EXPERT RULE VERSUS MAJORITY RULE UNDER PARTIAL INFORMATION, III

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Abstract

In this paper we deal with certain aspects of the dichotomous choice model. Our main purpose is clarifying the connections between some characteristics of the decision making body and the probability of its making correct decisions. A group of experts is required to select one of two alternatives, of which exactly one is regarded as correct. The alternatives may be related to a wide variety of areas. A decision rule translates the individual opinions of the members into a group decision. A decision rule is optimal if it maximizes the probability of the group to make a correct choice for all possible combinations of opinions.

We study the situation where only partial information on the probabilities of each expert in the group to choose the right decision is available. Specifically, we assume the expertise levels to be independent generalized Pareto distributed random variables. Moreover, the ranking of the members of the team is (at least partly) known. Thus, one can follow rules based on this ranking. The extremes are the expert rule and the simple majority rule. We show that, similarly to other previously studied cases, the expert rule is more likely to be optimal than the majority rule. The results are partly obtained theoretically and partly by simulation.

Section 1. Introduction and Model

We focus on the dichotomous choice model, where a group of n experts is required to select one of two alternatives, of which exactly one is regarded as correct. The experts may be humans or not (say, computers). Each of them has some probability to make the right choice. We assume the experts are independent in their choices. A *decision rule* is a rule for translating the individual opinions into a group decision. Such a rule is *optimal* if it maximizes the probability that the group will reach the correct decision for all possible combinations of opinions. If the members indexed by some subset $A \subseteq \{1, 2, ..., n\}$ of the group recommend the first alternative, while those indexed by $B = \{1, ..., n\} \setminus A$ recommend the second, then the first alternative should be chosen if and only if

$$\prod_{i \in A} \frac{p_i}{1 - p_i} > \prod_{i \in B} \frac{p_i}{1 - p_i}, \tag{1.1}$$

or, equivalently, if

$$\sum_{i \in A} \ln \left(\frac{p_i}{1 - p_i} \right) > \sum_{i \in B} \ln \left(\frac{p_i}{1 - p_i} \right), \tag{1.2}$$

where p_i is the correctness probability of the i-th expert (see Nitzan and Paroush (1982, 1984a ,1985) and Grofman et al (1983)). Thus, if the values p_i are known, then the optimal rule is a weighted majority rule, with weights $\ln \left(\frac{p_i}{1 - p_i} \right)$. In view of (1.1) and (1.2) it is natural to define the *expertise* of an individual, whose probability of being correct is p, as $\frac{p}{1 - p}$, and his *logarithmic expertise* as $\ln \left(\frac{p}{1 - p} \right)$.

However, the assumption of full information regarding the decision makers competences is very restricting and often far from being fulfilled. Our goal here is identifying the optimal decision rule under partial information on the decision skills. Specifically, we assume the correctness probabilities of the group members to be independent random variables distributed according to some given distribution rule. Moreover, while the values these variables take are unknown, we assume that the ranking of the members in terms of their individual correctness probabilities is known. Thus, one can follow rules based on this ranking (cf. Gradstein and Nitzan(1986)). The extremes are the expert rule – following the advice of the most qualified individual while ignoring all the rest, and the majority rule – always taking the majority advice, even when advocated by mostly less qualified group members. Clearly, there are numerous other decision rules in-between these two extremes.

The probabilities of the two extreme rules to be optimal were compared in a series of papers. This line was started by Nitzan and Paroush(1985), who dealt with the case of

normally distributed logarithmic expertise levels. Berend and Harmse (1993) regarded the case of uniformly distributed correctness probabilities, Berend and Sapir(1999) continued some of the results of Berend and Harmse (1993) for the case of generalized uniform distribution, and Sapir(1998) – the situation of Pareto distributed expertise levels.

In this paper we continue the above, assuming the expertise levels to be distributed according to some generalization of Pareto distribution.

Section 2. Main Results.

Let $F(p) = \dfrac{p}{1-p}$ be the expertise of an individual and $f(p) = \ln F(p)$ be his logarithmic expertise. In terms of the logarithmic expertise levels $f(p_i)$, the assumption of the expertise levels $F(p_i)$ being Pareto distributed is equivalent to $f(p_i)$ being exponentially distributed. In this paper we consider, more generally, the situation of gamma distributed logarithmic expertise, $f(p_i) \sim \Gamma(\lambda, \alpha)$. Namely, we assume that $f(p_i)$ are independent and distributed according to the same density function

$$\rho_{f(p_i)}(x) = \begin{cases} \dfrac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, & x \geq 0, \\ 0, & \text{otherwise}, \end{cases} \qquad (\lambda, \alpha > 0).$$

Equivalently, as one readily verifies:

$$g_{F(p_i)}(x) = \begin{cases} \dfrac{\lambda^\alpha \ln^{\alpha-1}(x)}{\Gamma(\alpha) x^{\lambda+1}}, & x \geq 1, \\ 0, & \text{otherwise}. \end{cases}$$

The special case $\alpha = 1$ yields Pareto distribution of $F(p_i)$. Denote by $P_{\mathrm{e}}(n)$ the probability of the expert rule to be optimal and by $P_{\mathrm{m}}(n)$ the probability of the majority rule to be optimal. In Sapir(1998) this case was studied extensively and it was found that

$$P_{\mathrm{e}}(n) = n \cdot \left(\frac{1}{2}\right)^{n-1}$$

and

$$P_{\mathrm{m}}(n) = \frac{1}{2^s (s+1)^s} \left\{ 1 - \frac{\binom{2s+1}{s}(s+1)}{(2s)^s} \sum_{i=0}^{s-1} \frac{(-1)^i (s-i-1)^s}{s+i+1} \binom{s}{i} \right\}, \qquad (2.1)$$

where $s = \dfrac{n-1}{2}$. (Recall that the majority rule is defined only for odd n.) The following table presents a few initial values of both of these probabilities .

n	3	5	7	9	11
$P_{\mathrm{e}}(n)$	0.75	0.31	0.11	0.04	0.01
$P_{\mathrm{m}}(n)$	0.25	0.01	$1.8{\cdot}10^{-4}$	$1.7{\cdot}10^{-6}$	$1.0{\cdot}10^{-8}$

Table 1. $P_{\mathrm{e}}(n)$ and $P_{\mathrm{m}}(n)$ for Pareto distributed expertise levels.

The table shows very distinctly that the expert rule is far more likely to be optimal than the majority rule already for quite small values of n.

In expressing our results, it will be convenient to introduce "binomial coefficients" with real entries, in analogy to the definition of regular binomial coefficients:

$$\left(\begin{array}{c} x \\ y \end{array} \right) = \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}, \qquad x,\ y \in \mathbf{R}, \qquad x, y, x-y \notin \mathbf{Z}_{-}.$$

Theorem 2.1: *The probability for the expert rule to be optimal is given by either of the following formulas:*

1) $P_{\mathrm{e}}(n) = \dfrac{n}{2^{\alpha n}} \cdot \displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k} \left(\begin{array}{c} \alpha n + k - 1 \\ \alpha - 1 \end{array} \right).$

2) $P_{\mathrm{e}}(n) = nG_{\beta}\left(\dfrac{1}{2}\right)$, where $G_{\beta}(x)$ is the distribution function of $\beta(\alpha n - \alpha, \alpha)$.

The following corollary yields the asymptotic behaviour of $P_{\mathrm{e}}(n)$ as the number of experts becomes large.

Corollary 2.1: *For any $\alpha > 0$, as $n \to \infty$,*

$$P_{\mathrm{e}}(n) = \frac{2\alpha^{\alpha}}{\Gamma(\alpha+1)} \left(\frac{n}{2^n}\right)^{\alpha} \left(1 + O(\frac{1}{n})\right).$$

For special values of α, the infinite sum in Theorem 2.1 reduces to a finite one, as follows.

Theorem 2.2: *If $\alpha = \frac{m}{n-2}$, where m is a positive integer, then:*

$$P_{\mathrm{e}}(n) = \frac{n}{2} \left(1 - 2^{1-\frac{2m}{n-2}} \sum_{j=0}^{m-1} \frac{1}{2^j} \left(\begin{array}{c} \frac{2m}{n-2} - 1 + j \\ \frac{m}{n-2} - 1 \end{array} \right)\right).$$

Theorem 2.3: *If $\alpha = \frac{m}{n-1}$, where m is a positive integer, then:*

$$P_{\mathrm{e}}(n) = n\left(1 - 2^{1-\frac{mn}{n-1}} \sum_{j=0}^{m-1} \left(\begin{array}{c} \frac{mn}{n-1} - 1 \\ j \end{array} \right)\right).$$

Notice that the binomial coefficients in the sum can be calculated without using the Γ function, since for non-negative integer j we have:

$$\left(\begin{array}{c} x \\ j \end{array} \right) = \frac{x(x-1)...(x-j+1)}{j!}, \qquad x \in \mathbf{R}, \quad j = 0, 1, 2, ... \quad .$$

For some special distributions, which are particular instances of Gamma distribution, such as $\chi^2(r)$ and Erlang(λ, d), we obtain simple expressions for $P_{\mathrm{e}}(n)$. Recall that the $\chi^2(r)$ distribution is $\Gamma(\frac{1}{2}, \frac{r}{2})$ for a positive integer r and Erlang(λ, d) coincides with $\Gamma(\lambda, d)$ for a positive integer d.

Corollary 2.2: *For $f(p_i) \sim \chi^2(r)$, the probability of the expert rule to be optimal is:*
1) *If $n = 2k$, where $k \geq 2$ is a positive integer, then:*

$$P_{\mathrm{e}}(n) = \frac{n}{2}\left(1 - \frac{1}{2^{r-1}} \cdot \sum_{j=0}^{r(k-1)-1} \left(\begin{array}{c} j+r-1 \\ \frac{r}{2}-1 \end{array} \right) \frac{1}{2^j}\right).$$

2) *If $n = 2k-1$, where $k \geq 2$ is a positive integer, then:*

$$P_{\mathrm{e}}(n) = n\left(1 - \frac{1}{2^{\frac{r}{2}}} \cdot \sum_{j=0}^{r(k-1)-1} \left(\begin{array}{c} j+\frac{r}{2}-1 \\ \frac{r}{2}-1 \end{array} \right) \frac{1}{2^j}\right).$$

The first part follows immediately from Theorem 2.2, and the second $-$ from the proof of Theorem 2.3. Note that the $\chi^2(r)$ distribution coincides with the Erlang$(\frac{1}{2}, d)$ distribution for an even number $r = 2d$ of degrees of freedom. In general, if $f(p_i) \sim$ Erlang(λ, d), Theorem 2.2 (or Theorem 2.3) assumes an even more convenient form for $P_{\mathrm{e}}(n)$.

Corollary 2.3: *If $f(p_i) \sim$ Erlang(λ, d), where s is a positive integer, the probability of the expert rule to be optimal is:*

$$P_e(n) = \frac{n}{2^{dn-1}} \sum_{j=0}^{d-1} \left(\begin{array}{c} dn-1 \\ j \end{array} \right).$$

The following table provides the likelihood of optimality of the expert rule, calculated by Corollary 2.3 for $d = 1, 2, 3, 4$:

d	1	2	3	4
$P_{\mathrm{e}}(n)$	$\frac{n}{2^{n-1}}$	$\left(\frac{n}{2^{n-1}}\right)^2$	$\frac{n}{2^{3n}}(9n^2-3n+2)$	$\frac{n^2}{3\cdot 2^{4n-4}}(4n^2-3n+2)$

Table 2. $P_{\mathrm{e}}(n)$ for Erlang(λ, d) distributed logarithmic expertise, $s = 1, 2, 3, 4$.

Estimating the probability $P_{\mathrm{m}}(n)$ of the majority rule to be optimal seems a much more formidable task. We have not found a way to estimate it in general. However, we did manage to replace formula (2.1), dealing with the special case of Pareto distributed expertise levels, by the following very simple formula, which readily provides the asymptotics as well.

Theorem 2.4: *Suppose the expertise levels $F(p_i)$, $i = 1, 2, ..., n$, are independent Pareto distributed random variables. For odd $n = 2s+1$, the probability that the majority rule is optimal is*

$$P_{\mathrm{m}}(n) = \frac{\binom{n-1}{s}}{(n^2-1)^s} \approx \frac{1}{\sqrt{\pi}}\left(\frac{2}{n}\right)^{n-\frac{1}{2}}\left(1+O\left(\frac{1}{n}\right)\right).$$

In the general case, we only estimated $P_{\mathrm{m}}(n)$ using Monte Carlo method (with 10^4 iterations). Table 3 provides the values of $P_{\mathrm{e}}(n)$ and $P_{\mathrm{m}}(n)$ for several values of α and n.

α \ n		3	5	7	9	11
0.1	$P_{\mathrm{e}}(n)$	0.9879	0.9393	0.8737	0.8083	0.7353
	$P_{\mathrm{m}}(n)$	0.0121	0	0	0	0
0.5	$P_{\mathrm{e}}(n)$	0.8787	0.5806	0.3488	0.1998	0.1113
	$P_{\mathrm{m}}(n)$	0.1213	0.0022	0.0001	0	0
1.0	$P_{\mathrm{e}}(n)$	0.7500	0.3125	0.1094	0.0352	0.0107
	$P_{\mathrm{m}}(n)$	0.2500	0.0104	0.0002	0	0
1.5	$P_{\mathrm{e}}(n)$	0.6467	0.1730	0.0357	0.0065	0.0011
	$P_{\mathrm{m}}(n)$	0.3533	0.0228	0.0003	0	0
2.0	$P_{\mathrm{e}}(n)$	0.5625	0.0977	0.0120	0.0012	0.0001
	$P_{\mathrm{m}}(n)$	0.4375	0.0425	0.0017	0	0
10	$P_{\mathrm{e}}(n)$	0.0921	0	0	0	0
	$P_{\mathrm{m}}(n)$	0.9079	0.4874	0.1223	0.0153	0.0011

Table 3. Optimality probability – expert versus majority rule.

The table hints that $P_{\mathrm{e}}(n)$ decreases and $P_{\mathrm{m}}(n)$ increases as functions of α. This is easily

explained by the fact that, as α increases, the distribution becomes relatively more and more concentrated. As functions of n, both converge to 0 as $n \to \infty$, with $P_{\mathrm{m}}(n)$ seemingly doing so much faster than $P_{\mathrm{e}}(n)$ (but less and less so as α increases).

Section 3. Proofs

We assume, without loss of generality, that $\lambda = 1$.

Proof of Theorem 2.1:

1)Put $T = \sum_{i=2}^{n} f(p_i)$. According to the properties of Gamma distribution, $T \sim \Gamma(1, \alpha(n-1))$. Hence:

$$\begin{aligned} P_e(n) &= n \cdot \mathrm{Prob}\left(f(p_1) \geq \sum_{i=2}^{n} f(p_i)\right) = n \cdot \mathrm{Prob}\left(f(p_1) \geq T\right) \\ &= n \int_0^{\infty} \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)} \int_0^{x} \frac{t^{\alpha(n-1)-1}e^{-t}}{\Gamma(\alpha(n-1))} dt dx. \end{aligned} \tag{3.1}$$

Using integration by parts in the internal integral we obtain:

$$\begin{aligned} \int_0^{x} t^{\alpha(n-1)-1}e^{-t}dt &= \frac{t^{\alpha(n-1)}e^{-t}}{\alpha(n-1)}|_0^{x} + \frac{1}{\alpha(n-1)} \int_0^{x} t^{\alpha(n-1)}e^{-t}dt \\ &= \frac{x^{\alpha(n-1)}e^{-x}}{\alpha(n-1)} + \frac{1}{\alpha(n-1)} \int_0^{x} t^{\alpha(n-1)}e^{-t}dt. \end{aligned}$$

Continuing this process we are led to the expression:

$$\int_0^{x} t^{\alpha(n-1)-1}e^{-t}dt = \sum_{k=0}^{\infty} \frac{x^{\alpha(n-1)+k}e^{-x}}{\alpha(n-1)(\alpha(n-1)+1)...(\alpha(n-1)+k)}.$$

Therefore:

$$\begin{aligned}
P_e(n) &= \frac{n}{\Gamma(\alpha)\Gamma(\alpha n-\alpha)}\cdot\sum_{k=0}^{\infty}\frac{\int_0^\infty x^{\alpha n+k-1}e^{-2x}dx}{\alpha(n-1)(\alpha(n-1)+1)...(\alpha(n-1)+k)} \\
&= \frac{n}{\Gamma(\alpha)\Gamma(\alpha n-\alpha)}\cdot\sum_{k=0}^{\infty}\frac{\Gamma(\alpha n+k)}{2^{\alpha n+k}\alpha(n-1)(\alpha(n-1)+1)...(\alpha(n-1)+k)} \\
&= \frac{n\Gamma(\alpha n)}{2^{\alpha n}(n-1)\alpha\Gamma(\alpha)\Gamma(\alpha n-\alpha)}\cdot\left(1+\sum_{k=1}^{\infty}\frac{1}{2^k}\prod_{j=1}^{k}\frac{(\alpha n+j-1)}{(\alpha(n-1)+j)}\right) \\
&= \frac{\binom{\alpha n}{\alpha}}{2^{\alpha n}}\cdot\sum_{k=0}^{\infty}\frac{1}{2^k}\prod_{j=1}^{k}\frac{(\alpha n+j-1)}{(\alpha(n-1)+j)} \\
&= \frac{\binom{\alpha n}{\alpha}\Gamma(\alpha n-\alpha+1)}{2^{\alpha n}\Gamma(\alpha n)}\cdot\sum_{k=0}^{\infty}\frac{1}{2^k}\frac{\Gamma(\alpha n+k)}{\Gamma(\alpha n-\alpha+1+k)}.
\end{aligned} \tag{3.2}$$

That expansion of $P_{\rm e}(n)$ gives the first part of the theorem.

2) We employ the following relationship between Beta and Gamma distributions: if $Z\sim\Gamma(1,v)$ and $Y\sim\Gamma(1,w)$ are independent random variables and $B=\dfrac{Z}{Z+Y}$, then $B\sim\beta(v,w)$. This gives:

$$P_{\rm e}(n)=n\cdot P\left(f(p_1)\geq T\right)=n\cdot P\left(f(p_1)+T\geq 2T\right)=n\cdot P\left(\frac{T}{T+f(p_1)}\leq\frac{1}{2}\right)=nG_\beta(\frac{1}{2}).$$

Proof of Corollary 2.1:

Start with the expression for $P_{\rm e}(n)$ given in (3.2). Denote $u_k(n)=\dfrac{1}{2^k}\displaystyle\prod_{j=1}^{k}q_j$, where $q_j=\dfrac{\alpha n+j-1}{\alpha(n-1)+j}$.

Then:

$$P_{\rm e}(n)=\frac{\Gamma(\alpha n+1)}{2^{\alpha n}\Gamma(\alpha n+1-\alpha)\Gamma(\alpha+1)}\sum_{k=0}^{\infty}u_k(n). \tag{3.3}$$

Since

$$q_{j+1}-q_j=\frac{1-\alpha}{(\alpha(n-1)+j)(\alpha(n-1)+j+1)},$$

the sequence q_j increases for $\alpha<1$, decreases for $\alpha>1$ and constant for $\alpha=1$. Thus, for $\alpha\geq 1$

$$\frac{1}{2^k}\leq\frac{1}{2^k}\left(\frac{\alpha n+k-1}{\alpha(n-1)+k}\right)^k\leq u_k(n)\leq\frac{1}{2^k}\left(\frac{\alpha n}{\alpha(n-1)+1}\right)^k,$$

whereas for $\alpha<1$

$$\frac{1}{2^k}\left(\frac{\alpha n}{\alpha(n-1)+1}\right)^k\leq u_k(n)\leq\frac{1}{2^k}.$$

Now $\sum_{k=0}^{\infty} \frac{1}{2^k} = 2$ and $\sum_{k=0}^{\infty} \left(\frac{\alpha n}{2\alpha(n-1)+2} \right)^k = 2 + \frac{2\alpha - 2}{\alpha n - 2\alpha + 2} \underset{n\to\infty}{\to} 2$, So that in any case

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} u_k(n) = 2. \tag{3.4}$$

By Stirling's formula, for large real values of x (cf. Artin (1964)),

$$\Gamma(x) = \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x+\frac{\theta}{12x}},$$

we have the following asymptotic behaviour of $\Gamma(\alpha n + b)$ for large n

$$\Gamma(\alpha n + b) = \sqrt{2\pi} e^{-\alpha n} (\alpha n)^{\alpha n + b - \frac{1}{2}} \left(1 + O(\frac{1}{n}) \right),$$

which gives:

$$\frac{\Gamma(\alpha n + 1)}{\Gamma(\alpha n - \alpha + 1)} = (\alpha n)^{\alpha} \left(1 + O(\frac{1}{n}) \right). \tag{3.5}$$

By (3.3), (3.4) and (3.5):

$$P_{\mathrm{e}}(n) = \frac{2\alpha^{\alpha}}{\Gamma(\alpha+1)} \left(\frac{n}{2^n} \right)^{\alpha} \left(1 + O(\frac{1}{n}) \right).$$

Proof of Theorem 2.2:
We start from the expression for $P_{\mathrm{e}}(n)$ given in (3.1). Since $\alpha = \frac{m}{n-2}$

$$\int_0^x t^{\alpha(n-1)-1} e^{-t} dt = \int_0^x t^{m-1+\alpha} e^{-t} dt.$$

Denote the right side by $\mathcal{I}_m$. Integrating by parts yields:

$$\mathcal{I}_m = -x^{m+\alpha-1} e^{-x} + (m + \alpha - 1)\mathcal{I}_{m-1}.$$

Continuing the process m times we obtain :

$$\mathcal{I}_m = -e^{-x} \sum_{i=0}^{m-1} \frac{\Gamma(m+\alpha)}{\Gamma(m+\alpha-i)} x^{m+\alpha-1-i} + \frac{\Gamma(m+\alpha)}{\Gamma(\alpha)} \mathcal{I}_0.$$

By (3.1):

$$\begin{aligned} P_e(n) &= n \int_0^{\infty} \frac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)\Gamma(\alpha(n-1))} \left(-e^{-x} \sum_{i=0}^{m-1} \frac{\Gamma(m+\alpha)}{\Gamma(m+\alpha-i)} x^{m+\alpha-1-i} + \frac{\Gamma(m+\alpha)}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t} dt \right) dx \\ &= \frac{n}{\Gamma(\alpha)\Gamma(\alpha(n-1))} \cdot \left(\sum_{j=0}^{m-1} - \frac{\Gamma(\alpha(n-1))\Gamma(j+2\alpha)}{\Gamma(\alpha+j+1) 2^{j+2\alpha}} + \frac{\Gamma(m+\alpha)}{\Gamma(\alpha)} \int_0^{\infty} \int_0^x t^{\alpha-1} e^{-t} x^{\alpha-1} e^{-x} dt dx \right). \end{aligned}$$

By symmetry:

$$\int_0^\infty \int_0^x t^{\alpha-1}e^{-t}x^{\alpha-1}e^{-x}dtdx = \frac{1}{2}\cdot\left(\int_0^\infty x^{\alpha-1}e^{-x}dx\right)^2.$$

Thus

$$\begin{aligned} P_{\mathrm{e}}(n) &= \frac{n}{\Gamma(\alpha)\Gamma(\alpha(n-1))}\cdot\left(\sum_{j=0}^{m-1} -\frac{\Gamma(\alpha(n-1))\Gamma(j+2\alpha)}{\Gamma(\alpha+j+1)2^{j+2\alpha}} + \frac{\Gamma(m+\alpha)}{2\Gamma(\alpha)}\left(\int_0^\infty x^{\alpha-1}e^{-x}dx\right)^2\right) \\ &= \frac{n}{2}\left(1-\sum_{j=0}^{m-1}\frac{\Gamma(j+2\alpha)}{\Gamma(\alpha)\Gamma(\alpha+j+1)2^{2\alpha+j-1}}\right) \\ &= \frac{n}{2}\left(1-\frac{1}{2^{2\alpha-1}}\sum_{j=0}^{m-1}\left(\begin{array}{c} j+2\alpha-1 \\ \alpha-1 \end{array}\right)\frac{1}{2^j}\right). \end{aligned}$$

Proof of Theorem 2.3:
We use again (3.1). Since $\alpha = \dfrac{m}{n-1}$, the internal integral in (3.1) gives:

$$\int_0^x \frac{t^{\alpha(n-1)-1}e^{-t}}{\Gamma(\alpha(n-1))}dt = \frac{1}{\Gamma(m)}\cdot\int_0^x t^{m-1}e^{-t}dt = 1-\sum_{i=0}^{m-1}\frac{e^{-x}x^i}{i!}.$$

Hence:

$$\begin{aligned} P_e(n) &= n\left(1-\frac{1}{\Gamma(\alpha)}\sum_{i=0}^{m-1}\frac{1}{i!}\int_0^\infty x^{\alpha+i-1}e^{-2x}dx\right) \\ &= n\left(1-\frac{1}{\Gamma\left(\frac{m}{n-1}\right)2^{\frac{m}{n-1}}}\sum_{i=0}^{m-1}\frac{\Gamma\left(\frac{m}{n-1}+i\right)}{i!2^i}\right) \\ &= n\left(1-\frac{1}{2^\alpha}\sum_{i=0}^{m-1}\frac{\Gamma(\alpha+i)}{\Gamma(\alpha)}\frac{1}{2^i i!}\right) \\ &= n\left(1-\frac{1}{2^\alpha}\sum_{i=0}^{m-1}2^{-i}\left(\begin{array}{c} \alpha-1+i \\ i \end{array}\right)\right). \end{aligned}$$

Now we claim that for any $a \in \mathrm{R}-\mathrm{Z}_-$ and non-negative integer r:

$$\sum_{k=0}^{r}2^{-k}\left(\begin{array}{c} a+k \\ k \end{array}\right) = \frac{1}{2^r}\sum_{k=0}^{r}\left(\begin{array}{c} a+1+r \\ k \end{array}\right). \tag{3.6}$$

In fact, this is trivial for $r=0$. Assuming the validity of (3.6) for $r=s-1$, namely

$$\sum_{k=0}^{s-1}2^{-k}\left(\begin{array}{c} a+k \\ k \end{array}\right) = \frac{1}{2^{s-1}}\sum_{k=0}^{s-1}\left(\begin{array}{c} a+s \\ k \end{array}\right),$$

we obtain for $r = s$:

$$\begin{aligned}
\sum_{k=0}^{s} 2^{-k} \left(\begin{array}{c} a+k \\ k \end{array} \right) &= \frac{1}{2^{s-1}} \sum_{k=0}^{s-1} \left(\begin{array}{c} a+s \\ k \end{array} \right) + \frac{1}{2^s} \left(\begin{array}{c} a+s \\ s \end{array} \right) \\
&= 2^{-s} \left(2 \sum_{k=0}^{s-1} \left(\begin{array}{c} a+s \\ k \end{array} \right) + \left(\begin{array}{c} a+s \\ s \end{array} \right) \right) \\
&= 2^{-s} \left(\sum_{k=0}^{s-1} \left(\begin{array}{c} a+s \\ k \end{array} \right) + \sum_{k=0}^{s} \left(\begin{array}{c} a+s \\ k \end{array} \right) \right) \\
&= 2^{-s} \left(2 + \left(\begin{array}{c} a+s \\ 1 \end{array} \right) + \sum_{k=1}^{s-1} \left(\left(\begin{array}{c} a+s \\ k \end{array} \right) + \left(\begin{array}{c} a+s \\ k+1 \end{array} \right) \right) \right) \\
&= 2^{-s} \left(1 + (a+1+s) + \sum_{k=1}^{s-1} \left(\begin{array}{c} a+s+1 \\ k+1 \end{array} \right) \right) \\
&= 2^{-s} \sum_{k=0}^{s} \left(\begin{array}{c} a+1+s \\ k \end{array} \right).
\end{aligned}$$

This proves (3.6) by induction. By (3.6)

$$P_e(n) = n \left(1 - \frac{1}{2^{\alpha+m-1}} \sum_{j=0}^{m-1} \left(\begin{array}{c} \alpha+m-1 \\ j \end{array} \right) \right),$$

which implies the required formula.

Proof of Corollary 2.2:

Let $f(p_i) \sim \chi^2(r)$, $i = 1, 2, ..., n$, with r degrees of freedom, where r is a positive integer. We will distinguish between two situations, depending on the parity of the number of experts:

1) If the number of experts is even, $n = 2k$, where $k \geq 2$ is an integer, then Theorem 2.2 immediately provides the first part of the corollary. Indeed, by substituting $\alpha = \frac{m}{2(k-1)} = \frac{r}{2}$ in Theorem 2.2 we obtain the first part.

2) If number of experts is odd, $n = 2k - 1$, where $k \geq 2$ is a positive integer, then Theorem 2.3 immediately provides the second part of the corollary. Indeed, substituting $\alpha = \frac{m}{2(k-1)} = \frac{r}{2}$ in Theorem 2.3 we obtain the second part.

Proof of Corollary 2.3:

Follows routinely from Theorem 2.3 or Theorem 2.2.

Proof of Theorem 2.4:
Let $Y_i,\ \ i = 1,\ 2,\ ...,\ n$, be the order statistics of $f(p_i) \sim \text{Exp}(1)$, namely:

$$Y_1 \leq Y_2 \leq ... \leq Y_n.$$

Obviously,

$$P_{\text{m}}(n) = P\left(\sum_{i=1}^{s+1} Y_i \geq \sum_{i=s+2}^{n} Y_i\right).$$

Denote:

$$Z_1 = Y_1,$$
$$Z_i = Y_i - Y_{i-1},\ \ i = 2,\ 3,\ ...,\ n.$$

Since $f(p_i),\ \ i = 1,\ 2,\ ..., n$, are independent exponentially distributed random variables, the differences $Z_i,\ \ i = 1,\ 2,\ ...,\ n$, are also independent exponentially distributed random variables and $Z_i \sim \text{Exp}(n-i+1)$(cf. Feller(1971)). Now we can represent the order statistics in terms of the Z_i's :

$$Y_i = \sum_{j=1}^{i} Z_j,\ \ \ i = 1,\ 2,\ ...,\ n.$$

Using this representation

$$\begin{aligned} P_{\text{m}}(n) &= P\left(\sum_{i=1}^{s+1}(s+2-i)Z_i \geq s\sum_{i=1}^{s+1} Z_i + \sum_{i=s+2}^{2s+1}(2s+2-i)Z_i\right) \\ &= P\left(Z_1 \geq \sum_{i=3}^{s+1}(i-2)Z_i + \sum_{i=s+2}^{2s+1}(2s+2-i)Z_i\right). \end{aligned}$$

Denote $W_i = (n-i+1)Z_i,\ \ i = 1,\ 2,\ ...,\ n$. Note that $W_i \sim \text{Exp}(1),\ \ i = 1,\ 2,\ ...,\ n$, are independent random variables, and using them $P_{\text{m}}(n)$ can be represented as follows:

$$\begin{aligned} P_{\text{m}}(n) &= P\left(\frac{1}{n}W_1 \geq \sum_{i=3}^{s+1}\frac{i-2}{n-i+1}W_i + \sum_{i=s+2}^{2s+1}\frac{n-i+1}{n-i+1}Z_i\right) \\ &= P\left(W_1 \geq \sum_{i=3}^{s+1}\frac{(i-2)n}{n-i+1}W_i + n\sum_{i=s+2}^{2s+1} W_i\right). \end{aligned}$$

Denote $b_i = \dfrac{(i-2)n}{n-i+1},\ \ i = 3,\ 4\ ,\ ...,\ (s+1)$. According to the lack-of-memory property of

the exponential distribution:

$$\begin{aligned} P_{\mathrm{m}}(n) &= P\left(W_1 \geq b_3 W_3\right) \cdot P\left(W_1 \geq b_3 W_3 + b_4 W_4 \mid W_1 \geq b_3 W_3\right) \cdot \ldots \\ &\quad \cdot P\left(W_1 \geq \sum_{i=3}^{s+1} \frac{(i-2)n}{n-i+1} W_i + n \sum_{i=s+2}^{2s+1} W_i \mid W_1 \geq \sum_{i=3}^{s+1} \frac{(i-2)n}{n-i+1} W_i + n \sum_{i=s+2}^{2s} W_i\right) \\ &= P\left(W_1 \geq b_3 W_3\right)\left(W_1 \geq b_4 W_4\right) \cdot \ldots \cdot P\left(W_1 \geq n W_n\right) \\ &= \prod_{i=3}^{s+1} \frac{1}{1+b_i} \cdot \frac{1}{(1+n)^s} = \frac{\frac{(n-2)!}{(n-s-1)!}}{s!(n-1)^{s-1}} \cdot \frac{1}{(1+n)^s} = \left(\begin{array}{c} n-1 \\ s \end{array} \right) \cdot \frac{1}{(n^2-1)^s}. \end{aligned}$$

To find the asymptotic behaviour of $P_{\mathrm{m}}(n)$ we use Stirling's formula:

$$\begin{aligned} \left(\begin{array}{c} n-1 \\ s \end{array} \right) &= \frac{\sqrt{2\pi} n^{n-\frac{1}{2}} e^{-n+\frac{\theta_1}{12n}}}{2\pi \left(\frac{n+1}{2}\right)^n e^{-n-1+\frac{\theta_2}{3(n+1)}}} \\ &= \sqrt{\frac{2}{\pi n}} \left(1+\frac{1}{n}\right)^{-n} \cdot 2^{n-1} e^{1+\frac{\theta_1}{12n}-\frac{\theta_2}{3(n+1)}} \qquad (3.7) \\ &\approx \sqrt{\frac{2}{\pi n}} \cdot 2^{n-1} \left(1+O\left(\frac{1}{n}\right)\right). \end{aligned}$$

A simple calculation shows that

$$\frac{1}{(n^2-1)^s} \approx \frac{1}{n^{n-1}} \left(1+O\left(\frac{1}{n^2}\right)\right). \qquad (3.8)$$

Combining (3.7) and (3.8) we obtain the result:

$$P_{\mathrm{m}}(n) \approx \frac{1}{\sqrt{\pi}} \left(\frac{2}{n}\right)^{n-\frac{1}{2}} \left(1+O\left(\frac{1}{n}\right)\right).$$

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