

Three-dimensional slow Rossby waves in rotating spherical density-stratified convectionT. Elperin,^{*} N. Kleeorin,[†] and I. Rogachevskii[‡]*The Pearlstone Center for Aeronautical Engineering Studies, Department of Mechanical Engineering,
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We develop a theory of three-dimensional slow Rossby waves in rotating spherical density stratified convection. The Rossby waves, with frequencies which are much smaller than the rotating frequency, are excited by a nonaxisymmetric instability from the equilibrium based on the developed convection. These waves interact with the inertial waves and the density stratified convection. The density stratification is taken into account using the anelastic approximation for very low-Mach-number flows. We study long-term planetary Rossby waves with periods which are larger than two years. We suggest that these waves are related to the southern oscillation and *El Niño*. The *El Niño* is an irregularly periodical variation in winds and sea surface temperatures over the tropical Pacific ocean, while the southern oscillation is an oscillation in surface air pressure between the tropical eastern and the western Pacific ocean. The strength of the southern oscillation is characterized by the southern oscillation index (SOI). The developed theory is applied for the interpretation of the observed periods of the SOI. This study demonstrates a good agreement between the theoretical predictions and the observations.

DOI: [10.1103/PhysRevE.96.033106](https://doi.org/10.1103/PhysRevE.96.033106)**I. INTRODUCTION**

Rossby waves [1,2], which are caused by the combined effect of rotation and curvature of the surface, exist in various hydrodynamic flows (see [3–5], and references therein). Rossby waves have been found in different geophysical phenomena (see [3,4,6–9], and references therein), e.g., they are observed in the atmosphere as the large meanders of the midlatitude jet stream that are responsible for the prevailing seasonal weather patterns and their day-to-day variations (see [3,6,7], and references therein). Mesoscale variability in the ocean, in scales of tens to hundreds of km and tens to hundreds of days, occurs as linear Rossby waves and as nonlinear vortices or eddies (see [10–15], and references therein).

The nonlinear dynamics of Rossby waves can cause the formation of zonal flows or zonal jets (see [5,16–18], and references therein). Zonal flows are coherent structures with a jetlike velocity profile in anisotropic flows due to the large aspect ratio and planetary rotation. Zonal flows are produced near the tropopause in the atmosphere (the polar and subtropical jet streams) and in the oceans (the Antarctic circumpolar current) [5,19]. The generation of zonal flows are caused by the generation of small-scale meridional Rossby waves by a modulational instability and nonlinear interaction of the Rossby waves [5]. Generation of zonal flows in the Earth's atmosphere occurs in unstably stratified flows.

Rossby waves also can be important in solar and stellar astrophysics, e.g., in the tachocline layer below the convection zone of the sun and solarlike stars [20–22]. In plasma physics, Rossby waves are analogous to the drift waves which are caused by the combined effect of the drift motions perpendicular to the electric and magnetic fields and large-scale density gradient (see [5,18], and references therein).

It was suggested that Rossby waves can be related to the *El Niño* and southern oscillation (see [23–26], and references therein). The *El Niño* is an irregularly periodical variation of winds and sea surface temperatures over the tropical Pacific ocean, while the southern oscillation is an oscillation of surface air pressure between the tropical eastern and the western Pacific ocean.

The strength of the southern oscillation is characterized by the southern oscillation index (SOI), and is determined from fluctuations of the surface air pressure difference between Tahiti and Darwin, Australia (see [27,28], and references therein). For example in Fig. 1 we show the time dependence of the SOI after a five month window averaging, where the time is measured in years, $t = 0$ corresponds to the year 1878, and the total time interval of the observations of the SOI is 138 years. Although this function looks like a random signal, the spectrum of the SOI shown in Fig. 2 contains several typical periods which are larger than two years.

In spite of many observations of the SOI, it is not clear how Rossby waves can explain these observations. The classical two-dimensional Rossby waves are usually considered in the β -plane approximation. In the framework of this approximation, the motions are studied in a quasi-two-dimensional incompressible thin fluid layer of constant density with a free surface in hydrostatic balance on the rotating plane with a variable Coriolis parameter, that depends linearly on the horizontal coordinate (see [3,4], and references therein). However, the periods of the classical planetary Rossby waves investigated in numerous studies do not exceed the time scales of about 100 days. On the other hand, the periods of oscillations related to the SOI are larger than two years [see Fig. 2].

The goal of the present study is to develop a theory of slow three-dimensional Rossby waves in rotating spherical density stratified convection, and apply the theory to explain the observations related to the southern oscillation. This paper is organized as follows. In Sec. II, we consider the classical two-dimensional Rossby waves in spherical geometry where we do not use the β -plane approximation. In Sec. III, we explain the basic ideas related to the theory of the three-dimensional

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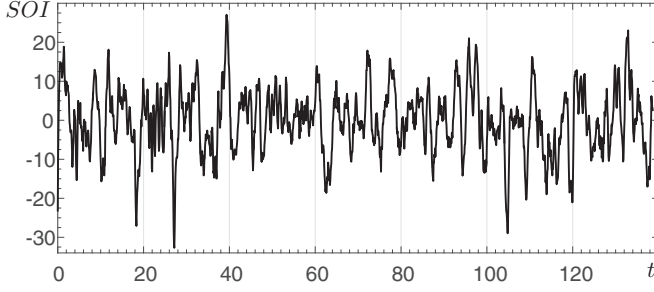


FIG. 1. Time dependence of the southern oscillation index (SOI) after 5 month window averaging, where the time is measured in years, $t = 0$ corresponds to the year 1878, and the total time interval of the observations of the SOI is 138 years. The data are taken from [29].

slow Rossby waves in rotating spherical density stratified convection. In this section we also apply the developed theory for the interpretation of the observed periods of the SOI. In Sec. IV, we discuss our results and draw some conclusions. In Appendices A and B we present detailed derivations related to the developed theory.

II. TWO-DIMENSIONAL ROSSBY WAVES

We start with traditional two-dimensional Rossby waves. To this end, we neglect viscosity in a Navier-Stokes equation for the velocity field $\mathbf{v}(t, \mathbf{r})$ and consider incompressible rotating flow:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla p}{\rho} + 2\mathbf{v} \times \boldsymbol{\Omega}, \quad (1)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (2)$$

where $\boldsymbol{\Omega}$ is the angular velocity, p is the fluid pressure, and ρ is the fluid density. We use spherical geometry, (r, ϑ, φ) , and consider the two-dimensional (2D) flow with $v_r = 0$. This implies that we study the two-dimensional flow on the spherical surface with the constant radius R . The equation for the radial component of vorticity $w_r(\vartheta, \varphi)$ is given by

$$\frac{\partial w_r}{\partial t} = -2(\mathbf{v} \cdot \nabla) \boldsymbol{\Omega}_r, \quad (3)$$

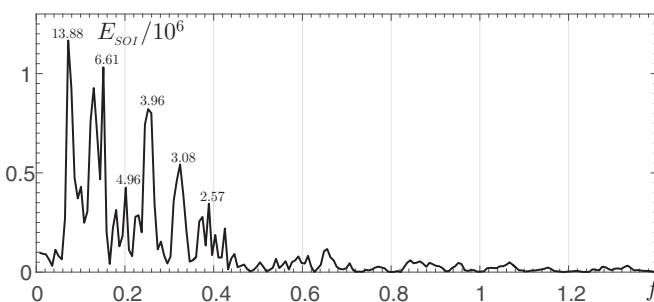


FIG. 2. The spectrum $E_{\text{SOI}}(f)$ of the SOI. The frequency is measured in the units of inverse years. The shown periods of oscillations (13.88; 6.61; 4.96; 3.96; 3.08; 2.57) are measured in years.

where $\mathbf{w} = \nabla \times \mathbf{v}$ is the vorticity and $\boldsymbol{\Omega} = \Omega(\cos \vartheta, -\sin \vartheta, 0)$. Equation (3) can be rewritten as follows:

$$\frac{\partial w_r}{\partial t} = \frac{2\Omega}{R} v_\vartheta \sin \vartheta. \quad (4)$$

For incompressible two-dimensional flow, we introduce the stream function $\Psi(\vartheta, \varphi)$ defined as

$$\mathbf{v} = \nabla \times [\mathbf{e}_r \Psi(\vartheta, \varphi) R]. \quad (5)$$

Equations (4) and (5) yield the following equation for the stream function $\Psi(t, \vartheta, \varphi)$:

$$\frac{\partial \Delta \Psi}{\partial t} = -\frac{2\Omega}{R^2} \frac{\partial \Psi}{\partial \varphi}, \quad (6)$$

where $w_r = -\Delta \Psi$ and $v_\vartheta = \sin^{-1} \vartheta (\partial \Psi / \partial \varphi)$. We seek for a solution of this equation in the following form:

$$\Psi(t, \vartheta, \varphi) = C_{\ell, m} P_\ell^m(\vartheta) \exp(i\omega t + im\varphi), \quad (7)$$

where $P_\ell^m(\vartheta)$ is the associated Legendre function (spherical harmonics) of the first kind, that is the eigenfunction of the operator $R^2 \Delta$, i.e.,

$$R^2 \Delta P_\ell^m(\vartheta) = -\ell(\ell + 1) P_\ell^m(\vartheta). \quad (8)$$

Substituting Eq. (7) into Eq. (6) and using Eq. (8), we obtain the well-known expression for the frequency of the classical two-dimensional Rossby waves in the incompressible flow [3]:

$$\omega_R^{(2D)} = \frac{2\Omega m}{\ell(\ell + 1)}. \quad (9)$$

III. THREE-DIMENSIONAL ROSSBY WAVES IN CONVECTION

In this section we study an excitation of slow three-dimensional (3D) Rossby waves with a frequency that is much smaller than the rotation frequency. We consider a spherical rotating layer with the radius R in the presence of a density stratified convection, use the anelastic approximation for very low-Mach-number flows, and neglect dissipative terms, for simplicity. The governing equations written in a rotating frame of reference read

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla \left(\frac{p}{\rho} \right) - \boldsymbol{\beta} \theta + 2\mathbf{v} \times \boldsymbol{\Omega}, \quad (10)$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{v} \cdot \nabla) \theta = -(\mathbf{v} \cdot \nabla) \theta_{\text{eq}}, \quad (11)$$

$$\nabla \cdot (\rho \mathbf{v}) = 0, \quad (12)$$

where $\boldsymbol{\beta} = \mathbf{g}/T_0$ is the buoyancy parameter, \mathbf{g} is the acceleration due to the gravity, and $\theta = T(p_0/p)^{1-1/\gamma}$ is the potential temperature with equilibrium value θ_{eq} . Here T is the physical temperature with the reference value T_0 , p is the fluid pressure with the reference value p_0 , ρ is the fluid density, and $\gamma = c_p/c_v$ is the ratio of specific heats.

We linearize Eqs. (10)–(12), take twice the curl from the equation of motion (10) to exclude the pressure term, use spherical coordinates r, ϑ, φ , and write the obtained equations

for the radial components of velocity and vorticity v_r, w_r :

$$(\mathbf{\Lambda} \cdot \nabla - \Delta) \frac{\partial v_r}{\partial t} = 2(\mathbf{\Omega} \cdot \nabla) w_r + 2(\mathbf{\Omega} \times \nabla)_r (\mathbf{\Lambda} \cdot \mathbf{v}) + 2r^{-1} \Omega_\vartheta \nabla_r v_\varphi - \beta \Delta_\perp \theta, \quad (13)$$

$$\frac{\partial w_r}{\partial t} = 2(\mathbf{\Omega} \cdot \nabla) v_r - 2\mathbf{\Omega}_r (\mathbf{\Lambda} \cdot \mathbf{v}) - 2(\mathbf{v} \cdot \nabla) \mathbf{\Omega}_r, \quad (14)$$

$$\frac{\partial \theta}{\partial t} = -(\mathbf{v} \cdot \nabla) \theta_{\text{eq}}, \quad (15)$$

where $\mathbf{\Lambda} = -\nabla \rho / \rho = \text{const}$, and the continuity equation (12) reads $\nabla \cdot \mathbf{v} = \mathbf{\Lambda} \cdot \mathbf{v}$. Let us rewrite Eqs. (13)–(15) using new variables: $\mathbf{U} = \sqrt{\rho} \mathbf{v}$, $\mathbf{W} = \sqrt{\rho} \mathbf{w}$ and $\Theta = \sqrt{\rho} \theta$:

$$\left(\frac{\Lambda^2}{4} - \Delta \right) \frac{\partial U_r}{\partial t} = (2\mathbf{\Omega} \cdot \nabla + \mathbf{\Omega} \cdot \mathbf{\Lambda}) W_r + 2(\mathbf{\Omega} \times \nabla)_r (\mathbf{\Lambda} \cdot \mathbf{U}) + r^{-1} \Omega_\vartheta (2\nabla_r + \Lambda) U_\varphi - \beta \Delta_\perp \Theta, \quad (16)$$

$$\frac{\partial W_r}{\partial t} = (2\mathbf{\Omega} \cdot \nabla - \mathbf{\Omega} \cdot \mathbf{\Lambda}) U_r + \frac{2\mathbf{\Omega}}{r} U_\vartheta \sin \vartheta, \quad (17)$$

$$\frac{\partial \Theta}{\partial t} = -\frac{\Omega_b^2}{\beta} U_r - U_\vartheta \nabla_\vartheta \Theta_{\text{eq}}, \quad (18)$$

$$\nabla \cdot \mathbf{U} = \frac{1}{2} \mathbf{\Lambda} \cdot \mathbf{U}, \quad (19)$$

where $\Omega_b^2 = \beta \nabla_r \Theta_{\text{eq}}$ and Δ_\perp is the angular part of the Laplacian Δ . We seek for a solution of Eqs. (16) to (19) using a basis of spherical vector functions that are eigenfunctions of the operator curl in spherical coordinates. In particular, \mathbf{U} is given by the following expression:

$$\mathbf{U} = \sum_{\ell, m} \exp(\lambda t - ik r) [\hat{A}_r \mathbf{Y}_r + \hat{A}_p \mathbf{Y}_p + \hat{A}_t \mathbf{Y}_t], \quad (20)$$

where the coefficients $\hat{A}_r, \hat{A}_p, \hat{A}_t$ depend on ℓ and m , and the vector basis functions (radial, poloidal, and toroidal) are given by

$$\begin{aligned} \mathbf{Y}_r &= \mathbf{e}_r Y_\ell^m(\vartheta, \varphi), & \mathbf{Y}_p &= r \nabla Y_\ell^m(\vartheta, \varphi), \\ \mathbf{Y}_t &= (\mathbf{r} \times \nabla) Y_\ell^m(\vartheta, \varphi), \end{aligned} \quad (21)$$

and $Y_\ell^m(\vartheta, \varphi)$ is the eigenfunction of the operator $r^2 \Delta_\perp$ with the eigenvalue $-\ell(\ell + 1)$.

Using the procedure described in Appendix A we obtain the following dispersion equation:

$$\begin{aligned} (\lambda - 2i\omega_R^{(2D)}) (\lambda + i\omega_R^{(3D)}) + 2\Omega^2 \left(1 + \frac{3i\sigma}{\ell^2(1 + \sigma^2)} \right) \\ + \frac{\Omega_b^2 H_\rho}{R} \left(\frac{i\omega_R^{(3D)}}{\lambda} + \frac{4\ell^2 H_\rho}{R(1 + \sigma^2)} \right) \\ - \omega_R^{(2D)} (\omega_R^{(2D)} + \omega_R^{(3D)}) = 0, \end{aligned} \quad (22)$$

where $\ell^2 \gg 1$, $\sigma = 2k H_\rho$, $H_\rho = 1/|\mathbf{\Lambda}|$,

$$\omega_R^{(3D)} = \frac{8m \Omega H_\rho}{R(1 + \sigma^2)}, \quad (23)$$

and $\omega_R^{(2D)}$ is the frequency of the two-dimensional Rossby waves determined by Eq. (9). Equation (22) determines three modes: the convective mode, the inertial waves, and the three-dimensional Rossby waves. These Rossby waves are due to the combined effect of rotation and the curvature of the surface in density stratified developed convection. The inertial waves are caused only by rotation and described by the following dispersion relation: $\omega_I = 2(\mathbf{\Omega} \cdot \mathbf{k})/k$, where \mathbf{k} is the wave vector.

Let us justify the assumption behind the condition $\ell^2 \gg 1$; m^2 that is used for the derivation of Eqs. (22) and (23). We study excitation of the three-dimensional slow Rossby waves from the equilibrium based on the developed convection. According to the observations (see, e.g., [7]), six large-scale convective cells are observed in the Earth's atmospheric large-scale circulation (the Hadley cell, the midlatitude cell, and the polar cell, i.e., there are three convective cells in the northern hemisphere and three convective cells in the southern hemisphere). These cells can be interpreted in terms of the solution of Eqs. (16) to (19) in the form of the spherical functions Y_ℓ^m with $\ell = 6$ and $m = 0$ (i.e., the axisymmetric modes). The solutions for the slow three-dimensional Rossby waves are not axisymmetric, and they have $m = 1$ or $m = 2$. This is the reason for the assumption that $\ell(\ell + 1) \gg 1; m^2$.

Let us determine the conditions for the existence of the slow 3D Rossby waves. The balance $2\Omega^2 + (\Omega_b^2 H_\rho / R) (i\omega_R^{(3D)} / \lambda) = 0$ with $\lambda = -i\omega_R^{(3D)}$ determines the Brunt-Väisälä frequency:

$$\Omega_b = \Omega \left(\frac{2R}{H_\rho} \right)^{1/2}, \quad (24)$$

so the frequency of the three-dimensional slow Rossby waves is given by

$$\omega_R = -\omega_R^{(3D)}. \quad (25)$$

The frequencies of the slow three-dimensional Rossby waves, $\omega_R^{(3D)}$, are smaller than some typical frequencies in the system: $\omega_R^{(3D)} \ll \omega_R^{(2D)} \ll \Omega \ll |\Omega_b|$. In particular, since $\ell \gg m$ the frequency of the 2D Rossby waves $\omega_R^{(2D)}$ is much smaller than the angular velocity Ω [see Eq. (9)]. Since $H_\rho \ll R$, the frequency of the 3D Rossby waves $\omega_R^{(3D)}$ is much less than the angular velocity Ω . The period of rotation, $T_\Omega = 2\pi/\Omega$, is 24 h; the periods of the classical planetary two-dimensional Rossby waves $T_R^{(2D)} = 2\pi/\omega_R^{(2D)}$ do not exceed the time scales of the order of 100 days. In the Earth's atmosphere, the periods of the slow three-dimensional Rossby waves, $T_R = 2\pi/\omega_R^{(3D)}$, vary in the range from 3 to 14 years (see below).

In Fig. 3 we show the period $T_R = 2\pi/|\omega_R^{(3D)}|$ of the slow Rossby waves as a function of the normalized characteristic scale L_z/H_ρ , where $L_z = \pi/k$. Let us invoke the three-dimensional slow Rossby waves for the interpretation of the observed periods of the southern oscillation. The southern oscillation is characterized by the SOI (see Fig. 2), that contains several typical periods which are larger than two years. For example, the period $T_R = 2\pi/|\omega_R^{(3D)}| = 13.88$ years is obtained for $L_z = 7.3$ km, where in Eqs. (23) and (25) we took into account that $m = 1$, $H_\rho = 8.3$ km, $\Omega = 2.27 \times 10^3$ rad/years, and $R = 6400$ km. Here $k = \pi/L_z$ corresponds

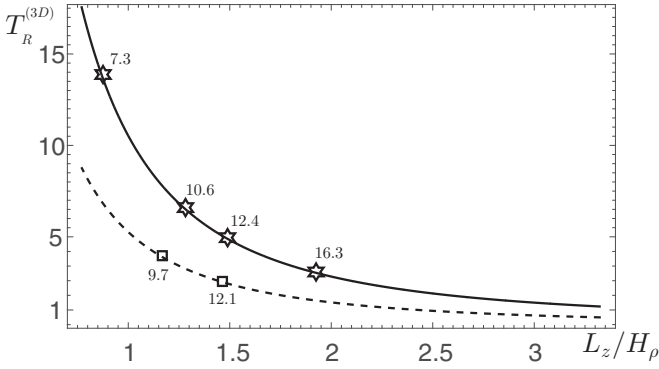


FIG. 3. The period $T_R = 2\pi/|\omega_R^{(3D)}|$ (measured in years) of the slow 3D Rossby waves vs the characteristic scale L_z/H_ρ , where $L_z = \pi/k$ for $m=1$ (solid) and $m=2$ (dashed). The stars (for $m=1$) and squares (for $m=2$) correspond to the observed periods of the SOI; see Fig. 2. The vertical sizes of these observed modes, $L_z = 7.3; 10.6; 12.4; 16.3$ km, determine the wave numbers k of the slow 3D Rossby waves; see Eqs. (23) and (25).

to the boundary conditions: $v_r(r=R) = v_r(r=R+L_z) = 0$ and $v_\theta(r=R) \approx -v_\theta(r=R+L_z)$. Similarly, for $m=1$, the period equals $T_R = 6.61$ years for $L_z = 10.6$ km; $T_R = 4.96$ years is obtained for $L_z = 12.4$ km; and $T_R = 3.96$ years is obtained for $L_z = 16.3$ km. On the other hand, for $m=2$, we obtain that the period equals $T_R = 3.96$ years for $L_z = 9.69$ km and $T_R = 2.57$ years is obtained for $L_z = 12.1$ km. Note that the vertical scales L_z are of the order of the height of the tropopause. These estimates and Fig. 3 show that the three-dimensional slow Rossby waves are able to describe the observed periods of the SOI.

Note that the frequency of the classical Rossby waves,

$$\omega = -\frac{\beta_f k_x}{k_x^2 + k_y^2 + R_d^{-2}}, \quad (26)$$

is obtained, e.g., using the β -plane approximation and assuming that the Coriolis parameter $f = f_0 + \beta_f y$ (see [3,4], and references therein). Here $f_0 = 2\Omega \sin \phi$, $R_d = \sqrt{gH}/f_0$ is the Rossby deformation radius, H is the average depth of the layer, ϕ is the contact point latitude, the coordinate y is in the meridional direction (northward in the geophysical context), the coordinate x is in the zonal direction (eastward), and the vertical coordinate z is in the direction normal to the tangent plane, i.e. opposite to the effective gravity \mathbf{g} , which includes a small correction due to the centrifugal acceleration. When β_f is small (i.e., the slope of a rotating container is sufficiently small), the frequency of the Rossby waves is small, i.e., the period of the Rossby waves is large. However, when Eq. (26) is applied to the Earth's atmosphere, it does not yield the time scales ranging from 3 to 14 years (which are the typical time scales for the SOI). In particular, the periods of the classical Rossby waves described by Eq. (26) do not exceed the time scales of the order of 100 days in the Earth's atmosphere.

Let us discuss the mechanism of excitation of the slow Rossby waves. The dispersion equation (22) allows us to determine the growth rate of the instability that excites the

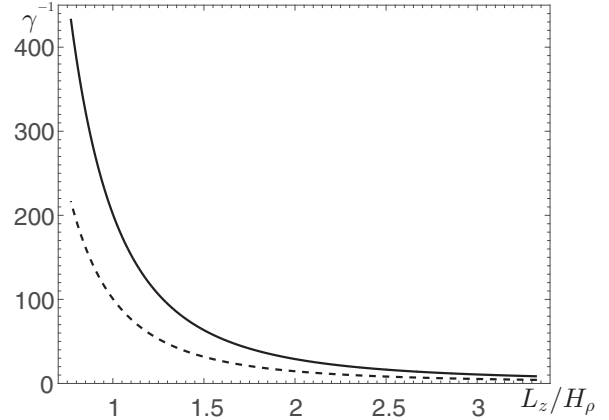


FIG. 4. The characteristic time $T_{\text{inst}} = \gamma_R^{-1}$ (measured in years) of the excitation of the 3D Rossby waves vs the characteristic scale L_z/H_ρ of the convective cells for $m=1$ (solid) and $m=2$ (dashed).

slow Rossby waves:

$$\gamma_R = \frac{3\sigma\omega_R^{(3D)}}{\ell^2(1+\sigma^2)}. \quad (27)$$

In Fig. 4 we show the characteristic time $T_{\text{inst}} = \gamma_R^{-1}$ of the excitation of the three-dimensional slow Rossby waves vs the characteristic scale L_z/H_ρ . This instability is a non-axisymmetric instability modified by rotation in a density stratified convection. The instability causes excitation of the 3D Rossby waves interacting with the convective mode and the inertial wave mode. The energy of this instability is supplied by thermal energy of convective motions. This instability is different from the convective instability in unstably stratified flows. In particular, the convective instability can be excited in the axisymmetric flows.

IV. DISCUSSION

In the present study we propose a theory of three-dimensional slow Rossby waves in rotating spherical density stratified convection. The frequency of these waves [see Eq. (23)] is approximately determined by the following balance in Eq. (16):

$$\left(\frac{\Delta^2}{4} - \Delta\right) \frac{\partial U_r}{\partial t} \sim 2(\boldsymbol{\Omega} \times \nabla)_r (\boldsymbol{\Lambda} \cdot \mathbf{U}). \quad (28)$$

This implies that the slow Rossby waves exist only in three-dimensional density stratified flows, while in the incompressible flows there occur classical two-dimensional Rossby waves (with vanishing radial velocity) which exist even without convection. The slow three-dimensional Rossby waves emerge when the convective mode is approximately balanced by the inertial mode, i.e., there is a balance between the Coriolis and buoyancy forces. The slow Rossby waves are excited by a nonaxisymmetric instability with the characteristic time that is much larger than the period of these waves. In this study we have not taken into account the effects of turbulence, e.g., turbulence causes dissipation of waves. In atmospheric flows, the strongest turbulence is located at the turbulent boundary layer at the height that is of the order of 1 km.

We apply the developed theory for the three-dimensional slow Rossby waves to interpret the observations of the southern oscillation characterized by the SOI. We found a good agreement between the theoretical results and observations of the SOI.

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APPENDIX A: DERIVATION OF THE DISPERSION EQUATION (22)

We seek for a solution of Eqs. (16) to (19) using a basis of spherical vector functions that are the eigenfunctions of the operator curl in spherical coordinates. The function \mathbf{U} is given by $\mathbf{U} = \sum_{\ell,m} \exp(\lambda t - ik r) [\hat{A}_r \mathbf{Y}_r + \hat{A}_p \mathbf{Y}_p + \hat{A}_t \mathbf{Y}_t]$, where the coefficients $\hat{A}_r, \hat{A}_p, \hat{A}_t$ depend on ℓ and m , and $\mathbf{Y}_r = \mathbf{e}_r Y_\ell^m(\vartheta, \varphi)$, $\mathbf{Y}_p = r \nabla Y_\ell^m(\vartheta, \varphi)$, $\mathbf{Y}_t = (\mathbf{r} \times \nabla) Y_\ell^m(\vartheta, \varphi)$, and $Y_\ell^m(\vartheta, \varphi)$ is given by Eq. (B1). The properties of these functions are given in Appendix B. Using Eqs. (B17) and (B18) given in Appendix B, we obtain

$$W_r = -r^{-1} \sum_{\ell,m} \ell(\ell+1) \hat{A}_t (\mathbf{e}_r \cdot \mathbf{Y}_t) \exp(\lambda t - ik r). \quad (\text{A1})$$

Using Eqs. (20), (A1), and (B15) we get

$$U_\vartheta = \left[\left(\frac{\Lambda}{2} - \nabla_r - \frac{2}{r} \right) \nabla_\vartheta \Delta_\perp^{-1} U_r - \nabla_\varphi \Delta_\perp^{-1} W_r \right], \quad (\text{A2})$$

$$U_\varphi = \left[\left(\frac{\Lambda}{2} - \nabla_r - \frac{2}{r} \right) \nabla_\varphi \Delta_\perp^{-1} U_r + \nabla_\vartheta \Delta_\perp^{-1} W_r \right]. \quad (\text{A3})$$

Using Eqs. (A2) and (A3), we rewrite Eqs. (16) and (17) in the following form:

$$\hat{\mathcal{L}} W_r = \hat{\Omega}^{(-)} U_r, \quad (\text{A4})$$

$$\hat{\mathcal{M}} U_r = \hat{\Omega}^{(+)} W_r - \beta \Delta_\perp \Theta, \quad (\text{A5})$$

where

$$\hat{\mathcal{M}} = \left(\frac{\Lambda^2}{4} - \Delta \right) \frac{\partial}{\partial t} + \frac{2\Omega}{r^2} \left[\left(\frac{\Lambda^2}{4} - \nabla_r^2 \right) \Delta_\perp^{-1} + \Lambda r \right] \frac{\partial}{\partial \varphi}, \quad (\text{A6})$$

$$\hat{\mathcal{L}} = \frac{\partial}{\partial t} + \frac{2\Omega}{r^2} \frac{\partial}{\partial \varphi} \Delta_\perp^{-1}, \quad (\text{A7})$$

$$\hat{\Omega}^{(\pm)} = 2(\Omega \cdot \nabla) \pm \left[-\frac{2\Omega \sin \vartheta}{r} \left(\nabla_r + \frac{\Lambda}{2} \right) \nabla_\vartheta \Delta_\perp^{-1} + \Omega \cdot \Lambda \right]. \quad (\text{A8})$$

The system of Eqs. (A4) and (A5) is reduced to the following equation:

$$\left[(\hat{\mathcal{L}} \hat{\mathcal{M}} - \Omega_b^2 \Delta_\perp \hat{H} - \hat{\Omega}^{(+)} \hat{\Omega}^{(-)}) \frac{\partial}{\partial t} - \Omega_b^2 \hat{H} \frac{2\Omega}{r^2} \frac{\partial}{\partial \varphi} - \Omega_{b\vartheta}^2 \hat{\Omega}^{(-)} \nabla_\varphi \right] \zeta_r = 0, \quad (\text{A9})$$

where $U_r = \partial \zeta_r / \partial t$ implies the definition of ζ_r ,

$$\hat{\mathcal{H}} = 1 + \left(\frac{\Omega_{b\vartheta}^2}{\Omega_b^2} \right) \left(-\nabla_r + \frac{\Lambda}{2} \right) \nabla_\vartheta \Delta_\perp^{-1}, \quad (\text{A10})$$

and $\Omega_{b\vartheta}^2 = \beta (\mathbf{e}_\vartheta \cdot \nabla_\vartheta) \Theta_{\text{eq}}$. Using Eqs. (A9), we arrive at the dispersion equation (22) for $\ell^2 \gg 1$.

Using Eqs. (A2)–(A4), we determine the components of the function \mathbf{U} and the radial component of the function \mathbf{W} for the three-dimensional slow Rossby waves:

$$U_r = i V_0 \exp(\gamma_R t) \exp[i(\omega_R t - k r)] Y_\ell^m, \quad (\text{A11})$$

$$U_\theta = \frac{R V_0 \exp(\gamma_R t)}{4 H_\rho \sin \theta} \exp[i(\omega_R t - k r)] (i - \sigma) \times (Y_{\ell-1}^m + Y_{\ell+1}^m), \quad (\text{A12})$$

$$U_\varphi = \frac{R V_0 \exp(\gamma_R t)}{8 m H_\rho \sin \theta} \exp[i(\omega_R t - k r)] \{ (3i\sigma - 1) Y_\ell^m + [i\sigma(\ell - 1) + \ell + 1] Y_{\ell-2}^m - [i\sigma(\ell + 2) + \ell] Y_{\ell+2}^m \}, \quad (\text{A13})$$

$$W_r = \frac{V_0 \exp(\gamma_R t)}{4 m H_\rho} \exp[i(\omega_R t - k r)] \{ (\ell - 1) [i\sigma(\ell - 1) + \ell + 1] Y_{\ell-1}^m + (\ell + 2) [i\sigma(\ell + 2) + \ell] Y_{\ell+1}^m \}, \quad (\text{A14})$$

where $V_0 = -i \hat{A}_r$. The equation for the function $P = \sqrt{\rho} p$ is obtained by calculating the divergence of Eq. (10) and using Eq. (A14):

$$P = -\frac{\rho V_0 \Omega H_\rho \exp(\gamma_R t)}{m(\sigma + i)^2} \exp[i(\omega_R t - k r)] \times \{ (\ell - 1) [i\sigma(\ell - 1) + \ell + 1] (Y_{\ell-2}^m + Y_\ell^m) + (\ell + 2) [i\sigma(\ell + 2) + \ell] (Y_{\ell+2}^m + Y_\ell^m) \}. \quad (\text{A15})$$

APPENDIX B: PROPERTIES OF THE SPHERICAL VECTOR FUNCTIONS

The spherical vector functions are the eigenfunctions of the operator curl in spherical coordinates. These functions can be presented as

$$Y_\ell^m(\vartheta, \varphi) = A_{\ell,m} P_\ell^{|m|}(\vartheta) \exp(im\varphi), \quad (\text{B1})$$

where the coefficient $A_{\ell,m}$ is determined by the normalization condition, $\int Y_\ell^m(\vartheta, \varphi) Y_n^{-m}(\vartheta, \varphi) \sin \vartheta d\vartheta d\varphi = \delta_{\ell n}$, and is given by

$$A_{\ell,m} = \frac{1}{2} \left[\frac{(\ell - |m|)!}{(\ell + |m|)!} \left(\frac{2\ell + 1}{\pi} \right) \right]^{1/2}, \quad (\text{B2})$$

(see [30,31], and references therein). Equation (B2) yields the following identity:

$$\frac{A_{\ell,m}}{A_{\ell+1,m}} \left[\frac{\ell + 1 - |m|}{2\ell + 1} \right] = \left[\frac{(\ell + 1)^2 - m^2}{4(\ell + 1)^2 - 1} \right]^{1/2}. \quad (\text{B3})$$

The associated Legendre function (spherical harmonics) of the first kind, $P_\ell^m(\vartheta)$, is determined by the following equation:

$$\left[(1 - Z^2) \frac{d^2}{dZ^2} - 2Z \frac{d}{dZ} + \left(\ell(\ell + 1) - \frac{m^2}{1 - Z^2} \right) \right] P_\ell^m(Z) = 0, \quad (\text{B4})$$

where $Z = \cos \vartheta$. The function $P_\ell^m(Z)$ has the following properties:

$$(2\ell + 1)Z P_\ell^m(Z) = (\ell - m + 1)P_{\ell+1}^m(Z) + (\ell + m)P_{\ell-1}^m(Z), \quad (\text{B5})$$

$$Z^2 P_\ell^m(Z) = I_{\ell,m} P_\ell^m(Z) + J_{\ell,m} P_{\ell+2}^m(Z) + L_{\ell,m} P_{\ell-2}^m(Z), \quad (\text{B6})$$

where

$$I_{\ell,m} = \frac{\ell^2 - m^2}{4\ell^2 - 1} + \frac{(\ell + 1)^2 - m^2}{4(\ell + 1)^2 - 1}. \quad (\text{B7})$$

When $\ell^2 \gg 1; m^2$, the function $I_{\ell,m} \approx 1/2$. We also took into account that

$$\int_0^\pi \sin^2 \vartheta \cos \vartheta P_\ell^m(\vartheta) \frac{\partial}{\partial \vartheta} P_\ell^m(\vartheta) d\vartheta = \frac{1}{2} \int_{-1}^1 (Z^3 - Z) \frac{\partial}{\partial Z} [P_\ell^m(Z)]^2 dZ = -\frac{1}{4}, \quad (\text{B8})$$

$$\int_0^\pi \sin^3 \vartheta P_\ell^m(\vartheta) \frac{\partial^2}{\partial \vartheta^2} P_\ell^m(\vartheta) d\vartheta = \frac{1}{2} [2m^2 - \ell(\ell + 1) - 1]. \quad (\text{B9})$$

Using Eqs. (B3) and (B5) we obtain the following identities for $\ell^2 \gg 1; m^2$:

$$ZY_\ell^m = \left[\frac{\ell^2 - m^2}{4\ell^2 - 1} \right]^{1/2} Y_{\ell-1}^m + \left[\frac{(\ell + 1)^2 - m^2}{4(\ell + 1)^2 - 1} \right]^{1/2} Y_{\ell+1}^m \approx \frac{1}{2} (Y_{\ell-1}^m + Y_{\ell+1}^m), \quad (\text{B10})$$

$$(Z^2 - 1) \frac{dY_\ell^m}{dZ} \equiv r \sin \vartheta \nabla_\vartheta Y_\ell^m = \left[\frac{(\ell + 1)^2 - m^2}{4(\ell + 1)^2 - 1} \right]^{1/2}$$

$$\times \ell Y_{\ell+1}^m - \left[\frac{\ell^2 - m^2}{4\ell^2 - 1} \right]^{1/2} (\ell + 1) Y_{\ell-1}^m \approx \frac{1}{2} (\ell Y_{\ell+1}^m - (\ell + 1) Y_{\ell-1}^m), \quad (\text{B11})$$

$$Z^2 Y_\ell^m \approx \frac{1}{4} (Y_{\ell-2}^m + 2Y_\ell^m + Y_{\ell+2}^m), \quad (\text{B12})$$

$$(\sin \vartheta \nabla_\vartheta \Delta_\perp^{-1}) Y_\ell^m \approx \frac{r}{2\ell} \left(Y_\ell^m - \frac{\ell}{\ell + 1} Y_{\ell+1}^m \right), \quad (\text{B13})$$

$$\begin{aligned} & (\sin \vartheta \nabla_\vartheta \Delta_\perp^{-1}) (\sin \vartheta \nabla_\vartheta \Delta_\perp^{-1}) Y_\ell^m \\ & \approx \frac{r^2}{4} \left[\frac{Y_{\ell-2}^m}{\ell(\ell - 1)} + \frac{Y_{\ell+2}^m}{(\ell + 1)(\ell + 2)} - Y_\ell^m [\ell^{-2} + (\ell + 1)^{-2}] \right]. \end{aligned} \quad (\text{B14})$$

Since $\nabla \cdot (\rho \mathbf{v}) = 0$, the coefficient \hat{A}_p is related with the coefficient \hat{A}_r :

$$\hat{A}_p = -\frac{ikr - 2 + r\Delta/2}{\ell(\ell + 1)} \hat{A}_r, \quad (\text{B15})$$

where we used the following properties of the spherical functions:

$$\begin{aligned} \nabla \cdot \mathbf{Y}_r &= \frac{2}{r} (\mathbf{e}_r \cdot \mathbf{Y}_r), \quad \nabla \cdot \mathbf{Y}_p = -\frac{\ell(\ell + 1)}{r} (\mathbf{e}_r \cdot \mathbf{Y}_r), \\ \nabla \cdot \mathbf{Y}_t &= 0. \end{aligned} \quad (\text{B16})$$

To derive Eq. (A1) we took into account that

$$\begin{aligned} \nabla \times \mathbf{Y}_r &= -\nabla \times \mathbf{Y}_p = -\frac{1}{r} \mathbf{Y}_t, \\ \nabla \times \mathbf{Y}_t &= -\frac{1}{r} [\ell(\ell + 1) \mathbf{Y}_r + \mathbf{Y}_p], \end{aligned} \quad (\text{B17})$$

$$\mathbf{Y}_p = -\mathbf{e}_r \times \mathbf{Y}_t, \quad \mathbf{Y}_t = \mathbf{e}_r \times \mathbf{Y}_p. \quad (\text{B18})$$

Using Eqs. (B16)–(B18) we obtain:

$$\begin{aligned} \nabla \cdot \mathbf{U} &= -\frac{1}{r} \sum_{\ell,m} \exp(\lambda_{\ell,mt} - ikr) [(ikr - 2) \hat{A}_r \\ & \quad + \ell(\ell + 1) \hat{A}_p] (\mathbf{e}_r \cdot \mathbf{Y}_r), \end{aligned} \quad (\text{B19})$$

$$\begin{aligned} \nabla \times \mathbf{U} &= -\frac{1}{r} \sum_{\ell,m} \exp(\lambda_{\ell,mt} - ikr) [(\hat{A}_r - ikr \hat{A}_p) \mathbf{Y}_t \\ & \quad + (ikr + 1) \hat{A}_t \mathbf{Y}_p + \ell(\ell + 1) r^{-1} \hat{A}_t \mathbf{Y}_r]. \end{aligned} \quad (\text{B20})$$

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