# Between the expert and majority rules

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**Abstract.** Our starting point is Sapir (1998), where the probabilities of the expert rule and of the simple majority rule being optimal were calculated under the assumption of exponentially distributed logarithmic expertise levels. Here we find the analogous probabilities for the family of restricted majority rules, including the above two extreme rules as special cases, and the family of balanced expert rules. We compare the two families, the rules within each family, and all rules of the two families with the extreme rules.

**Keywords:** dichotomous choice model, experts, decision rule, balanced expert rules, restricted majority rules, probability being optimal, logarithmic expertise levels, partial information.

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## 1. Introduction

We study the uncertain dichotomous choice model, which goes back as far as Condorcet (1785). In this model a group of $n$ decision makers is required to select one of two alternatives, only one of which is correct. We assume that the alternatives are symmetric. Each expert $i$ selects independently of the others and has his own correctness probability $p_i$, indicating his ability to identify the correct alternative. A decision rule translates all the individual opinions of the members into a group decision. A decision rule is *optimal* if it maximizes the probability of the group to make a correct choice, for all possible combinations of opinions.

Nitzan and Paroush (1982, 1984a, 1985) obtained a criterion to identify the optimal decision rule for known values of correctness probabilities. They proved that the optimal decision rule is always one of the weighted majority rules.

However, in a variety of situations the values of the correctness probabilities are unknown. Thus we assume that the correctness probabilities $p_i$ or, equivalently, the *logarithmic expertise levels* $f(p_i) = \ln \frac{p_i}{1-p_i}$ are independent random variables, distributed according to some known distribution function. We assume that $p_i \in [\frac{1}{2}, 1], \quad i = 1, \ldots, n$ (cf. Nitzan and Paroush (1982, 1984a, 1985)). If the ranking of the members in the group is known, then one can follow rules based on this ranking. The extremes are the expert and majority rules. The expert rule assigns zero weights to all members of the group but the most qualified one, so that the group always follows his decision. The majority rule assigns equal weights to all decision makers, so that the group always follows the majority opinion.

A general comprehensive study of weighted majority rules is a very complicated task, since the class of such rules becomes very large as the number of group members increases. For example, for committee size $n = 3$ it includes 2 weighted majority rules, for size four $-$ 3 rules, for size five $-$ 7, for size six $-$ 21 (von Neumann and Morgenstern (1944)), for size seven $-$ 135 (Isbell (1959) and Fishburn and Gehrlein (1977)), for size eight $-$ 2470 (cf. Karotkin (1993)) and for size nine $-$ 172958 rules (Muroga *et al* (1967)). Karotkin (1994, see also. 1998) coded a general algorithm to identify all the weighted majority rules for any group size. Nitzan and Paroush (1985), by Monte Carlo simulation, calculated the probabilities of such rules to be optimal for $n \leq 5$, under the assumption of uniform distribution on $[\frac{1}{2}, 1]$ of the $p_i$'s. Table I provides their results.

Table I. Optimality probabilities of all weighted majority rules for $n \leq 5$ for $p_i \sim$ U$[1/2, 1]$

| n | Rules | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | (1,0,0)<br>0.675 | (1,1,1)<br>0.325 | | | | | |
| 4 | (1,0,0,0)<br>0.373 | | (1,1,1,0)<br>0.277 | | (2,1,1,1)<br>0.350 | | |
| 5 | (1,0,0,0,0)<br>0.199 | (1,1,1,1,1)<br>0.022 | (1,1,1,0,0)<br>0.175 | (3,1,1,1,1)<br>0.107 | (2,1,1,1,0)<br>0.229 | (3,2,2,1,1)<br>0.194 | (2,2,1,1,1)<br>0.074 |

Table II represents the optimality probabilities of the same rules under the assumption of exponentially distributed logarithmic expertise levels $f(p_i)$, as found in Sapir (1998).

In both cases the expert rule is far more likely to be optimal than the majority rule. Note that for $n = 5$ the "leaders" were the expert rule, $(1, 0, 0, 0, 0)$, for the exponential distribution and some rule "close" to the expert rule, called balanced expert rule, $(2, 1, 1, 1, 0)$, for the

Table II. Optimality probabilities of all weighted majority rules for $n \leq 5$ for $f(p_i) \sim \mathrm{E}(\lambda)$

| n | Rules | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | (1,0,0)<br>0.75 | (1,1,1)<br>0.25 | | | | | |
| 4 | (1,0,0,0)<br>0.5 | | (1,1,1,0)<br>0.25 | | (2,1,1,1)<br>0.25 | | |
| 5 | (1,0,0,0,0)<br>0.312 | (1,1,1,1,1)<br>0.010 | (1,1,1,0,0)<br>0.157 | (3,1,1,1,1)<br>0.104 | (2,1,1,1,0)<br>0.208 | (3,2,2,1,1)<br>0.157 | (2,2,1,1,1)<br>0.052 |

uniform distribution, while the "loser" in both of the situations was the majority rule.

The probabilities $P_{\mathrm{e}}(n)$ and $P_{\mathrm{m}}(n)$ of the expert and majority rules being optimal were calculated or estimated in a series of papers for a variety of distributions (Nitzan and Paroush (1985), Berend and Harmse (1993), Berend and Sapir (2001, 2002a, 2002b), Sapir (1998, 1999, 2002)). The comparison of $P_{\mathrm{e}}(n)$ and $P_{\mathrm{m}}(n)$ shows that typically the expert rule has a much better chance of being optimal than the majority rule, especially for large $n$ (Berend and Harmse (1993), Berend and Sapir (2002a), Sapir (1998, 1999, 2002)). This conclusion is not valid for any distribution, though (see Berend and Sapir (2002b), where the range of possible asymptotic behaviour of $P_{\mathrm{e}}(n)$ and $P_{\mathrm{m}}(n)$ is explored).

The two extreme rules attracted most of the attention in many respects, and in particular regarding their probabilities of being optimal. However, other (families of) rules are also mentioned in the literature. Gradstein and Nitzan (1986) explored the families of the balanced expert rules and restricted majority rules. For given correctness probabilities, they found a simple criterion for optimality of such rules. They explain in length how these rules may be viewed as mixtures of the extreme rules. Thus, the study of these rules, interesting in its own sake, may also shed more light on the two extreme rules. For any group size, Karotkin (1998) arranged all the weighted majority rules in a graph, where the nodes are the rules and the edges represent voting profiles. In this graph, the restricted majority rules are always the leaves. The graph provides the ranking of all rules by their efficiency.

In this paper we are concerned with the abovementioned two families of rules – balanced expert rules and restricted majority rules. (These rules will be defined rigorously in Section 2.) In each families, the rule is determined by the number of group members having an influence on the group decision. However, under the restricted majority rules, each of these members is equally influential, the balanced expert rule gives the top member almost all the power, and he is outvoted only if opposed by all other influential members. It is important to note that the family of restricted majority rules contains the expert and majority rules as special instances.

As mentioned earlier, we consider the situation of incomplete information about the decisional skills. More specifically, we assume that the logarithmic expertise levels are independent exponentially distributed random variables. This situation was considered by Sapir (1998, 1999), who calculated the probabilities of the expert, simple majority and so-called balanced expert rule of order $n$ being optimal. The following theorem was obtained:

THEOREM A. *Suppose* $f(p_i)$, $i = 1, 2, ..., n$, *are i.i.d. exponential variables. The probability of:*

(i) *the simple majority rule being optimal ( for odd* $n = 2s + 1$ *) is*

$$P_{\mathrm{m}}(n) = \frac{\binom{n-1}{s}}{(n^2 - 1)^s}.$$

(ii) *the expert rule being optimal is*

$$P_{\mathrm{e}}(n) = \frac{n}{2^{n-1}}.$$

(iii) *the balanced expert rule of order* $n$ *being optimal is*

$$P_{\mathrm{be}}(n, n) = \frac{n}{(n-2)2^{n-1}}.$$

In this paper we generalize the results of Sapir (1998, 1999) obtaining explicit formulae for the probabilities of being optimal of the rules from the above families. It is interesting to note that our result for the family of restricted majority rules contains the expert and simple majority rules as special cases, and in particular we provide a unified proof which covers both extremes. In addition, we rank the rules in each family according to optimality probabilities, compare between the families and compare all rules with the two extremes.

In Section 2 we present the definitions of the rules of the two families and the criteria for the optimality. Section 3 contains the main results, and Section 4 is devoted to the proofs.

## 2. Restricted Majority and Balanced Expert Rules

The criterion obtained by Nitzan and Paroush (1982, 1984a, 1985) to identify the optimal decision rule for known values of correctness probabilities $p_i$ is as follows:

*The group recommends the first alternative if and only if*

$$\sum_{i \in A} f(p_i) \geq \sum_{i \in B} f(p_i),$$

*where* $A \subseteq \{1, 2, ..., n\}$ *is the set of group members recommending the first alternative, and* $B = \{1, ..., n\} \setminus A-$ *of those recommending the second.*

Thus, the optimal decision rule is always a weighted majority rule, with weights $f(p_i), \ \ i = 1, 2, \ldots \ , n$.

Now we define the rules which are the subject of the paper − the restricted majority rules and the balanced expert rules.

DEFINITION 1. *The restricted majority rule of* (*odd*) *order* $k = 2s+1$ (*where* $1 \leq s < n/2$) *is characterized by assigning equal weights to the* $k$ *most competent group members and zero weights to the remaining members.*

In the particular cases $k = 1$ and $k = n$ we obtain the expert rule and the simple majority rule, respectively. The rules obtained for $1 < k < n$ lie in-between the extreme two. If the values of the logarithmic expertise levels are $f(p_1), f(p_2), \ldots \ , f(p_n)$, then the restricted majority rule of order $k = 2s + 1$ is optimal if and only if

$$\sum_{i=n-2s}^{n-s} w_i \geq \sum_{i=1}^{n-2s-1} w_i + \sum_{i=n-s+1}^{n} w_i, \tag{1}$$

where $w_1 \leq \ldots \leq w_n$ are the ordered values of $f(p_1), \ldots \ , f(p_n)$ (Gradstein and Nitzan, 1986).

DEFINITION 2. *The balanced expert rule of order* $k$ (*where* $1 \leq k \leq n$) *is characterized by assigning weight of* $k-2$ *to the most competent member, weight* 1 *to each of the next* $k-1$ *members, and weight* 0 *to the remaining* $n-k$ *members.*

The balanced expert rule of order $k$ is optimal if and only if

$$\sum_{i=1}^{n-k} w_i + \sum_{i=n-k+2}^{n-1} w_i - w_{n-k+1} \leq w_n \leq \sum_{i=n-k+1}^{n-1} w_i - \sum_{i=1}^{n-k} w_i, \tag{2}$$

(Gradstein and Nitzan, 1986).

Thus, the rule is identical to the expert rule, except that, if all members, ranking from the second up to the $k$-th place oppose the top expert, we follow their opinion. Clearly, the balanced expert rule of order 3 coincides with the restricted majority rule of order 3.

## 3. Main Results

### 3.1. Restricted majority rule

THEOREM 1. *Suppose $f(p_i)$, $i = 1, 2, ..., n$, are i.i.d. exponential variables. For odd $k = 2s + 1$, $1 \le k \le n$, the probability of the restricted simple majority rule of order $k$ being optimal is:*

$$P_{\mathrm{rm}}(n, s) = \frac{n}{2^{n-1}} \cdot \frac{\binom{n-1}{s, s, n-2s-1}}{(2s+1)(s+1)^{n-s-1} s^s}.$$

The special cases $k = 1$ and $k = n$, where we have the expert and the majority rules, are not new; see Theorem A.(i) and A.(ii) in the Introduction. In Table III we present a few initial values. Note that the first row entries relate to the expert rule, and the leftmost entries at the other rows – to the majority rule.

Table III. Optimality probability of the restricted majority rule

| $s$ \ $n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.75 | 0.5 | 0.3125 | 0.1875 | 0.1094 | 0.0625 | 0.0352 | 0.0195 |
| 1 | 0.25 | 0.25 | 0.1563 | 0.0781 | 0.0342 | 0.0137 | 0.0051 | 0.0018 |
| 2 | | | 0.0104 | 0.0104 | 0.0061 | 0.0027 | 0.0010 | 0.0003 |
| 3 | | | | | 0.0002 | 0.0002 | 0.0001 | $4 \cdot 10^{-5}$ |
| 4 | | | | | | | $1.7 \cdot 10^{-6}$ | $1.7 \cdot 10^{-6}$ |

The table illustrates very distinctly that the probability of the optimality of the decision rule decreases upon moving from the expert to the majority rule. Formally, this is expressed in

PROPOSITION 1. *For any fixed $n$ and $0 \le s \le \frac{n-3}{2}$ we have:*

$$P_{\mathrm{rm}}(n, s) > P_{\mathrm{rm}}(n, s+1).$$

REMARK 1. A curiosity which stands out upon looking at Table III is that:

$$P_{\mathrm{rm}}(2s+1, s) = P_{\mathrm{rm}}(2s+2, s), \qquad s \geq 1.$$

The equality follows immediately from Theorem 1. We mention that one can also give an independent proof, relying on the lack-of-memory property of the exponential distribution. Thus, the property is not valid for a general distribution. For example, it fails for the uniform distribution (see Table I).

### 3.2. Balanced expert rule

THEOREM 2. *Suppose* $f(p_i)$, $i = 1, 2, ..., n$, *are i.i.d. exponential variables. For* $3 \leq k \leq n$ *the probability of the balanced expert rule of order* $k$ *being optimal is:*

$$P_{\mathrm{be}}(n,k) = \begin{cases} \dfrac{\binom{n}{3}}{4^{n-2}}, & k = 3, \\[2ex] \dfrac{k}{2^{n-1}} \cdot \dfrac{\binom{n}{k}}{(k-2)(k-1)^{n-k}}, & 4 \leq k \leq n. \end{cases}$$

REMARK 2. The case $k = n$ yields Theorem A.(iii).

REMARK 3. As mentioned earlier, the restricted majority rule of order 3 coincides with the balanced expert rule of the same order. Indeed, for this rule we obtain either by Theorem 1 or Theorem 2:

$$P_{\mathrm{be}}(n,3) = P_{\mathrm{rm}}(n,1) = \frac{\binom{n}{3}}{4^{n-2}}.$$

Table IV presents a few initial values of $P_{\mathrm{be}}(n,k)$. Of course, the row corresponding to $k = 3$ coincides with the row corresponding to $s = 1$ in Table III.

For fixed $n$ and varying $k$ the information provided by Table IV is less clear than that given by Table III for the restricted majority rules. The following propositions give full information in this regard.

PROPOSITION 2. *We have*

$$\begin{cases} P_{\mathrm{be}}(n,3) \leq P_{\mathrm{be}}(n,4), & 4 \leq n \leq 7, \\[1ex] P_{\mathrm{be}}(n,3) > P_{\mathrm{be}}(n,4), & n \geq 8. \end{cases}$$

Table IV. Optimality probability of the balanced expert rule

| $k$ \ $n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| 3 | 0.25 | 0.25 | 0.1563 | 0.0781 | 0.0342 | 0.0137 | 0.0051 | 0.0018 |
| 4 | | 0.25 | 0.2083 | 0.1042 | 0.0405 | 0.0135 | 0.0041 | 0.0011 |
| 5 | | | 0.1042 | 0.0781 | 0.0342 | 0.0114 | 0.0032 | 0.0008 |
| 6 | | | | 0.0469 | 0.0328 | 0.0131 | 0.0039 | 0.0010 |
| 7 | | | | | 0.0219 | 0.0146 | 0.0055 | 0.0015 |
| 8 | | | | | | 0.0104 | 0.0067 | 0.0024 |
| 9 | | | | | | | 0.0050 | 0.0031 |
| 10 | | | | | | | | 0.0024 |

PROPOSITION 3. *For sufficiently large $n$*

$$\begin{cases} P_{\mathrm{be}}(n,k) > P_{\mathrm{be}}(n,k+1), & 3 \le k < k_0, \\ P_{\mathrm{be}}(n,k) \le P_{\mathrm{be}}(n,k+1), & k = k_0, \\ P_{\mathrm{be}}(n,k) < P_{\mathrm{be}}(n,k+1), & k_0 < k \le n-2, \end{cases}$$

*where $k_0 = k_0(n)$ belongs to the interval $\left(\frac{n}{1+\ln n}, \frac{n}{\ln n}\right)$.*

PROPOSITION 4.

$$\begin{cases} P_{\mathrm{be}}(n,n-1) = P_{\mathrm{be}}(n,n), & n = 4, \\ P_{\mathrm{be}}(n,n-1) > P_{\mathrm{be}}(n,n), & n \ge 5. \end{cases}$$

PROPOSITION 5. *For sufficiently large $n$*

$$P_{\mathrm{be}}(n,n-1) = \max_{3 \le k \le n} P_{\mathrm{be}}(n,k).$$

### 3.3. Comparing the two families

For both of the above families of rules, the order is the number of members having an influence on the group decision. Hence it is interesting to compare, for any given order, the behavior of the restricted majority rule and the balanced expert rule of that order. (Of course, this comparison is relevant only for the odd orders.) As mentioned in Remark 3, for $k = 3$ the two rules coincide, and for other values of $k$ the following proposition answers the question.

PROPOSITION 6. *For odd $k \geq 5$ and sufficiently large $n$*

$$\begin{cases} P_{\mathrm{rm}}\left(n, \frac{k-1}{2}\right) > P_{\mathrm{be}}(n,k), & k < k_1, \\ P_{\mathrm{rm}}\left(n, \frac{k-1}{2}\right) \leq P_{\mathrm{be}}(n,k), & k = k_1, \\ P_{\mathrm{rm}}\left(n, \frac{k-1}{2}\right) < P_{\mathrm{be}}(n,k), & k_1 < k \leq n, \end{cases}$$

*where $k_1 = k_1(n)$ belongs to the interval $\left(\frac{n \ln 2}{\ln n}, \frac{n \ln 2}{\ln n} + \frac{n \ln 2}{\ln^{3/2} n}\right)$.*

Another question raised in the introduction regards a comparison of the rules of the two families with the two extreme rules – expert and simple majority. As the extreme rules are special cases of restricted simple majority rules, Proposition 1 provides an answer to the question for this family. Namely, the probability of the expert rule being optimal is higher that that of all other members of the family, while that of the majority rule is smaller than that of all others. The following proposition provides this comparison for the other family.

PROPOSITION 7.

(i) *For $n \geq 3$*

$$P_{\mathrm{be}}(n,k) < P_{\mathrm{e}}(n), \qquad k = 3, \dots, n.$$

(ii) *For sufficiently large $n$:*

$$P_{\mathrm{be}}(n,k) > P_{\mathrm{m}}(n), \qquad k = 3, \dots, n.$$

Figure 1 provides a schematic drawing of the graphs of the functions $P_{\mathrm{be}}(n,k)$ and $P_{\mathrm{rm}}\left(n, \frac{k-1}{2}\right)$ as functions of $k$ for large $n$. It combines the information contained in Proposition 1 through 7 − the behaviour of each function separately (Proposition 1 for $P_{\mathrm{rm}}$ and Proposition 2- 5 for $P_{\mathrm{be}}$) a comparison of the functions (Proposition 6) and a comparison of the two extreme rules with all other rules of the two families (Proposition 1 and Proposition 7). We emphasize that the graphs are designed to give only qualitative information but not to depict accurate quantitative information regarding the values of the functions. In particular, the functions are drawn as if defined for any real $k$, whereas

$k$ is actually an integer (odd for $P_{\rm rm}$). The points $a, b, c$ and $d$ are the endpoints of the intervals containing the points $k_1$ of Proposition 6 and $k_0$ of Proposition 3.

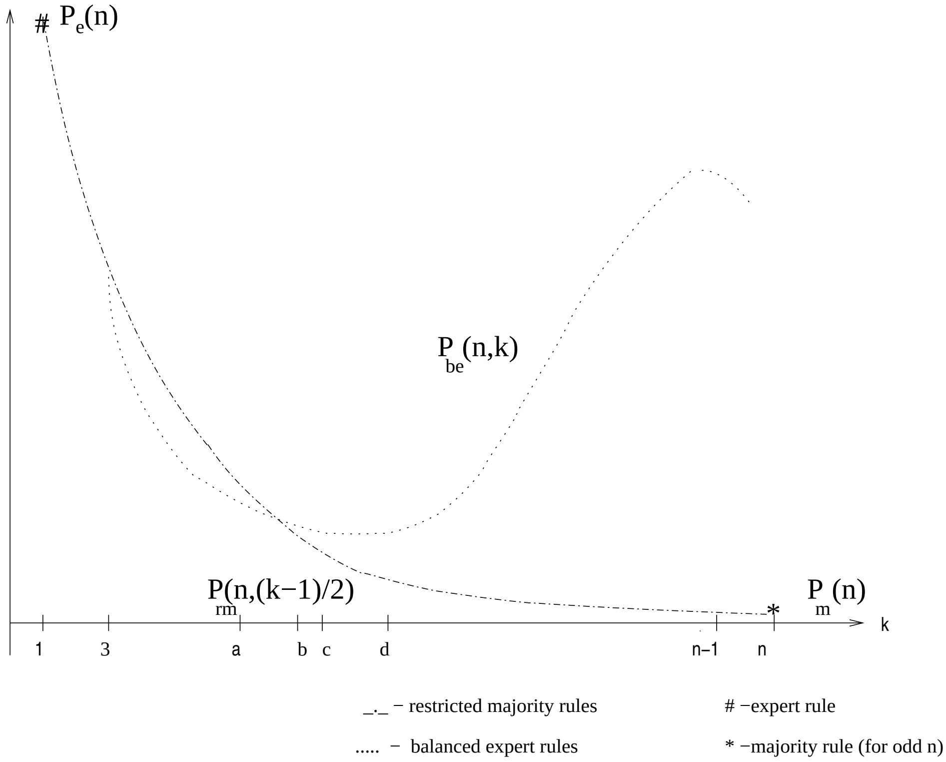


*Figure 1.* Comparing the two families

## 4. Proofs

### 4.1. Restricted majority rule

*Proof of Theorem 1.* Let $Y_i, \;\; i = 1,\ 2,\ ...,\ n$, be the order statistics of $f(p_i) \sim \mathrm{Exp}(1)$:

$$Y_1 \leq Y_2 \leq ... \leq Y_n. \tag{3}$$

According to (1):

$$P_{\rm rm}(n, s) = P\left(\sum_{i=n-2s}^{n-s} Y_i \geq \sum_{i=1}^{n-2s-1} Y_i + \sum_{i=n-s+1}^{n} Y_i\right).$$

Denote:

$$Z_1 = Y_1,$$
$$Z_i = Y_i - Y_{i-1}, \qquad i = 2,\ 3,\ ...,\ n. \tag{4}$$

Since $f(p_i)$, $i = 1,\ 2,\ ..., n$, are independent exponentially distributed random variables, the differences $Z_i$, $i = 1,\ 2,\ ...,\ n$, are also independent exponentially distributed random variables and $Z_i \sim \mathrm{Exp}(n-i+1)$ (cf. Feller (1971) Sec. 1.6).

Now we can represent the order statistics in terms of the $Z_i$'s:

$$Y_i = \sum_{j=1}^{i} Z_j, \qquad i = 1,\ 2,\ ...,\ n.$$

Using this representation

$$\begin{aligned} P_{\mathrm{rm}}(n,s) &= P\left((s+1)\sum_{i=1}^{n-2s} Z_i + \sum_{i=n-2s+1}^{n-s} (n-i+1-s)Z_i \right. \\ &\qquad \left. \geq \sum_{i=1}^{n-2s-1} (n-2s-i)Z_i + s\sum_{i=1}^{n-s+1} Z_i + \sum_{i=n-s+2}^{n} (n+1-i)Z_i \right) \\ &= P\left( Z_{n-2s} \geq \sum_{j=1}^{n-2s-2} jZ_{n-2s-1-j} + \sum_{j=1}^{s-1} jZ_{n-2s+1+j} + \sum_{j=1}^{s} jZ_{n-j+1} \right). \end{aligned}$$

Denote:

$$W_i = (n-i+1)Z_i, \qquad i = 1,\ 2,\ ...,\ n. \tag{5}$$

Note that $W_i \sim \mathrm{Exp}(1)$, $i = 1,\ 2,\ ...,\ n$, are independent random variables. Using these variables $P_{\mathrm{rm}}(n,s)$ can be represented as follows:

$$\begin{aligned} P_{\mathrm{rm}}(n,s) = \ & P\left( W_{n-2s} \geq \sum_{j=1}^{n-2s-2} \frac{j(2s+1)}{2s+2+j} W_{n-2s-1-j} \right. \\ & \left. + \sum_{j=1}^{s-1} \frac{j(2s+1)}{2s-j} W_{n-2s+1-j} + (2s+1)\sum_{j=1}^{s} W_{n-j+1} \right). \end{aligned} \tag{6}$$

Employing the lack-of-memory property of the exponential distribution, it is easy to see that for any positive numbers $a_1, a_2, \ldots, a_m$ we have

$$P(a_1W_1 \geq a_2W_2 + a_3W_3 + \ldots + a_mW_m) = \prod_{i=2}^{m} \frac{1}{1+\frac{a_i}{a_1}}. \tag{7}$$

Hence (6) yields:

$$\begin{aligned} P_{\mathrm{rm}}(n,s) &= \frac{1}{(2s+2)^s} \prod_{j=1}^{n-2s-2} \frac{1}{1+\frac{j(2s+1)}{2s+2+j}} \cdot \prod_{j=1}^{s-1} \frac{1}{1+\frac{j(2s+1)}{2s-j}} \\ &= \frac{1}{2^s(s+1)^s} \prod_{j=1}^{n-2s-2} \frac{2s+2+j}{2(j+1)(s+1)} \cdot \prod_{j=1}^{s-1} \frac{2s-j}{2(j+1)s}. \end{aligned}$$

Routine calculations give

$$\begin{aligned} P_{\mathrm{rm}}(n,s) &= \frac{n!}{s!s!(n-2s-1)!2^{n-1}} \cdot \frac{1}{(s+1)^s} \cdot \frac{1}{(2s+1)s^s(s+1)^{n-2s-1}} \\ &= \frac{n}{2^{n-1}} \cdot \frac{\binom{n-1}{s,s,n-2s-1}}{(2s+1)(s+1)^{n-s-1}s^s}\,, \end{aligned}$$

which completes the proof.

*Proof of Proposition 1.* For fixed $n$ and $0 \leq s \leq \dfrac{n-3}{2}$ we have:

$$\begin{aligned} \frac{P_{\mathrm{rm}}(n,s+1)}{P_{\mathrm{rm}}(n,s)} &= \frac{2s+1}{2s+3}\frac{(n-2s-1)(n-2s-2)}{(s+2)^2}\left(\frac{s+1}{s+2}\right)^{n-4}\left(\frac{s(s+2)}{(s+1)^2}\right)^s \\ &\leq \left(\frac{n-2s}{s+2}\right)^2\left(\frac{s+1}{s+2}\right)^{n-4}. \end{aligned}$$

For $s = 0, 1$ we can easily give a direct proof. Therefore, for $s \geq 2$ we have

$$\begin{aligned} \ln\frac{P_{\mathrm{rm}}(n,s+1)}{P_{\mathrm{rm}}(n,s)} &\leq 2\ln\frac{n-2s}{s+2} + (n-4)\ln\left(1-\frac{1}{s+2}\right) \\ &\leq 2\ln\frac{n-4}{s+2} - \frac{n-4}{s+2} \\ &\leq 2(\ln 2 - 1) < 0. \end{aligned}$$

### 4.2. Balanced expert rule

*Proof of Theorem 2.* Since for $k = 3$ the result is contained in Theorem 1, we shall carry out the proof only for $k \geq 4$.

Let $Y_i, Z_i$ and $W_i$ $i = 1, 2, ..., n$, be as (3), (4) and (5), respectively. According to (2):

$$P_{\mathrm{be}}(n,k) = P\left(\sum_{i=1}^{n-k} Y_i + \sum_{i=n-k+2}^{n-1} Y_i - Y_{n-k+1} \le Y_n \le \sum_{i=n-k+1}^{n-1} Y_i - \sum_{i=1}^{n-k} Y_i\right).$$

In terms of the $Z_i'$s the inequalities (2) may be written as

$$\sum_{i=1}^{n-k+1} (n-3-i)Z_i \le Z_n - \sum_{i=n-k+2}^{n-2} (n-1-i)Z_i \le \sum_{i=1}^{n-k+1} (2k-n-3+i)Z_i.$$

Using the lack-of-memory property, we obtain:

$$\begin{aligned} P_{\mathrm{be}}(n,k) = {} & P\left(Z_n \ge \sum_{i=n-k+2}^{n-2} (n-1-i)Z_i\right) \\ & \cdot P\left(\sum_{i=1}^{n-k+1} (n-3-i)Z_i \le Z_n \le \sum_{i=1}^{n-k+1} (2k-n-3+i)Z_i\right). \end{aligned} \tag{8}$$

By (7), the first factor on the right hand side of (8) is

$$\begin{aligned} P\left(Z_n \ge \sum_{i=n-k+2}^{n-2} (n-1-i)Z_i\right) & = P\left(W_n \ge \sum_{i=n-k+2}^{n-2} \frac{n-1-i}{n+1-i} W_i\right) \\ & = \prod_{i=n-k+2}^{n-2} \frac{1}{1+\frac{n-1-i}{n+1-i}} = \frac{k-1}{2^{k-2}}. \end{aligned} \tag{9}$$

For the second factor we have:

$$\begin{aligned} & P\left(\sum_{i=1}^{n-k+1} (n-3-i)Z_i \le Z_n \le \sum_{i=1}^{n-k+1} (2k-n-3+i)Z_i\right) \\ & = P\left(\sum_{i=1}^{n-k+1} \frac{n-3-i}{n+1-i} W_i \le W_n \le \sum_{i=1}^{n-k+1} \frac{2k-n-3+i}{n+1-i} W_i\right). \end{aligned} \tag{10}$$

For arbitrary fixed values $w_i$ of $W_i$ satisfying

$$\sum_{i=1}^{n-k+1} \frac{n-3-i}{n+1-i} w_i \le \sum_{i=1}^{n-k+1} \frac{2k-n-3+i}{n+1-i} w_i,$$

denote

$$A = \left\{\sum_{i=1}^{n-k+1} \frac{n-3-i}{n+1-i} W_i \le W_n \le \sum_{i=1}^{n-k+1} \frac{2k-n-3+i}{n+1-i} W_i\right\}$$

and

$$P_j = P\left(A \mid W_i = w_i,\ 1 \le i \le j\right), \qquad j = 0, \dots, n-k+1.$$

(Note that $P_0$ is exactly the left hand side of (10).) Clearly,

$$P_{n-k+1} = e^{-\sum_{i=1}^{n-k+1} \frac{n-3-i}{n+1-i} w_i} - e^{-\sum_{i=1}^{n-k+1} \frac{2k-n-3+i}{n+1-i} w_i}. \tag{11}$$

Continuing this process, we find that

$$P_{n-k} = \int_D P_{n-k+1} \cdot e^{-x} dx,$$

where $D = \left\{ W_{n+1-k} \ge \sum_{i=1}^{n-k} \frac{k(n-k-i)}{n+1-i} w_i \right\}$. Hence (11) yields

$$P_{n-k} = \frac{k}{2(k-1)(k-2)} e^{-\sum_{i=1}^{n-k} \frac{w_i}{n+1-i}(2(k-2)(n-k)+n-3-i(2k-3))}. \tag{12}$$

Continuing this process, we finally obtain:

$$P_0 = \int_0^\infty \dots \int_0^\infty P_{n-k}(w_1, \dots, w_{n-k}) e^{-\sum_{i=1}^{n-k} w_i} dw_1 \dots w_{n-k}. \tag{13}$$

Substituting (12) into (13) we obtain

$$\begin{aligned} P_0 &= \frac{k}{2(k-1)(k-2)} \prod_{i=1}^{n-k} \int_0^\infty e^{-\left(\frac{2(k-2)(n-k)+n-3-i(2k-3)}{n+1-i}+1\right)x} dx \\ &= \frac{k}{2(k-1)(k-2)} \prod_{i=1}^{n-k} \frac{n+1-i}{2(k-2)(n-k)+2n-2-i(2k-2)} \\ &= \frac{k}{2(k-1)(k-2)} \prod_{i=1}^{n-k} \frac{n+1-i}{2(k-1)(n-k+1-i)} \\ &= \frac{n!}{2^{n-k+1}(k-1)^{n-k+1}(k-2)(n-k)!(k-1)!} \\ &= \frac{k}{(k-2)2^{n-k+1}(k-1)^{n-k+1}} \binom{n}{k}. \end{aligned} \tag{14}$$

Combining (9) and (14), (8) takes on form

$$P_{\mathrm{be}}(n,k) = \frac{k\binom{n}{k}}{2^{n-1}(k-2)(k-1)^{n-k}}, \qquad k \ge 4,$$

which proves the theorem.

Note that one can prove the theorem for $k = 3$ in a similar way (instead of relying on Theorem 1), but some of the details are different.

Instead of (8) we have

$$P_{\text{be}}(n,3) = P\left(0 \le Z_n \le \sum_{i=1}^{n-2}(3-n+i)Z_i\right),$$

and (11) becomes

$$P_{n-k+1} = 1 - e^{-\sum_{i=1}^{n-2}\frac{3-n+i}{n+1-i}w_i}.$$

The rest of the computations remain basically the same.

*Proof of Proposition 2.* A routine calculation yields:

$$\frac{P_{\text{be}}(n,4)}{P_{\text{be}}(n,3)} = (n-3)\left(\frac{2}{3}\right)^{n-4}.$$

For $n \ge 5$ the right hand side decreases with $n$. Checking the values for $n = 4, 5, 6, 7, 8,$ we obtain the required inequalities.

*Proof of Proposition 3.* For arbitrary fixed $n$ denote:

$$h(k) = \frac{P_{\text{be}}(n,k+1)}{P_{\text{be}}(n,k)} = \frac{(k-2)(n-k)}{(k-1)\left(1+\frac{1}{k-1}\right)^{n-k}}, \qquad k \ge 4. \tag{15}$$

Then

$$\begin{aligned}
\ln h(k) &= \ln\left(1-\tfrac{1}{k-1}\right) + \ln n + \ln\left(1-\tfrac{k}{n}\right) - (n-k)\ln\left(1+\tfrac{1}{k-1}\right)\\
&= \ln n - \left(\tfrac{1}{k-1} + O\left(\tfrac{1}{k^2}\right)\right) - \left(\tfrac{k}{n} + O\left(\tfrac{k^2}{n^2}\right)\right)\\
&\quad -(n-k)\left(\tfrac{1}{k-1} - \tfrac{1}{2(k-1)^2} + O\left(\tfrac{1}{k^3}\right)\right)\\
&= A(k) + B(k) + C(k),
\end{aligned} \tag{16}$$

where for $x \in [4, n/2]$:

$$A(x) = \ln n + 1 - \tfrac{n}{x-1},$$

$$B(x) = \tfrac{n-x}{2(x-1)^2} - \tfrac{x}{n},$$

$$C(x) = O\left(\tfrac{1}{x^2}\right) + O\left(\tfrac{x^2}{n^2}\right) + O\left(\tfrac{n}{x^3}\right) < 0.$$

Taking $c = \left[\frac{n}{1+\ln n}\right]$ and $d = \left[\frac{n}{\ln n}\right]$, we will show that for sufficiently large $n$

$$h(c) < 1 \tag{17}$$

and

$$h(d) > 1. \tag{18}$$

Indeed, for sufficiently large $n$

$$A(c) \leq A\left(\frac{n}{1+\ln n}\right) = \ln n + 1 - \frac{n}{\frac{n}{1+\ln n} - 1} = -\frac{(1+\ln n)^2}{n - \ln n - 1} < 0$$

and

$$\begin{aligned} B(c) &\leq B\left(\tfrac{n}{1+\ln n} - 1\right) = \frac{(1 + n\ln n + \ln n)(1+\ln n)}{2(n-2-2\ln n)^2} - \frac{n - \ln n - 1}{n(1+\ln n)} \\ &= \Theta\left(\tfrac{\ln^2 n}{n}\right) - \Theta\left(\tfrac{1}{\ln n}\right) < 0, \end{aligned}$$

which implies (17). Similarly

$$A(d) \geq A\left(\frac{n}{\ln n} - 1\right) = \ln n + 1 - \frac{n}{\frac{n}{\ln n} - 2} = 1 - \frac{2\ln^2 n}{n - 2\ln n} \underset{n\to\infty}{\longrightarrow} 1,$$

$$B(d) \geq B\left(\frac{n}{\ln n}\right) = \frac{n\ln^2 n - n\ln n}{2(n-\ln n)^2} - \frac{1}{\ln n} = O\left(\frac{1}{\ln n}\right) \underset{n\to\infty}{\longrightarrow} 0,$$

and

$$C(d) = O\left(\frac{1}{\ln^2 n}\right) \underset{n\to\infty}{\longrightarrow} 0,$$

which implies (18).

Now by (15) we have

$$\frac{h(k+1)}{h(k)} = \frac{(k-1)(n-k-1)}{(k-2)(n-k)} \left(\frac{k^2}{k^2-1}\right)^{n-k-1}.$$

Since for $k \leq \frac{n+1}{2}$

$$\frac{(k-1)(n-k-1)}{(k-2)(n-k)} = \frac{(n-k)(k-1) - k + 1}{(n-k)(k-1) - n + k} \geq 1$$

we obtain:

$$h(k+1) > h(k), \qquad k \leq \frac{n+1}{2}. \tag{19}$$

Clearly, for sufficiently large $n$ we have $b < \frac{n+1}{2}$. Combining (17), (18), (19) and Proposition 2 we obtain for sufficiently large $n$

$$\begin{cases} P_{\mathrm{be}}(n,k) > P_{\mathrm{be}}(n,k+1), & 3 \le k < k_0, \\ P_{\mathrm{be}}(n,k) \le P_{\mathrm{be}}(n,k+1), & k = k_0, \\ P_{\mathrm{be}}(n,k) < P_{\mathrm{be}}(n,k+1), & k_0 < k \le \frac{n+1}{2}, \end{cases}$$

for some $k_0 \in (a,b)$.

To complete the proof, it remains to show that

$$P_{\mathrm{be}}(n,k+1) > P_{\mathrm{be}}(n,k), \qquad \frac{n+1}{2} \le k \le n-2. \tag{20}$$

Indeed, by (16) we have for large $n$ and $k \le n-3$

$$\begin{aligned} \ln h(k) &\ge \ln(n-k) - \frac{n-k}{k-1} + \ln\frac{n-1}{n+1} \\ &\ge \ln(n-k) - 1.01 \ge \ln 3 - 1.01 > 0, \end{aligned}$$

while for $k = n-2$ we verify directly that $\ln h(n-2) > 0$. This completes the proof.

*Proof of Proposition 4.* The inequality for $n = 4$ is immediate. For $n \ge 5$

$$\frac{P_{\mathrm{be}}(n,n)}{P_{\mathrm{be}}(n,n-1)} = \frac{n-3}{n-1} < 1.$$

*Proof of Proposition 5.* We have

$$\frac{P_{\mathrm{be}}(n,3)}{P_{\mathrm{be}}(n,n-1)} = \frac{(n-2)^2(n-3)}{3 \cdot 2^{n-2}} < 1$$

for sufficiently large $n$. By Proposition 3, this proves the proposition.

### 4.3. Comparisons between the two families and the two extremes

*Proof of Proposition 6.* For arbitrary fixed $n$ and $k = 2s+1 \ge 5$ denote:

$$g(s) = \frac{P_{\mathrm{rm}}(n,s)}{P_{\mathrm{be}}(n,2s+1)}.$$

By Theorems 1 and 2

$$g(s) = \binom{2s}{s} 2^{n-2s-1} \frac{(2s-1)s^{n-3s-1}}{(2s+1)(s+1)^{n-s-1}}. \tag{21}$$

Denote:

$$r(x) = \frac{2^n}{x^{2x+\frac{1}{2}}} \cdot \frac{1}{e^{n/x}}, \qquad \sqrt{n} \le x \le \frac{n-1}{2}.$$

By Stirling's formula it is easy to show that for $s \ge \sqrt{n}$ we have

$$g(s) \asymp r(s), \tag{22}$$

where $y(x) \asymp z(x)$ if $0 < c \le \frac{y(x)}{z(x)} \le C < \infty$ as $x \to \infty$. It will usually be convenient to use $\ln r(x)$:

$$\ln r(x) = n(\ln 2 - 1/x) - (2x + 1/2)\ln x. \tag{23}$$

First we will show that for sufficiently large $n$

$$g\left(\left[\frac{n\ln 2}{2\ln n}\right]\right) > 1 \tag{24}$$

and

$$g\left(\left[\frac{n\ln 2}{2\ln n} + \frac{n\ln 2}{2\ln^{3/2} n} - \frac{1}{2}\right]\right) < 1. \tag{25}$$

Indeed, substituting $s = \left[\frac{n\ln 2}{2\ln n}\right]$ in (23) and using the inequality

$$\ln r([x]) \ge n(\ln 2 - 1/(x-1)) - (2x + 1/2)\ln x,$$

we have

$$\begin{aligned} \ln r\left(\left[\frac{n\ln 2}{2\ln n}\right]\right) &\ge n\left(\ln 2 - \frac{2\ln n}{n\ln 2 - 2\ln n}\right) - \left(\frac{n\ln 2}{\ln n} + \frac{1}{2}\right)\ln\frac{n\ln 2}{2\ln n} \\ &= \ln 2 \cdot \frac{n\ln\ln n}{\ln n} + O\left(\frac{n}{\ln n}\right) \underset{n\to\infty}{\longrightarrow} \infty\,, \end{aligned}$$

which by (22) implies (24). Similarly, substituting $s = \left[\frac{n\ln 2}{2\ln n} + \frac{n\ln 2}{2\ln^{3/2} n} - \frac{1}{2}\right]$ in (23) and using the inequality

$$\ln r([x]) \le n(\ln 2 - 1/x) - (2(x-1) + 1/2)\ln(x-1),$$

we obtain

$$\begin{aligned} \ln r\left(\left[\frac{n\ln 2}{2\ln n} + \frac{n\ln 2}{2\ln^{3/2} n} - \frac{1}{2}\right]\right) &\le n\ln 2 - \left(\frac{n\ln 2}{\ln n} + \frac{n\ln 2}{\ln^{3/2} n} - \frac{5}{2}\right)\ln\frac{n\ln 2}{2\ln n} \\ &= -\frac{n\ln 2}{\sqrt{\ln n}} + O\left(\frac{n\ln\ln n}{\ln^{3/2} n}\right) \underset{n\to\infty}{\longrightarrow} -\infty \end{aligned}$$

for sufficiently large $n$. Thus we have (25).

Now we will show that $g$ decreases for $2 \le s \le \sqrt{\frac{n}{\ln n}} - 1$ and increases for $\sqrt{\frac{n}{\ln n - \ln\ln n}} \le s \le \frac{n-1}{2}$. Indeed, by (21)

$$\frac{g(s+1)}{g(s)} = h(s)\left(1 + \frac{1}{s(s+2)}\right)^{n-3s-4}\left(1 + \frac{1}{s+1}\right)^{-2(s+1)},$$

where $h(s) = \dfrac{(2s+1)^3}{2s^3(2s-1)(2s+3)}$. Denote $q(s) = \ln \dfrac{g(s+1)}{g(s)}$. A technical calculation shows that

$$\frac{n}{(s+1)^2} - 2\ln s - 4 \le q(s) \le \frac{n}{s^2} - 2\ln s + \ln 2 - 2, \qquad s \ge 2. \tag{26}$$

If $2 \le s \le \sqrt{\frac{n}{\ln n}} - 1$, then by the left inequality in (26)

$$q(s) \ge \frac{n}{(s+1)^2} - 2\ln(s+1) - 4 \ge \ln\ln n - 4 > 0$$

for sufficiently large $n$, and therefore

$$g(s+1) > g(s), \qquad 2 \le s \le \sqrt{\frac{n}{\ln n}} - 1.$$

It is easy to check that $g(2) > 1$, so that the last inequality yields

$$g(s) > 1, \qquad 2 \le s \le \sqrt{\frac{n}{\ln n}} - 1. \tag{27}$$

If $\sqrt{\frac{n}{\ln n - \ln\ln n}} \le s \le \frac{n-1}{2}$, then by the right inequality in (26)

$$q(s) \le \ln(\ln n - \ln\ln n) - \ln\ln n + \ln 2 - 2 < 0,$$

which implies

$$g(s+1) < g(s), \qquad \sqrt{\frac{n}{\ln n - \ln\ln n}} \le s \le \frac{n-1}{2}. \tag{28}$$

Next we show that for sufficiently large $n$

$$g(s) > 1, \qquad \sqrt{\frac{n}{\ln n}} - 1 \le s \le \sqrt{\frac{n}{\ln n - \ln\ln n}}\,. \tag{29}$$

Indeed, for large $n$

$$\left[\sqrt{\frac{n}{\ln n}} - 1, \sqrt{\frac{n}{\ln n - \ln\ln n}}\,\right] \subseteq \left[\frac{1}{2}\sqrt{\frac{n}{\ln n}}, \frac{3}{2}\sqrt{\frac{n}{\ln n}}\,\right].$$

Thus we may take $\frac{1}{2}\sqrt{\frac{n}{\ln n}} \le s \le \frac{3}{2}\sqrt{\frac{n}{\ln n}}$, and for such $s$ we have by (23)

$$\ln r(s) \ge (\ln 2 - 1/2)\, n - O(\sqrt{n \ln n}) \underset{n\to\infty}{\longrightarrow} \infty \,,$$

which by (22) implies (29).

Since for sufficiently large $n$

$$\sqrt{\frac{n}{\ln n - \ln\ln n}} < \left[\frac{n \ln 2}{2 \ln n}\right], \tag{30}$$

combining (24),(25), (27), (28), (29), and (30) we complete the proof.

*Proof of Proposition 7.*

(i) For $k = 3$ the proposition follows from Proposition 1 (for $s = 0$). For $k \ge 4$ we have

$$\begin{aligned}\frac{P_{\mathrm{be}}(n,k)}{P_{\mathrm{e}}(n)} &= \frac{1}{(k-1)^{n-k}(k-2)}\binom{n-1}{k-1} \\ &= \frac{k}{(k-2)(k-1)}\prod_{i=1}^{n-k-1}\frac{k+i}{(i+1)(k-1)} < 1\,,\end{aligned}$$

which implies the proof of the first part.

(ii) It suffices to show that $P_{\mathrm{m}}(n) < P_{\mathrm{be}}(n,k_0)$, where $k_0$ is as in Proposition 3, and that $P_{\mathrm{m}}(n) < P_{\mathrm{be}}(n,n)$. By Proposition 6 we have $P_{\mathrm{rm}}(n,(k_0-1)/2) < P_{\mathrm{be}}(n,k_0)$ (since $k_1 \approx k_0 \ln 2$). From Proposition 1 if follows that $P_{\mathrm{m}}(n) < P_{\mathrm{rm}}(n,(k_0-1)/2)$. Also:

$$\frac{P_{\mathrm{m}}(n)}{P_{\mathrm{be}}(n,n)} = O\left(\sqrt{n}\left(\frac{4}{n}\right)^n\right) < 1$$

for sufficiently large $n$.

REMARK 4. Part (i) of the proposition follows readily from Proposition 5 for sufficiently large $n$, as we have to prove only

$$P_{\mathrm{e}}(n) > P_{\mathrm{be}}(n,n-1).$$

We needed a direct proof to show it for all $n \ge 3$.

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