Spectral and Algebraic Instabilities in Thin Keplerian Disks: I – Linear Theory

Edward Liverts

Michael Mond

Yuri Shtemler

Accretion disks – A classical astrophysical problem

- Magneto rotational instability (MRI) *Velikhov*, 1959, Chandrasekhar, 1960 – a major player in the transition to turbulence scenario, *Balbus and Hawley*, 1992.

- Original theory is for infinite axially uniform cylinders. Recent studies indicate stabilizing effects of finite small thickness. *Coppi and Keyes*, 2003 ApJ., *Liverts and Mond, 2008 MNRAS* have considered modes contained within the thin disk.

- Weakly nonlinear studies indicate strong saturation of the instability. *Umurhan et al., 2007, PRL.*

- Criticism of the shearing box model for the numerical simulations and its adequacy to full disk dynamics description. *King et al., 2007 MNRAS*, *Regev & Umurhan, 2008 A&A.*

Algebraic Vs. Spectral Growth

spectral analysis

$$f(\mathbf{r},t) \xrightarrow[t\to\infty]{} \sum f_n(\mathbf{r}) e^{-i\omega_n t}$$

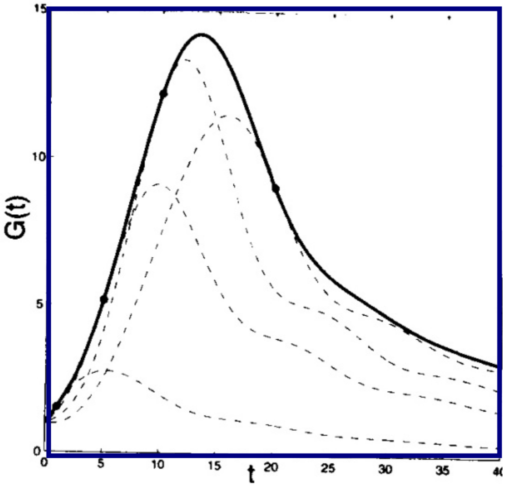
ω_n - Eigenvalues of the linear operator

Spectral stability if $\mathrm{Im}\,\omega_n < 0$ for all n s

Short time behavior

$$G(t) = \underset{f(0)}{Max} \frac{\|f(t)\|}{\|f(0)\|}$$

Schmid and Henningson, "Stability and Transition in Shear Flows"



Transient growth for Poiseuille flow.

$R_e = 1000$

Goals of the current research

To investigate the linear as well as nonlinear development of small perturbations in realistic thin disk geometry.

- Structure of the spectrum in stratified (finite) disk. Generalization to include short time dynamics. Serve as building blocks to nonlinear analysis.
- Derive a set of reduced model nonlinear equations
- Follow the nonlinear evolution.
- Have pure hydrodynamic activities been thrown out too hastily ?

Magneto Rotational Instability (MRI)

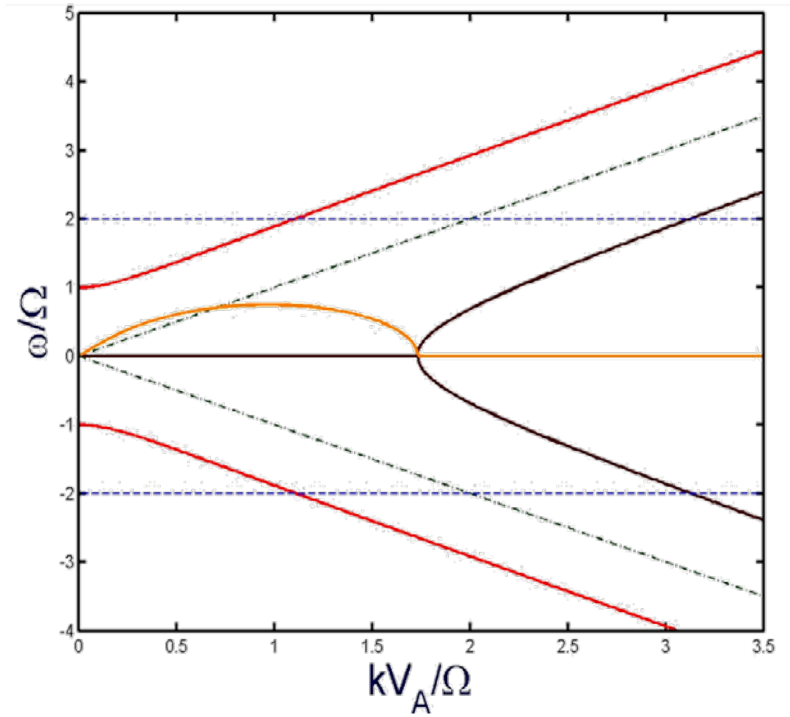
Governing equations

$$\frac{d\vec{v}}{dt} = -\frac{c_s^2}{\rho}\vec{\nabla}\rho - \Omega^2\vec{r} + \frac{1}{c\rho}\left[\vec{j}\times\vec{B}\right],$$

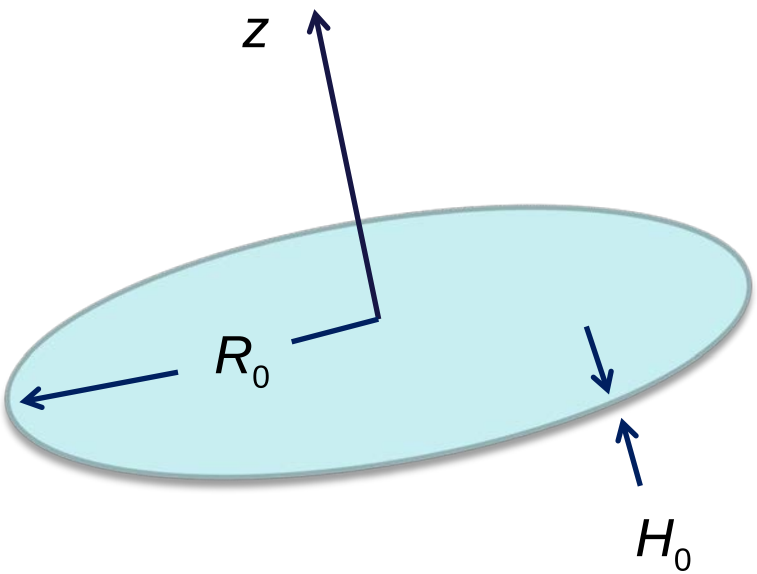
$$\frac{d\vec{B}}{dt} = \left(\vec{B}\cdot\vec{\nabla}\right)\vec{v} - \vec{B}\left(\vec{\nabla}\cdot\vec{v}\right), \quad \vec{j} = \frac{c}{4\pi}\vec{\nabla}\times\vec{B},$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{v} = 0, \quad \vec{\nabla}\cdot\vec{B} = 0.$$

$$\vec{B}\parallel\vec{z}, \quad \vec{v},\vec{b} = \vec{v},\vec{b}(z,t)$$



Thin Disk Geometry



Vertical stretching

$$\varepsilon = \frac{H_0}{R_0} \qquad \zeta = \frac{z}{\varepsilon}$$

$$\Rightarrow \quad \frac{\partial}{\partial z} = \frac{1}{\varepsilon}\frac{\partial}{\partial \zeta}$$

$$\frac{\partial}{\partial \zeta} \sim \frac{\partial}{\partial r}$$

Supersonic rotation:

$$M_s = \frac{1}{\varepsilon}$$

Vertically Isothermal Disks

$$\frac{d\mathbf{V}}{dt} = -\frac{\varepsilon^2}{n}\nabla P + \nabla \boldsymbol{\Phi} + \frac{\varepsilon^2}{\beta^2}\mathbf{j}\times\mathbf{B}$$

$$P(r,\zeta) = n(r,\zeta)T(r)$$

$$\beta \equiv 4\pi \frac{P_*}{B_*^2}$$

Vertical force balance: $n(r,\zeta) = N(r)e^{-\zeta^2/2H^2(r)}$

Radial force balance: $V_\vartheta(r) = r\boldsymbol{\Omega}(r) \;\; , \;\; \boldsymbol{\Omega}(\mathrm{r}) = r^{-3/2}$

Free functions: $\begin{cases} T(r) \\ N(r) \end{cases}$ The thickness $H(r)$ is determined by $H(r) = T(r)/\boldsymbol{\Omega}(r)$.

Two magnetic configurations

- I. Dominant axial component
- II. Dominant toroidal component

I.

$$B_z = \varepsilon^0 \bar{B}_z(r,\zeta)$$
$$B_\vartheta = \varepsilon \bar{B}_\vartheta(r,\zeta) = 0$$

Free function: $B_z(r)$

II.

$$B_z = \varepsilon \bar{B}_z(r,\zeta)$$
$$B_\vartheta = \varepsilon^0 \bar{B}_\vartheta(r,\zeta) = \bar{B}_\vartheta(r)$$

Free function: $B_\vartheta(r)$

Magnetic field does not influence lowest order equilibrium but makes a significant difference in perturbations' dynamics.

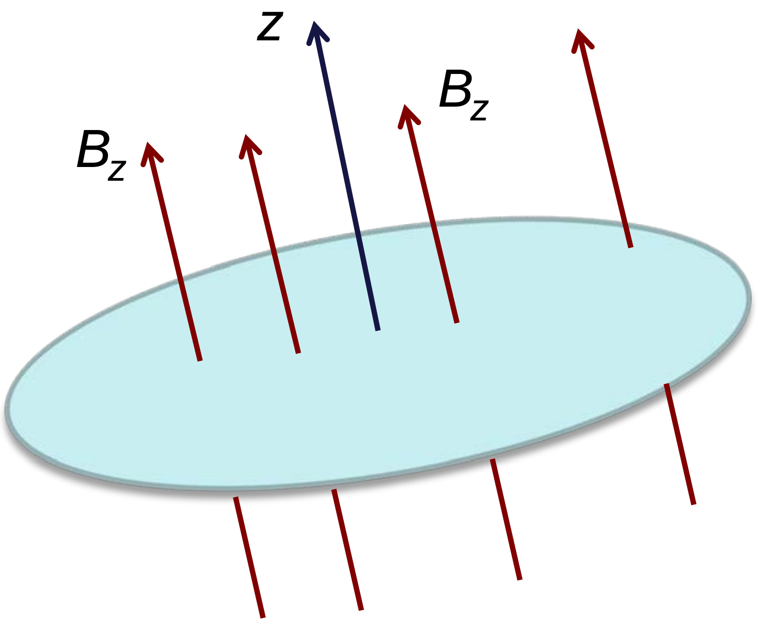
Linear Equations: Case I

$$\bar{n}(r,\zeta) = N(r)e^{-\zeta^2/2H^2(r)}$$

Self-similar variable

$$\eta \equiv \frac{\zeta}{H(r)}$$

Consider $B_\vartheta = 0$



$$\bar{\beta}(r) = \beta \frac{N(r)H^2(r)\Omega^2(r)}{\bar{B}_z^2(r)}$$

$$\left(H^2(r)\Omega^2(r) = C_s^2(r)\right)$$

The In-Plane Coriolis-Alfvén System

$$\frac{\partial V_r'}{\partial t} - 2\Omega(r)V_\vartheta' - \frac{\Omega(r)}{\beta(r)}\frac{1}{\tilde{n}(\eta)}\frac{\partial B_r'}{\partial \eta} = 0$$

$$\frac{\partial V_\vartheta'}{\partial t} + \frac{1}{2}\frac{\chi^2}{\Omega}V_r' - \frac{\Omega(r)}{\beta(r)}\frac{1}{\tilde{n}(\eta)}\frac{\partial B_\vartheta'}{\partial \eta} = 0$$

$$\frac{\partial B_r'}{\partial t} - \Omega(r)\frac{\partial V_r'}{\partial \eta} = 0$$

$$\frac{\partial B_\vartheta'}{\partial t} - \Omega(r)\frac{\partial V_\vartheta'}{\partial \eta} - \frac{d\ln\Omega}{d\ln r}B_r' = 0$$

$$\tilde{n}(\eta) = e^{-\eta^2/2}$$

$$\beta(r) = \beta\frac{N(r)H^2(r)\Omega^2(r)}{\bar{B}_z^2(r)}$$

Boundary conditions

$$B_{r,\vartheta} \xrightarrow[\zeta\to\pm\infty]{} 0$$

No radial derivatives ⟹ Only spectral instability possible

The Coriolis-Alfvén System

① Approximate profile

$$\tilde{n}(\eta) = e^{-\eta^2/2}$$

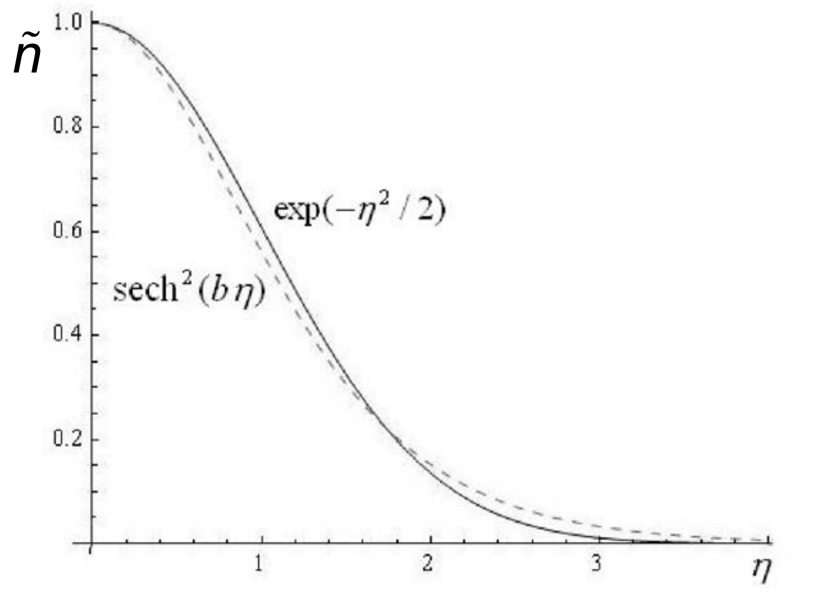
⇓

$$\tilde{n}(\eta) = \operatorname{sech}^2(b\eta)$$

$$\int_{-\infty}^{\infty} \operatorname{sech}^2(b\eta)d\eta = \int_{-\infty}^{\infty} e^{-\eta^2/2}d\eta$$

⇓

$$b = \sqrt{\frac{2}{\pi}}$$



② Change of independent variable

$$\xi \equiv \tanh(b\eta)$$

Spitzer, (1942)

The Coriolis-Alfvén Spectrum

③ $$f(r,\xi,t) = \hat{f}(r,\xi)e^{-i\omega t} \quad , \quad \omega = \lambda(r)\Omega(r)$$

⇒ $$(\mathrm{L}+K^{+})(\mathrm{L}+K^{-})\hat{V}_{\vartheta,r} = 0$$

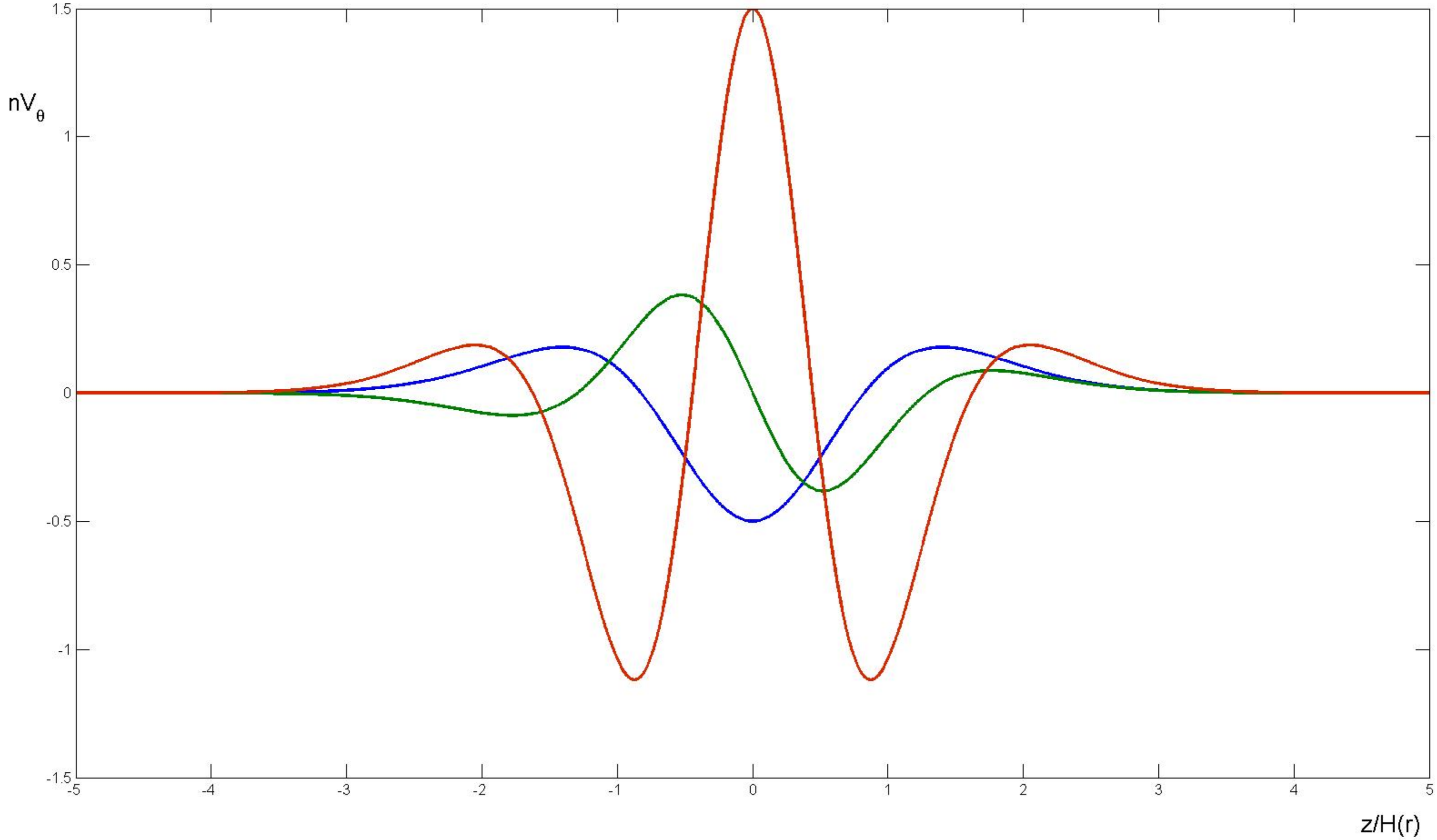
$$\mathrm{L} \equiv \frac{d}{d\xi}[(1-\xi^2)\frac{d}{d\xi}]$$ Legendre operator

$$K^{\pm} = \frac{\beta(r)}{2b^2}\left(3+2\lambda^2 \pm \sqrt{9+16}\lambda^2\right)$$

⇒ $$\hat{V}_{\vartheta,r} = P_k(\xi)$$

$$K^{\pm} = k(k+1) \quad , \quad k = 1,2,\ldots$$

The Coriolis-Alfvén Spectrum



The Coriolis-Alfvén Dispersion Relation

$$K^{\pm} = \frac{\beta(r)}{2b^2}\left(3 + 2\lambda^2 \pm \sqrt{9+16\lambda^2}\right) = k(k+1)$$

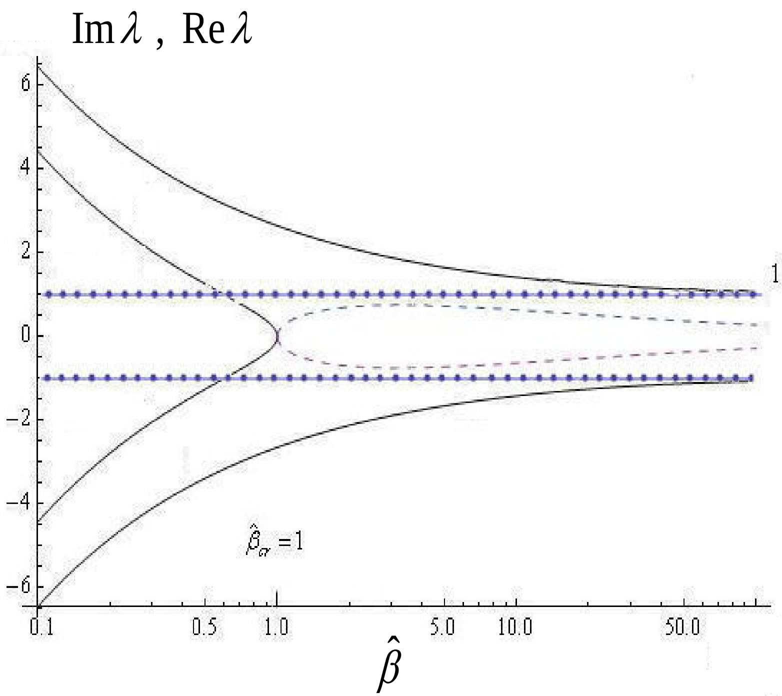
$$\Rightarrow \quad \hat{\beta}^2\lambda^4 - \lambda^2\hat{\beta}(\hat{\beta}+6) + 9(1-\hat{\beta}) = 0 \quad , \quad \hat{\beta} \equiv \frac{\beta(r)}{\beta_{cr}}$$

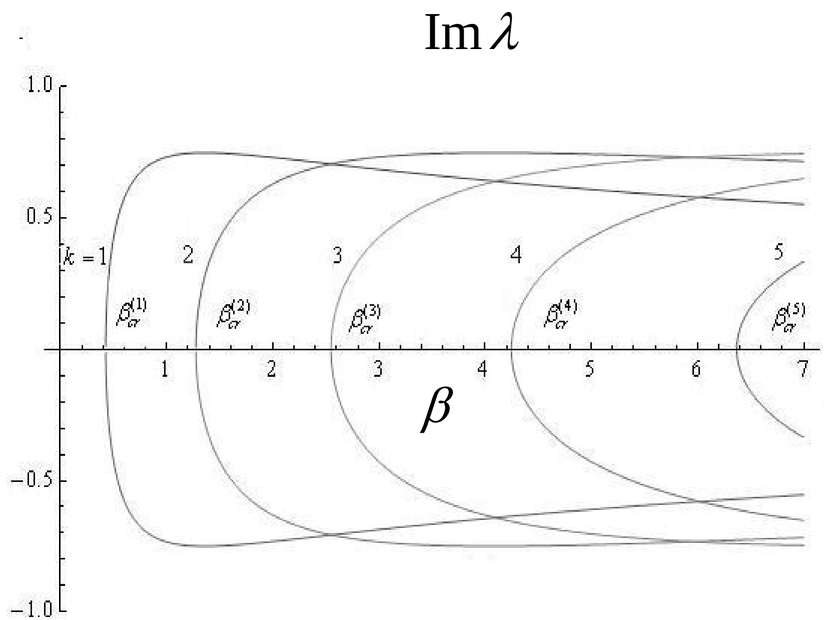
$$\beta_{cr} \equiv \frac{b^2}{3}k(k+1)$$

Universal condition for instability: $\hat{\beta} > 1$ $\Rightarrow$ $\beta > \beta_{cr}$

$\Rightarrow$ Number of unstable modes increases with β .

The Coriolis-Alfvén Dispersion Relation-The MRI





$\left(\mathrm{Im}\,\lambda\right)^{\max}$ for $\hat{\beta} \approx 3$

$\beta_{cr}(k=1) = 0.42$

The Vertical Magnetosonic System

$$\frac{\partial V_z'}{\partial t} + \Omega(r)\frac{\partial}{\partial \eta}\left[\frac{n'}{\tilde{n}(\eta)}\right] = 0$$

$$\frac{\partial n'}{\partial t} + \Omega(r)\frac{\partial}{\partial \eta}\left[\tilde{n}(\eta) V_z'\right] = 0$$

$$\tilde{n}(\eta) = \sec h^2(b\eta)$$

$$f(r,\xi,t) = \hat{f}(r,\xi)e^{-i\omega t}$$

$$\omega = \lambda(r)\Omega(r)$$

$$\xi \equiv \tanh(b\eta)$$

$$\Rightarrow$$

$$(1-\xi^2)\frac{d^2\hat{n}}{d\xi^2} + \left[2 + \frac{\omega^2}{1-\xi^2}\right]\hat{n} = 0$$

$$\hat{n}(\eta = \pm\infty) = \hat{n}(\xi = \pm 1) = 0$$

Continuous acoustic spectrum

Define: $$n'(\xi) = \sqrt{1-\xi^2}\, g(\xi)$$

$$(1-\xi^2)\frac{d^2 g}{d\xi^2} - 2\xi \frac{d^2 g}{d\xi^2} + \left[2 - \frac{1-\omega^2}{1-\xi^2}\right] g = 0$$

Associated Legendre equation

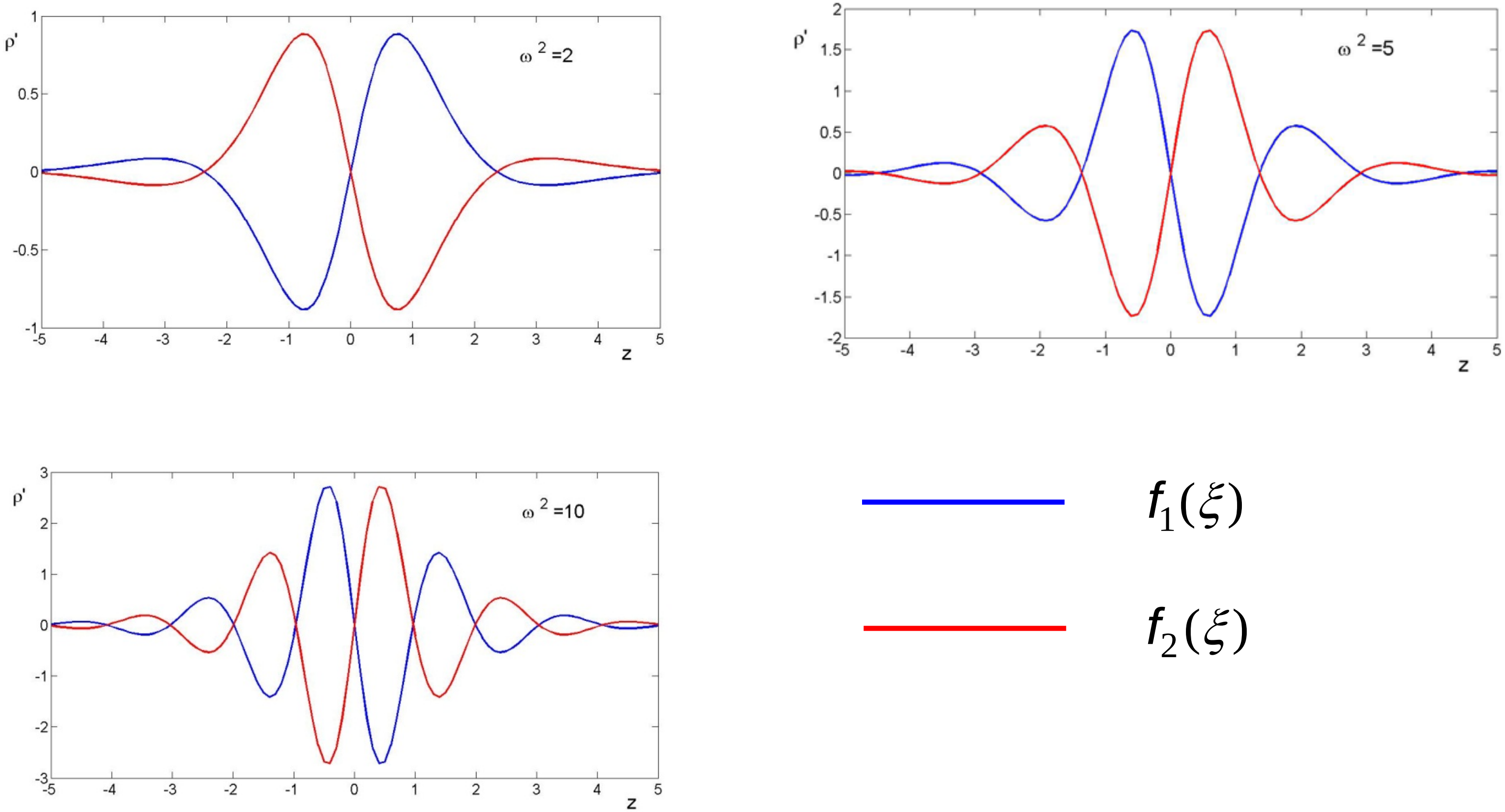
$$n'(\xi) = \sqrt{1-\xi^2}\left[c_1 f_1(\xi) + c_2 f_2(\xi)\right]$$

$$f_1(\xi) = \left(\frac{1-\xi}{1+\xi}\right)^{-\frac{\mu}{2}} (\mu - \xi)$$

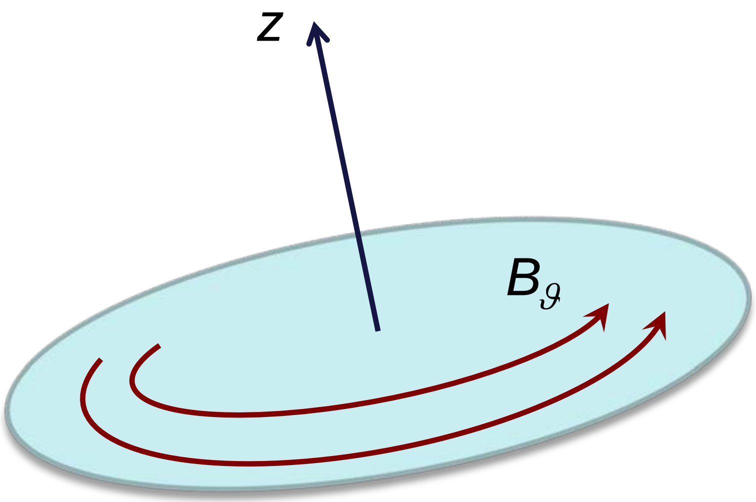
$$f_2(\xi) = \left(\frac{1-\xi}{1+\xi}\right)^{\frac{\mu}{2}} (\mu + \xi)$$

$$\mu^2 = 1 - \omega^2$$

Continuous eigenfunctions



Linear Equations: Case II



$$B_z = \varepsilon \bar{B}_z(r,\zeta)$$

$$B_\vartheta = \bar{B}_\vartheta(r,\zeta)$$

$$\bar{n}(r,\zeta) = N(r)e^{-\zeta^2/2H^2(r)}$$

Axisymmetric perturbations are spectrally stable.

⇒ Examine non-axisymmetric perturbations.

$$\tau \equiv \Omega(r)t \ , \ \Theta \equiv \vartheta\text{-}\tau \ , \ \eta \equiv \frac{\zeta}{H(r)} \ , \ \rho \equiv \int^{r} \frac{dr'}{H(r')}$$

Two main regimes of perturbations dynamics

I. Decoupled inertia-coriolis and MS modes

II. Coupled inertia-coriolis and MS modes

A. Inertia-coriolis waves driven algebraically by acoustic modes

B. Acoustic waves driven algebraically by inertia-coriolis modes

I.A – Inertial-coriolis waves driven by MS modes

The magnetosonic eigenmodes

$$\frac{\partial V_z'}{\partial \tau}+\frac{\partial n'}{\partial \eta}+\frac{1}{\beta_\vartheta(\rho)}\frac{1}{\bar{n}(\eta)}\frac{\partial b_\vartheta}{\partial \eta}=0$$

$$\beta_\vartheta(\rho)\equiv\beta\frac{N(\rho)c_s^2(\rho)}{B_\vartheta^2(\rho)}$$

$$\frac{\partial n'}{\partial \tau}+\frac{1}{\bar{n}(\eta)}\frac{\partial[\bar{n}(\eta)V_z']}{\partial \eta}=0$$

$$f(\rho,\eta,\Theta,\tau)=\hat{f}(\rho,\eta)e^{-i\lambda\tau+ik\Theta}$$

$$\frac{\partial b_\vartheta}{\partial \tau}+\frac{\partial V_z'}{\partial \eta}$$

$$\Downarrow$$

$$\frac{d^2\hat{b}_\vartheta}{d\eta^2}-\eta\frac{\beta_\vartheta\bar{n}(\eta)-1}{\beta_\vartheta\bar{n}(\eta)+1}\frac{d\hat{b}_\vartheta}{d\eta}+(\lambda^2-2)\frac{\beta_\vartheta\bar{n}(\eta)}{\beta_\vartheta\bar{n}(\eta)+1}\hat{b}_\vartheta=0$$

WKB solution for the eigenfunctions

Turning points at $\eta^* = \pm|\lambda|$

The Bohr-Sommerfeld condition

$$\int_{-\eta^*}^{\eta^*} \kappa(\eta)d\eta = \frac{\pi}{2}(m+1) \quad , \quad \kappa(\eta) \approx -\frac{1}{4}(\eta^2 - \eta^{*2})$$

$$\Rightarrow \quad \lambda_{MS} = \pm\sqrt{m+1} \qquad f(\rho,\eta,\Theta,\tau) = \hat{f}(\rho,\eta)e^{-i\lambda\tau+ik\Theta}$$

The driven inertia-coriolis (IC) waves

$$\frac{\partial V_r'}{\partial \tau} - 2V_\vartheta' = \tau L_1\{n', V_z', b_\vartheta\} + L_0\{n', V_z', b_\vartheta\}$$

$$\frac{\partial V_\vartheta'}{\partial \tau} + \frac{1}{2}V_r' = K_0\{n', V_z', b_\vartheta\}$$

IC operator   Driving MS modes

$$v_r = \tau \frac{i\lambda_{MS}}{\lambda_{MS}^2 - 1}\bar{D}_\Omega(\rho)\left\{ik\left(\hat{v} + \frac{1}{\bar{\beta}_\theta}\frac{1}{\bar{v}(\eta)}\hat{b}_\theta\right) + \frac{1}{\bar{v}(\eta)}\frac{d[\bar{v}(\eta)\hat{v}_z]}{d\eta}\right\}\exp[-i\lambda_{MS}\tau + ik\Theta]$$

$$\bar{D}_\Omega(\rho) = \frac{d\ln\Omega}{d\rho} \qquad b_r = \frac{1}{\bar{S}(\rho)}\frac{dv_r}{d\eta}$$

II – Coupled regime

Need to go to high axial and azimuthal wave numbers

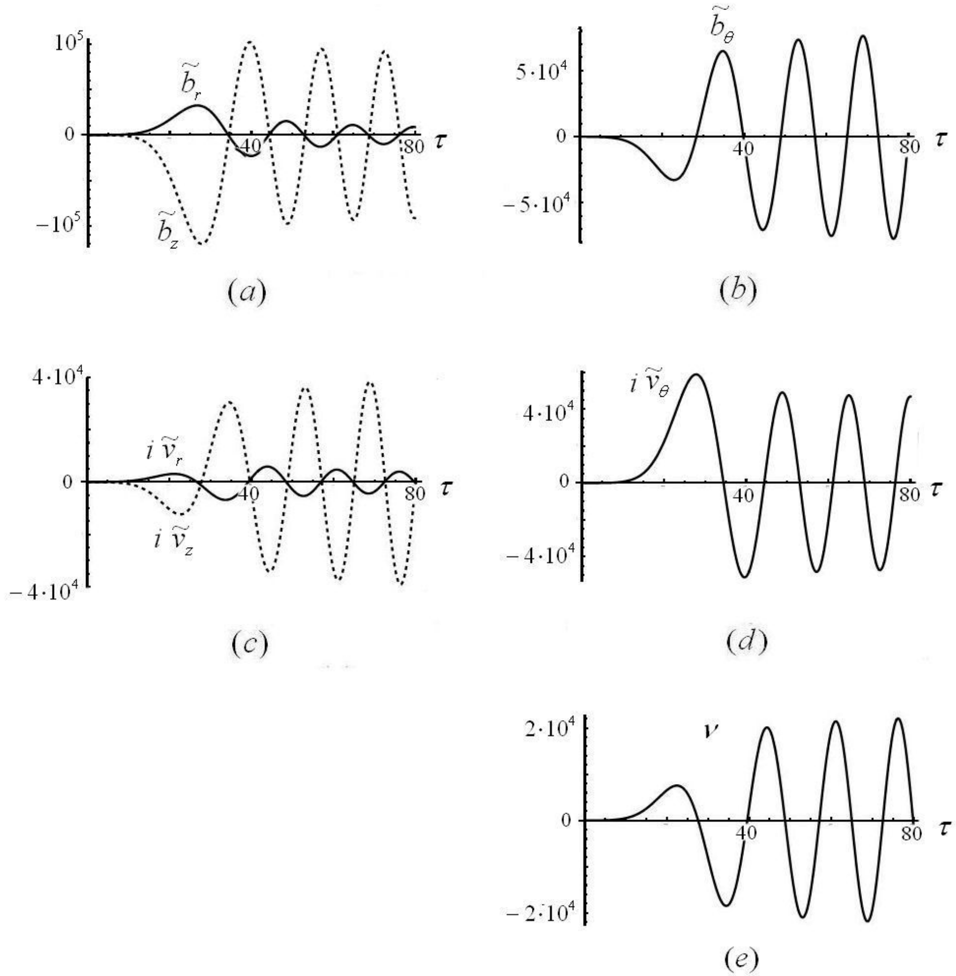
$$k_{\vartheta},\ k_z\ \sim\ \varepsilon^{-1}$$

$$\frac{d^2\tilde{b}_r}{d\tau^2} = -2\tilde{k}_\eta \frac{d\tilde{b}_z}{d\tau} - 2\Delta\tilde{k}_r(\tau)\frac{d\tilde{b}_r}{d\tau} + \Delta\tilde{k}_r(\tau)\tilde{\nu} - \frac{1}{\tilde{\beta}_\theta}\{[\Delta\tilde{k}_r^2(\tau)+1]\tilde{b}_r + \Delta\tilde{k}_r(\tau)\tilde{k}_\eta\tilde{b}_z\}$$

$$\frac{d^2\tilde{b}_z}{d\tau^2} = \tilde{k}_\eta\tilde{\nu} - \frac{1}{\tilde{\beta}_\theta}[(1+\tilde{k}_z^2)\tilde{b}_z + \Delta\tilde{k}_r(\tau)\tilde{k}_\eta\tilde{b}_r]$$

$$\Delta\tilde{k}_r(\tau,\rho) = \tilde{k}_r - \rho\bar{D}_{\Omega}(\rho)\tau \equiv \tilde{k}_r + \frac{3}{2}\tau$$

Instability



Hydrodynamic limit

Both regimes have clear hydrodynamic limits.

Similar algebraic growth for decoupled regime.

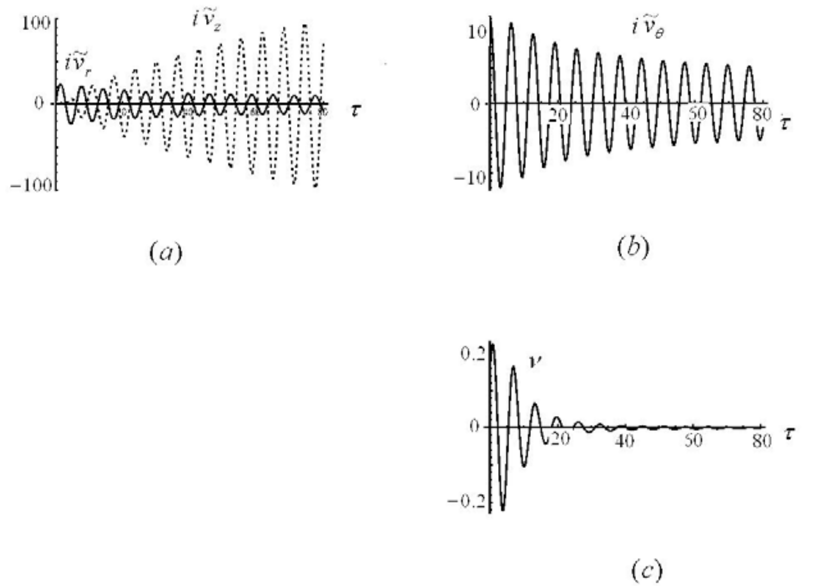
Much reduced growth is obtained for the coupled regime.

Consistent with the following hydrodynamic calculations:

Umurhan et al. 2008

Rebusco et al. 2010

Shtemler et al. 2011



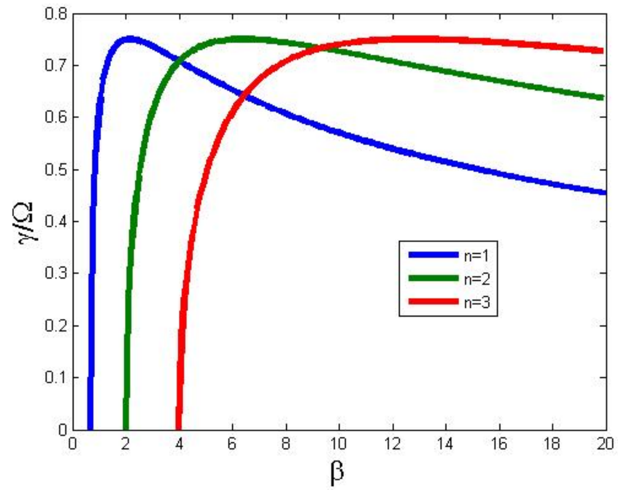
II – Nonlinear theory

$$\frac{\partial}{\partial t}\begin{pmatrix} v_r \\ v_\vartheta \end{pmatrix} = \widehat{L}_{A1}\begin{pmatrix} v_r \\ v_\vartheta \end{pmatrix} + (1-\xi^2)\left(\frac{1}{4\pi((1-\xi^2)+\rho}b_z\frac{\partial}{\partial\xi}\begin{pmatrix} b_r \\ b_\vartheta \end{pmatrix} - v_z\frac{\partial}{\partial\xi}\begin{pmatrix} v_r \\ v_\vartheta \end{pmatrix}\right) + O(\varepsilon),$$

$$\frac{\partial}{\partial t}\begin{pmatrix} b_r \\ b_\vartheta \end{pmatrix} = \widehat{L}_{A2}\begin{pmatrix} b_r \\ b_\vartheta \end{pmatrix} + (1-\xi^2)\left(b_z\frac{\partial}{\partial\xi}\begin{pmatrix} v_r \\ v_\vartheta \end{pmatrix} - \frac{\partial}{\partial\xi}\left(v_z\begin{pmatrix} b_r \\ b_\vartheta \end{pmatrix}\right)\right) + O(\varepsilon),$$

$$\frac{\partial}{\partial t}\begin{pmatrix} v_z \\ \rho \end{pmatrix} = \widehat{L}_s\begin{pmatrix} v_z \\ \rho \end{pmatrix} - (1-\xi^2)\begin{pmatrix} \dfrac{1}{8\pi\beta(1-\xi^2+\rho)} \\ 1 \end{pmatrix}\frac{\partial}{\partial\xi}\begin{pmatrix} b_r^2 + b_\vartheta^2 \\ \rho v_z \end{pmatrix} + O(\varepsilon),$$

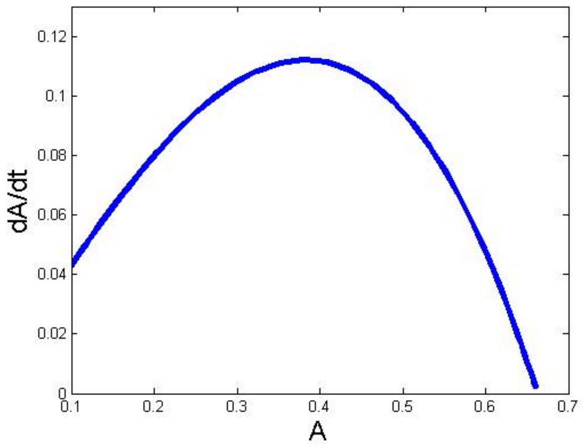
Weakly nonlinear asymptotic model



$$\beta_c = \frac{2}{3}, \quad \omega = i\gamma, \quad \gamma = \sqrt{\tfrac{27}{14}(\beta - \beta_c)}.$$

$$A \sim \sqrt{\gamma}$$

Phase plane

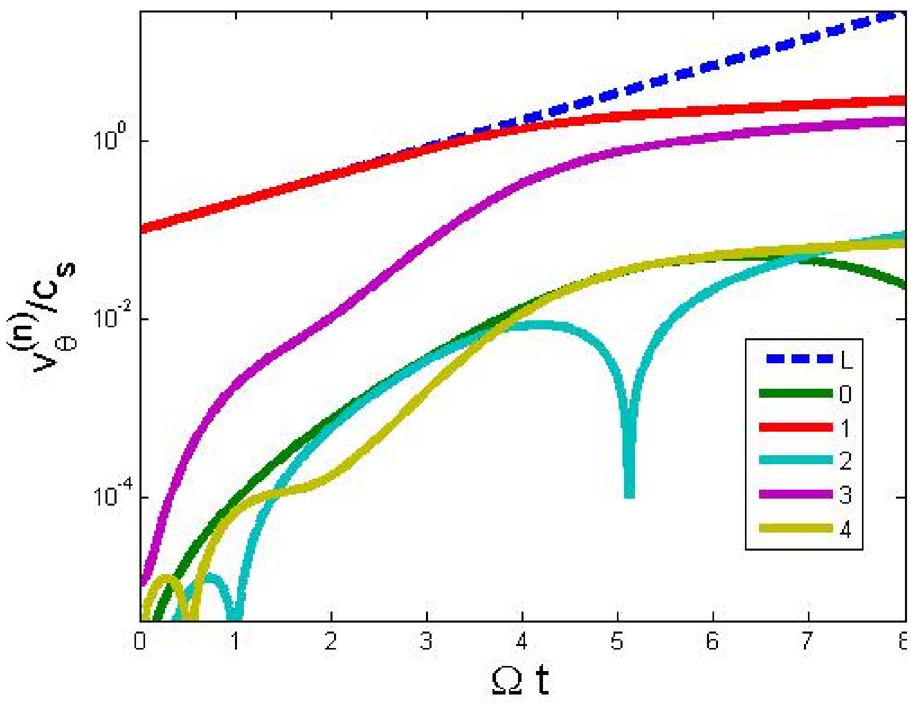


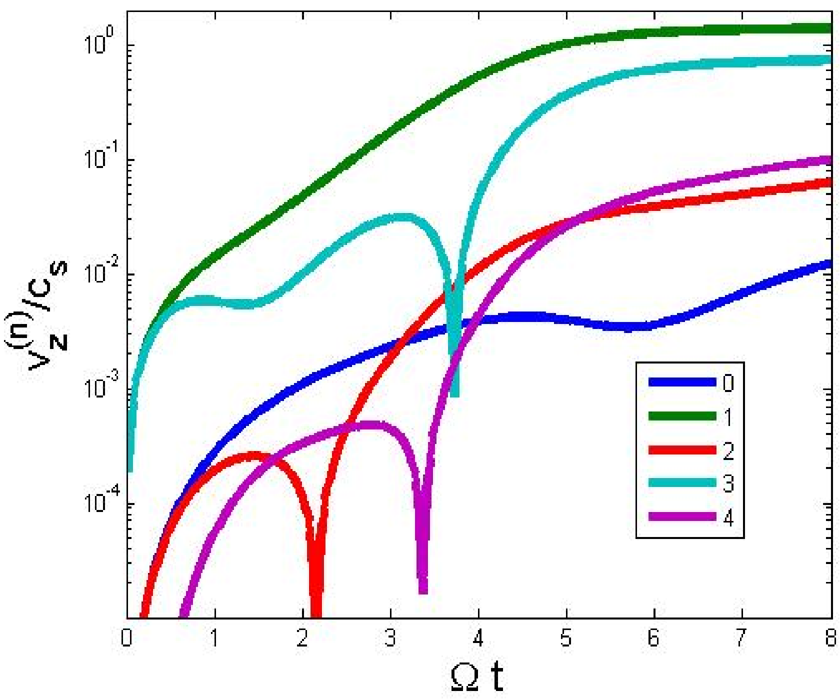
Ginzburg-Landau equation

$$\frac{\partial A}{\partial t} = \gamma A - \alpha A^3$$

Landau, (1948)

Numerical Solution





$$v_{\vartheta,z}(t,y) = v_{\vartheta,z}^{(1)}(t)P_1(y) + v_{\vartheta,z}^{(2)}(t)P_2(y) + v_{\vartheta,z}^{(3)}(t)P_3(y) + \ldots$$

Weakly nonlinear theory, range I

$$[2-\omega^2\beta][2-(3+\omega^2)\beta]-4\omega^2\beta=0,\quad \beta_c=\frac{2}{3},\quad \omega=i\gamma,\quad \gamma=\sqrt{\tfrac{27}{14}(\beta-\beta_c)}.$$

$$\begin{pmatrix} v_r \\ v_\vartheta \end{pmatrix} = A(t)\begin{pmatrix} \sqrt{\tfrac{6}{7}(\beta-\beta_c)} \\ 1 \end{pmatrix} P_1(y),\quad \begin{pmatrix} b_r \\ b_\vartheta \end{pmatrix} = A(t)\begin{pmatrix} \tfrac{2}{3}+\tfrac{4}{7}(\beta-\beta_c) \\ -\sqrt{\tfrac{8}{21}(\beta-\beta_c)} \end{pmatrix} [P_0(y)-P_2(y)].$$

$$v_z = B(t)\xi(1-\xi^2)$$

$$\rho = C(t)(1-6\xi^2+5\xi^4)$$

$$\int_{-1}^{1} P_n(\xi)P_m(\xi)d\xi = \frac{2}{2n+1}\delta_{nm}$$

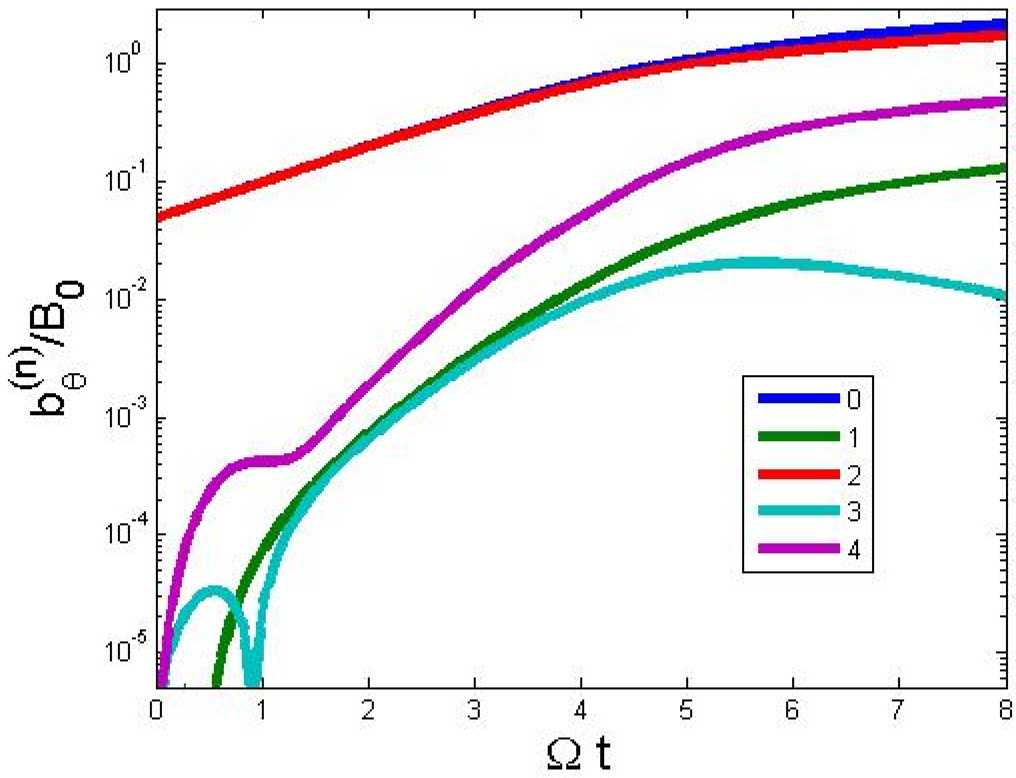
$$\begin{cases} \dfrac{dA}{dt} = \gamma A - \dfrac{8}{35}AB + O(A^4), \\ \dfrac{dB}{dt} = C + \dfrac{4}{9}A^2 + O(A^3), \\ \dfrac{dC}{dt} = -\dfrac{B}{10} + O(A^3). \end{cases}$$

Nonlinear theory, range II

$$v_z = B_1(t)P_1(\xi) - B_3(t)P_3(\xi),$$
$$\rho = C_2(t)P_2(\xi) - C_4(t)P_4(\xi)$$

$$\begin{cases} \dfrac{dA}{dt} = \gamma A + A(\dfrac{8}{75}B_1 + \dfrac{8}{175}B_2) + O(A^3), \\ \dfrac{dB_1}{dt} = \dfrac{385}{104}C_2 + \dfrac{105}{52}C_4 + \dfrac{49}{17}A^2 + O(A^3), \\ \dfrac{dB_3}{dt} = -\dfrac{315}{104}C_2 + \dfrac{245}{52}C_4 - \dfrac{35}{13}A^2 + O(A^3), \\ \dfrac{dC_2}{dt} = \dfrac{1}{5}B_1 + \dfrac{6}{5}B_3 + O(A^3), \\ \dfrac{dC_4}{dt} = \dfrac{20}{7}B_3 + O(A^3). \end{cases}$$

Generation of toroidal magnetic field



$$b_{\vartheta} = A(t)[P_0(\xi) - P_2(\xi)]$$

$$b_{\vartheta} \propto b_r \frac{d\Omega}{dr} rt$$

$$b_{\vartheta}(t, y) = b_{\vartheta}^{(1)}(t)P_1(y) + b_{\vartheta}^{(2)}(t)P_2(y) + b_{\vartheta}^{(3)}(t)P_3(y) + \ldots$$

Conclusions

- It is shown that the spectrum in a stratified disk (the finite thickness) contains a discrete part and a continuous one for AC and MS waves respectively. In the linear approximation, the AC and MS waves are decoupled. There is a special critical value of parameter plasma beta β_c for which the system becomes unstable if $\beta - \beta_c \succ 0$

- Both main regimes of perturbations dynamics, namely decoupled (coupled) AC and MS modes, are discussed in detail and compared with hydrodynamic limits.

- It was found from numerical computations that the generalization to include nonlinear dynamics leads to saturation of the instability.

- The nonlinear behavior is modeled by Ginzburg-Landau equation in case of weakly nonlinear regime (small supercriticality) and by truncated system of equations in the case of fully nonlinear regime. The latter is obtained by appropriate representation of the solutions of the nonlinear problem using a set of eigenfunctions for linear operator corresponding to AC waves