between N_p and I_u with edges randomly weighted from the distribution $\mathcal{D}$.

Consider adding a new item $\hat{\jmath}^q$ to I_u and edges from $\hat{\jmath}^q$ to all agents in N_p with randomly weights from the distribution whose density function is $f(x) = \lambda e^{-\lambda(1-x)}$ ($x \in (-\infty, 1]$). This is same process as the proof of Claim 1. Let $\hat{M}^u_r(N_p, I_u)$ denote the maximum weight matching with r edges on the complete bipartite graph between N_p and $I_u \cup \{\hat{\jmath}^q\}$.

From the same discussion as the proof of (1), we have

$$\mathbb{E}[v_p(I_{u,r})] - \mathbb{E}[v_p(I_{u,r-1})] = 1 - \frac{1}{r}\lim_{\lambda \to 0}\frac{1}{\lambda}\mathbf{Pr}[\hat{j}^q \in \hat{M}^u_r(N_p, I_u)]. \tag{10}$$

Now we consider the augmenting path algorithm to find a unique maximum weight matching. In each round $r = 1, 2, ..., \min(n_p, n_q)$, class p finds an augmenting path and updates a matching between N_p and $\hat{I}_u$ by selecting a new item from the remaining items. In round r, let denote the remaining items by $\hat{I}_{u,r}$. Here $|\hat{I}_{u,r}| = n_q - r + 1$. Also, we denote by $\hat{j}^q_r$ an item which is selected by class p in round r when we consider $I_u \cup \{\hat{j}^q\}$ instead of I_u.

Then, as like we did in the proof of Claim 1, we consider the augmenting path by $P = (i^q_r, ..., i^q_a, \hat{j}^q_r)$. We condition on (i) i^q_a, (ii) the weights of all edges other than the edges $\{i^q_a, j\}$ $(j \in \hat{I}_{u,r})$ and (iii) $t^q_r = \max\{u_{i^q_a}(j) \mid j \in \hat{I}_{u,r}\}$. Then, we have

$$\begin{aligned}
&\mathbf{Pr}\left[\hat{j}^q \in \hat{M}^u_r(N_p, I_u) \;\middle|\; \max\{u_{i^q_a}(j) \mid j \in \hat{I}_{u,r}\} = t^q_r\right] \\
&= \sum_{r'=1}^{r} \lambda \mathrm{e}^{\lambda(1-t^q_r)} \int_0^{t^q_{r'}} \mathrm{e}^{-\lambda(1-x)} \left(\frac{F_{\mathcal{D}}(x)}{F_{\mathcal{D}}(t^p_{r'})}\right)^{n_q - r'} \mathrm{d}x + o(\lambda).
\end{aligned}$$

From this,

$$\begin{aligned}
\mathbf{Pr}\left[\hat{j}^q \in \hat{M}^u_r(N_p, I_u)\right] &= \mathbb{E}_{t^q_r}\left[\mathbf{Pr}\left[\hat{j}^q \in \hat{M}^u_r(N_p, I_u) \;\middle|\; \max\{u_{i^q_a}(j) \mid j \in \hat{I}_{u,r}\} = t^q_r\right]\right] \\
&= \mathbb{E}_{t^q_r}\left[\sum_{r'=1}^{r} \lambda \mathrm{e}^{\lambda(1-t^q_r)} \int_0^{t^q_{r'}} \mathrm{e}^{-\lambda(1-x)} \left(\frac{F_{\mathcal{D}}(x)}{F_{\mathcal{D}}(t^p_{r'})}\right)^{n_q - r'} \mathrm{d}x\right] + o(\lambda).
\end{aligned}$$

So, we get

$$\begin{aligned}
\frac{1}{r}\lim_{\lambda \to 0}\frac{1}{\lambda}\mathbf{Pr}\left[\hat{j}^q \in \hat{M}^u_r(N_p, I_u)\right] &= \mathbb{E}_{t^q_r}\left[\frac{1}{r}\sum_{r'=1}^{r} \int_0^{t^q_{r'}} \left(\frac{F_{\mathcal{D}}(x)}{F_{\mathcal{D}}(t^p_{r'})}\right)^{n_q - r'} \mathrm{d}x\right] \\
&= \mathbb{E}_{t^q_r}\left[\frac{1}{r}\sum_{r'=1}^{r} \int_0^{1} \frac{F_{\mathcal{D}}(t^q_{r'})}{F'_{\mathcal{D}}(x)} y^{n_q - r'} \mathrm{d}y\right] \\
&\geq \mathbb{E}_{t^q_r}\left[\frac{1}{r}\sum_{r'=1}^{r} \frac{F_{\mathcal{D}}(t^q_{r'})}{\beta} \cdot \frac{1}{n_q - r' + 1}\right] \\
&\geq \mathbb{E}_{t^q_r}\left[\frac{\alpha}{\beta} \cdot \frac{1}{r}\sum_{r'=1}^{r} \frac{t^q_{r'}}{n_q - r' + 1}\right] \\
&= \frac{\alpha}{\beta} \cdot \frac{1}{r}\sum_{r'=1}^{r} \frac{\mathbb{E}_{t^q_r}[t^q_{r'}]}{n_q - r' + 1}.
\end{aligned}$$

Here we use $F_{\mathcal{D}}(x) = \int_0^x f_{\mathcal{D}}(x)\mathrm{d}x \geq \int_0^x \alpha \mathrm{d}x \geq \alpha x$ and $1/F'_{\mathcal{D}}(x) \geq 1/\beta$ for all $0 \leq x \leq 1$.

From (c) in Lemma 4 described in Section 3.2 and shown in Appendix F.3, no edge of weight at most $1 - c_u \frac{(\log \min(n_p, n_q))^2}{\min(n_p, n_q)}$ appears in the maximum weight perfect matching with high probability. From this and Lemma 8 in Frieze [13], we can show that no edge of weight at most $1 - c_u \frac{(\log \min(n_p, n_q))^2}{\min(n_p, n_q)}$ appears in the maximum weight matching with r edges for all $r = 1, 2, ..., \min(n_p, n_q)$ with high probability. In each round $r + 1$, the edge $\{i^q_q, \hat{j}^q_r\}$ is employed in the maximum weight matching with $r + 1$ edges between N_p and $I_u \cup \{\hat{j}^q\}$.

If $\hat{j}^q \neq \hat{j}$, we can say that the edge $\{i^q_q, j^q_r\}$ is employed in the maximum weight matching with $r+1$ edges between N_p and I_u. Therefore, we have, with high probability, for all $r = 1, 2, ..., \min(n_p, n_q)$, there exists a constant $c_u > 0$ such that

$$t^q_r \geq 1 - c_u \frac{(\log \min(n_p, n_q))^2}{\min(n_p, n_q)}.$$

From Markov's inequality, we have, for all $r = 1, 2, ..., \min(n_p, n_q)$,

$$\begin{aligned}
\mathbb{E}_{t^q_r}[t^q_{r'}] &\geq \left(1 - c_u \frac{(\log \min(n_p, n_q))^2}{\min(n_p, n_q)}\right) \cdot \mathbf{Pr}\left[t^q_{r'} \geq 1 - c_u \frac{(\log \min(n_p, n_q))^2}{\min(n_p, n_q)}\right] \\
&\geq \left(1 - c_u \frac{(\log \min(n_p, n_q))^2}{\min(n_p, n_q)}\right) \cdot (1 - o(1)) \\
&= 1 - o(1).
\end{aligned}$$

Here, we use the condition (c). That is, since there exists a constant $C > 0$ such that $n^{2/3} \leq C \cdot \min_{p'}(n_{p'}) \leq C \cdot \min(n_p, n_q)$ if we ignore the logarithm factor in the condition (c), $1 - c_u \frac{(\log \min(n_p, n_q))^2}{\min(n_p, n_q)} = 1 - o(1)$. Then, we have

$$\frac{1}{r} \lim_{\lambda \to 0} \frac{1}{\lambda} \mathbf{Pr}\left[\hat{j}^q \in \hat{M}^u_r(N_p, I_u)\right] \geq \frac{\alpha}{\beta}(1 - o(1)) \frac{1}{r} \sum_{r'=1}^{r} \frac{1}{n_q - r' + 1}.$$

From this and (10), we get

$$\begin{aligned}
\mathbb{E}[v_p(M(N_q))] &\leq \mathbb{E}[v_p(I_u)] \\
&= \sum_{r=1}^{\min(n_p, n_q)} \mathbb{E}\left[v_p(I_{u,r}) - v_p(I_{u,r-1})\right] \\
&= \min(n_p, n_q) - \sum_{r=1}^{\min(n_p, n_q)} \frac{1}{r} \lim_{\lambda \to 0} \frac{1}{\lambda} \mathbf{Pr}\left[\hat{j}^q \in \hat{M}^u_r(N_p, I_u)\right] \\
&\leq \min(n_p, n_q) - \sum_{r=1}^{\min(n_p, n_q)} \frac{\alpha}{\beta}(1 - o(1)) \frac{1}{r} \sum_{r'=1}^{r} \frac{1}{n_q - r' + 1}.
\end{aligned}$$

□

E.3 Proof of Lemma 3

Lemma 3. *Suppose that $\mathcal{D}$ is (α, β)-PDF-bounded and $k \cdot \max_{p \in [k]}(n_p + 1) \leq m$. Then we have*

$$\mathbb{E}[v_p(M(N_p))] - \mathbb{E}[v_p(M(N_q))] \geq n_p - \min(n_p, n_q) + \left(\frac{\alpha}{\beta} - \frac{1}{\alpha k}\right) \frac{\min(n_p, n_q)}{n_q} - o(1).$$

Proof of Lemma 3. First, we consider the lower bound on the number of remaining items when class p selects a new item in round r; m^p_r. We have $m^p_r = |I^p_r| \geq m - (r-1) \cdot k - (p-1)$ since the number of items already taken when class p chooses an item in round r is at most $(r-1) \cdot k + (p-1)$. From the assumption $k \cdot (n_q + 1) \leq m$ for all class p, we get

$$\begin{aligned}
m^p_{r'} + 1 &\geq m - (r' - 1) \cdot k - (p - 1) + 1 \\
&\geq k \cdot (n_q + 1) - (r' - 1) \cdot k - p + 2 \\
&\geq (n_q - r' + 1)k.
\end{aligned}$$

Then, from inequalities in Claims 1 and 2, we have,

$$\begin{aligned}
&\mathbb{E}[v_p(M(N_p))] - \mathbb{E}[v_p(M(N_q))] \\
&\geq n_p - \frac{1}{\alpha}\sum_{r=1}^{n_p}\frac{1}{r}\sum_{r'=1}^{r}\frac{1}{m_{r'}^{p}+1} - \min(n_p, n_q) + \sum_{r=1}^{\min(n_p,n_q)}\frac{\alpha}{\beta}(1-o(1))\frac{1}{r}\sum_{r'=1}^{r}\frac{1}{n_q - r' + 1} \\
&\geq n_p - \min(n_p, n_q) + \frac{\alpha}{\beta}\sum_{r=1}^{\min(n_p,n_q)}\frac{1}{r}\sum_{r'=1}^{r}\frac{1}{n_q - r' + 1} - \frac{1}{\alpha}\sum_{r=1}^{n_p}\frac{1}{r}\sum_{r'=1}^{r}\frac{1}{m_{r'}^{p}+1} - o(1) \\
&\geq n_p - \min(n_p, n_q) + \frac{\alpha}{\beta}\sum_{r=1}^{\min(n_p,n_q)}\frac{1}{r}\sum_{r'=1}^{r}\frac{1}{n_q - r' + 1} - \frac{1}{\alpha k}\sum_{r=1}^{n_p}\frac{1}{r}\sum_{r'=1}^{r}\frac{1}{n_q - r' + 1} - o(1).
\end{aligned}$$

In the case of $n_p \leq n_q$, we have

$$\begin{aligned}
\mathbb{E}[v_p(M(N_p))] - \mathbb{E}[v_p(M(N_q))] &= \left(\frac{\alpha}{\beta} - \frac{1}{\alpha k}\right)\sum_{r=1}^{n_p}\frac{1}{r}\sum_{r'=1}^{r}\frac{1}{n_q - r' + 1} - o(1) \\
&\geq \left(\frac{\alpha}{\beta} - \frac{1}{\alpha k}\right)\sum_{r=1}^{n_p}\frac{1}{r}\sum_{r'=1}^{r}\frac{1}{n_q} - o(1) \\
&= \left(\frac{\alpha}{\beta} - \frac{1}{\alpha k}\right)\frac{n_p}{n_q} - o(1).
\end{aligned}$$

In the case of $n_p > n_q$, we have

$$\begin{aligned}
&\mathbb{E}[v_p(M(N_p))] - \mathbb{E}[v_p(M(N_q))] \\
&= n_p - n_q + \left(\frac{\alpha}{\beta} - \frac{1}{\alpha k}\right)\sum_{r=1}^{n_q}\frac{1}{r}\sum_{r'=1}^{r}\frac{1}{n_q - r' + 1} - \frac{1}{\alpha k}\sum_{r=n_q}^{n_p}\frac{1}{r}\sum_{r'=1}^{r}\frac{1}{n_q - r' + 1} - o(1) \\
&\geq n_p - n_q + \left(\frac{\alpha}{\beta} - \frac{1}{\alpha k}\right)\sum_{r=1}^{n_p}\frac{1}{r}\sum_{r'=1}^{r}\frac{1}{n_q} - o(1) \\
&\geq n_p - n_q + \frac{\alpha}{\beta} - \frac{1}{\alpha k} - o(1).
\end{aligned}$$

The extra parts are summarized in $o(1)$.

Finally, we can get

$$\mathbb{E}[v_p(M(N_p))] - \mathbb{E}[v_p(M(N_q))] \geq n_p - \min(n_p, n_q) + \left(\frac{\alpha}{\beta} - \frac{1}{\alpha k}\right)\frac{\min(n_p, n_q)}{n_q} - o(1).$$

□

F Omitted Proofs from Section 3.2

F.1 Proof of Lemma 5

Lemma 5. *Consider the complete bipartite graph H^* with left vertices L and right vertices R with $|L| \geq |R| = a$ and its saturated matching M'. Suppose that $H = (L \cup R, E_H)$ is a θ-expanding subgraph of H^* for some constant $\theta > 1$. Then, for any vertex $i \in L$ that participates in M', there exists an alternating cycle C of length $\ell = O_a(\log a)$ in H^* such that C includes i, uses an edge in H when it goes from a left vertex to a right vertex, and uses an edge in M' when it goes from a right vertex to a left vertex.*

Proof of Lemma 5. Fix $i \in L$. We define a sequence of sets $(S_t)_{t\geq 0}$ as follows. First, we set $S_0 = \{i\}$. Recall that $\Gamma_H(S_0) \subseteq R$ is the neighborhood of S_0 in H. Let S_1 be the set of vertices in L which are matched to some vertex in $\Gamma_H(S_0)$ under matching M'. Here, we have $|S_1| = |\Gamma_H(S_0)|$ since M' is a saturated matching in H^* and $|L| \geq |R|$, implying that all vertices in R appear in matching M'. Similarly, for $t \geq 2$, we let $S_t \subseteq L$ be the set of vertices which are matched to some vertex in $\Gamma_H(S_{t-1}) \subseteq R$ under matching M'. By the same argument, we have $|S_t| = |\Gamma_H(S_{t-1})|$ for every $t \geq 2$.

It is not difficult to see that $S_t \setminus S_{t-1}$ is the set of vertices in L such that there exists an alternating cycle C of length $2t$ from i that uses an edge in H when C goes from left to right and uses an edge in M' when C goes from right to left.

From the first inequality in the definition of a θ-expanding bipartite graph (Definition 4), for every $t \geq 0$, if $|S_t| \leq \frac{a}{2}$, then $\theta|S_t| \leq |\Gamma_H(S_t)| = |S_{t+1}|$. This means that the size of S_t increases by a factor of θ until it exceeds $\frac{a}{2}$. Thus, for $t \geq 1$ with $|S_{t-1}| \leq \frac{a}{2}$, we have

$$\theta^{t-1} = \theta^{t-1}|S_0| \leq \cdots \leq \theta|S_{t-1}| \leq |\Gamma(S_{t-1})| = |S_t|.$$

Let t_1 be the first time when the size of S_t becomes strictly larger than $\frac{a}{2}$. Observe that $t_1 \leq \left\lceil \frac{\log(a/2)}{\log\theta} \right\rceil + 2$ since $\theta^{\frac{\log(a/2)}{\log\theta}+1-1} \geq \frac{a}{2}$ and $\theta > 1$.

Now consider $t \geq t_1$ with $|S_t| > \frac{a}{2}$. Recall that from the second inequality of Definition 4, if $|S_t| > \frac{a}{2}$, then $a - |\Gamma_H(S_t)| \leq \frac{1}{\theta}(a - |S_t|)$. Thus, for $t \geq t_1$ with $|S_t| > \frac{a}{2}$,

$$a - |S_{t+1}| = a - |\Gamma_H(S_t)| \leq \frac{1}{\theta}(a - |S_t|),$$

where the first equality holds since $|S_{t+1}| = |\Gamma_H(S_t)|$. This means that the difference between the number of right vertices and $|S_t|$ decreases by a factor of $\frac{1}{\theta}$ as long as $|S_t| > \frac{a}{2}$. Observe further that $|S_{t+1}| > \frac{a}{2}$ if $|S_t| > \frac{a}{2}$ since by the above inequality and by $\theta > 1$, we get

$$a - |S_{t+1}| \leq \frac{1}{\theta}(a - |S_t|) < \frac{1}{\theta}\left(a - \frac{a}{2}\right) < \frac{a}{2},$$

which implies $|S_{t+1}| > \frac{a}{2}$. Now, let $t_2 = \lceil \log\frac{a}{2}/\log\theta \rceil + 1$. Applying the second inequality of Definition 4 repeatedly for $t = t_1, t_1+1, \ldots, t_1+t_2-1$, we get

$$a - |S_{t_1+t_2}| \leq \frac{1}{\theta^{t_2}}(a - |S_{t_1}|) \leq \frac{1}{\theta} \cdot \theta^{-\log\frac{a}{2}/\log\theta} \cdot \frac{a}{2} = \frac{1}{\theta} < 1.$$

This means that $|S_{t_1+t_2}| = a$ and $S_{t_1+t_2}$ includes all the vertices in L which participate in matching M'. Thus, $i \in S_{t_1+t_2}$. Therefore, we find a desired alternating cycle of length $\ell = t_1 + t_2 = O_a(\log a)$. □

F.2 Proof of Lemma 6

Lemma 6. *Let H^* be a heavy θ-expanding complete bipartite graph with left vertices L and right vertices R with $|L| \geq |R| = a$ and with edge weights w where $w(i,j) \in (0,1]$ for every edge $\{i,j\}$. Then, the weights of all edges in a maximum weight matching in H^* are $1 - O_a\left(\frac{(\log a)^2}{a}\right)$.*

Proof of Lemma 6. Let M^* be a maximum weight matching in the bipartite graph H^*. Since H^* is a complete bipartite graph and $|L| \geq |R|$, all vertices in R are incident to some edge in M^*. Let $H = (L \cup R, E_H)$ be a θ-expanding bipartite subgraph such that $\{i,j\} \in E_H \Leftrightarrow w(i,j) \geq 1 - za^{-1}\log a$, where $z > 0$ is a constant. Consider any $i \in L$ that participates in M^*, i.e., i is incident to some edge in M^*. From Lemma 5, there exists an alternating cycle C of M^* such that $C = (i_1 = i, M^*(i_2), i_2, \ldots, M^*(i_\ell), i_\ell, M^*(i_1), i_1)$, $\ell = O_a(\log a)$, and $\{i_t, M^*(i_{t+1})\} \in E_H$ for every $t = 1, 2, \ldots, \ell$.

Now, we construct a new matching M' from M^* by replacing the edges in $C \cap M^*$ with those in $C \cap H$, namely,

$$M' = (M^* \setminus \{\{i_1, M^*(i_1)\}, \ldots, \{i_\ell, M^*(i_\ell)\}\}) \cup \{\{i_1, M^*(i_2)\}, \ldots, \{i_\ell, M^*(i_1)\}\}.$$

Then, since M^* is a maximum weight matching in the bipartite graph H, the sum of the edge weights in M^* is at least that in M'. Also since $M'(i') = M^*(i')$ for every $i' \notin \{i_1, ..., i_\ell\}$ that participates in the matching M', we have $\sum_{t=1}^{\ell} w(i_t, M^*(i_t)) \geq \sum_{t=1}^{\ell} w(i_t, M'(i_t))$. Recall $w(i,j) \leq 1$ for all i and j, which implies that $\sum_{t=2}^{\ell} w(i_t, M^*(i_t)) \leq \ell - 1$. Thus,

$$\begin{aligned} w(i, M^*(i)) &\geq \sum_{t=1}^{\ell} w(i_t, M^*(i_t)) - \ell + 1 \\ &\geq \sum_{t=1}^{\ell} w(i_t, M'(i_t)) - \ell + 1 \\ &\geq \ell \cdot \left(1 - z \cdot \frac{\log a}{a}\right) - \ell + 1 \\ &= 1 - \ell \cdot z \cdot \frac{\log a}{a}. \end{aligned}$$

Since $\ell = O_a(\log a)$, we get $w(i, M^*(i)) = 1 - O_a\left(\frac{(\log a)^2}{a}\right)$. □

F.3 Proof of Lemma 4

We first introduce the Chernoff bound, which can be used to bound the tails of the distribution for sums of independent random variables.

Lemma 7 (Chernoff bound). *Given independent random variables $X_1, X_2, ..., X_d$ on $[0,1]$. Let $X = \sum_{i=1}^{d} X_i$. Then, for all $\varepsilon > 0$, we have the followings:*

$$\text{(i) } \mathbf{Pr}[X \geq (1+\varepsilon)\mathbb{E}[X]] \leq \exp\left(-\frac{2\varepsilon^2 \mathbb{E}[X]^2}{d}\right),$$
$$\text{(ii) } \mathbf{Pr}[X \leq (1-\varepsilon)\mathbb{E}[X]] \leq \exp\left(-\frac{\varepsilon^2 \mathbb{E}[X]^2}{d}\right).$$

Lemma 4 (No light edge lemma). *Suppose that distributions are (α, β)-PDF-bounded. With high probability, we have the followings:*

(a) *No edge of weight at most $1 - c_p \frac{(\log n_p)^2}{n_p}$ appears in the matching M^p. Here $c_p > 0$ is a constant.*

(b) *No edge of weight at most $1 - c_q \frac{(\log \min(n_p, n_q))^2}{\min(n_p, n_q)}$ appears in a maximum weight matching in the bipartite subgraph of G induced by N_p and $M(N_q)$ with edge weights $u_i(j)$ for $i \in N_p$ and $j \in M(N_q)$. Here $c_q > 0$ are sufficiently large constant.*

(c) *No edge of weight at most $1 - c_u \frac{(\log \min(n_p, n_q))^2}{\min(n_p, n_q)}$ appears in a maximum weight matching in the complete bipartite graph between N_p and I_u with edge weights drawn from $\mathcal{D}$ independently. Here $c_u > 0$ is a constant.*

Proof of Lemma 4. First we show the third part (c) in Lemma 4 for reasons of simplicity of proof. Recall that the bundle I_u denotes a bundle whose size is n_q which is uniformly at random selected from I. Consider the randomly weighted complete bipartite graph H_u composed of N_p and I_u. The weight of edge $\{i,j\}$ ($i \in N_p$, $j \in I_u$) can be considered as it is drawn independently from $\mathcal{D}$ since I_u is selected uniformly at random. If we can show that H_u is heavy 2-expanding bipartite graph, we can say that there is no light edge in the maximum weight matching on H_u from Lemma 6. In the proof, we consider two case $n_p > n_q$ and $n_p \leq n_q$.

First, we consider the case $n_p > n_q$. Set c_1 is the sufficiently large constant. For all $i \in N_p$ and all $j \in I_u$, we have $\mathbf{Pr}\left[u_i(j) \geq 1 - c_1 \frac{\log n_q}{n_q}\right] \geq \alpha c_1 \frac{\log n_q}{n_q}$ since $\mathcal{D}$ is (α, β)-PDF-bounded. Consider $S \subseteq N_p$ with $1 \leq |S|$. Let $\Gamma_{H_u}(S) = \{j \in I_u \mid u_i(j) \geq 1 - c_1 \frac{\log n_q}{n_q} \text{ for some } i \in S\}$ and $\rho = \mathbf{Pr}\left[j \in \Gamma_{H_u}(S)\right]$ for $j \in I_u$. Then we have $1-\rho \leq \left(1 - \alpha c_1 \frac{\log n_q}{n_q}\right)^{|S|}$. Note that this definition of $\Gamma_{H_u}(S)$ is slightly different from that in Section 3.2. Here we only consider the edges in the subgraph in the definition of $\Gamma_{H_u}(S)$. Also, from the independence, $|\Gamma_{H_u}(S)|$ is distributed as the binomial distribution $\mathrm{Bin}(n_q, \rho)$.

Now our goal is to show that the complete bipartite graph H_u is heavy 2-expanding bipartite graph with high probability. So, we show that $\Gamma(S)$ defined above satisfies the two conditions in Definition 4 with high probability. For the proofs, we use Chernoff bound.

In the case of $|S| \leq \frac{n_q}{2\alpha c_1 \log n_q}$, we have

$$\begin{aligned}
\mathbb{E}\left[|\Gamma_{H_u}(S)|\right] &= n_q \rho \\
&\geq n_q \left(1 - \left(1 - \alpha c_1 \frac{\log n_q}{n_q}\right)^{|S|}\right) \\
&\geq n_q \left(1 - \exp\left(-\alpha c_1 \frac{\log n_q}{n_q}|S|\right)\right) \\
&\geq \left(1 - \mathrm{e}^{-1}\right) \alpha c_1 |S| \frac{n_q}{n_q} \log n_q \\
&\geq \frac{\alpha c_1}{2} \log n_q |S|.
\end{aligned}$$

Here we use two inequalities: $(1-t)^x \leq \mathrm{e}^{-tx}$ for $x \geq 1$ and $0 \leq t \leq 1$ and, $1-\mathrm{e}^{-x} \geq (1-\mathrm{e}^{-1})x$ for $0 \leq x \leq 1$. Here, since $|S| \leq \frac{n_q}{2\alpha c_1 \log n_q}$, we have $\alpha c_1 \frac{\log n_q}{n_q}|S| \leq \frac{1}{2} \leq 1$. Then, by Chernoff bound, we obtain

$$\begin{aligned}
\mathbf{Pr}\left[\exists S : |\Gamma_{H_u}(S)| < 2|S|\right] &\le \mathbf{Pr}\left[\exists S : |\Gamma_{H_u}(S)| < \frac{1}{2}\mathbb{E}[|\Gamma_{H_u}(S)|]\right] \\
&\le \sum_{S\subseteq N_p : |S| \le \frac{n_q}{2c_p \log n_q}} \binom{n_p}{|S|} \cdot \mathbf{Pr}\left[|\Gamma_{H_u}(S)| \le \frac{1}{2}\mathbb{E}[|\Gamma_{H_u}(S)|]\right] \\
&\le \sum_{S\subseteq N_p : |S| \le \frac{n_q}{2c_p \log n_q}} n_p^{|S|} \cdot \exp\left(-\frac{\mathbb{E}[|\Gamma_{H_u}(S)|]}{8}\right) \\
&\le \sum_{s=1}^{\frac{n_q}{2\alpha c_1 \log n_q}} n_p^s \cdot \exp\left(-\frac{\alpha c_1 s \log n_q}{16}\right) \\
&\le \sum_{s=1}^{\frac{n_q}{2\alpha c_1 \log n_q}} n_p^s n_q^{-\frac{\alpha c_1}{16} s} \\
&\le \sum_{s=1}^{\frac{n_q}{2\alpha c_1 \log n_q}} (\log n_q)^{-\frac{5}{2}s} n_q^{\left(\frac{5}{2} - \frac{\alpha c_1}{16}\right)s}.
\end{aligned}$$

We use the condition (c′): $n_p \le n_q^{3/2}(\log n_q)^{-5/2}$. Since the constant $c_1 > 0$ is sufficiently large, $\frac{5}{2} - \frac{\alpha c_1}{16} < 0$. Then, as $n_q \to \infty$, the probability that there exists $S \subseteq N_p$ such that $|\Gamma_{H_u}(S)| < 2|S|$ approaches 0. From the condition (d), $n_q \to \infty$ as $n \to \infty$. So, we prove that $|\Gamma_{H_u}(S)| \ge 2|S|$ for all $S \subseteq N_p$ such that $|S| \le \frac{n_q}{10\alpha c_1 \log n_q}$ with high probability.

In the case of $\frac{n_q}{2\alpha c_1 \log n_q} < |S| \le \frac{n_q}{2}$, we have

$$\begin{aligned}
&\mathbf{Pr}\left[\exists S \subseteq N_p : |\Gamma_{H_u}(S)| < 2|S|\right] \\
&\le \sum_{s=\frac{n_q}{10\alpha c_1 \log n_q}}^{\frac{n_q}{2}} \mathbf{Pr}\left[\exists S \subseteq N_p : |S| = s, |\Gamma_{H_u}(S)| \le 2s-1\right] \\
&= \sum_{s=\frac{n_q}{2\alpha c_1 \log n_q}}^{\frac{n_q}{2}} \mathbf{Pr}\left[\exists S \subseteq N_p, T \subseteq I_u : |S| = s, |T| = 2s-1, |\Gamma_{H_u}(S)| \subseteq T\right] \\
&\le \sum_{s=\frac{n_q}{2\alpha c_1 \log n_q}}^{\frac{n_q}{2}} \sum_{\substack{\exists S\subseteq N_p, T\subseteq I_u \\ |S|=s, |T|=2s-1}} \prod_{i\in N_p, j\in I_u} \mathbf{Pr}\left[\{i,j\} \notin E\right] \\
&\le \sum_{s=\frac{n_q}{2\alpha c_1 \log n_q}}^{\frac{n_q}{2}} \binom{n_p}{s} \cdot \binom{n_q}{2s-1} \cdot \left(\alpha c_1 \frac{\log n_q}{n_q}\right)^{s(n_q-2s+1)} \\
&\le \sum_{s=\frac{n_q}{2\alpha c_1 \log n_q}}^{\frac{n_q}{2}} n_p^s \cdot n_q^{2s-1} \cdot \left(\frac{1}{2}\right)^{s(n_q-2s+1)} \\
&\le \sum_{s=\frac{n_q}{2\alpha c_1 \log n_q}}^{\frac{n_q}{2}} n_q^{\frac{7}{2}s-1} (\log n_q)^{-5/2} \cdot \left(\frac{1}{2}\right)^{s(n_q-2s+1)}
\end{aligned}$$

Here, we use the condition (c′): $n_p \le n_q^{3/2}(\log n_q)^{-5/2}$. From simple calculation, we conclude the probability that there exists $S \subseteq N_p$ such that $|\Gamma(S)| < 2|S|$ approaches 0 as $n_q \to \infty$ from the inequality above.

In the case of $\frac{n_q}{2} < |S|$, we have $1-\rho \le \left(1-\alpha c_1 \frac{\log n_q}{n_q}\right)^{|S|} \le \left(1-\alpha c_1 \frac{\log n_q}{n_q}\right)^{\frac{n_q}{2}} \le \exp\left(-\frac{\alpha c_1 \log n_q}{2}\right) = n_q^{-\frac{\alpha c_1}{2}}$. Moreover, we have $\binom{n_p}{|S|} \le \binom{n_p}{n_p/2} \le (2\mathrm{e})^{n_p/2}$. Let $t = n_q - |S|$ and so

$$\begin{aligned}\mathbf{Pr}\left[\exists S : n_q - |\Gamma_{H_u}(S)| > \frac{1}{2}(n_q - |S|)\right] &\le \sum_{t=1}^{\frac{n_p}{2}} \binom{n_p}{t} \cdot \binom{n_q}{t/2} \cdot (1-\rho)^{n_q - n_p + t/2} \\ &\le n_p \cdot (2\mathrm{e})^{n_p/2} \cdot (2\mathrm{e})^{n_q/2} \cdot \left(n_q^{-\frac{\alpha c_1}{2}}\right)^{n_q - n_p + \frac{n_p}{4}}.\end{aligned}$$

So, as $n_q \to \infty$, the probability that there exists $S \subseteq N_p$ such that $n_q - |\Gamma(S)| > \frac{1}{2}(n_q - |S|)$ approaches 0. Finally, in the case of $n_p > n_q$, we can show the complete bipartite graph between N_p and I_u with edge weights drawn from $\mathcal{D}$ independently is heavy 2-expanding. Moreover, from Lemma 6, we conclude that, with high probability, no edge of weight at most $1 - O_{n_q}((\log n_q)^2/n_q)$ appears in the maximum weight matching on the complete bipartite graph between N_p and I_u.

Next, we consider the case of $n_p \le n_q$. In this case, we also consider the complete bipartite graph H_u between I_u and N_p. We see I_u to be the left-hand vertex set and N_p to be the right-hand vertex set. Consider $S \subseteq I_u$ and $\Gamma_{H_u}(S) := \{i \in N_p \mid u_i(j) \ge 1 - c_1 \frac{\log n_p}{n_p}$ for some $j \in S\}$. Since $\mathcal{D}$ is (α, β)-PDF-bounded, we also have

$$\mathbf{Pr}\left[u_i(j) \ge 1 - c_1 \frac{\log n_p}{n_p}\right] \ge \alpha c_1 \frac{\log n_p}{n_p}.$$

From the same discussion as in the case of $n_p > n_q$, we can show that the complete bipartite graph between N_p and I_u with edge weights drawn from $\mathcal{D}$ independently is also heavy 2-expanding in the the case of $n_p \le n_q$. Furthermore, we can show that, with high probability, all weights of edges in the maximum weight matching on the complete bipartite graph are $1 - O_{n_p}((\log n_p)^2/n_p)$.

Thus, with high probability, there exists a constant $c_u > 0$ such that no edge of weight at most $1 - c_u \frac{\log \min(n_p, n_q)^2}{\min(n_p, n_q)}$ appears in the maximum weight matching between N_p and I_u.

Next, we show the first part (a) in Lemma 4. We consider the weight of the edges in M^p. By the algorithm, M^p is the maximum weight matching on the complete bipartite graph between N_p and $M(N_p)$. Here $|N_p| = |M(N_p)| = n_p$. Let $c_2 > 0$ a sufficiently large constant. By the construction of the matching M^p in which class p selects only their favorite items, and $\mathcal{D}$ is (α, β)-PDF-bounded, we have, for all p and for all $i \in N_p$,

$$\mathbf{Pr}\left[u_i(j) \ge 1 - c_2 \frac{\log n_p}{n_p} \;\middle|\; j \in M(N_p)\right] \ge \mathbf{Pr}\left[u_i(j) \ge 1 - c_2 \frac{\log n_p}{n_p}\right] \ge \alpha c_2 \frac{\log n_p}{n_p}.$$

From this, with conditioning $j \in M(N_p)$, we can use the same technique as that we used in the proof of the third part (c) in Lemma 4. Then we can show that the complete bipartite graph between N_p and $M(N_p)$ is heavy 2-expanding. From Lemma 6, with high probability, there exists a constant $c_p > 0$ such that no edges of weights at most $1 - c_p \frac{\log n_p}{n_p}$ in M^p.

Finally, we show the second part (b) in Lemma 4 The weight of the edge $\{i, j\}$ is $u_i(j)$. Since the size of N_p and the size of $M(N_q)$ are different, we need to consider two cases: $n_p \le n_q$ and $n_p > n_q$ as like we did in the proof of the third part (c).

We first consider the case $n_p \leq n_q$. In this case, class p selects items before class q does. Let c_3 be a large constant such that $c_3 \geq 2 \cdot c_p > 0$. Like we did in the case of N_p and M_p, we consider bounding from below the probability that, for $i \in N_p$ and $j \in M(N_q)$, $u_i(j)$ is larger than $1 - c_3 \cdot \log n_p / n_p$ with conditioning $j \in M(N_q)$. However we have several careful points.

Recall that $j_r^q \in M(N_q)$ is an item which is chosen by class q at round r in the algorithm. Since j_r^q was not selected by class p at same round r, the utility of j_r^q for all $i \in N_p$ is bounded as $u_i(j_r^q) \leq t_r$. Then we have, for all $i \in N_p$,

$$\begin{aligned}
&\mathbf{Pr}\left[u_i(j_r^q) \geq 1 - c_3\frac{(\log n_p)^2}{n_p} \;\middle|\; j_r^q \in M(N_q)\right] \\
&\geq \mathbf{Pr}\left[u_i(j_r^q) \geq 1 - c_3\frac{(\log n_p)^2}{n_p} \bigwedge j_r^q \in M(N_q)\right] \\
&\geq \mathbf{Pr}\left[u_i(j_r^q) \geq 1 - c_3\frac{(\log n_p)^2}{n_p} \bigwedge u_i(j_r^q) \leq t_r\right],
\end{aligned}$$

where $u_i(j_r^q)$ is generated from $\mathcal{D}$. From the first part (a) in Lemma 4, we have $t_r \geq 1 - c_p\frac{(\log n_p)^2}{n_p}$. Then, for all $i \in N_p$ and for $r = 1, 2, ..., n_p$, we have

$$\begin{aligned}
&\mathbf{Pr}\left[u_i(j_r^q) \geq 1 - c_3\frac{(\log n_p)^2}{n_p} \bigwedge u_i(j_r^q) \leq t_r\right] \\
&\geq \mathbf{Pr}\left[u_i(j_r^q) \geq 1 - c_3\frac{(\log n_p)^2}{n_p} \bigwedge u_i(j_r^q) \leq 1 - c_p\frac{(\log n_p)^2}{n_p}\right] \\
&= \mathbf{Pr}\left[1 - c_3\frac{(\log n_p)^2}{n_p} \leq u_i(j_r^q) \leq 1 - c_p\frac{(\log n_p)^2}{n_p}\right] \\
&\geq \alpha(c_3 - c_p)\frac{(\log n_p)^2}{n_p} \\
&\geq \alpha c_p\frac{(\log n_p)^2}{n_p}.
\end{aligned}$$

We use here the density function $f_{\mathcal{D}}(x)$ of $\mathcal{D}$ is lower bounded by $\alpha \cdot x$. Therefore, we get

$$\mathbf{Pr}\left[u_i(j_r^q) \geq 1 - c_3\frac{(\log n_p)^2}{n_p} \;\middle|\; j_r^q \in M(N_q)\right] \geq \alpha c_p\frac{(\log n_p)^2}{n_p}.$$

Moreover, for $r = n_p + 1, ..., n_q$, similarly we have

$$\begin{aligned}
&\mathbf{Pr}\left[u_i(j_r^q) \geq 1 - c_3\frac{(\log n_p)^2}{n_p} \;\middle|\; j_r^q \in M(N_q)\right] \\
&\geq \mathbf{Pr}\left[u_i(j_r^q) \geq 1 - c_3\frac{(\log n_p)^2}{n_p} \bigwedge j_r^q \in M(N_q)\right] \\
&\geq \mathbf{Pr}\left[u_i(j_r^q) \geq 1 - c_3\frac{(\log n_p)^2}{n_p} \bigwedge u_i(j_r^q) \leq t_{n_p}\right] \\
&\geq \alpha c_p\frac{(\log n_p)^2}{n_p}.
\end{aligned}$$

From these, we can use same technique to that in the proof of the third part (c) in Lemma 4 and then we can show that, in the case of $n_p \leq n_q$, the complete bipartite graph between N_p and $M(N_q)$ is heavy 2-expanding bipartite graph with high probability. Therefore, with high probability, there exists a constant $c_q > 0$ such that no edge of weight at most $1 - c_q((\log n_p)^2/n_p)$ appears in the maximum weight matching on the complete bipartite graph between N_p and $M(N_q)$.

Second, we consider the case of $n_p > n_q$. In this case class q chooses items before class p does. Here we consider the fact that item $j_r^q \in M(N_q)$ with $r \geq 1$ is not selected by class p in round $r-1$. Then the weight of edge $\{i, j_r^q\}$ for $i \in N_p$ are conditioned as the weight is less than t_{r-1}. From the discussion which is similar which we consider the bound on t_r^q in the proof of Claim 2, we get $t_{r-1} \geq 1 - c_p \frac{(\log n_p)^2}{n_p}$. Let c_3 be a large constant such that $c_3 \geq 2 \cdot c_p$. Then, for all $i \in N_p$ and for each $r = 2, 3, ..., n_q$,

$$\begin{aligned}
&\mathbf{Pr}\left[u_i(j_r^q) \geq 1 - c_3\frac{(\log n_q)^2}{n_q} \;\middle|\; j_r^q \in M(N_q)\right] \\
&\geq \mathbf{Pr}\left[u_i(j_r^q) \geq 1 - c_3\frac{(\log n_q)^2}{n_q} \;\bigwedge\; j_r^q \in M(N_q)\right] \\
&\geq \mathbf{Pr}\left[u_i(j_r^q) \geq 1 - c_3\frac{(\log n_q)^2}{n_q} \;\bigwedge\; u_i(j_r^q) \leq t_{r-1}\right] \\
&\geq \mathbf{Pr}\left[u_i(j_r^q) \geq 1 - c_3\frac{(\log n_q)^2}{n_q} \;\bigwedge\; u_i(j_r^q) \leq 1 - c_p\frac{(\log n_p)^2}{n_p}\right] \\
&\geq \mathbf{Pr}\left[1 - c_3\frac{(\log n_q)^2}{n_q} \leq X \leq 1 - c_p\frac{(\log n_p)^2}{n_p}\right] \\
&\geq \alpha\left(c_3\frac{(\log n_q)^2}{n_q} - c_p\frac{(\log n_p)^2}{n_p}\right) \\
&\geq \alpha c_p\frac{(\log n_q)^2}{n_q}.
\end{aligned}$$

Moreover, for $r = 1$,

$$\begin{aligned}
&\mathbf{Pr}\left[u_i(j_1^q) \geq 1 - c_3\frac{(\log n_q)^2}{n_q} \;\middle|\; j_1^q \in M(N_q)\right] \\
&\geq \mathbf{Pr}\left[u_i(j_1^q) \geq 1 - c_3\frac{(\log n_q)^2}{n_q} \;\bigwedge\; j_1^q \in M(N_q)\right] \\
&= \mathbf{Pr}\left[u_i(j_r^q) \geq 1 - c_3\frac{(\log n_q)^2}{n_q}\right] \\
&\geq \alpha c_3\frac{(\log n_q)^2}{n_q},
\end{aligned}$$

and, for $r = n_q + 1, ..., n_p$,

$$\begin{aligned}
&\mathbf{Pr}\left[u_i(j_r^q) \geq 1 - c_3\frac{(\log n_q)^2}{n_q} \;\middle|\; j_r^q \in M(N_q)\right] \\
&\geq \mathbf{Pr}\left[u_i(j_r^q) \geq 1 - c_3\frac{(\log n_q)^2}{n_q} \;\bigwedge\; j_r^q \in M(N_q)\right] \\
&= \mathbf{Pr}\left[u_i(j_r^q) \geq 1 - c_3\frac{(\log n_q)^2}{n_q} \;\bigwedge\; u_i(j_r^q) \leq t_{n_q}\right] \\
&\geq \alpha c_3\frac{(\log n_q)^2}{n_q}.
\end{aligned}$$

From these, similarly we can show that the complete bipartite graph between N_p and $M(N_q)$ is heavy 2-expanding bipartite graph with high probability in the case of $n_p > n_q$. Hence, in this case, with high probability, there exists a constant $c_q > 0$ such that no edge of weight at most $1 - c_q((\log n_q)^2/n_q)$ appears in the maximum weight matching on the complete bipartite graph between N_p and $M(N_q)$.

Finally, with high probability, there exists a constant $c_q > 0$ such that no edge of weight at most $1 - c_q \frac{\log \min(n_p,n_q)^2}{\min(n_p,n_q)}$ appears in the maximum weight matching between N_p and $M(N_q)$.

□

G Omitted Proofs from Section 3.3

G.1 Proofs of (c′) and (d)

In the proof of Theorem 1 in Section 3.3, we have used the following facts (c′) there exists a constant $c' > 0$ such that, for all $p, q \in [k]$, $\max(n_p, n_q) \leq c' \cdot \min(n_p, n_q)^{3/2} \cdot (\log \min(n_p, n_q))^{-5/2}$, and (d) $n_p \to \infty$ as $n \to \infty$ for all $p \in [k]$.

Here, we prove these facts by assuming the condition (c) in Theorem 1, namely, there exists a constant $c > 0$ such that $n \leq c \cdot \min_p(n_p)^{3/2} \cdot (\log \min_p(n_p))^{-5/2}$.

- We first prove (c′). From the condition (c), for all $p, q \in [k]$, we have $\max(n_p, n_q) \leq n \leq c \cdot (\min_{p'} n_{p'})^{3/2} \cdot (\log \min_{p'} n_{p'})^{-5/2}$. Also, for all $p, q \in [k]$, we have $c \cdot (\min_{p'} n_{p'})^{3/2} \cdot (\log \min_{p'} n_{p'})^{-5/2} \leq c \cdot \min(n_p, n_q)^{3/2} \cdot (\log \min(n_p, n_q))^{-5/2}$ since the function $f(x) = x^{3/2}(\log x)^{-5/2}$ is monotonically increasing for sufficiently large x. Thus, combining these, we get, for all $p, q \in [k]$, $\max(n_p, n_q) \leq c \cdot \min(n_p, n_q)^{3/2} \cdot (\log \min(n_p, n_q))^{-5/2}$.
- Second, we show (d). Observe that $n \leq c \cdot (\min_p n_p)^{3/2} \cdot (\log \min_p n_p)^{-5/2}$ and that the function $f(x) = x^{3/2}(\log x)^{-5/2}$ has a unique inflection point $x^* = \mathrm{e}^{5/3}$ where $f(x)$ is monotonically decreasing for $x \leq x^*$ and monotonically increasing for $x \geq x^*$. If n approaches infinity, there are two cases: $\min_p n_p$ goes to infinity, or goes to 0; however, the latter case is impossible since $\min_p n_p$ is at least one. Thus, $\min_p n_p \to \infty$ as $n \to \infty$. Hence, for all $p \in [k]$, $n_p \to \infty$ as $n \to \infty$.