# Nonlinear waves (zonons) in zonostrophic turbulence


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Waves and instabilities in space and astrophysical plasmas

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# Overview:

❖A two-dimensional flow on the surface of a rotating sphere presents a simple model of planetary turbulence (or **drift waves turbulence in plasma theory**)

❖With finite Rossby deformation radius, the flow is known as geostrophic turbulence (Charney).

❖Even in its simplified, barotropic version (infinite Rossby radius), the commingling of strong nonlinearity, strong anisotropy and Rossby waves gives rise to complicated dynamics

❖In flows with small-scale forcing, the inherent anisotropic inverse energy cascade may lead to the regime of zonostrophic turbulence

❖It is distinguished by an anisotropic spectrum and stable systems of alternating zonal jets

❖Another important attribute of zonostrophic turbulence is a new class of nonlinear waves – zonons

❖Zonons may form coherent structures observable in physical space (solitons).

## 2D turbulence on the surface of a rotating sphere (BVE)

The flow is forced on small scales and linearly damped on large scales

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta + f) = \nu \nabla^{2p} \zeta - \lambda \zeta + \xi,$$

$\zeta = \Delta\psi$ - vorticity; $\psi$ - stream function; $f = 2\Omega\sin\theta$ - Coriolis parameter; $\Omega$ - angular velocity of the sphere's rotation; $\theta$ - latitude; $\phi$ - longitude; $\nu$ - hyperviscosity coefficient; $\lambda$ - linear friction coefficient which sets the large-scale friction wave number $n_{fr}$.

**In plasma theory, this model describes drift waves turbulence in nonuniform, finite *β* plasma with infinite gyroradius**

The small-scale forcing $\xi$ acts on the scales $n_\xi^{-1}$ and pumps energy into the system at a constant rate. This energy feeds the inverse cascade at a rate $\varepsilon$.

**Beta-plane** approximates a curved spherical surface by a tangential plane,

**β** – gradient of Coriolis parameter ($f = f_0 + \beta y$, $y$ – northward, $x$ – eastward )

$$\frac{\partial \zeta}{\partial t} + \frac{\partial(\nabla^{-2}\zeta, \zeta)}{\partial(x,y)} + \beta \frac{\partial(\nabla^{-2}\zeta)}{\partial x} = \nu \nabla^{2p} \zeta - \lambda \zeta + \xi$$

# Turbulence and Rossby waves: The basics

Conservation of potential vorticity on a rotating sphere leads to generation of Rossby-Haurwitz waves (RHWs) with the dispersion relation

$$\omega_R(m,n) = -\frac{2\Omega}{R}\frac{m}{n(n+1)}$$

(spherical harmonics decomposition, $m$ – zonal, $n$ – total wave-number)

On a beta-plane: $$\omega_R(\mathbf{k}) = -\beta k_x / k^2$$

**RHW** are solutions of barotropic vorticity equation (BVE) on a rotating sphere without nonlinear term.

Fully nonlinear BVE without rotation describes classical 2D turbulence with inverse energy cascade.

Variation of Coriolis parameter with latitude (beta-effect) introduces anisotropy and Rossby waves which give rise to complicated dynamics.

Characteristic feature of such dynamics – generation of zonal jets.

# Forced 2D turbulence - simulations

In order to study nonlinear dynamics, we performed DNS of BVE on a rotating sphere. Spectral model is employed

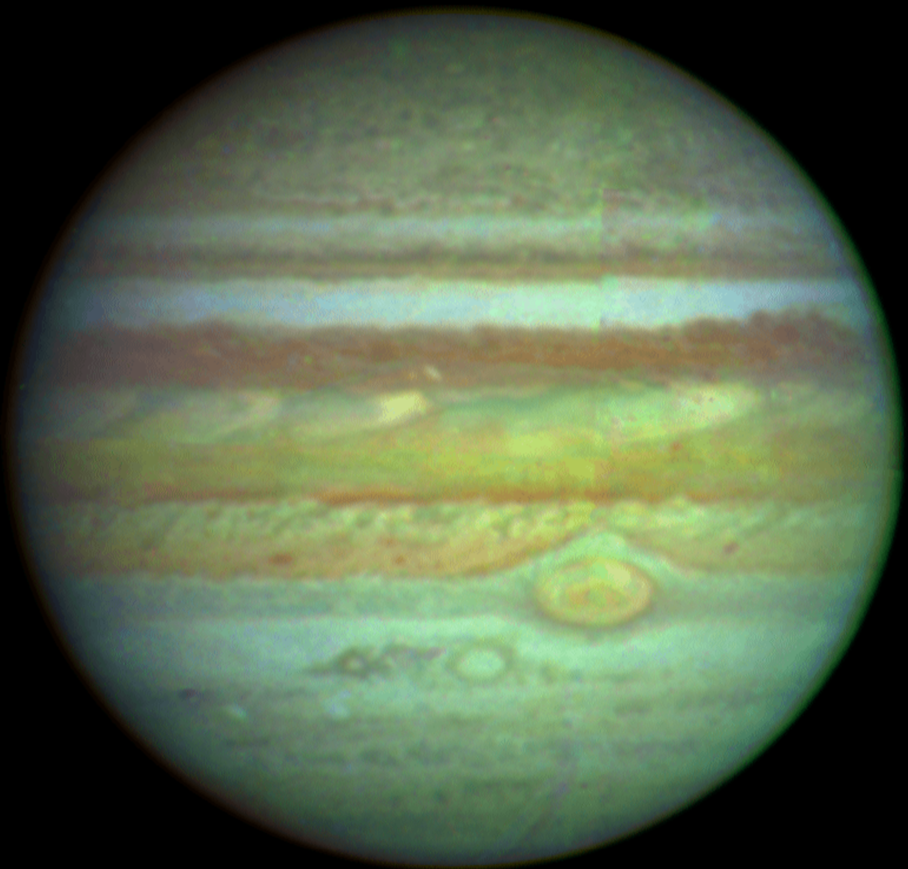
$$\Psi(\mu,\phi,t) = \sum_{n=1}^{N} \sum_{m=-n}^{n} \psi_n^m(t) Y_n^m(\mu,\phi)$$

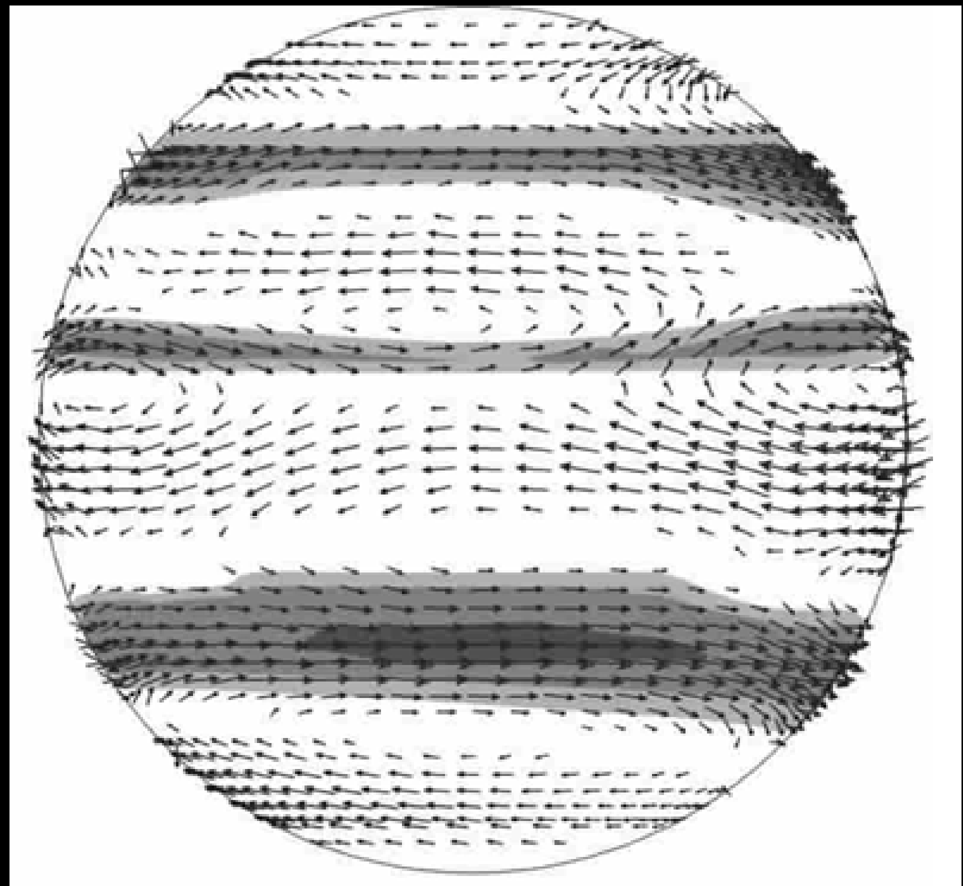
- R-truncation; R133 and R240 resolutions
- Random energy injection with the constant rate $\varepsilon$ at about $n_\xi = 100$
- Very long-term integrations in a steady-state to compile long records for statistical analysis
- Analyze anisotropic spectrum

$$E(n) = \frac{n(n+1)}{4R^2} \sum_{m=-n}^{n} \langle |\psi_n^m|^2 \rangle \;\; = E_Z(n) + E_R(n)$$

**zonal (m=0)** **residual**

# Typical flow field





# Processes of turbulence on β-plane/rotating sphere

1. How is energy delivered to $k_x \to 0$ modes?
2. How much energy those modes retain?

For answers we look at spectral energy transfer

$$\left[\frac{\partial}{\partial t} + 2\nu_o k^2\right]\Omega(\mathbf{k},t) = \mathcal{T}_\Omega(\mathbf{k},t)$$

$$\mathcal{T}_\Omega(\mathbf{k},t) = \int\int_D T(\mathbf{k},\mathbf{p},\mathbf{q},t)d\mathbf{p}d\mathbf{q}.$$

Second order spectral closures yield

$$T(\mathbf{k},\mathbf{p},\mathbf{q}) = \Theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}(p^2-q^2)\sin\alpha$$
$$\left[\frac{p^2-q^2}{p^2q^2}\Omega(\mathbf{p})\Omega(\mathbf{q}) - \frac{k^2-q^2}{k^2q^2}\Omega(\mathbf{q})\Omega(\mathbf{k}) + \frac{k^2-p^2}{k^2p^2}\Omega(\mathbf{p})\Omega(\mathbf{k})\right] + \textbf{similar terms.}$$

where $\Theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$ is triad relaxation time

## **Further insight:** from triad relaxation time

$$\Theta_{-\mathbf{k},\mathbf{p},\mathbf{q}} = \frac{\mu_{\mathbf{k}} + \mu_{\mathbf{p}} + \mu_{\mathbf{q}}}{[\mu_{\mathbf{k}} + \mu_{\mathbf{p}} + \mu_{\mathbf{q}}]^2 + [\omega_{-\mathbf{k}} + \omega_{\mathbf{p}} + \omega_{\mathbf{q}}]^2},$$

$\mu_{\mathbf{k}} \propto \tau_{tu}^{-1}$, eddy frequency scale at wave-number $k$

$$\theta_{\mathbf{-k,p,q}} \to \pi\delta(\omega_{-\mathbf{k}} + \omega_{\mathbf{p}} + \omega_{\mathbf{q}}) \text{ in the limit } \frac{(\mu_{-\mathbf{k}} + \mu_{\mathbf{p}} + \mu_{\mathbf{q}})^2}{(\omega_{-\mathbf{k}} + \omega_{\mathbf{p}} + \omega_{\mathbf{q}})^2} \to 0$$

Thus, at large scales, resonance condition is needed for effective energy transfer, $\omega_{-\mathbf{k}} + \omega_{\mathbf{p}} + \omega_{\mathbf{q}} \to 0$

Nonlinear interactions of turbulence and waves modifies the flow dynamics

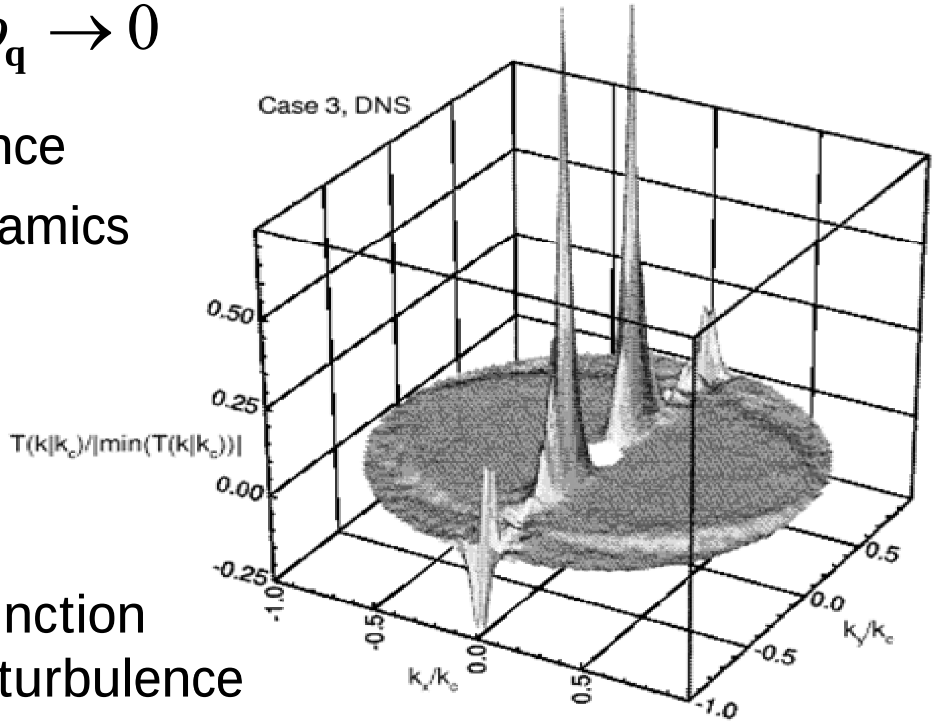



**Figure:** Spectral energy transfer function computed from DNS of beta-plane turbulence

# Development of kinetic energy spectra

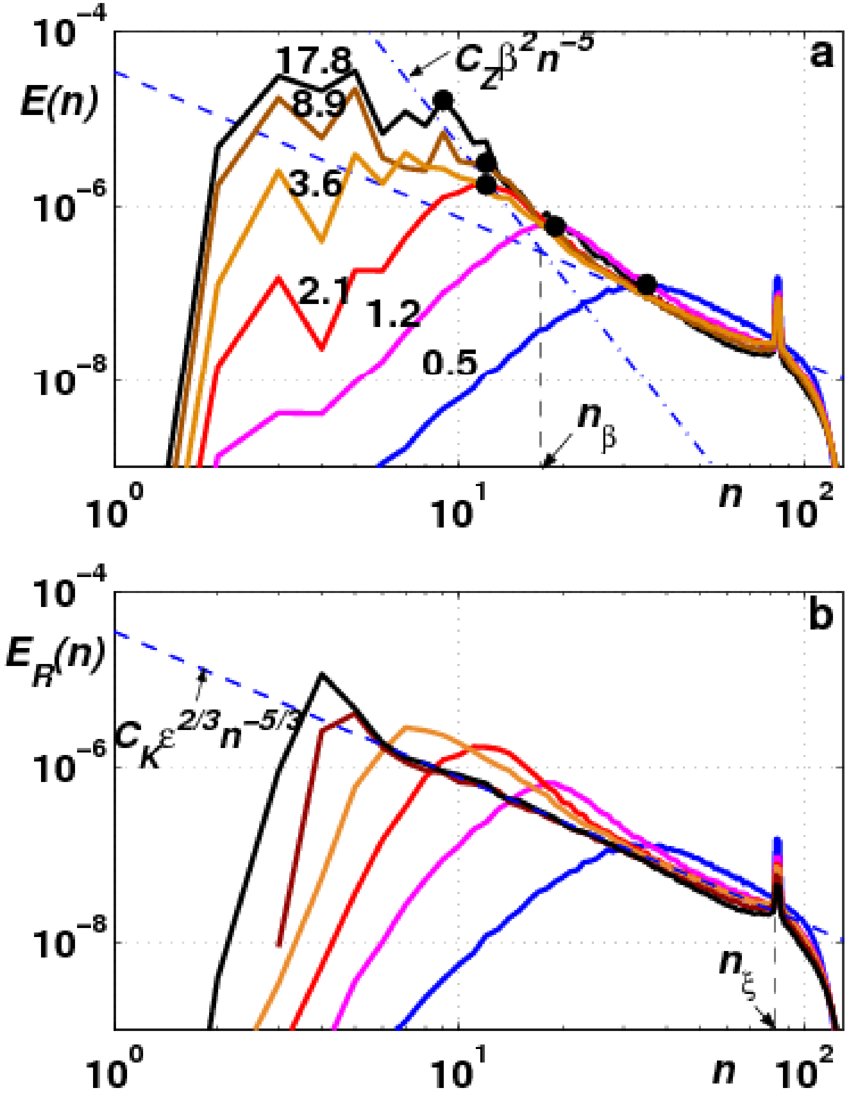



**Figure:** Total (a) and nonzonal (b) energy spectra at different times

# The transitional wave number, $n_\beta$ and Rhines's wave number, $n_R$

- The characteristic time scale of turbulence is $\tau_t = [n^3E(n)]^{-1/2}$
- The characteristic time of RHWs is $\tau_R = [\omega_R(n,m)]^{-1}$
- Turbulent processes prevail on small scales where $\tau_t < \tau_R$
- RHWs are dominant on large scales.
- The transitional wave is at the scale with $\tau_t \sim \tau_R$

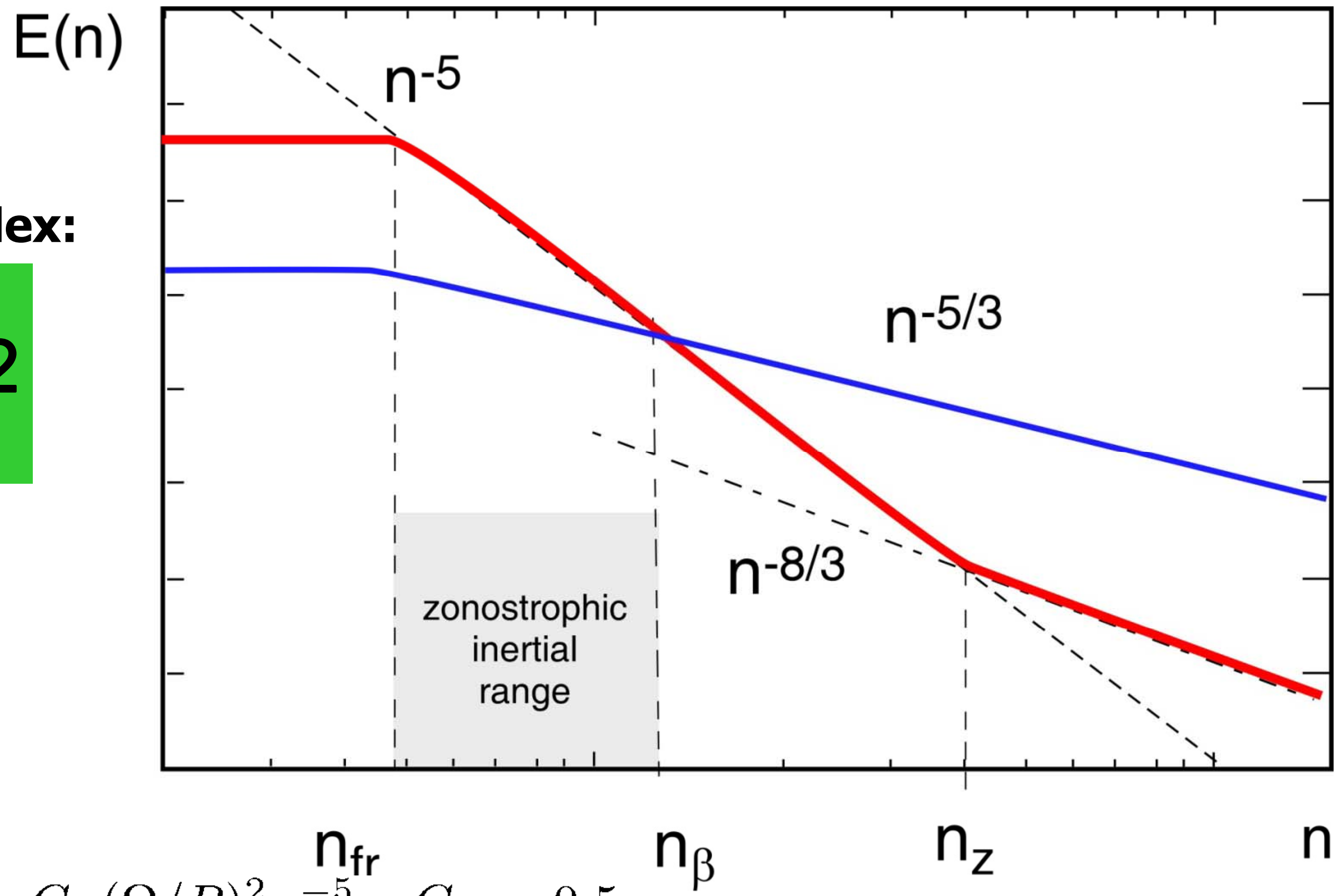
  leading to $n_\beta = 0.5\ (\beta^3/\varepsilon)^{1/5}$, $\beta = \Omega/R$
- On larger scales, we observe anisotropization of the inverse energy cascade
- In flows with large-scale drag the "final", stationary destination of the energy front is identified with the friction wave number $n_{fr}$ which coincides with the Rhines's wavenumber $n_R = (2V/\beta)^{1/2}$, $V$ is the rms velocity

# Zonostrophic turbulence

(from Greek ζωνη - band, belt, and στροφη - turning)

**Zonostrophy index:**

$$R_\beta = \frac{n_\beta}{n_R} > 2$$




Intersection of -5 and -8/3 zonal spectra

$$E_Z(n) = C_Z(\Omega/R)^2 n^{-5}, \quad C_Z \simeq 0.5$$

$$E_R(n) = C_K \epsilon^{2/3} n^{-5/3}, \quad C_K \simeq 4 \text{ to } 6$$

# Examples of zonostrophic turbulence - the ocean-Jupiter connection



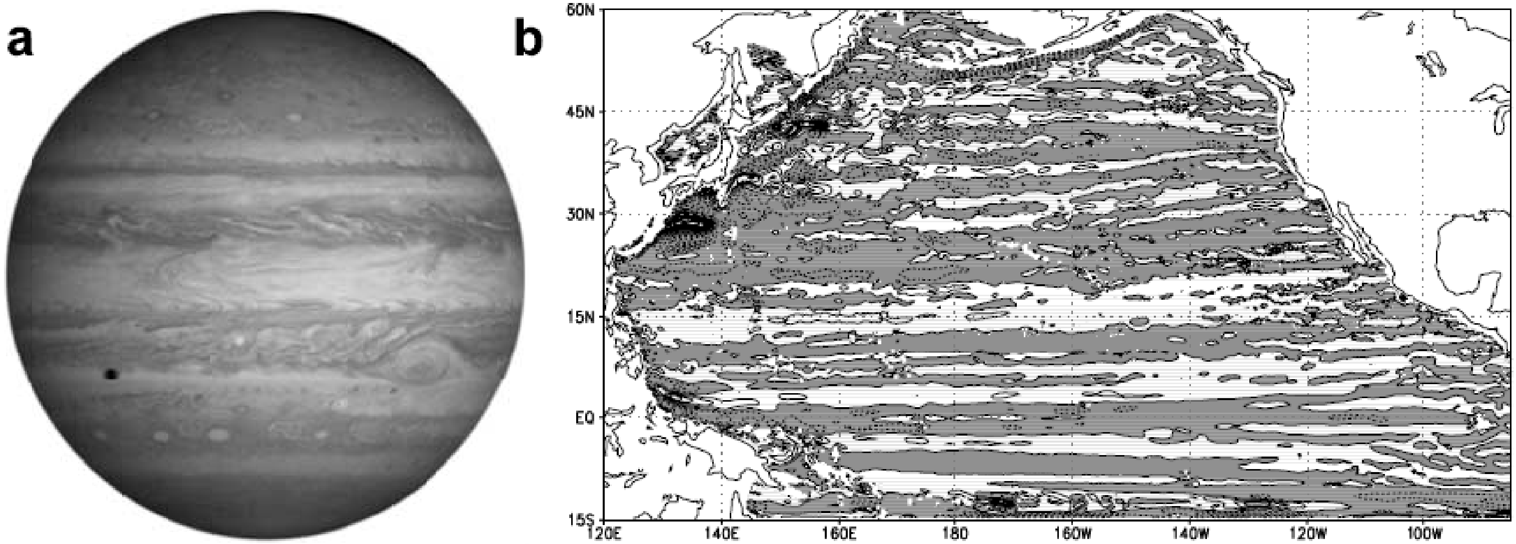
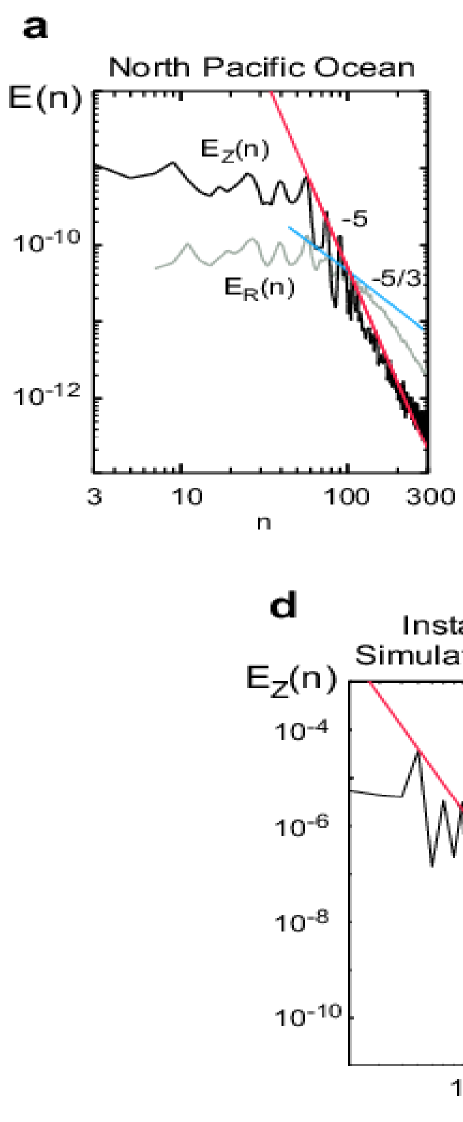



**Figure 1.** (a) Composite view of the banded structure of the disk of Jupiter taken by NASA's Cassini spacecraft on December 7, 2000 (image credit: NASA/JPL/University of Arizona); (b) zonal jets at 1000 m depth in the North Pacific Ocean averaged over the last five years of a 58-year long computer simulation. The initial flow field was reconstructed from the Levitus climatology; the flow evolution was driven by the ECMWF climatological forcing. Shaded and white areas are westward and eastward currents, respectively; the contour interval is 2 cm $s^{-1}$.

# The zonal and residual spectra in the ocean, on giant planets and in simulations are indicative of zonostrophic turbulence



# Rossby-Haurwitz waves and turbulence

Are RHWs present in the fully nonlinear equation and, if yes, how are they affected by the nonlinearity?

Fourier-transform of the velocity autocorrelation function

$$U(\omega, m, n) = \frac{n(n+1)}{4R^2}\langle|\psi_n^m(\omega)|^2\rangle$$

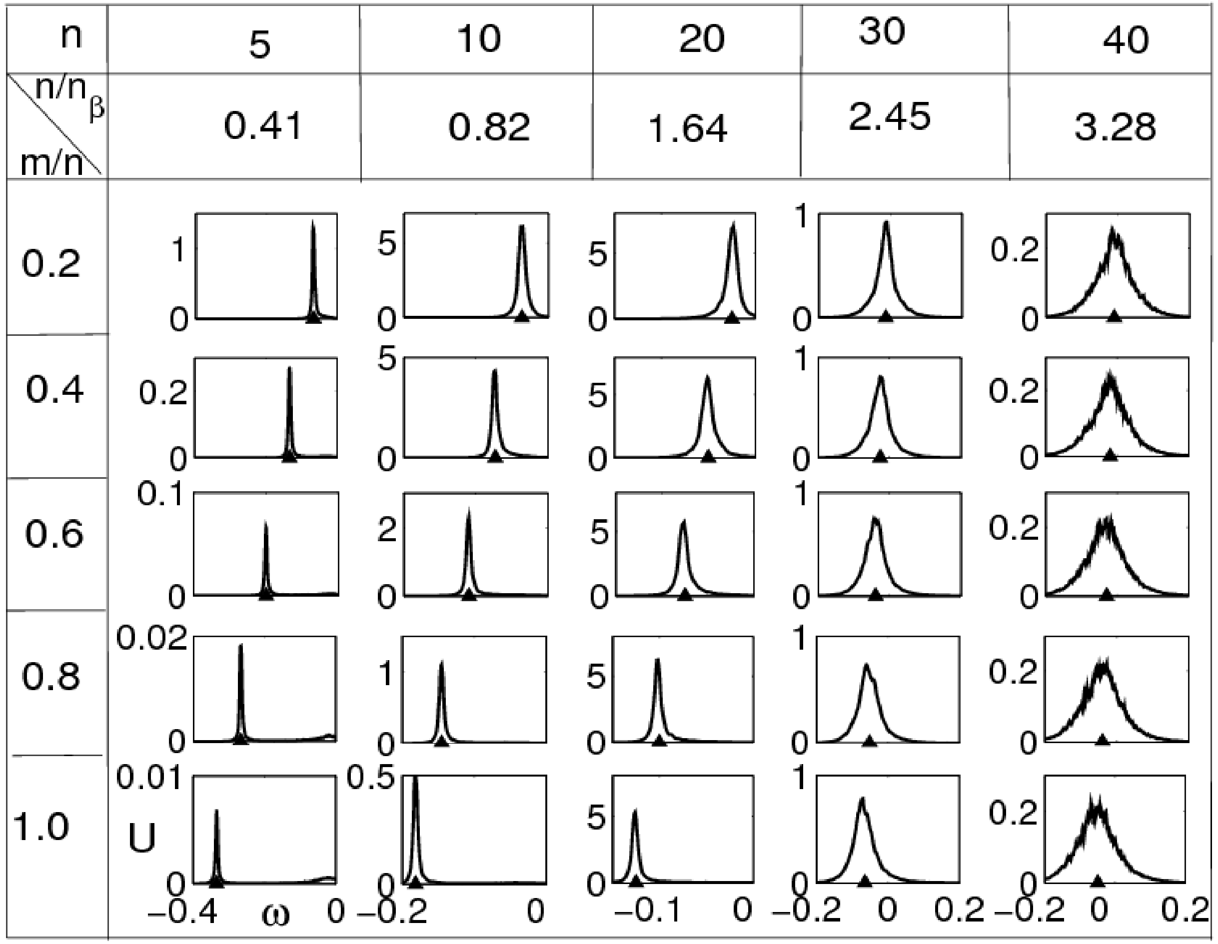
$\psi(\omega)$ is a time Fourier transformed spectral coefficient $\psi(t)$

**Spikes of $U(\omega,m,n)$ correspond to the dispersion relation → the correlator $U(\omega,m,n)$ is a convenient diagnostic tool for finding waves in data and in simulations**

# Waves in friction-dominated regime

$n_R = 9.2$
$n_\beta = 12.3$
$R_\beta = 1.34$

The filled triangles correspond to the RHWs dispersion relation.




❖The large-scale modes are populated by linear RHWs

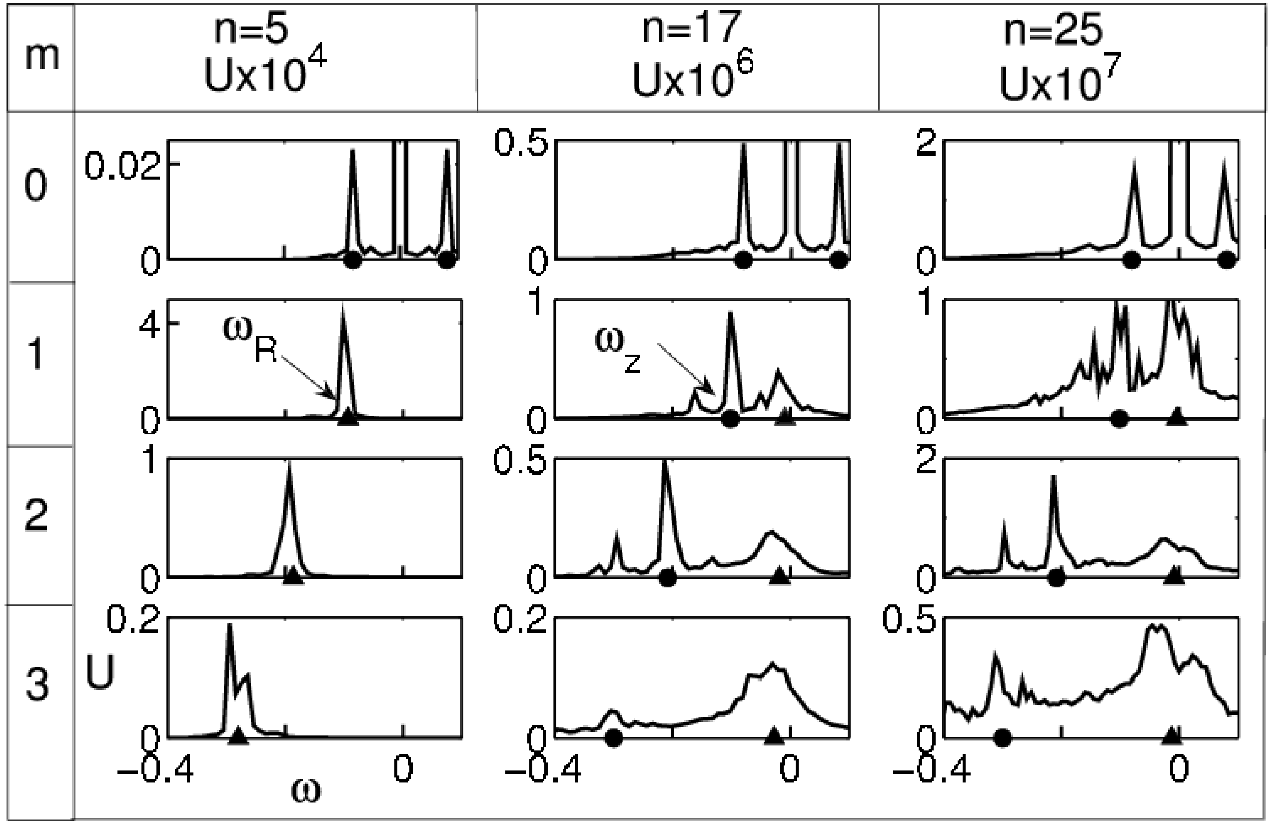
❖A strong RHW signature is present even on scales with $n/n_\beta > 2$

❖On the smallest scales, the RHW peaks are broadened by turbulence

❖Even though the flow dynamics is dominated by strong nonlinearity the flow features linear RHWs

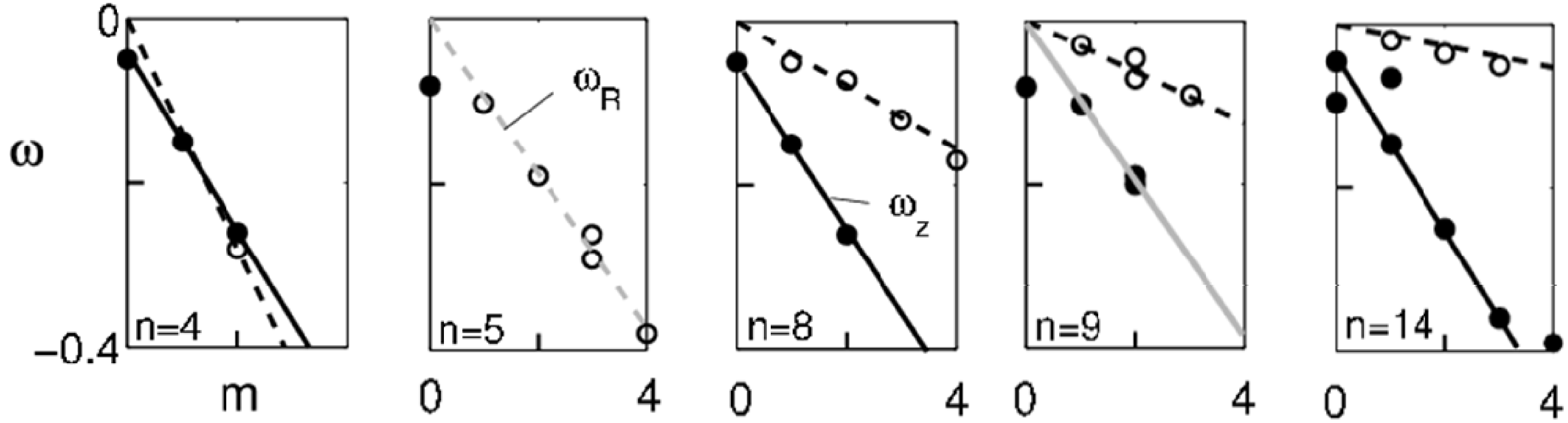
# Waves in zonostrophic turbulence

$n_R = 5.5$
$n_\beta = 16.2$
$R_\beta = 2.95$




❖U(ω,n,m) is the velocity correlator
❖Filled triangles ➔ RHWs dispersion relation
❖Filled circles ➔ zonons.

# Frequencies of RHWs and zonons as functions of m for different n




❖RHWs are evident in all modes including $n > n_\beta$

❖Along with RHWs, one discerns zonons excited by the most energetic RHWs with n = 4 and 5.

❖$\omega_z(n,m) \propto m$ and independent of n for all zonons.

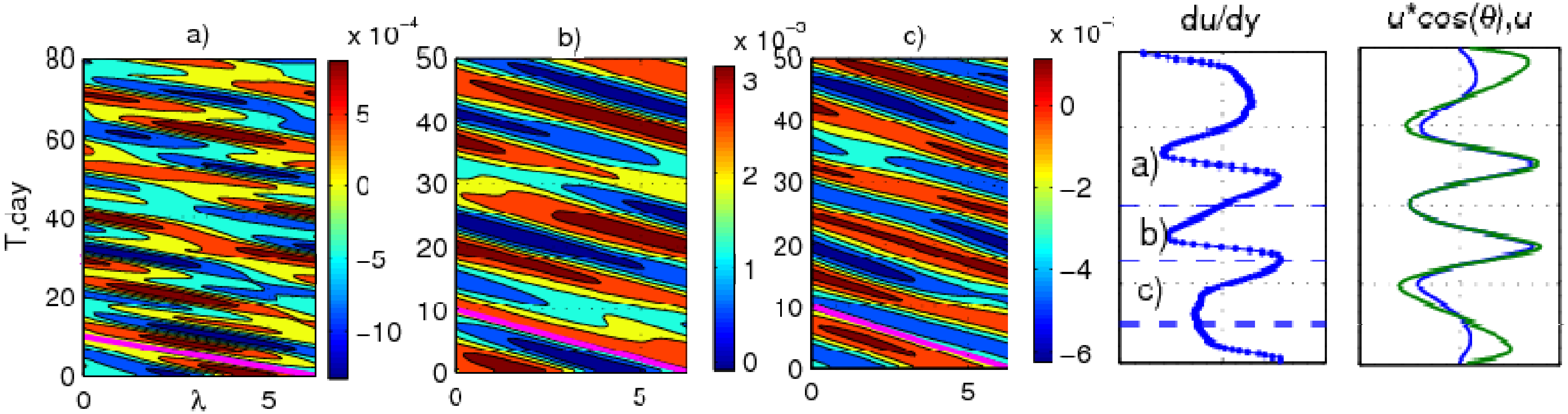
❖Zonons form wave packets

❖Their zonal speeds are $c_z = \omega_z(n,m)/m$

❖$c_z$ are equal to the zonal phase speeds of the corresponding master RHWs

- All zonons are ‘‘slave’’ waves excited by RHWs
- Their dispersion relations differ from RHWs ➔ zonons should be recognized as an entity completely different from RHWs
- How do the zonons appear in the physical space? The RHWs with n = 4 (denoted $n_E$) are the most energetic ➔ their respective packets of zonons are dominant in physical space and are easiest to observe
- The zonal speed of these packets is $\omega_R(n_E,m)/m = c_{RE}$
- In physical space, these wave packets are expected to form westward propagating eddies detectable in the Hovmoller diagrams
- The slope of the demeaned diagrams yields a velocity of the zonally propagating eddies relative to local zonal flows
- If eddies are indeed comprised of zonons, their zonal phase speed should be equal to $c_z = c_{RE}$

# Zonons are Rossby wave solitons

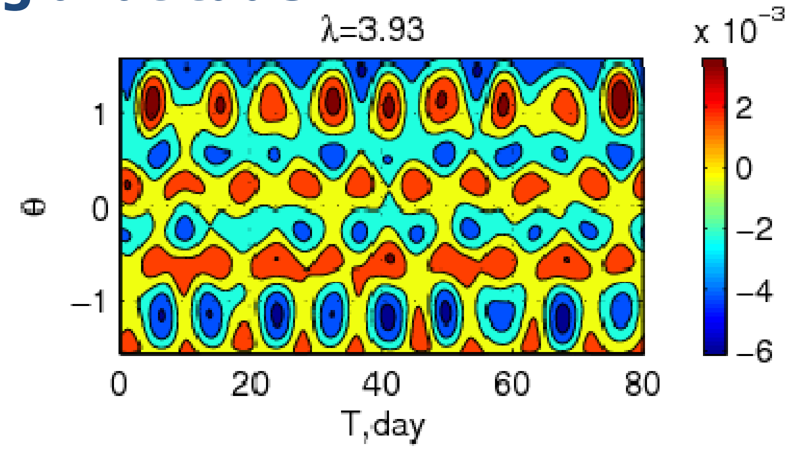



❖The Hovmoller diagrams reveal westward propagating eddies at three different latitudes at which the zonal jets have their maximum, minimum, and zero velocity

❖The slope of the diagrams yields a velocity of the zonally propagating eddies relative to local zonal flows. The figure demonstrates that $c_z = c_{RE}$ at all latitudes

## Homvoler diagram along a latitude

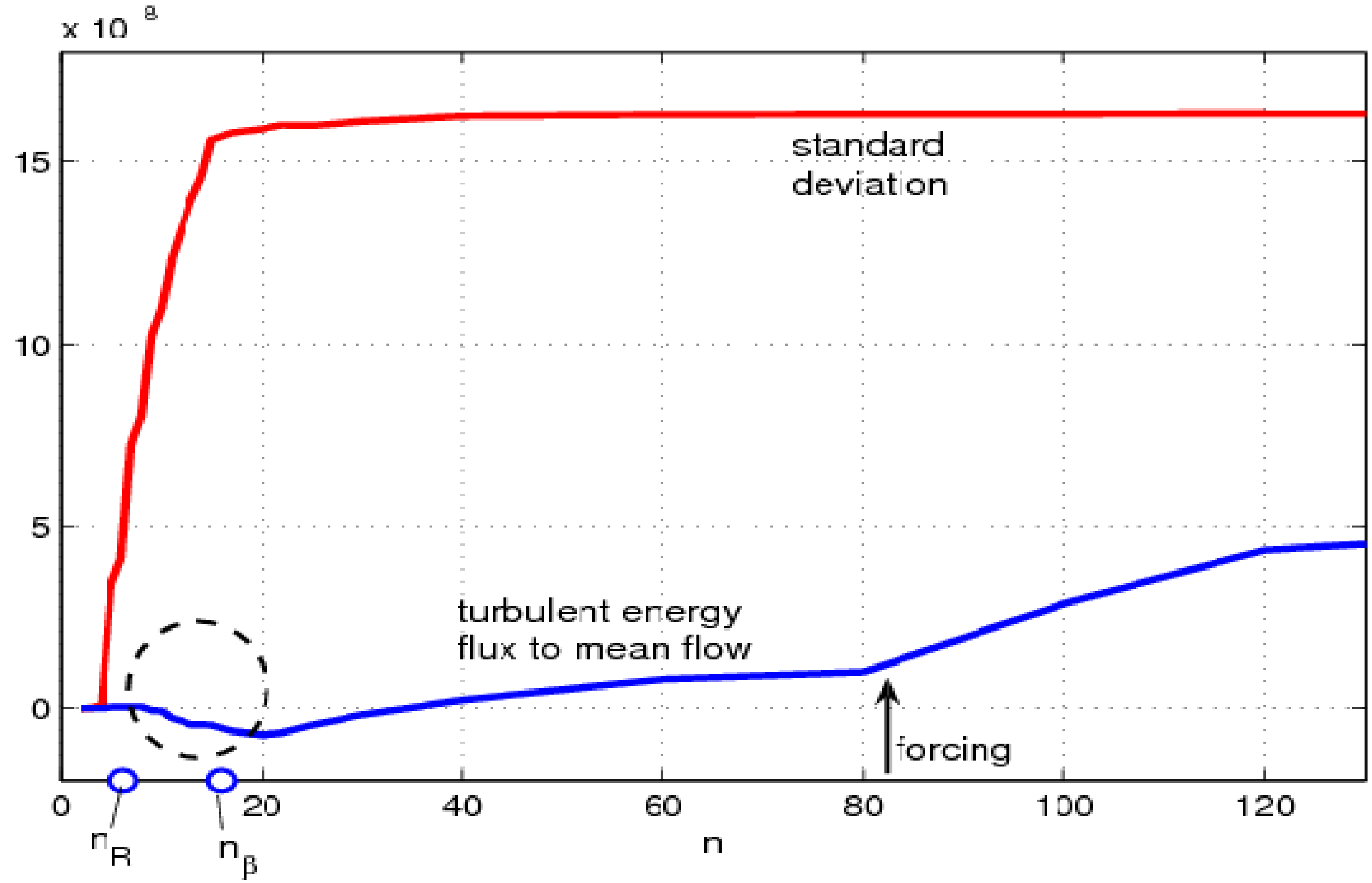
This diagram shows that zonons propagate along a latitude with maximal shear → zonal flow forms a waveguide, in which the solitary waves propagate



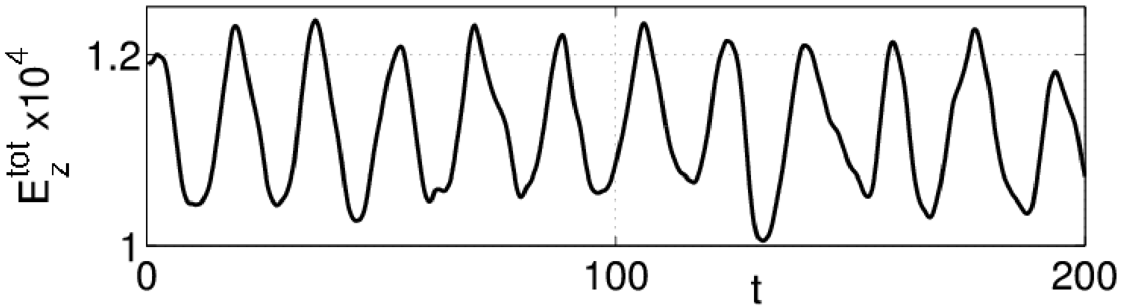
# Energy exchange between jets and waves/turbulence

T(m=0|n)

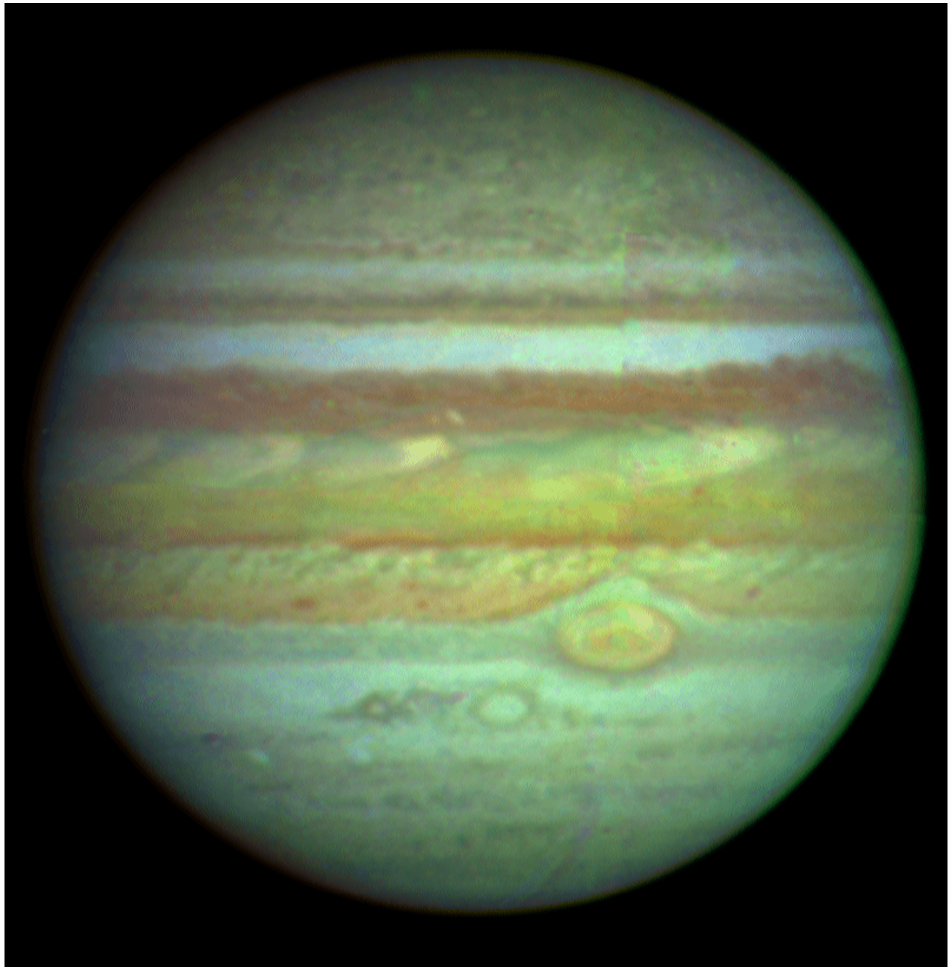
Energy flux from all nonzonal modes < n to mean flow

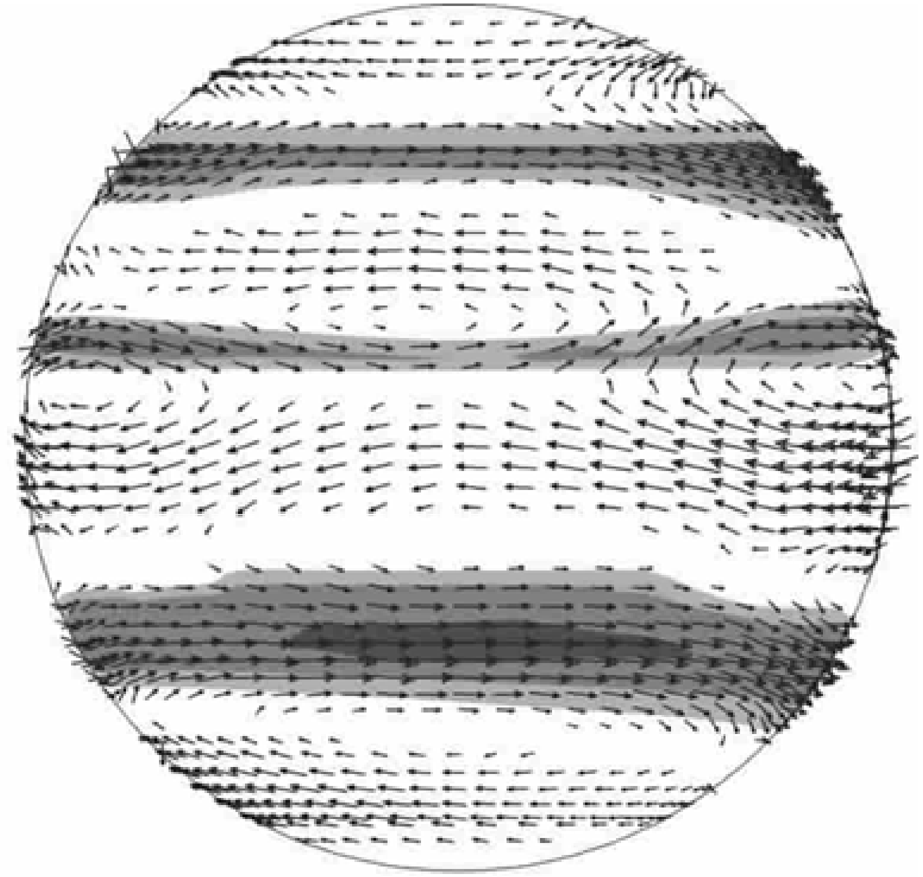



Nonlinear interactions between jets and zonons cause energy oscillations



# Jets-Zonons Symbiosis





The zonal flow forms a waveguide, in which the solitary waves propagate.
Zonons and mean flow (jets) continuously exchange large amount of energy.

# Conclusions

❖A new class of nonlinear waves in 2D turbulence with a $\beta$-effect, zonons, is presented

❖Zonons are forced oscillations excited by RHWs in other modes via non-linear interactions

❖Zonons are an integral part of the zonostrophic regime. They emerge in the process of energy accumulation in the large-scale modes, formation of the steep $n^{-5}$ spectrum and generation of zonal jets

❖Zonons have characteristic features of solitary waves. The zonal flow forms a waveguide, in which the solitary waves propagate.

❖Future research should clarify zonons' roles in planetary circulations and their relation to large oceanic eddies detected in satellite altimetry  (provided that the oceanic circulation is marginally zonostrophic)

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