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Bridges between mathematics, natural sciences, architecture and art: case of fullerenes

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The discovery of C_{60}, a third variety of carbon, in addition to the more familiar diamond and graphite forms, has generated enormous interest in many areas of physics, chemistry and material science. Furthermore, it turns out that C60 is only the first of an entire class of closed-cage *polyhedral* molecules consisting of only carbon atoms - the fullerenes (C_{20}, C_{24}, C_{26}, … C_{60}, …C_{70}, … $C_{1000000}$-carbon nanotubes). This paper presents concepts and terms of fullerene science in a historical context, with main emphasis of its interdisciplinary character and interrelationships of various branches of cognition and, in particular, exploration of *polyhedra* in mathematics and fine art. It is discussed how Nature uses fullerene-like structures to minimize energy and matter resources in molecules and nanoclusters, viruses and living organisms. Examples of achievement of such goals in architecture are also presented.

1. History of Discoveries of Fullerenes. Long before experimental discovery of fullerenes, few scientists from different countries predicted an existence of molecules, which would consist only of carbon atoms, located in the vertices of a polyhedron, in particular, a truncated icosahedron (fig. 1). Exploration history of such polyhedron lasts more than two millenniums and will be discussed below.

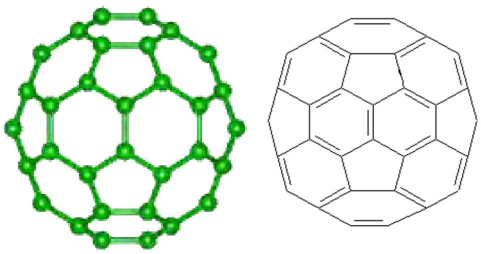

Figure 1: Molecule of buckminsterfullerene, C_{60}.

In 1966, D. Jones had conjectured that if pentagonal disclinations could be introduced into a graphene sheet, consisting of regular hexagons, the sheet would close into a giant hollow molecule of carbon [1]. In 1970, Japanese chemist E. Osawa - published a short article in Japanese [2] on a possible existence of a molecule of 60 carbon atoms, C_{60}, in a shape of truncated icosahedron. Osawa named the molecule 's*occer-ball'*. In 1973 D. Bochvar (1903-1990), E. Galpern and I. Stankevich, performed computer simulation of electronic structure of the C_{60} molecule, which quantitatively proved its stability [3].

…This way or another, by 1985, none of the theoretical predictions mentioned above was truly appreciated by scientific community. It did happen so many times in a history of a human thought in general and in a history of science in particular. The destiny of ideas that are ahead of their time is usually quite tragic. From one point, the society, or in this case scientific community, should be mature for these ideas. On the other hand, the demand for an idea, hypothesis or theory has to evolve as well. In our case, a "trigger" that caused a tremendous, practically explosive "crystallization" of a common interest in the fullerene-like molecules was an experimental discovery of self-generated molecule of C_{60} in a hot carbon plasma by a joint group of Harold Kroto, Richard Smalley (1943-2005) and Robert Curl in 1985 [4].

Kroto's idea was to compare radio-astronomy data with the spectroscopic characterization of carbon clusters produced in the laboratory. This was a motivation for the joint experiment performed by Kroto, Kearl and Smolly in September 1985, that led to an absolutely unexpected result – the discovery of the novel C_{60} molecule. Studying the evaporation of graphite disk under the impulse laser radiation in helium atmosphere, and analyzing the mass-distribution of the generated carbon clusters by a mass spectrometer, scientists detected a dominating peak with the mass of 720 atom units (a. u.). It meant nothing else but self-creation of a stable molecule consisting of 60 atoms of carbon, C_{60} (remember that mass of 1 carbon atom = 12 a. u.). The peak of C_{60} always neighbored by a less intense peak of C_{70}.

Researchers took a risk and answered much more complex question – how to construct a molecule of C_{60} in a way so that all carbon atoms would have satisfied normal bonding (four bonds per carbon atom). Such a requirement would follow a fact of the chemical stability of the molecule. The team (remember they didn't know about the early theoretical predictions!), suggested a structure of truncated icosahedron This polyhedron has 32 faces (20 regular hexagon and 12 regular pentagons), 60 vertices (carbon atoms) and 90 edges. In such a structure, each carbon atom is in equivalent position, but carbon-carbon chemical bonds are of 2 types (fig. 1b): (i) single bonds (C-C), represented by edges between hexagon and pentagon; (ii) double bond (C=C), represented by edges between 2 hexagons.

Figure 2: Buckminster Fuller. The US pavilion at the EXPO-1967, Montreal.

In 13 days after the beginning of experiments, the article about the discovery of C_{60} was sent to "Nature" magazine [4]. The article suggested a name for a new molecule - buckminsterfullerene - after an American architect Buckminster Fuller, the author of a concept of *geodesic domes* – polyhedra buildings (Fig.2). In 1996 Kroto, Smalley and Curl got a Nobel Prize in chemistry for the discovery.

2. Polyhedra in science and art.

Mathematics is the key and door to the sciences.

Galileo Galilei

2.1. Platonic and Archimedean solids

Let no one unskilled in geometry enter [here].

Inscription over the entrance to Plato's Academy.

Archimedes (287 b.c. - 212 b.c.) is often acknowledged as a 1st researcher of truncated icosahedron though one may reasonably suspect that icosahedra had been truncated long before Archimedes.

Truncated icosahedron is one of 13 semiregular polyhedra. These polyhedra are called Archimedean because they were described by Archimedes, even if we have only "second hand" references to his writings on this topic from Heron of Alexandria and Pappus of Alexandria.

The knowledge of Platonic and Archimedean polyhedra was disseminated through the Arabic culture by means of translations made during the VIIIth and IXth centuries, among which the most outstanding one is the account of Abu'l–Wafa (Baghdad, 940–988). Most of Archimedean polyhedra were rediscovered, described and incorporated into the world of Art during the Renaissance, except for the snub dodecahedron, that was described later by Johannes Kepler who in 1619 in his book "Harmonice Mundi" ("Harmony of the Worlds") described the entire class[1] of Archimedean polyhedra or Archimedean solids (fig. 3) - highly symmetric, semi-regular convex polyheda composed of two or more types of regular polygons meeting in identical vertices (please recall equivalent positions of carbon atoms in C_{60} molecule).

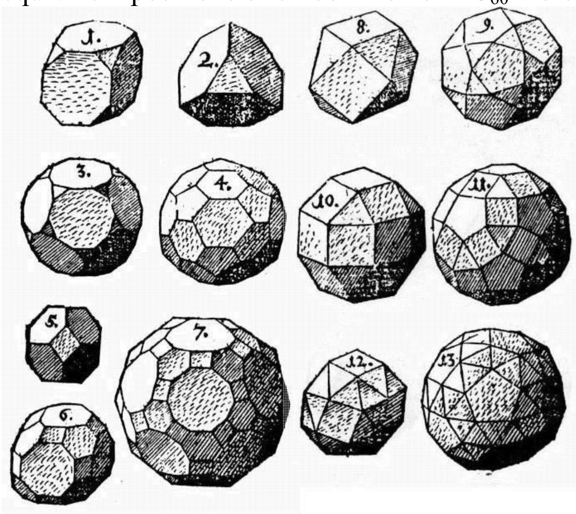

Figure 3: Images of Archimedean polyhedra from Johannes Kepler "Harmonice Mundi" ("Harmony of the Worlds") (1619): polyhedron # 4 is truncated icosahedron.

The fact that Archimedean solids consist of at least 2 different types of polygons that makes them distinct from the *regular polyhedra* or *Platonic solids* which are polyhedra bounded by a number of congruent polygonal faces, so that the same number of faces meet at each vertex, and in each face all the sides and angles are equal (i.e. faces are regular polygons). Only five platonic solids could be constructed. They are tetrahedron, cube, octahedron, icosahedron and dodecahedron. (fig. 4).

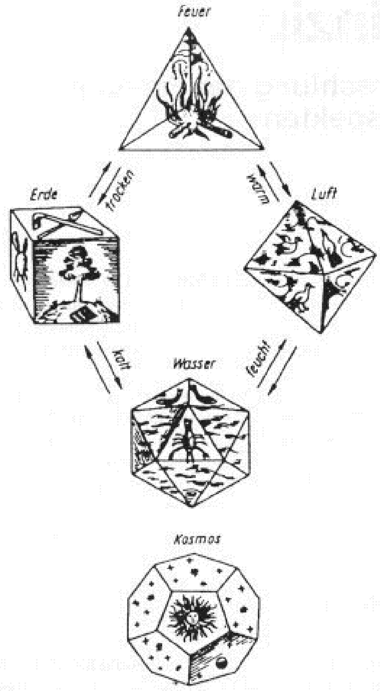

Figure 4: Icons of Platonic solids and the corresponding classical elements from Johannes Kepler "Harmonice Mundi" ("Harmony of the Worlds") (1619).

Plato, and, like him, many other philosophers, including Kepler, associated platonic solids with the classical elements. The tetrahedron, icosahedron, cube and octahedron correspond to FIRE, WATER, EARTH and AIR, respectively. The dodecahedron corresponds to the Quinta Essentia – the UNIVERSE.

Platonic Polyhedra	Four classical elements	Four states of matter (modern view)
Tetrahedron	Fire	Plasma
Cube	Earth	Solid
Icosahedron	Water	Liquid
Octahedron	Air	Gas
Dodecahedron	UNIVERSE	

Table 1: Relationship between Platonic polyhedra, classical elements and states of matter (in modern view).

[1] The 14^{th} Archimedean polyhedron discovered just in the second part of XX century.

The most important property of Platonic polyheda is their high symmetry. These polyhedra belong to the most symmetric point groups: tetrahedral, octahedral or icosahedral.[2] They are the very Platonic solids and classical elements from which H. Kroto started the “symmetry” part of his Nobel Lecture:

“Symmetry appears to be fundamental to our perception of the physical world and it also plays a major role in our attempts to explain everything about it. As far as structural symmetry is concerned it goes back to ancient times, as indicated by the (pre-) Platonic structures exhibited in the Ashmolean Museum in Oxford. The most famous examples are of course to be found in "The Timaeus", where in the section relating to 'The Elements" Plato says: "In the first place it is clear to everyone (!) that fire, earth, water and air are bodies and all bodies are solids" (!!).Plato goes on to discuss chemistry in terms of these elements and associates them with the four Platonic... Although this may at first sight seem like a somewhat naive philosophy it indicates a very deep understanding of the way Nature actually functions”.

Is it a true “deep understanding”? Table 1 alludes to a positive answer.

Nature widely uses Platonic solids for elementary forms of crystals (tetrahedron, cube and octahedron) and quasicrystals (icosahedron and dodecahedron) as well as some viruses and simplest micro-organisms (icosahedron). Below we demonstrate that the smallest fullerene, C_{20}, has a shape of dodecahedron and C_{20}, C_{60}, and other important fullerene molecules exhibit icosahedral symmetry.

Many man-made polyhedral objects in the form of Platonic solids may come easily to the mind of the reader. We can meet them in the arts and architecture across the centuries and in different cultures. In the Neolithic period some carved stones (around 2000 B. C.), discovered in Scotland, were shaped like polyhedra (including *icosahedron* and *dodecahedron*). Below we will demonstrate a number of examples of artistic depictions of polyhedra. Here we will show just two fascinating sculptural images of icosahedron: a splendid sundial in the courtyard of the Palace of Holyroodhouse in Edinburgh (fig. 5) and the monument to Baruch Spinoza in Amsterdam (Fig. 6).

Figure 5: Sundial by John Mylne (1633). Courtyard of the Palace of Holyroodhouse, Edinburgh. Photo by E.A. Katz.

Figure 6: Nicolas Dings. Monument to Baruch de Spinoza (2008). Amsterdam. Photo by E.A. Katz.

Let's now come back to truncated icosahedron (C_{60}). That is the very name “truncated icosahedron” that tells that this polyhedron can be obtained from the Platonic icosahedron by *truncation* which is a geometrical operation that consists in cutting off the vertices of a polyhedron,

[2] The Archimedean polyhedra belong to the same point groups as the Platonic solids, with which they bear close relationships.

thus generating a new polyhedron with more faces. Fig. 7 shows a particular truncation of the icosahedron, in which all the resulting edges have the same length, giving the Archimedean truncated icosahedron. We may think of truncation as the replacement of each vertex by a polygon, perpendicular to the radial direction, with the restriction that such polygons must have as many sides as the number of edges meeting at the vertex. The data from Table 2 may help us to understand this transformation. Indeed, 5 edges meet at every of 12 vertices of the icosahedron. Cutting off the vertices will generate 12 new pentagonal faces. At the same time, original 20 faces will transform to hexagons and together with 12 new pentagonal faces will constitute 32 faces of Archimedean truncated icosahedron which will also have 90 edges and 60 vertices.

Table 2: Characteristics of Platonic solids.

Polyhedron	Number of edges per each face, *m*	Number of edges that connect in each vertex, *n*	Number of faces, F	Number of edges, E	Number of vertices, V	*F-E+V*
Tetrahedron	3	3	4	6	4	2
Cube	4	3	6	12	8	2
Octahedron	3	4	8	12	6	2
Icosahedron	3	5	20	30	12	2
Dodecahedron	5	3	12	30	20	2

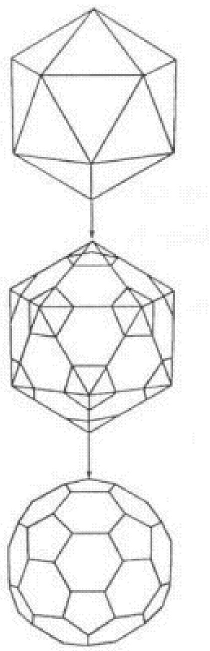

Figure 7: How to built Archimedean truncated icosahedron from Platonic icosahedron.

2.2. Mathematical Discoveries Made by Artists and Polyhedra in Art: Leonardo, Luca Pacioli, Albrecht Dürer, Piero della Francesca and Others…

Mathematics possesses not only truth but supreme beauty,
a beauty cold and austere, like that a sculpture,
sublimely pure and capable of a stern perfection,
such as only the greatest art can show.
Bertrand Russel

For me it remains an open question whether [my work]
pertains to the realm of mathematics or to that of art.
M.C. Escher

In fullerene literature it is often acknowledged an original way to display a truncated icosahedron, suggested by Leonardo Da Vinci. Figure 8 shows Leonardo's drawing of truncated icosahedron from a book "De divina prportione" ("The Divine Proportion") [5] written by Franciscan friar and mathematician Luka Pacioli (1445 - 1514) and illustrated by Leonardo. The book was published in 1509. We like to believe, that Leonardo's "involvement" in the research of truncated icosahedrons (or in other words, in the prehistory of C_{60} discovery) was not accidental. This connection is deeply symbolic. The titan of Renaissance, artist, sculptor, scientist and inventor, Leonardo da Vinci (1452 – 1519) is a symbol of continuum of art and science, and, therefore, his interest in the objects like beautiful and highly symmetrical polyhedra, and particularly in truncated icosahedron is not accidental (here is the place to remind once again the title of Pacioli's book – "The Devine Proportion").

Figure 8: Leonardo's drawing of truncated icosahedron from Pacioli's book "The Divine Proportion" (1509).

Leonardo precedes his print of a truncated icosahedron with a scripture in Latin 'Ycocehedron Abscisus Vacuus'. Term 'Vacuus' means, that all the faces of this polyhedron are shown as 'hollow'. Actually, faces are not shown at all, they exist only in our imagination. On the other hand, the edges of the polyhedron are not shown using geometrical lines, which have neither width nor thickness, but with solid segments. These two features of this print constitute a base for the representation of polyhedra, which was invented by Leonardo for illustration of Pacioli's book, and is called today '*method of solid segments*'.

Pacioli's book strongly influenced on the geometry at that time. Furthermore, Luca Pacioli is considered as one of the biggest European algebraists of the 15th century, and - not less important - inventor of the double book-keeping method, which is used by today in all the accounting systems with no exceptions. He definitely deserved a title of "Father of Modern Accounting". Even though, till now the mysterious and controversial Pacioli causes ferocious discussions among historians of science.

It is known for certain, that Luca Pacioli was born in 1445, in Italian town Borgo San Sepolero (now Sansepolcro). During his childhood, he assisted to a local merchant and also studied in a workshop and studio of the great artist and mathematician Piero della Francesca.

Some of the historians (the first was Giorgio Vasari who published a biography of Piero della Francesca in 1550) blame the author of "The Divine Proportion" in plagiarism of unpublished manuscripts, that belong to his teacher Piero della Francesca. This matter is not really clear. However, what we know exactly is that how Pacioli looked, thanks to his portraits by Jacopo de Barbari (1440 - 1515) and Piero della Francesca.

Figure 9: Jacopo de Barbari, 1495. Luca Pacioli.

In the Barbari's painting (fig. 9) Pacioli, in a robe of Franciscan monk, is shown standing in front of the table with geometrical tools and books (in a right lower corner we see a model of dodecahedron). Pacioli and a handsome young man behind him are looking at the artist (and at us, spectators). At the same time we realize, that the attention of both is focused on the polyhedron glass model. The choice of polyhedron is not random – it's a *rhombicuboctahedron*, rediscovered (after Archimedes) by Pacioli**.**

The figure of young man on the same portrait, standing by Pacioli, is still under discussion of art historians. Some of them suspect this is Barbari's self-portrait, some of them argue that this is young Albrecht Dürer (1471 - 1528). Though it is an open question, it is well known that Dürer was fascinated by the artistic style of Barbari, who created his compositions based on a mathematically defined system of proportions. After his meeting with Jacopo de Barbari in around 1500 in Nurnberg, Dürer started to study the laws of perspective. He dreamed to meet other famous Italian masters, to learn from them, to compete with them. For this purpose, in 1505-1507 Dürer took his second trip to Italy (the first one was in 1494-1495). It is not known for sure who were his teachers in this school of perspective (among others, the names of Luca Pacioli and Piero della Franchesca are considered), but Dürer continued to study in such a school till the end of his life.

Starting since 1525, i.e. in three years before he died, the Master hurried to share with next generations the secrets of perspective, that he had being acquiring all his life. He published two treatises, one of which, "Underweysung der Messung" ("Painter's Manual"), is a serious input in a theory of perspective and geometry of polyhedra. For instance, Dürer was the first to describe few archimedean solids unknown at his time. The book contains a very interesting discussion of perspective and other techniques and it typifies the renaissance idea that polyhedra are models worthy of an artist's attention. More importantly, this book presents the earliest known examples of polyhedral nets, i.e., polyheda unfolded to lie flat for printing (fig. 10). The net for truncated icosahedron is shown in Fig. 10a. One should notice the wedge-shaped gaps between hexagons and pentagons when these faces are made to lie in a plain (the pentagons are required in the structure of truncated icosahedron to generation the curvature of the fullerene shell).

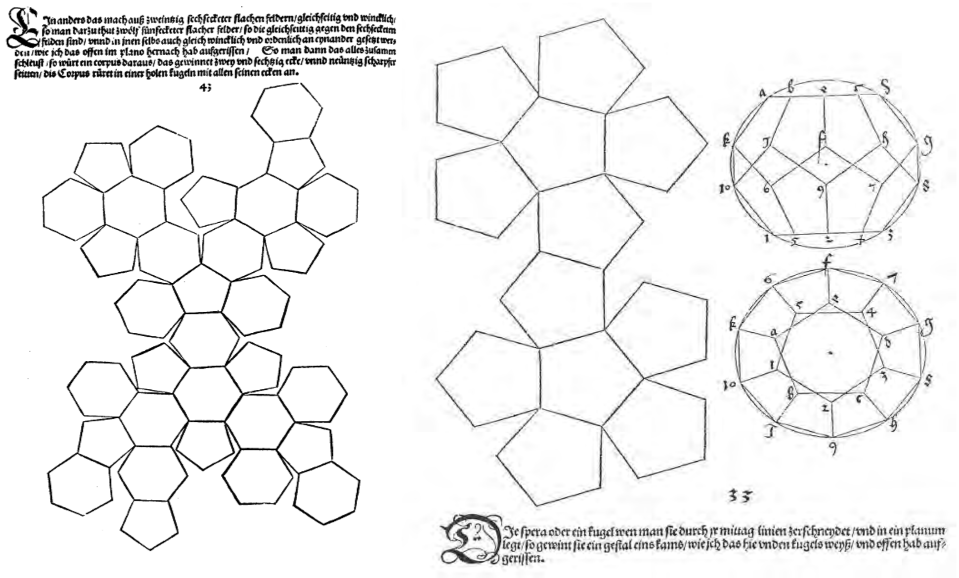

Figure 10: Net for truncated icosahedron (a) and dodecahedron (b) from Albrecht Dürer's treatise "Underweysung der Messung" ("Painter's Manual") (1525).

Nowadays such nets are widely used in studies of elementary forms of crystals, molecular structures (fullerenes, for instance), viruses, etc. We suggest a reader to make a model of C_{60} molecule using a net of truncated icosahedron shown in Fig. 11 [6].

Figure 11: Net for making truncated icosahedron [6].

It should be noted that contrary to the Dürer's net one shown in fig. 11 does not contain any pentagon. This resembles one possible mechanism by which real C_{60} molecules may form in nature using solely graphite hexagonal building blocks.

Dürer's theoretical works, as well as the art of the Renaissance in general, are filled with a thirst for a knowledge. The greatest artists of that time often became outstanding natural scientists. The idea of unity of artistic inspiration and mathematical theory is reflected in one of the most famous Dürer's masterpieces, woodcut "Melancholia 1" (fig. 12). It contains the first magic square to be seen in Europe, cleverly including the date 1514 as two entries in the middle of the bottom row. Of course, of our interest is the polyhedron in the picture. This is truncated rhombohedron that is now known as Dürer's solid. In the mathematical field of graph theory a skeleton of this polyhedron is known as the Dürer graph. S.Alavarez suggested [add15] name "*melancholyhedron"* for this polyhedron and demonstrated inorganic cluster having this shape: a shell of twelve As atoms around Ni (at a distance of 4.3 Å) in the solid state structure of $NiAs_{12}$ (fig. 12). We would suggest name "*durerene*" for such a cluster.

Not only Dürer, Leonardo and Barbari, but many other artists of different epochs and countries were interested in studying and drawing polyhedra. Peak of this interest was, of course, during the Renaissance. Studying Nature, the Renaissance artists tried to find scientific ways of drawing it. Built on geometry, optics and anatomy theories of perspective, proportions and treatment of light and shade became a base for a new art. They allowed an artist to create a three-dimensional space on a flat surface, saving an impression of relief of the objects. For some masters of Renaissance polyhedrons were just a convenient model for practicing the laws of perspective. Some of them were fascinated by their symmetry and laconic beauty. The others, following Plato, were attracted by philosophical and mystical symbols of polyhedra. List of the greatest Renaissance artists, that often draw and seriously studied a geometry of polyhedra (beside Leonardo and Dürer mentioned above), should be started with Uccello (1397-1475) and, first of all, Piero della Francesca (~1420 - 1492).

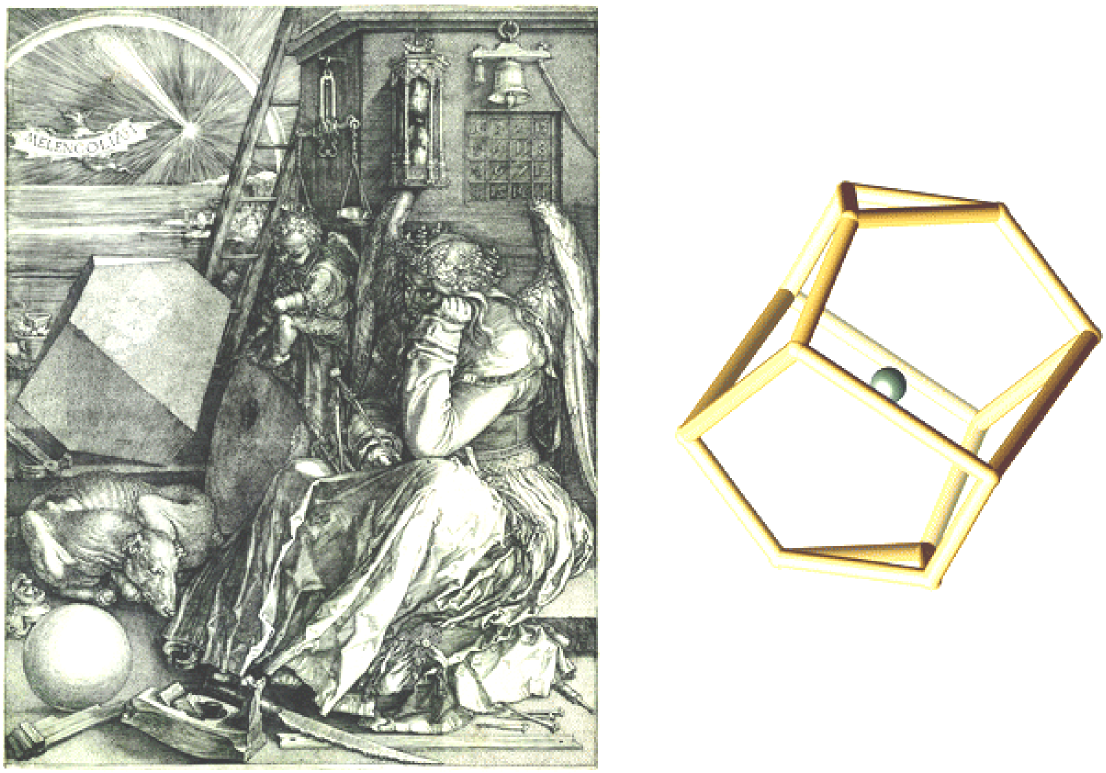

Figure 12: (a)Albrecht Dürer. “Melancholia I ” (1514); (b) structure of $NiAs_{12}$ cluster.

We have very little knowledge on life of Piero della Francesca, a genius artist, serious theorist of art and outstanding mathematician. We know that he was born in 1420 in a family of crafters in a small town of Sansepolcro in Tuscany (as well as his controversial pupil Pacioli). He studied in Florence where he showed a great interest in the art works of Masaccio, Uccello, Brunelleschi and Alberti. Afterwards, he worked in cities of Rimini, Arezzo, Urbino, Ferrara and Rome. Work of Piero della Francesca went beyond the limits of local art schools and influenced the art of the whole Italian Renaissance.

Piero della Francesca was a great mathematician, who contributed a lot to algebra, geometry, science of perspective and, in particular, theory of polyhedra. However, after his death, the name of Piero della Francesca-scientist was forgotten for a long time. It happened, probably, due to the fact, that none of Piero's mathematical work was published under his own name in the Renaissance, but it seems to have circulated quite widely in manuscript and became influential through its incorporation into the works of others. Much of Piero's algebra appears in Pacioli's “Summa”, much of his work on the archimedean solids appears in Pacioli's “The Divine Proportion” and the simpler parts of Piero's perspective treatise were incorporated into almost all subsequent treatises on perspective addressed to painters.

Luckily, in the beginning of the 20th century, the originals of three mathematical manuscripts by Piero della Francesca were found. Nowadays they are in the Vatican Library. After five centuries of oblivion, the fame of famous mathematician got back to the Master. Today we know for certain, that Piero della Francesca was the first who rediscovered and described in details 5 truncated Archimedean solids (without knowing, of course, that Archimedes already discovered them): truncated tetrahedron, truncated cube, truncated octahedron, truncated dodecahedron, and mostly important for our story, truncated icosahedron. His manuscript “Libbelus de quinque corporibus regularibus” (“Short book on the five regular solids”) written in 1480 has the oldest known image of truncated icosahedron (fig. 13).

During following years, many artists and sculptors from different countries used polyhedra as fine art objects. Fig. 14 displays two beautiful samples of *intarsia*, a special kind of mosaics made of thousands little pieces of inlaid wood. Intarsia art reached a peak in northern Italy in the late XV and early XVI centuries. Many outstanding examples of this period feature polyhedra. Both examples shown in Fig. 14 are intarsia panels by Fra Giovanni da Verona (1457 - 1525), constructed around 1520 for the church of Santa Maria in Organo, Verona. The appearance of open cupboard doors creates a strong three-dimensional effect of the masterful perspective in this flat panel which is amplified by polyhedra imaging (including icosahedron and truncated icosahedron) by Leonardo's method of solid segments.

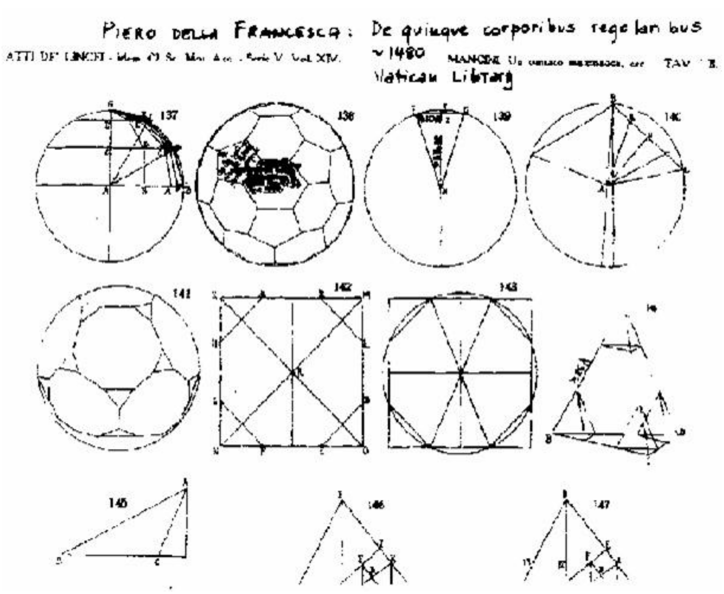

Figure 13: The oldest known picture of truncated icosahedron: drawing by Piero della Francesca from his manuscript “Libbelus de quinque corporibus regularibus” (“Short book on the five regular solids”) (1480).

Figure 14: Intarsia panels by Fra Giovanni da Verona, ~ 1520, the church of Santa Maria in Organo, Verona.

2.3. Harmony of Johannes Kepler.

At ubi materia, ibi Geometria.
Where there is matter, there is geometry.
Johannes Kepler

Johannes Kepler (1571 - 1630) has a special place among those scientists, who have been exploring polyhedrons. Unlike the Renaissance artists that have discovered and even mathematically described some polyhedra, Kepler defined the entire classes of polyhedra, particularly that of Archimedean solids. In 1619, in his book “Harmony of the Worlds” Kepler derived that there are only 13 such polyhedra, fully described each of them and coined the names by which they are known today. In Kepler’s drawing illustrating the Archimedean solids (fig. 3), polyhedron # 13 (icosidodecahedron) was discovered by Kepler himself.

The correspondence between the titles of Kepler’s and Pacioli’s books gets our attention: “Harmony of the Worlds” and “The Divine Proportion”. Nevertheless, unlike the exclusively geometrical work of Pacioli, Kepler was trying to formulate the major principles of structure of the Universe, all the aspects of our world: geometrical, astronomical, astrological, metaphysical, musical (!), social (!!). Kepler understood a marvelous harmony ruling the world not just in an abstract sense. It sounded in his poetic soul as a real music. One could hear this music only by entering into a world of Kepler’s ideas, feeling his powerful enthusiasm for an enchanted structure of the Universe and

Pythagorean admiration for correlation between numbers. In fact, isn't it amazing, that "beautiful" for an ear depends on a strict numeric correlation, for instance, correlation between lengths of the strings, that produce sounds, - the correlation discovered by Pythagoras? Kepler's soul, without a doubt, hosted a part of Pythagoras's soul, and naturally, he saw Pythagorean correlations in the planetary cosmos.

Another Kepler's input to polyhedra geometry is description of two of four regular stellated solids known today as Kepler- Poinsot. Kepler discovered the great stellated dodecahedron and the small stellated dodecahedron (fig. 14a). Both have regular pentagrams as their faces and the full symmetry of dodecahedron. The other two regular star polyhedra (the great dodecahedron and the great icosahedron) were described by Louis Poinsot in 1809. Stellated fullerene-like structures are shown below (Fig. 15).

Figure 14: Small stellated dodecahedron. a: Kepler's drawing from "Harmony of the Worlds" (1619); (b) Detail of a marble inlay in the floor of the Basilica of St. Mark in Venice. Attributed to Paolo Uccello (about 1420's ?); (c) Regards from Kepler to Leonardo. Monument by Mimmo Paladino near to the entrance to Leonardo da Vinci Museum in the village of Vinci (Leonardo's birthplace, Tuscany). Photo by E.A. Katz.

Fig. 14b reproduces a marble inlay which features the small stellated dodecahedron, located in the floor of the Basilica of St. Mark in Venice. Many references attribute this to the great Renaissance painter and mosaicist Paolo Uccello (1397-1475). If so, it is remarkable, for this would be two hundred years before Kepler's 1619 mathematical description of this polyhedron.

3. Euler relation for convex polyhedra helps to understand molecular structure of fullerenes

Read Euler, read Euler, he is our master in everything.
Pierre-Simon Laplace

The universe is a grand book written
in the language of mathematics.
Galileo

Leonard Euler (1707-1783), mathematician, physicist, mechanist and astronomer, author of more than 800 scientific papers is one of the most outstanding scientists in history. There was no brunch of science that would not be of some interest to this great man. Mathematical analysis, geometry, numbers theory, theory of approximation, mechanics, astronomy, optics, ballistics, ship building, music theory, graph theory or mathematical topology. The birth dates of the latter two branches of mathematics were, correspondingly, 1736 when Euler solved the problem known as “the Seven Bridges of Königsberg” and 1758 when he formulated [7] and proved [8] a theorem on correlation between the number of vertices (V), edges (E) and faces (F) of a convex polyhedron:

$$V - E + F = 2 \quad (1)$$

This is the Euler relation which is widely and effectively used in nowadays fullerene studies.

One of the consequences from the Euler theorem dictates that a hypothetical polyhedron formed only by hexagons is not feasible. The latter means in turn that it is impossible to design a close-cage molecule using only graphite hexagonal building blocks. Therefore hexagonal and pentagonal faces in C_{60} molecule. This is true for any polyhedral molecule, cluster, living organism, architectural or any other construction. A nice application of such a principle to the analysis of the shape of radiolaria was given by D'Arcy Thompson in his classical book “On Growth and Form” [9]: “No system of hexagons can enclose space; whether the hexagons be equal or unequal, regular or irregular, it is still under all circumstances mathematically impossible. Neither our *reticulum plasmatique* nor what seems to be the very perfection of hexagonal symmetry in *Aulonia* are as we are wont to conceive them; hexagons indeed predominate in both, but a certain number of facets are and must be other than hexagonal”.

Assuming an existence of the entire family of carbon molecules - *fullerenes* – in the shape of convex polyhedra consisting only of f_6 hexagonal and f_5 pentagonal faces, one gets.
For such molecules:

$$F = f_5 + f_6 \quad (2)$$

Since all vertices of the network have degree of 3 and each edge is shared by exactly two faces, the following expressions can be derived

$$3V = 2E = 5f_5 + 6f_6 \quad (3)$$

or $$6(F - E + V) = 6f_5 + 6f_6 - 15f_5 - 18f_6 + 10f_5 + 12f_6 = f_5 \quad (4)$$

Together with *Euler relation* it yields

$$f_5 = 12 \quad (5)$$

$$3V = 60 + 6f_6 \quad (6)$$

or

$$V = 20 + 2f_6 = 2(10 + f_6) \quad (7)$$

Thus, there must be 12 pentagons in any of such molecules!

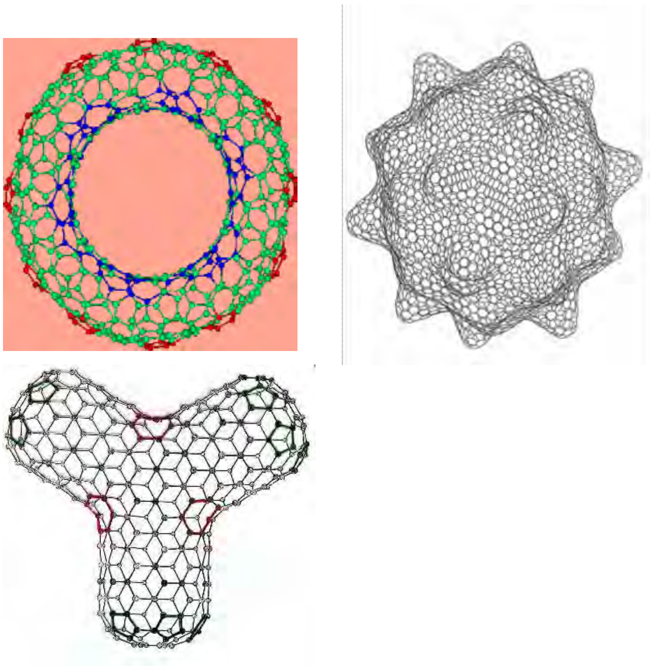

Figure 15: Fullerene-like structures with negative curvature: a – toroidal clusters, b - stellated carbon clusters, c - Y-junctions of carbon nanotubes. Pentagons and heptagons providing positive and negative curvatures, respectively, are indicated.

Number of hexagonal faces f_6 can be varied while evidently V (number of carbon atoms in the molecules) must be always even. Accordingly, the smallest fullerene has a shape of polyhedron with f_6 = 0, formed only by pentagons, is nothing else than the dodecahedron. The next fullerene is C_{24}, затем C_{26}, C_{28}, …, C_{60}, C_{70}, $C_{2(10+h)}$ ...

Finally, fullerene-like structures with negative curvature can be realized by implanting, heptagon disclinations into hexagon/pentagon fullerene nets.[3]

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[3] More generally, introduction of any n-gonal face (where n ≥ 7) will result in negative curvature in fullerene-like structures.

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