SOLAR IRRADIANCE VARIATIONS AND NONLINEAR MEAN FIELD DYNAMO

N. KLEEORIN
Department of Mechanical Engineering, Ben-Gurion University of Negev, POB 653, 84105 Beer-Sheva, Israel

I. ROGACHEVSKII
The Racah Institute of Physics, The Hebrew University of Jerusalem, 91904 Jerusalem, Israel

and

A. RUZMAIKIN
Department of Physics and Astronomy, California State University Northridge, Northridge, CA 91330, U.S.A.

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Abstract. By using a nonlinear model of an axisymmetric $\alpha - \Omega$ dynamo, an analytical expression which gives the magnitude of the mean magnetic field as a function of rotation and other parameters for a solar-type convective zone is obtained. The mean magnetic field varies as the $\frac{3}{4}$ power of the rotation rate. The resulting theoretical relationship of the X-ray luminosity as a function of the angular velocity is in agreement with observations by Fleming, Gioia, and Maccacaro (1989).

1. Introduction

The magnetic fields of the Sun and of solar-type stars are believed to be generated by a dynamo process in their convective zones (Moffatt, 1978; Parker, 1979; Zeldovich, Ruzmaikin, and Sokoloff, 1983). It is known that kinematic dynamo models give no estimate of magnitude for the generated magnetic field. In order to find the magnitude of the field, the nonlinear effects which limit the field growth must be taken into account.

The first theoretical attempts to study the relationship between the magnitude of the magnetic field and the angular velocity and spectral type of the star were made by Robinson and Durney (1982, see also references therein). In these papers a simplified nonlinear dynamo model was used and crude scaling arguments were made.

In the present paper, we obtain a new analytical expression for the magnitude of the mean magnetic field near the stellar surface as a function of the angular velocity of the star and the parameters characterizing the convective zone. By use of this expression, we find a relation between X-ray luminosity and the stellar rotational velocity.

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2. Nonlinear Dynamo Model

The evolution of the mean magnetic field, $\mathbf{B}$, is described by the standard equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times [\mathbf{V} \times \mathbf{B} + \alpha \mathbf{B} - (\eta_T + \eta_m)\nabla \times \mathbf{B}] \tag{1}$$

(Moffatt, 1978; Parker, 1979; Zeldovich, Ruzmaikin, and Sokoloff, 1983), where α determines the effect of the mean helicity of turbulent motions, $\mathbf{V}$ is a mean velocity (the differential rotation), and η_T and η_m are the turbulent and molecular magnetic diffusion. We will use an $(\alpha - \Omega)$-approximation .

Let us split the α-effect into two parts:

$\alpha = \alpha_0 + \alpha_m$, where $\alpha_0 = -(\tau/3)\langle \mathbf{u}\nabla \times \mathbf{u}\rangle$ is the hydrodynamic part of the α-effect, and $\alpha_m = (\tau/12\pi\rho)\langle \mathbf{h}\nabla \times \mathbf{h}\rangle$ is the magnetic part of the α-effect, where $\mathbf{u}$ and $\mathbf{h}$ are the turbulent velocity and magnetic field, ρ is the density, $\tau \approx l_0^2/\eta_T$ is the lifetime of the turbulent eddy, l_0 is the characteristic scale of turbulent motions at the depth of the convective zone where the turbulent magnetic diffusion η_T is a maximum. The splitting of the total α-effect into the hydrodynamic and magnetic parts was first suggested by Frisch *et al.* (1975).

The type of nonlinearity we use includes the effect of delayed back action of the magnetic field on the magnetic part of the α-effect. It is described by an evolutionary equation (see Appendix):

$$\frac{\partial \alpha_m}{\partial t} = \frac{\mu}{4\pi\rho}\left(\mathbf{B}\cdot(\nabla \times \mathbf{B}) - \frac{\alpha \mathbf{B}^2}{\eta_T}\right) - \frac{\alpha_m}{T}\,, \tag{2}$$

where $T = l_s^2\mu_*/8\pi^2\eta_m$, l_s is the characteristic scale of the turbulent motions near the top of the convective zone where the magnetic part of the α-effect is a maximum, and the values $\mu \approx 0.1$ and $\mu_* \approx 1$. Equation (2) was derived by Kleeorin and Ruzmaikin (1982). The closed system of Equations (1) and (2) for $\mathbf{B}$ and α_m represents the nonlinear dynamo model under consideration.

Note that in the case $T \ll T_c$ and $\mathbf{B}\cdot(\nabla \times \mathbf{B}) \ll \alpha_0/(\xi\eta_T)$ Equation (2) yields the well-known result for the total α-effect: $\alpha = \alpha_0/(1 + \xi B^2)$ (see, e.g., Roberts and Soward, 1975) where $\xi = \mu T/(4\pi\rho\eta_T)$, T_c is the period of the stellar activity.

For the sake of simplicity we consider only the axisymmetric case. (Note that the axisymmetry refers only to the mean magnetic field, the fluctuating fields are basically non-axisymmetric so that there is no contradiction to the Cowling theorem.) Then the mean magnetic field can be represented by the poloidal, $\mathbf{B}_p = \nabla \times A(t,r,\theta)\mathbf{e}_\varphi$ and toroidal $\mathbf{B}_t = B(t,r,\theta)\mathbf{e}_\varphi$ components evolving according to Equation (1),

$$\frac{\partial}{\partial t}\begin{pmatrix} A \\ B \end{pmatrix} = (\hat{L} + \hat{N})\begin{pmatrix} A \\ B \end{pmatrix}, \tag{3}$$

where r, θ, φ are the spherical coordinates, and

$$\hat{L} = \begin{pmatrix} \eta\Delta_* & \alpha_0(r,\,\theta) \\ D\hat{\Omega} & \eta\Delta_* \end{pmatrix}, \qquad \hat{N} = \begin{pmatrix} 0 & \alpha_m(r,\,\theta) \\ 0 & 0 \end{pmatrix},$$

$$\hat{\Omega}A = \frac{1}{r}\frac{\partial}{\partial r}\frac{\partial(\Omega,\, A_r \sin\theta)}{\partial(r,\,\theta)}, \qquad \alpha_0(r,\,\theta) = -\alpha_0(r,\,\pi - \theta)\,,$$

$$\Delta_* = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}r\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\right) \equiv \Delta - \frac{1}{r^2\sin^2\theta}\,.$$

Combination of Equations (2) and (3) yields

$$\frac{\partial\alpha_m}{\partial t} + \frac{\alpha_m}{T} = -\frac{B}{\rho}\frac{\partial A}{\partial t} + \frac{\hat{M}(B,\,A)}{\rho}\,, \tag{4}$$

where

$$\hat{M}(B,\,A) = 2B\Delta_* A + \frac{1}{r^2}\frac{\partial}{\partial r}(rA)\frac{\partial}{\partial r}(rB) + \\ + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}(A\sin\theta)\frac{\partial}{\partial\theta}(B\sin\theta)\,.$$

Equations (3) and (4) are written in dimensionless variables: the coordinate r and time t are measured in the units of the stellar radius, R_* and R_*^2/η_T, respectively; the magnetic diffusivity η is measured in the units η_T; α is measured in the units α_*; the angular velocity $\Omega(r,\,\theta)$ is measured in the units Ω_*; the vector-potential of the poloidal field $A(t,\,r,\,\theta)$ and the toroidal magnetic field $B(t,\,r,\,\theta)$ are measured in units of $R_\alpha R_* B_*$ and B_*; the density $\rho(r,\,\theta)$ is measured in the units ρ_*, where $R_\alpha = \alpha_* R_*/\eta_T$,

$$B_* = \left(\frac{4\pi\rho_*}{\mu}\right)^{1/2}\frac{\eta_T}{R_*}\,.$$

The terms of the order of $O(R_\alpha/R_\Omega)$ are dropped in Equations (3) and (4). This assumption corresponds to the usual assumptions of $\alpha - \Omega$ dynamos. Equations (3) and (4) describe a closed nonlinear system.

The most important parameter in the theory is the dynamo number $D = R_\alpha R_\Omega$, where $R_\Omega = \Omega_* R_*^2/\eta_T$. The dynamo number in solar-type convective zones is estimated not to be much larger than the critical dynamo number D_{cr} (a threshold of dynamo excitation), so that only a few modes are expected to be excited.

By using Equations (3) and (4) we estimate the magnitude of the mean magnetic field as a function of rotation and other parameters for a solar-type convective zone. The mechanism of $\alpha - \Omega$-dynamo operates in the following way. The differential rotation generates the toroidal magnetic field from the poloidal field, whereas the α-effect excites a poloidal component of the field from the toroidal field. Nonlinearity results in saturation of the magnetic field, because the total α-effect decreases as the field grows. Therefore the nonlinearity supresses the generation of the poloidal component from the toroidal field.

Let the total α-effect be reduced to the value $\alpha_{nl} = K\alpha_a$, where the nondimensional function α_a is of the order of 1. We have to estimate the value K. The reduction of the α-effect can be described by means of a decrease of the Dynamo number. Indeed, let us provide the following replacements in Equation (3): $A \to KA_a$, $B \to B_a$ and we take into account that $\alpha \to K\alpha_a$. The system obtained coincides with that of the linear dynamo (see Equation (3) with $\hat{N} = 0$) only if we replace $D \to KD$ and $\alpha \to \alpha_a$. Therefore in the saturation regime the nonlinear problem is reduced to the linear one with an effective Dynamo number $D_{\text{eff}} = KD$. At saturation there is no growth of the magnetic field. This means that $D_{\text{eff}} \approx D_{cr}$. It follows from here that

$$K = D_{cr}/D \,. \tag{5}$$

Now we estimate the ratio A_a/B_a. Comparison of terms in Equation (3) which describe the generation and dissipation of the rnagnetic field yields

$$D_{\text{eff}}\hat{\Omega}A_a \approx \Delta_* B_a \,, \qquad \alpha_a B_a \approx \Delta_* A_a \,. \tag{6}$$

It follows from this that

$$A_a/B_a \approx D_{\text{eff}}^{-1/2} \approx D_{cr}^{-1/2} \,. \tag{7}$$

Here we take into account that $|\hat{\Omega}| \approx |\alpha_a| \approx 1$. Using the replacements $A = KA_a$, $B = B_a$ and Equations (5) and (7) we obtain

$$\frac{A}{B} = K\frac{A_a}{B_a} \approx \frac{\sqrt{D_{cr}}}{D} \,. \tag{8}$$

Equation (8) is in agreement with results of direct numerical simulations (see, Ivanova and Ruzmaikin, 1977).

Note that here we have not yet used any quenching mechanism. However for the estimation of the value B we have to choose an explicit form of the quenching. Averaging Equation (4) over a time which is much longer than the period T_c of the activity cycle we obtain

$$\langle \alpha_M \rangle \approx 3\frac{T}{\rho} B \Delta_* A \,.$$

Here we take into account that $|\Delta_* A| \gg (2\pi/T_c)A$. The averaged total α-effect is given by $\langle\alpha\rangle = \langle\alpha_m\rangle + \alpha_0$.

The nonlinear problem has been reduced to a linear one with an effective dynamo number $D_{\rm eff} \approx D_{cr}$. Now we average the value α over the volume of the convective zone. We take into account that

$$\langle\tilde{\alpha}\rangle \equiv \frac{3}{V}\int \frac{TB\Delta_* A}{\rho} d^3r \approx \frac{4\pi}{V} C_0 T B_s \overline{\Delta_* A_s} \Lambda_\rho \approx -\frac{4\pi}{V} C T A_s B_s \Lambda_\rho \, , \tag{9a}$$

$$\tilde{\alpha}_0 \equiv \frac{1}{V}\int \alpha_0 d^3r \approx 4\pi L_0 \, , \tag{9b}$$

where $\tilde{S}$ denotes averaging over the volume of the convective zone, V is the volume of the convective zone, $\overline{P}$ denotes averaging over a spherical surface, the coefficient C_0 depends on both the distribution, α_0, in the convective zone and the structure of the nonlinear solution for the magnetic field, A_s and B_s are the toroidal components of the vector potential and the magnetic field near the top of the convective zone, L_0 is the depth of the convective zone, Λ_ρ is the density height scale and $C = -C_0\overline{\Delta_* A_s}/A_s$. The minus sign in Equation (9a) (as well as in the expression for C) is due to the fact that the second spatial derivative of A is negative. This is because the toroidal component of the vector potential has a maximum inside the convective zone. The main contribution to the integral (9a) is from the region near the surface of the Sun, because the density has a minimum near the surface and strongly increases in the direction towards the bottom of the convective zone. Therefore the value $\langle\tilde{\alpha}_m\rangle$ is determined by the fields A_s and B_s near the surface. The distribution of the nondimensional density in the convective zone is given by

$$\rho(r) = \exp\left\{ \int_r^{R_\odot} \Lambda_\rho(r')\, {\rm d}r' \right\} .$$

This yields $\int \rho^{-1}(r){\rm d}^3r \approx 4\pi\Lambda_\rho(r)$. At saturation the total α-effect is reduced to the value $\langle\tilde{\alpha}\rangle \approx K\tilde{\alpha}_0$ which yields

$$K \approx 1 - CT\frac{\Lambda_\rho}{L_0} A_s B_s \, . \tag{10}$$

Combining Equations (5), (8), (10) we obtain, in dimensional variables, the magnitude of the mean toroidal magnetic field near the stellar surface:

$$B_s \approx \kappa D_{cr}^{1/4} \left(\frac{D}{D_{cr}} - 1\right)^{1/2} \left(\frac{L_0}{l_s}\right) \left(\frac{\pi\rho_*\eta_m\eta_T}{L_0\Lambda_\rho\mu}\right)^{1/2} , \tag{11}$$

where $\kappa = \pi(32/C)^{1/2} \approx 1$. The result (11) is in agreement with that obtained by means of a quantitative analysis (Kleeorin, Rogachevskii, and Ruzmaikin, 1994).

Note that, whereas the growth rate in the kinematic regime is independent of the molecular magnetic diffusivity, η_m, the field magnitude in the stationary state is proportional to $\eta_m^{1/2}$. Therefore, in a perfectly conducting fluid, the magnitude B_s vanishes. The value (11) for the magnetic field also correctly vanishes when the turbulent diffusivity, η/T or the convective zone depth, L_0, goes to zero.

Now we specify the dependence of the dynamo number on the angular velocity and other parameters of the convective zone. We use a spatial distribution of the hydrodynamic part of the α-effect of the form

$$\alpha_0 \approx \begin{cases} l(z)\Omega_*(z) & \text{for } l\Omega_*/u \ll 1\,, \\ u(z) & \text{for } l\Omega_*/u \gg 1 \end{cases}$$

(Zeldovich, Ruzmaikin, and Sokoloff, 1983). This function has a maximum at the depth $z = z_m$ determined by the condition $l_m(z_m) = u_0(z_m)/\Omega_*(z_m)$. The turbulent magnetic diffusivity is $\eta_T \approx l_m(z_m)u_0(z_m)$. It follows then that $l_m(z_m) \approx (\eta_T/\Omega_*)^{1/2}$. The maximum value α_* of the hydrodynamic part of the α-effect is given by

$$\alpha_* \approx u_0(z_m) \approx \eta_T/l_m(z_m) \approx (\eta_T\Omega_*)^{1/2}\,.$$

Now the dependence of the dynamo number on the angular velocity and other parameters of the convective zone is given by

$$D \approx (\Omega_*/\eta_T)^{3/2}R_*^3\,.$$

Then the magnitude of the mean magnetic field near the surface is

$$B_s \approx \kappa(\Omega_*\tau)^{3/4}D_{cr}^{-1/4}\left(\frac{R_*}{l_0}\right)^{3/2}\left(\frac{L_0}{l_s}\right)\left(\frac{\pi\rho_*\eta_m\eta_T}{L_0\Lambda_\rho\mu}\right)^{1/2}\,. \tag{12}$$

Here $\tau = l_0^2/\eta_T$, and it is assumed that $D/D_{cr} - 1 \approx D/D_{cr}$. The magnitude B_s depends both on $\Omega_*\tau$ and the parameters of the stellar convective zone.

Let us estimate, as an example, the mean toroidal magnetic field near the surface of the Sun. The parameters of the solar convective zone at the depth $\approx 2\times10^7$ cm are: $\eta_m \approx 4\times10^6$ cm^2 s^{-1}, $l_s \approx 2.6\times10^7$ cm, $\rho_* \approx 4.5\times10^{-7}$ g cm^{-3}, $\Lambda_\rho \approx 3.6\times10^7$ cm (Spruit, 1974). We use here also $\mu \approx 0.1$, $\eta_T \approx 10^{12}$ cm^2 s^{-1}, $L_0 = 0.3\ R_\odot$, $D_{cr} \approx 10^4$, $D/D_{cr} \approx 2$ to 5. This gives, according to (11), $B_s \approx (1\text{–}3)\times10^2$ G. This value is the same as the mean toroidal magnetic field usually estimated from solar observations (Parker, 1979).

Note that we do not take into account the terms $\sim \alpha^2$ in the mean field dynamo equations for the Sun. Indeed the α^2 terms are not essential if

$$\left|\frac{1}{r}\frac{\partial(\Omega,\, Ar\sin\theta)}{\partial(r,\,\theta)}\right| \gg |\alpha\Delta A|\ .$$

This inequality is reduced to the following sufficient condition for neglecting the α^2 effect in the dynamo equation:

$$(\delta\Omega)L_0 \gg \alpha\ .$$

Let us estimate the terms in this inequality. As follows from Libbrecht (1988) and Küker, Rüdiger, and Kitchatinov (1993) the maximum value of the radial drop of $\delta\Omega$ for the Sun is the following: for $30° < \theta < 90°$ the radial drop $\delta\Omega \approx (0.15–0.2)\Omega_*$; at $\theta = 30°$ the value $\delta\Omega \approx 0.05\Omega_*$; for $\theta < 10°$ the radial drop $\delta\Omega \to 0$. The angle $\theta = 90°$ corresponds to the pole. We use also the fact that $\alpha \approx (\Omega_*\eta_T)^{1/2} \approx 10^3$ cm s^{-1}, and $L_0 \approx 0.3\ R_\odot$. Then for $\delta\Omega \approx (0.15–0.2)\Omega_*$, $(\delta\Omega)L_0 \approx 10^4$ cm s^{-1}. Note also in the nonlinear stage of the magnetic field evolution, α decreases. Therefore the α^2 effect in the scale $\sim L_0$ is not essential for the Sun and we can use the $\alpha - \Omega$ dynamo as a good approximation. However the α^2 effect can be important in scales much smaller than the thickness of the solar convective zone (see Gilman, Morrow, and DeLuca, 1989).

3. Solar X-Ray Luminosity

As an application of (12), we relate the amplitude of X-ray variability to stellar rotation. Observations of soft X-rays in stellar coronae show a correlation between the X-ray luminosity, L_x, and rotational velocity V_Ω: $L_x \sim V_\Omega^\beta$. Fleming, Gioia, and Maccacaro (1989) analyze an X-ray-selected sample of 128 late-type (F-M) single stars and find the exponent $\beta \approx 1.05 \pm 0.08$.

Let us find a theoretical dependence of the X-ray luminosity on the magnetic field and express the magnetic field using our formula (12) to obtain the luminosity dependence on the rotational velocity. Theoretically, the X-ray luminosity is estimated by assuming that the necessary energy comes from reconnections of the magnetic fields in the coronae. Roughly, if the magnetic energy accumulated in a unit volume, $B^2/8\pi$, is released in a reconnection time τ_c, the rate of release is proportional to $B^2/(8\pi\tau_c)$. The accumulation of magnetic energy is due to electric currents generated by convective motions on the stellar surface (see, e.g., Priest, 1982). More accurately, the rate of magnetic energy release in ergs per second, Q_x, is determined by a time-averaged Poynting flux:

$$Q_x \approx \frac{B_c^2\tau_c}{8\pi l}\int\frac{\mathrm{d}\omega}{1+(\omega\tau_r)^2}\int W(\omega)\,\mathrm{d}S \tag{13}$$

(Vekshtein, 1987), where l is a characteristic size of the magnetic region, $W(\omega)$ is the spectrum of the hydrodynamic energy of the turbulent pulsations near the surface of the star, and τ_r is a characteristic time describing the relaxation of the magnetic field to a minimum energy state. The time τ_r can be estimated as the time scale for the onset of the tearing-mode instability: $\tau_r \sim \tau_d^{1-\delta}\tau_a^{\delta}$, where δ varies from 0 to 1 depending on the regime of the tearing instability (see, e.g., White, 1983), $\tau_a \sim l_c/C_{\rm A}$ is the Alfvén time, $\tau_d \sim l_c^2/\eta_c$ is the diffusive time, l_c is the thickness of the current layer in the reconnection region, and $C_{\rm A} = B_c/(4\pi\rho_c)^{1/2}$ is the Alfvén speed. Parameters with the subscript 'c' correspond to the corona and upper chromosphere. The spectrum $W(\omega)$ can be chosen in the form

$$W(\omega) = \int (q-1)\left(\frac{u_0^2}{k_0}\right)\left(\frac{k}{k_0}\right)^{-q}\left(\frac{\nu k^2}{\pi}\right)\frac{1}{\omega^2+\nu^2k^4}\mathrm{d}k\,,$$

where $\nu(k)$ is the turbulent viscosity in the scale k^{-1}, k is the wave number, k_0 and u_0 are the wave number and characteristic velocity in the maximum scale, l_0, the turbulent motions. For example, for the Kolmogorov spectrum of hydrodynamic pulsations, $q = \frac{5}{3}$. We use here the Lorenz profile of the frequency component of the spectrum. Integration in ω and k space in (13) yields

$$Q_x \approx \frac{B_c^2\tau_r}{8\pi l}\int u_0^2\,\mathrm{d}S\,. \tag{14}$$

Here we assume that $\nu k^2 \sim 1/\tau_*(k)$ and $\tau_r < \tau_0$, where $\tau_*(k) = 2\tau_0(k/k_0)^{1-q}$, $\tau_0 \sim l_0/u_0$. It follows from (14) that the rate of the released magnetic energy Q_x depends on magnetic field as

$$Q_x \approx B_c^{2-\delta}\,,$$

where we take into account that $\tau_r \sim B_c^{-\delta}$. The X-ray luminosity L_x is defined as $L_x \approx Q_x N_m$, where N_m is the number of magnetic regions in the corona. Therefore, the X-ray luminosity L_x is given by

$$L_x \sim N_m B_c^{2-\delta}\,.$$

We estimate the exponent β, in the X-ray luminosity vs stellar rotation, assuming that the basic contribution to the X-ray luminosity comes from the magnetic fields outside sunspots and active regions. In this case the number of magnetic regions, N_m, is independent of the mean magnetic field, B_s, and is determined only by the number of convective cells at the photosphere. On the other hand, the rate of magnetic energy release in ergs per second, $Q_x \sim B_s^{2-\delta}$. This is due to the magnitude $B_c \sim k_c B_s$, $k_c \approx 10^{-1}$. Therefore, the dependence of the X-ray luminosity on the mean magnetic field is given by $L_x \sim B_s^{2-\delta}$. By use of the expression (12) for the field B_s near the surface we can express the X-ray luminosity in terms of the rotational velocity, V_Ω. The result is given by: $L_x \sim V_\Omega^{\beta}$, where $\beta = 3(2-\delta)/4$. Since $0 < \delta < 1$ we obtain that $\frac{3}{4} < \beta < \frac{3}{2}$, in satisfactory agreement with observations (see Fleming, Gioia, Maccacaro, 1989).

4. Conclusions

By use of a nonlinear model of an axisymmetric $\alpha - \Omega$ dynamo we have found an expression for the magnitude of the mean magnetic field as a function of the stellar rotation rate and other parameters of the solar-type convective zone. This expression predicts that the field varies as the $\frac{3}{4}$ power of the rotation rate. This permitted the study of the relation between stellar rotation rates and the X-ray luminosity. The exponent in the relation between the X-ray luminosity and the stellar rotation is $\frac{3}{4} < \beta < \frac{3}{2}$, in satisfactory agreement with observations.

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Appendix. Evolutionary Equation for the Magnetic Part of the α-Effect

Let us derive the equation for the magnetic part α_m of the α-effect (for details see Kleeorin and Ruzmaikin, 1982). The mean magnetic helicity of the turbulent field is given by

$$\chi \equiv \langle \mathbf{a} \cdot \mathbf{h} \rangle = \int \chi_*(k)\, \mathrm{d}k \,,$$

where k is the wave number, $\chi_*(k)$ is the spectral density of the magnetic helicity, and $\mathbf{a}$ is the fluctuating part of the vector potential.

For $\nabla \cdot \mathbf{A} = \nabla \cdot \langle \mathbf{A} \rangle = \nabla \cdot \mathbf{a} = 0$ we obtain

$$\langle \mathbf{h} \cdot (\nabla \times \mathbf{h}) \rangle = \int k^2 \chi_*(k)\, \mathrm{d}k \,, \tag{A1}$$

$$\alpha_m \equiv \frac{\tau}{12\pi\rho} \langle \mathbf{h} \cdot (\nabla \times \mathbf{h}) \rangle = \frac{1}{12\pi\rho} \int k^2 \tau_*(k) \chi_*(k)\, \mathrm{d}k \,, \tag{A2}$$

where $\mathbf{A} = \langle \mathbf{A} \rangle + \mathbf{a}$ is the total vector potential. The induction equation for the total magnetic field $\mathbf{H} = \mathbf{B} + \mathbf{h}$ is given by

$$\frac{\partial \mathbf{H}}{\partial t} = \nabla \times [\mathbf{u} \times \mathbf{H} - \eta_m \nabla \times \mathbf{H}] \,, \tag{A3}$$

where we consider the case with zero mean velocity. It follows from (A3) that the equation for the vector potential, $\mathbf{A}$, is given by

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \times \mathbf{H} - \eta_m \nabla \times (\nabla \times \mathbf{A}) + \nabla \Phi \, . \qquad \text{(A4)}$$

Here Φ is an arbitrary function. Let us multiply Equation (A3) by $\mathbf{a}$ and Equation (4) by $\mathbf{h}$, add them, and average over the ensemble of turbulent pulsations. The result is given by

$$\frac{\partial \chi}{\partial t} = -2 \langle (\mathbf{u} \times \mathbf{h}) \cdot \mathbf{B} \rangle - 2\eta_m \langle \mathbf{h} \cdot (\nabla \times \mathbf{h}) \rangle - \langle \nabla \cdot [\mathbf{a} \times (\mathbf{u} \times \mathbf{h})] \rangle \, . \qquad \text{(A5)}$$

Here we omit the term $\sim \langle \mathbf{h} \cdot \nabla \Phi \rangle$ which is small in a medium with zero mean velocity. Taking into account that the mean velocity $\mathbf{V}$ yields an additional term $\nabla \cdot (\chi \mathbf{V})$ in Equation (A5), the effective electric field $\mathbf{E}_{\text{eff}} \equiv \langle \mathbf{u} \times \mathbf{h} \rangle$ is given by (Moffatt, 1978; Parker, 1979; Zeldovich, Ruzmaikin, and Sokoloff, 1983)

$$\mathbf{E}_{\text{eff}} \equiv \langle \mathbf{u} \times \mathbf{h} \rangle = \alpha \mathbf{B} - \eta (\nabla \times \mathbf{B}) \, ,$$

where $\alpha = \alpha_0 + \alpha_m$ and $\eta = \eta_m + \eta_T$. The term $\langle \nabla \cdot [\mathbf{a} \times (\mathbf{u} \times \mathbf{h})] \rangle$ in Equation (A5) vanishes as a result of averaging of $\langle \text{div} \rangle$ over the volume. It follows from this that Equation (A5) is reduced to

$$\frac{\partial \chi}{\partial t} = 2[\eta \mathbf{B} \cdot (\nabla \times \mathbf{B}) - \alpha \mathbf{B}^2 - \eta_m \langle \mathbf{h} \cdot (\nabla \times \mathbf{h}) \rangle] \, . \qquad \text{(A6)}$$

Now let us assume that in the inertial range $k_0 < k < k_1$ the spectrum of the helicity $\chi_*(k)$ is given by

$$\chi_*(k) = \chi \frac{P}{k_0} \left(\frac{k}{k_0} \right)^{-q} , \quad P = (q-1) \left[1 - \left(\frac{k_0}{k_1} \right)^{q-1} \right]^{-1} , \qquad \text{(A7)}$$

where $|\chi| = |\langle \mathbf{a} \cdot \mathbf{h} \rangle| \sim B^2/k_0$, k_0^{-1} is the maximum scale of the turbulence, k_1^{-1} is the scale of the cutoff of the helicity spectrum. The parameter q is assumed to be known. For example, for Kolmogorov's spectrum, $q = \frac{5}{3}$ (developed hydrodynamic turbulence) and for Kraichnan's spectrum, $q = \frac{3}{2}$ (turbulence of interacting Alfvén waves). Substitution of (A6) into (A1) yields

$$\alpha_m = I \chi \, , \qquad \text{(A8)}$$

where

$$I = \frac{\mu}{4\pi \rho \eta_T} \, ,$$

$$\mu = \frac{1}{18}\frac{q-1}{2-q}\left[\left(\frac{k_1}{k_0}\right)^{4-2q} - 1\right]\left[1 - \left(\frac{k_0}{k_1}\right)^{q-1}\right]^{-1} .$$

Here we take into account that $\tau_*(k) = 2\tau_0(k/k_0)^{1-q}$. The turbulent magnetic diffusivity, η_T, for turbulence, which is far from the equipartition of the energy of hydrodynamic pulsations and magnetic fluctuations, is given by $\eta_T = (12\tau_0 k_0^2)^{-1}$.

Multiplying Equation (A6) by I and using (A1) we obtain Equation (2), where

$$\mu_* = \frac{3-q}{q-1}\left[1 - \left(\frac{k_0}{k_1}\right)^{q-1}\right]\left[\left(\frac{k_1}{k_0}\right)^{3-q} - 1\right]^{-1}$$

(see Kleeorin and Ruzmaikin, 1982).

Spectral properties of the magnetic helicity, $\chi_*(k)$, satisfy the realizability condition: $|\chi_*(k)| \leq 2k^{-1}M(k)$ (Moffatt, 1978), where $M(k)$ is the spectrum of the magnetic fluctuations. It follows that

$$\left(\frac{k}{k_0}\right)^{q_m+1-q} \leq \frac{2}{P} . \tag{A9}$$

Here we use a magnetic spectrum of the form

$$M(k) \approx \frac{B^2}{k_0}\left(\frac{k}{k_0}\right)^{-q_m} .$$

For the case $q_m = 1$ this spectrum was obtained using different techniques by Ruzmaikin and Shukurov (1982), Kleeorin, Rogachevskii, and Ruzmaikin (1990), Kleeorin and Rogachevskii (1994), Brandenburg *et al.* (1994a, b). It is seen from (A9) that for $q_m \geq 1$ and $q < 2$ the wave number k and therefore k_1 is close to k_0. Indeed, k_1 is determined from

$$\left(\frac{k_1}{k_0}\right)^{q_m+1-q} = \frac{2}{P(k_1)} .$$

It follows from this that $P(k_1) \approx 1$ and $k_1 \approx k_0$. In this case, which seems to be typical for the solar type convective zones, $\mu \approx \frac{1}{9}$ and $\mu_* \approx 1$, and $T \approx \tau_0 R_m$.

Note that Equation (2) can also be regarded as a consequence of the conservation of total magnetic helicity $\int \langle \mathbf{A} \cdot \mathbf{H} \rangle \, d^3r$ in the limit of $R_m \to \infty$.

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