

Mean-field dynamos in random Arnold-Beltrami-Childress and Roberts flows

Nathan Kleorin* and Igor Rogachevskii†

Department of Mechanical Engineering, The Ben-Gurion University of the Negev, POB 653, Beer-Sheva 84105, Israel

Dmitry Sokoloff‡

Department of Physics, Moscow State University, Moscow 119992, Russia

Dmitry Tomin§

Department of Mechanics and Mathematics, Moscow State University, Moscow 119992, Russia

(Received 18 November 2008; published 2 April 2009)

We study magnetic field evolution in flows with fluctuating in time governing parameters in electrically conducting fluid. We use a standard mean-field approach to derive equations for large-scale magnetic field for the fluctuating Arnold-Beltrami-Childress (ABC) flow as well as for the fluctuating Roberts flow. The derived mean-field dynamo equations have growing solutions with growth rate of the large-scale magnetic field which is not controlled by molecular magnetic diffusivity. Our study confirms the Zeldovich idea that the nonstationarity of the fluid flow may remove the obstacle in the large-scale dynamo action of classic stationary flows.

DOI: [10.1103/PhysRevE.79.046302](https://doi.org/10.1103/PhysRevE.79.046302)

PACS number(s): 47.65.Md

I. INTRODUCTION

Many celestial bodies including the Earth, the Sun, and the Milky Way have magnetic fields with spatial scales which are much larger than the basic (maximum) scale of turbulence or turbulent convection. It has been widely recognized that these magnetic fields originate due to the mean-field dynamo based on a joint action of differential rotation and α effect operating in the mirror-asymmetric turbulence or turbulent convection (see, e.g., Refs. [1–8]). More recent models of large-scale dynamos include also such phenomena as the shear-current effect, kinetic helicity fluctuations, etc. (see, e.g., Refs. [9–14]). Corresponding dynamo models for particular celestial bodies result in magnetic fields configurations which are compatible with available phenomenology at least in a crude approximation (see, e.g., Ref. [15]).

On the other hand, many important features in astrophysical dynamos remain unclear (see, e.g., Ref. [16]). In particular, many realistic dynamo models contain a concept of turbulence or turbulent convection which is considered as a flow random in space and time. This concept that originated in the Kolmogorov theory (see, e.g., Refs. [17–19]), is accepted by many experts. However, we have to appreciate the fact that from many points of view it is at least helpful to reserve a possibility to present astrophysical flows as deterministic solutions of the Navier-Stokes equation.

Arnold suggested to mimic the magnetic field generation in a random flow by a dynamo action of a special flow known today as ABC (Arnold-Beltrami-Childress) flow (see, e.g., Refs. [20–24]). The point is that a conventional mean-field dynamo requires a nonzero kinetic helicity, i.e., a correlation between velocity field \mathbf{v} and $\text{curl } \mathbf{v}$. On the other

hand, there are deterministic flows (Beltrami fields) whereby \mathbf{v} is parallel to $\text{curl } \mathbf{v}$. If we accept the concept of the α effect and the kinetic helicity for deterministic flows, their kinetic helicity is maximum so these flows can be suitable for a generation of a large-scale magnetic field. Simultaneously, Arnold stressed that the Beltrami velocity fields provide an effective magnetic lines stretching. The ABC flow is a simple example of Beltrami flows. Speaking in modern terms, the ABC flow provides a simple example of dynamical chaos.

The problem, however, is that this flow is complicated enough so that the induction equation cannot be generally solved analytically. Arnold with coauthors suggested an artificial example of chaotic flow in a specially chosen Riemannian space for which the induction equation can be solved analytically (see Ref. [25]). It was demonstrated in [25] that a large-scale magnetic field can be generated by this flow. On the other hand, numerical experiments (see, e.g., Refs. [26–28]) also show that the ABC flow can excite a magnetic field. However, the generated magnetic field is quite remote from that which astrophysicists are inclined to refer to as a large-scale magnetic field. Its spatial configuration looks like a combination of cigarlike structures, which spatial scale in the cross section is controlled by magnetic Reynolds number.

In spite of various impressive results in solving particular problems for dynamos in stationary deterministic flows (see, e.g., Refs. [20,21,28–33]) no particular deterministic flow is yet known to mimic a self-excitation of the large-scale magnetic field which widely occurs in random flow. This situation looks obviously unsatisfactory.

It was suggested by Zeldovich that a nonstationarity of a fluid flow is an important feature required for the generation of the mean magnetic field (see, e.g., Ref. [4]). Since the induction equation for a nonstationary flow is even more complicated for analytical study, it was quite difficult to provide a convincing support for this idea. It looks reasonable in this context to introduce time dependence in the ABC flow (or other deterministic velocity field) as a noise added to numerical parameters (here A , B , and C) which govern the

*nat@bgu.ac.il

†gary@bgu.ac.il; URL: <http://www.bgu.ac.il/~gary>

‡sokoloff@dds.srcc.msu.su

§dtomin@gmail.com

flow. This idea as well as possible approaches have been known for years, however, the bulk of analytical work required was quite large, so no particular solution has been suggested until now.

The goal of this study is to revise this old and intriguing problem. This paper is organized as follows. In Sec. II the mean-field approach for a flow with random coefficients is formulated. In Sec. III the electromotive force is determined for random ABC and Roberts flows. Finally, we discuss obtained results and draw conclusions in Sec. IV. In Appendixes A and B we discuss spatial averaging of the mean-field equation and present details of calculations of the electromotive force for the ABC flow as well as for the Roberts flow with random coefficients.

II. MEAN-FIELD APPROACH FOR A FLOW WITH RANDOM COEFFICIENTS

The problem under consideration requires quite a lot of simple, however, bulky algebra so we describe here the strategy of the research undertaken and present results for particular flows. To be specific, we describe the strategy using ABC flow which is determined as

$$\begin{aligned} v_x &= B \cos y + C \sin z, & v_y &= C \cos z + A \sin x, \\ v_z &= A \cos x + B \sin y, \end{aligned} \quad (1)$$

where x, y, z are measured in units of the characteristic scale l of the velocity field variations. We consider the flow (1) as given and study a kinematic dynamo problem. We do not discuss here how the flow can be excited in practice or what is a nonlinear stage of dynamo action. We highly appreciate the importance of these questions (see, e.g., Ref. [8] and references therein). From our point of view a straightforward way to resolve these questions is to return to the concept of turbulence as a field that is random in space and time. We appreciate as well that in practice a kinetic helicity is far from the maximum possible (i.e. realistic velocity fields have kinetic helicity that is much lower than that for the Beltrami field).

A first approach applied here is to consider the flow determined by Eq. (1) with random coefficients A, B, C as a particular example of inhomogeneous and anisotropic random velocity field, and to apply corresponding expressions for electromotive force \mathcal{E} known in the literature (see, e.g., Refs. [34–36]). This approach is practical in the sense that it requires a minimum bulk of algebra and allows a clear identification of various terms which appear in the resulting mean-field equation. This is a reason why we use this approach in the present study. On the other hand, we recognize that the above approach is not fully convincing from the viewpoint of high-brow probability theory. The point is that we accept here that the flow under consideration can mimic a turbulent flow while it is something what we have to prove.

This is a reason why we also need to use another approach consisting in a direct averaging of the induction equation and not using the known expressions for electromotive force \mathcal{E} . In the second approach we have to use a particular model of random A, B, C in the form of the δ -correlated in

time random processes and reproduce just from the beginning the procedure of derivation of mean-field equations for the magnetic field in such flow. The second approach requires much more algebra and creates various problems with identifications of terms in the resulting equation. However, the final results occur to be identical to those obtained using the first approach. Comparing the results of these two approaches, one has to bear in mind that the equations exploited in both cases suppose different normalization for velocity field (up to the factor 1/2 in resulting expressions). We avoid here the bulky, however, important calculations used in the second approach and present them in a separate study [37].

The main result in both approaches is a governing equation for the magnetic field averaged over the ensemble of realizations of A, B, C . Because the velocity field investigated is far from being statistically homogeneous and isotropic, the resulting equation is quite bulky and hardly can be solved analytically. A numerical approach to this equation does not look easier than the one for initial equations. We are interested, however, in large-scale solutions for the equations. To this end, we perform an additional spatial averaging to remove small-scale noise from the solutions and keep large-scale properties of the solutions only. In practice, we remove terms like $\sin x$ and $\cos x$ and replace terms like $\sin^2 x$ and $\cos^2 x$ by 1/2. As a result, we arrive at simple equations with constant coefficients which we solve in Fourier space.

We perform our analysis for two velocity fields which look instructive for our problem. One is the above discussed ABC flow where fluctuations of governing parameters are implied. We choose as a counterpart another well-known flow suggested for the dynamo problem by Roberts [38]:

$$\begin{aligned} v_x &= -C \sin x \cos y, & v_y &= C \cos x \sin y, \\ v_z &= C \sin x \sin y, \end{aligned} \quad (2)$$

and consider the governing parameter C as random.

The ABC flow and the Roberts flow demonstrate properties which are to some extent opposite from the viewpoint of topological fluid dynamics (see, e.g., Ref. [39,40]). In particular, the ABC flow looks as a most advantageous velocity field in the sense that the large-scale field generation becomes independent on details of the flow geometry. The ABC flow is a Beltrami flow, i.e., vectors \mathbf{v} and $\text{curl } \mathbf{v}$ are parallel. Such a velocity field provides exponential stretching of magnetic lines.

In contrast, the Roberts flow is not a Beltrami flow, i.e., two-component vectors $(\text{curl}_x \mathbf{v}, \text{curl}_y \mathbf{v})$ and (v_x, v_y) are parallel, however, $\text{curl}_z \mathbf{v} = 0$ while $v_z \neq 0$. It means that the stationary Roberts flow (i.e., constant in time governing parameter C) does not provide exponential stretching of magnetic lines. On the other hand, we demonstrate in this study that the Roberts flow becomes a dynamo when the coefficient C fluctuates in time (see Sec. III).

In the next section we investigate how fluctuations of the governing parameters in these velocity fields affect the large-scale dynamo action.

III. ELECTROMOTIVE FORCE IN ANISOTROPIC VELOCITY FIELD

The electromotive force in anisotropic velocity field is determined by the following equation:

$$\mathcal{E}_i = \alpha_{ij} b_j - \eta_{ij} (\nabla \times \mathbf{b})_j + (\mathbf{V}^{\text{eff}} \times \mathbf{b})_i - \kappa_{ijk} (\partial b)_{jk}, \quad (3)$$

where $\mathbf{b} = \langle \mathbf{H} \rangle$ (in the context of the ABC flow the angular brackets denote averaging over random coefficients A, B, C), and

$$\alpha_{ij} = -\frac{\tau}{2} (\varepsilon_{imn} \langle v_n \nabla_j v_m \rangle + \varepsilon_{jmn} \langle v_n \nabla_i v_m \rangle), \quad (4)$$

$$\eta_{ij} = \frac{\tau}{2} (\langle v^2 \rangle \delta_{ij} - \langle v_i v_j \rangle), \quad (5)$$

$$\mathbf{V}_i^{\text{eff}} = -\frac{1}{2} \nabla_m \tau \langle v_m v_i \rangle, \quad (6)$$

$$\kappa_{ijk} = -\frac{\tau}{2} (\varepsilon_{ijm} \langle v_m v_k \rangle + \varepsilon_{ikm} \langle v_m v_j \rangle), \quad (7)$$

where τ is the characteristic time scale of the random velocity field, δ_{ij} is the Kronecker tensor, and ε_{ijk} is the fully antisymmetric Levi-Civita tensor. Equations (3)–(7) have been derived by the following procedures: (i) the second order correlation approximation (or first-order smoothing) in [34] that is valid for small Reynolds numbers; (ii) the path integral approach in [35] for the delta-correlated in time random velocity field; (iii) the tau approach in [36] that is valid for large Reynolds numbers.

The spatial averaging $\langle \dots \rangle^{(sp)}$ of the induction equation for the field \mathbf{b} yields the equation for the mean magnetic field $\mathbf{B} = \langle \mathbf{b} \rangle^{(sp)}$. In spatial averaging the effects, $\langle \tilde{\alpha}_{ij} \tilde{b}_j \rangle^{(sp)}$, $\langle \tilde{\eta}_{ij} (\nabla \times \tilde{\mathbf{b}})_j \rangle^{(sp)}$, $\langle \tilde{\mathbf{V}}^{\text{eff}} \times \tilde{\mathbf{b}} \rangle^{(sp)}$, and $\langle \tilde{\kappa}_{ijk} (\partial \tilde{b})_{jk} \rangle^{(sp)}$, should be also taken into account. Here $\eta_{ij} = \langle \eta_{ij} \rangle^{(sp)} + \tilde{\eta}_{ij}$ and $\langle \tilde{\eta}_{ij} \rangle^{(sp)} = 0$, and similarly for other tensors (a tilde denotes the spatially variable part of the tensor). The magnetic field $\tilde{\mathbf{b}}$ is determined by the following equation:

$$\varepsilon_{mni} \nabla_n [\langle \alpha_{ij} \rangle^{(sp)} \tilde{b}_j - \langle \eta_{ij} \rangle^{(sp)} (\nabla \times \tilde{\mathbf{b}})_j + \langle (\mathbf{V}^{\text{eff}})^{(sp)} \times \tilde{\mathbf{b}} \rangle_i - \langle \kappa_{ijk} \rangle^{(sp)} (\partial \tilde{b})_{jk}] = I_m, \quad (8)$$

$$I_m = -\varepsilon_{mni} \nabla_n [\tilde{\alpha}_{ij} B_j - \tilde{\eta}_{ij} (\nabla \times \mathbf{B})_j + (\tilde{\mathbf{V}}^{\text{eff}} \times \mathbf{B})_i - \tilde{\kappa}_{ijk} (\partial B)_{jk}]. \quad (9)$$

The solution of Eq. (8) allows us to determine the second moments $\langle \tilde{\alpha}_{ij} \tilde{b}_j \rangle^{(sp)}$, $\langle \tilde{\eta}_{ij} (\nabla \times \tilde{\mathbf{b}})_j \rangle^{(sp)}$, and $\langle \tilde{\kappa}_{ijk} (\partial \tilde{b})_{jk} \rangle^{(sp)}$ and to derive equation for the mean magnetic field \mathbf{B} .

A. The ABC flow with random coefficients

Let us consider the ABC flow that is determined by Eq. (1), where the coefficients A, B, C are the random variables. Using Eq. (4) we determine the tensor α_{ij} for the ABC flow:

$$\alpha_{xx} = -\frac{\tau \sigma_A}{l}, \quad \alpha_{yy} = -\frac{\tau \sigma_B}{l}, \quad \alpha_{zz} = -\frac{\tau \sigma_C}{l}, \quad (10)$$

and $\alpha_{ij} = 0$ for $i \neq j$, where $\sigma_A = \langle A^2 \rangle$, $\langle A \rangle = 0$, and similarly for random variables B, C . The tensor α_{ij} determines the generation of magnetic field. For the ABC flow with random coefficients $\tilde{\alpha}_{ij} = 0$ and the effective drift velocity of the magnetic field $\mathbf{V}^{\text{eff}} = 0$ [see Eq. (6), and Eq. (A1) in Appendix A]. Note that the condition $\langle A \rangle = 0$ implies that we consider a fluctuating ABC flow with vanishing mean velocity. In other words, we choose parametrization of the random flow in such a way as to stress the difference between the flow under consideration and the conventional stationary ABC flow.

Using Eqs. (5) and (A1) we get the symmetric tensor η_{ij} for the ABC flow:

$$\eta_{xx} = \frac{\tau}{2} [\sigma_A + \sigma_B \sin^2 y + \sigma_C \cos^2 z],$$

$$\eta_{yy} = \frac{\tau}{2} [\sigma_B + \sigma_C \sin^2 z + \sigma_A \cos^2 x],$$

$$\eta_{zz} = \frac{\tau}{2} [\sigma_C + \sigma_A \sin^2 x + \sigma_B \cos^2 y],$$

$$\eta_{xy} = -\frac{\tau \sigma_C}{4} \sin(2z), \quad \eta_{xz} = -\frac{\tau \sigma_B}{4} \sin(2y),$$

$$\eta_{yz} = -\frac{\tau \sigma_A}{4} \sin(2x). \quad (11)$$

Equation (7) allows us to determine the tensor $\kappa_{ijk} (\partial b)_{jk}$:

$$\kappa_{xjk} (\partial b)_{jk} = \frac{\tau}{2} [u_{yk} (\partial b)_{zk} - u_{zk} (\partial b)_{yk}],$$

$$\kappa_{yjk} (\partial b)_{jk} = \frac{\tau}{2} [u_{zk} (\partial b)_{xk} - u_{xk} (\partial b)_{zk}],$$

$$\kappa_{zjk} (\partial b)_{jk} = \frac{\tau}{2} [u_{xk} (\partial b)_{yk} - u_{yk} (\partial b)_{xk}], \quad (12)$$

where the velocity fluctuations, $u_{ij} = \langle v_i v_j \rangle$, are determined by Eq. (A1). The tensors η_{ij} and κ_{ijk} contribute to the magnetic eddy diffusivity caused by random motions of the conducting fluid. This effect limits the dynamo growth rate of magnetic field.

The spatial averaging $\langle \dots \rangle^{(sp)}$ of the electromotive force yields the following equation (see Appendix A):

$$\langle \mathcal{E} \rangle^{(sp)} = -\tau \sigma_A (\mathbf{B} l^{-1} + \nabla \times \mathbf{B}), \quad (13)$$

where we considered the case $\sigma_A = \sigma_B = \sigma_C$. Note that there are generally two contributions to the electromotive force caused by the spatial averaging $\langle \dots \rangle^{(sp)}$. Indeed, the direct contribution to the electromotive force, $\mathcal{E}^{(1)}$, is due to the spatial averaging of Eqs. (10)–(12), while the second contribution to the electromotive force, $\mathcal{E}^{(2)}$, is caused by the second moments $\langle \tilde{\alpha}_{ij} \tilde{b}_j \rangle^{(sp)}$, $\langle \tilde{\eta}_{ij} (\nabla \times \tilde{\mathbf{b}})_j \rangle^{(sp)}$, etc. However, for

the ABC flow $\mathcal{E}^{(2)}$ vanishes (see Appendix A), while for the Roberts flow this contribution is not zero. Therefore the equation for the mean magnetic field reads

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{\tau\sigma_A}{l}(\nabla \times \mathbf{B}) + (\eta + \tau\sigma_A)\Delta \mathbf{B}, \quad (14)$$

where η is the magnetic diffusion due to the electrical conductivity of the fluid.

B. The random Roberts flow

Now let us consider the random Roberts flow (2), that can be rewritten in the form $\mathbf{v} = C(\mathbf{e} + \mathbf{e} \times \nabla) \sin x \sin y$, where \mathbf{e} is the unit vector in the z direction and C is the random variable. Using Eq. (4) we get the symmetric tensor α_{ij} :

$$\begin{aligned} \alpha_{xx} &= \frac{\tau\sigma}{l} \sin^2 y, & \alpha_{yy} &= \frac{\tau\sigma}{l} \sin^2 x, \\ \alpha_{xz} &= \frac{\tau\sigma}{4l} \sin(2y), & \alpha_{yz} &= \frac{\tau\sigma}{4l} \sin(2x), \\ \alpha_{xy} &= \alpha_{zz} = 0, \end{aligned} \quad (15)$$

where $\sigma = \langle C^2 \rangle$ and $\langle C \rangle = 0$. We choose parametrization of this flow in a way which stresses the difference with the conventional stationary Roberts flow.

The effective drift velocity of the magnetic field \mathbf{V}^{eff} is given by

$$\begin{aligned} V_x^{\text{eff}} &= -\frac{\tau\sigma}{4l} \sin(2x), & V_y^{\text{eff}} &= -\frac{\tau\sigma}{4l} \sin(2y), \\ V_z^{\text{eff}} &= 0, \end{aligned} \quad (16)$$

[see Eq. (6), and Eq. (B1) in Appendix B]. The effective drift velocity of the magnetic field determines the diamagnetic (or paramagnetic) effect. Using Eqs. (5) and (B1) we determine the symmetric tensor η_{ij} for the Roberts flow:

$$\begin{aligned} \eta_{xx} &= \frac{\tau\sigma}{2} \sin^2 y, & \eta_{xy} &= \frac{\tau\sigma}{8} \sin(2x)\sin(2y), \\ \eta_{yy} &= \frac{\tau\sigma}{2} \sin^2 x, & \eta_{yz} &= -\frac{\tau\sigma}{4} \sin^2 y \sin(2x), \\ \eta_{zz} &= \frac{\tau\sigma}{4} [1 - \cos(2x)\cos(2y)], \\ \eta_{xz} &= \frac{\tau\sigma}{4} \sin^2 x \sin(2y). \end{aligned} \quad (17)$$

Equation (7) yields the tensor $\kappa_{ijk} = -\tau\epsilon_{ijm}u_{mk}$, where the second moment for the velocity field $u_{ij} = \langle v_i v_j \rangle$ is determined by Eq. (B1).

The spatial averaging $\langle \dots \rangle^{(sp)}$ of the electromotive force yields $\langle \mathcal{E} \rangle^{(sp)} = \mathcal{E}^{(1)} + \mathcal{E}^{(2)}$, where

$$\mathcal{E}^{(1)} = \frac{\tau\sigma}{4} [2(\delta_{ij} - e_i e_j) B_j l^{-1} - \nabla \times \mathbf{B}], \quad (18)$$

and the second contribution to the electromotive force $\mathcal{E}^{(2)}$ is given by Eq. (B7) in Appendix B. This contribution causes a decrease of the total turbulent magnetic diffusion and an increase of the total α effect. However, these effects are small [see Eqs. (B8) and (B9)]. This implies that the equation for the mean magnetic field evolution for the random Roberts flow reads

$$\frac{\partial B_i}{\partial t} = \frac{\tau\sigma}{2l} (\delta_{ij} - e_i e_j) (\nabla \times \mathbf{B})_j + \left(\eta + \frac{\tau\sigma}{4} \right) \Delta B_i. \quad (19)$$

IV. DISCUSSION

A natural step in further analysis is to solve the mean-field equations with transport coefficients obtained for the random ABC flow and Roberts flow. Straightforward calculations in Fourier space yield the growth rates of the large-scale magnetic field for the ABC flow:

$$\gamma = K \left[\frac{\tau\sigma_A}{l} - (\tau\sigma_A + \eta)K \right] \quad (20)$$

[see Eq. (14)], and for the Roberts flow:

$$\gamma = K \left[\frac{\tau\sigma}{2l} - \left(\frac{\tau\sigma}{4} + \eta \right) K \right] \quad (21)$$

[see Eq. (19)], where $K = 2\pi/L_B$ is the wave number and L_B is the characteristic scale of the magnetic field variations. Equations (20) and (21) imply that the large-scale magnetic field grows in the same time scale for both flows. Moreover, the growth rates obtained are similar to that for the mean-field dynamo in developed mirror asymmetric turbulence with α effect.

Therefore our study confirms the Zeldovich idea that the time dependence (e.g., in the form of random in time fluctuations) removes the obstacle in the large-scale dynamo action of classic stationary flows and provides dynamos. Moreover, we do not see any necessity for an instantaneous flow geometry to be topologically complex. Of course, trajectories of the flow particles remain chaotic due to the random nature of governing parameters.

In this study we choose parametrization of the fluctuating ABC and Roberts flows in a way to vanish the mean velocities and stress the difference with conventional stationary ABC and Roberts flows. Therefore the dynamo effects isolated are associated with fluctuations only. When $\langle A \rangle \neq 0$ (and $\langle C \rangle \neq 0$) the standard $\mathbf{U} \times \mathbf{B}$ term in the mean-field dynamo equation vanishes, where $\mathbf{U} = \langle \langle \mathbf{v} \rangle \rangle^{(sp)}$ and $\mathbf{B} = \langle \langle \mathbf{H} \rangle \rangle^{(sp)}$ are the mean velocity and magnetic fields. On the other hand, there can be a nonvanishing contribution to the mean-field dynamo equation caused by the term $\langle \langle \mathbf{v} \rangle \times \tilde{\mathbf{b}} \rangle^{(sp)}$, where $\tilde{\mathbf{b}}$ is determined by Eq. (8) with an additional source term $\propto -\nabla \times [\langle \mathbf{v} \rangle \times \mathbf{B}]$. However, an account for the effect of the mean fluid velocity on the mean-field dynamo is out of the scope of the present study.

Note that stationary deterministic flows (like the ABC flow with constant coefficients) cannot cause a dynamo of the large-scale magnetic field, i.e., stationary flows cannot generate magnetic field in the scales which are larger than the characteristic scales of the fluid flow field. On the other hand, numerical experiments [26,28] show that the ABC flow with constant coefficients excites a magnetic field in the scales which are smaller than the scales of the fluid flow field. The generation of the magnetic field by the stationary ABC flow requires the magnetic Reynolds numbers, Rm , which are larger than 9. In particular, as follows from numerical experiments [26,28] the magnetic field is excited by the ABC flow within two ranges of the magnetic Reynolds numbers: the first one is for $9 < Rm < 17.5$ and the second range is for $Rm > 27$. Stationary Roberts flow cannot generate magnetic field. In contrast, in the present study we have shown that the generation of large-scale magnetic fields by random ABC or Roberts flows can be possible even for small magnetic Reynolds numbers.

The analysis undertaken in the present study is not addressed to clarify the problems of nonlinear dynamo saturation and dynamics of small-scale magnetic fields associated with dynamo action (see, e.g., Refs. [8,41]). We highly appreciate the importance of these problems, however, this is out of the scope of this paper. The main goal of the present study is to demonstrate a possibility for a mean-field dynamo action (in the scales which are larger than the characteristic scales of the fluid flow field) in random ABC and Roberts flows in the framework of the kinematic approach.

Based on the Roberts flow in a finite cylindrical domain [38,42], one of the first successful laboratory dynamo experiments was performed in Karlsruhe [43–45]. These experiments demonstrate a generation of magnetic field in a cylindrical container filled with liquid sodium in which by means of guide tubes counter-rotating and counter-current spiral vortices are established. The dynamo in the Karlsruhe-type experiment is self-exciting and the magnetic field saturates at a mean value for fixed supercritical flow rates [43–45]. The random Roberts flow might be created in the Karlsruhe-type experimental set up when the pumping of energy in flow tubes has a random component. The present study might be relevant to such dynamo experiment.

Solar supergranular flow structures [46] can be considered as ensemble of random cells. Their collective effect on generation of the solar magnetic field at scales which are larger than the sizes of supergranulations is very important. It is plausible to suggest that the dynamo in random ABC or Roberts flows can mimic the effect of random supergranulations on the large-scale solar dynamos.

D.T. and D.S. are grateful to RFBR for financial support under Grant No. 07-02-00127.

APPENDIX A: THE ELECTROMOTIVE FORCE FOR RANDOM ABC FLOW

Using Eq. (1) we get the tensor $u_{ij} = \langle v_i v_j \rangle$ for random ABC flow:

$$u_{xx} = \sigma_B \cos^2 y + \sigma_C \sin^2 z, \quad u_{xy} = \frac{\sigma_C}{2} \sin(2z),$$

$$u_{yy} = \sigma_C \cos^2 z + \sigma_A \sin^2 x, \quad u_{yz} = \frac{\sigma_A}{2} \sin(2x),$$

$$u_{zz} = \sigma_A \cos^2 x + \sigma_B \sin^2 y, \quad u_{xz} = \frac{\sigma_B}{2} \sin(2y),$$

$$\langle \mathbf{v}^2 \rangle = \sigma_A + \sigma_B + \sigma_C. \quad (\text{A1})$$

Now we consider for simplicity the case $\sigma_A = \sigma_B = \sigma_C$. Equation (8) yields the equation for the magnetic field $\tilde{\mathbf{b}}$ for the random ABC flow that is in Fourier space reads

$$D_{ij} \tilde{b}_j = - \frac{l^2}{(\eta + \tau \sigma_A) k^2} I_i, \quad (\text{A2})$$

where

$$D_{ij} = \delta_{ij} - \frac{i}{k^2} \varepsilon_{ijm} k_m, \quad (\text{A3})$$

\mathbf{k} is the dimensionless wave vector measured in the units of l^{-1} , and l is the characteristic scale of the velocity field variations, I_i is determined by Eq. (9), and we have taken into account Eqs. (13) and (14). The solution of Eq. (A2) reads

$$\tilde{b}_i = - \frac{l^2}{(\eta + \tau \sigma_A) k^2} D_{ij}^{-1} I_j, \quad (\text{A4})$$

where the inverse operator D_{ij}^{-1} is given by

$$D_{ij}^{-1} = (k^2 - 1)^{-1} \left(k^2 \delta_{ij} + i \varepsilon_{ijm} k_m - \frac{k_i k_j}{k^2} \right). \quad (\text{A5})$$

Straightforward calculations using Eqs. (A4) and (A5) yield the second moments $\langle \tilde{\eta}_{ij} (\nabla \times \tilde{\mathbf{b}})_j \rangle^{(sp)}$ and $\langle \tilde{\kappa}_{ijk} (\partial \tilde{b})_{jk} \rangle^{(sp)}$ for random ABC flow:

$$\langle \tilde{\eta}_{ij} (\nabla \times \tilde{\mathbf{b}})_j \rangle^{(sp)} = - \frac{\eta_A^{\text{eff}}}{6} (\nabla \times \mathbf{B})_i, \quad (\text{A6})$$

$$\langle \tilde{\kappa}_{ijk} (\partial \tilde{b})_{jk} \rangle^{(sp)} = \frac{\eta_A^{\text{eff}}}{6} (\nabla \times \mathbf{B})_i, \quad (\text{A7})$$

where $\eta_A^{\text{eff}} = \tau^2 \sigma_A^2 / (\eta + \tau \sigma_A)$, and we have taken into account that $k_x = k_y = 2$. Note also that for the random ABC flow $\langle \tilde{\alpha}_{ij} \tilde{b}_j \rangle^{(sp)} = 0$ and $\langle \tilde{\mathbf{v}}^{\text{eff}} \times \tilde{\mathbf{b}} \rangle^{(sp)} = 0$, because $\tilde{\alpha}_{ij} = 0$ and $\tilde{\mathbf{v}}^{\text{eff}} = 0$.

APPENDIX B: THE ELECTROMOTIVE FORCE FOR RANDOM ROBERTS FLOW

Using Eq. (5) we get the tensor $u_{ij} = \langle v_i v_j \rangle$ for random Roberts flow:

$$u_{xx} = \sigma \sin^2 x \cos^2 y, \quad u_{xy} = - \frac{\sigma}{4} \sin(2x) \sin(2y),$$

$$u_{yy} = \sigma \cos^2 x \sin^2 y, \quad u_{yz} = \frac{\sigma}{2} \sin^2 y \sin(2x),$$

$$u_{zz} = \sigma \sin^2 x \sin^2 y, \quad u_{xz} = -\frac{\sigma}{2} \sin^2 x \sin(2y),$$

$$\langle \mathbf{v}^2 \rangle = \sigma (\sin^2 x + \cos^2 x \sin^2 y). \quad (\text{B1})$$

The solution of Eq. (8) yields the magnetic field $\tilde{\mathbf{b}}$ for random Roberts flow that is determined by the following equation in Fourier space:

$$\tilde{b}_i = \frac{l^2}{k^2} \left(\eta + \frac{\tau\sigma}{4} \right)^{-1} \left\{ I_i + \frac{2}{k^2} \left[(\mathbf{e} \cdot \nabla)(\mathbf{e} \times \mathbf{I})_i - (\mathbf{e} \times \nabla)_i(\mathbf{e} \cdot \mathbf{I}) + \frac{2}{k^2} (\mathbf{e} \cdot \nabla) \nabla_i(\mathbf{e} \cdot \mathbf{I}) \right] \right\}, \quad (\text{B2})$$

where I_i is given by Eq. (9), \mathbf{k} is the dimensionless wave vector measured in the units of l^{-1} . Here we have taken into account Eqs. (18) and (19). Straightforward calculations yield

$$\begin{aligned} \langle \tilde{\alpha}_{xj} \tilde{b}_j \rangle^{(sp)} &= \frac{\eta^{\text{eff}}}{32} \nabla_y B_z, & \langle \tilde{\alpha}_{yj} \tilde{b}_j \rangle^{(sp)} &= \frac{\eta^{\text{eff}}}{32} \nabla_x B_z, \\ \langle \tilde{\alpha}_{zj} \tilde{b}_j \rangle^{(sp)} &= \frac{\eta^{\text{eff}}}{16} [2(\nabla \times \mathbf{B})_z + \nabla_y B_x], \end{aligned} \quad (\text{B3})$$

$$\langle (\tilde{\mathbf{V}}^{\text{eff}} \times \tilde{\mathbf{b}})_x \rangle^{(sp)} = -\frac{\eta^{\text{eff}}}{32} \nabla_y B_z,$$

$$\langle (\tilde{\mathbf{V}}^{\text{eff}} \times \tilde{\mathbf{b}})_y \rangle^{(sp)} = \frac{\eta^{\text{eff}}}{32} \nabla_x B_z,$$

$$\langle (\tilde{\mathbf{V}}^{\text{eff}} \times \tilde{\mathbf{b}})_z \rangle^{(sp)} = \frac{\eta^{\text{eff}}}{32} (\nabla \times \mathbf{B})_z, \quad (\text{B4})$$

$$\langle \tilde{\eta}_{xj} (\nabla \times \tilde{\mathbf{b}})_j \rangle^{(sp)} = -\frac{\eta^{\text{eff}}}{64} [6(\nabla \times \mathbf{B})_x + 5\nabla_z B_y],$$

$$\langle \tilde{\eta}_{yj} (\nabla \times \tilde{\mathbf{b}})_j \rangle^{(sp)} = \frac{\eta^{\text{eff}}}{64} [(\nabla \times \mathbf{B})_y - 5\nabla_x B_z],$$

$$\langle \tilde{\eta}_{zj} (\nabla \times \tilde{\mathbf{b}})_j \rangle^{(sp)} = \frac{\eta^{\text{eff}}}{64} [(\nabla \times \mathbf{B})_z + 4\nabla_y B_x - 8l^{-1} B_z], \quad (\text{B5})$$

and

$$\langle \tilde{\kappa}_{xjk} (\partial \tilde{b})_{jk} \rangle^{(sp)} = -\frac{\eta^{\text{eff}}}{64} [3(\nabla \times \mathbf{B})_x + 6\nabla_z B_z + \nabla_x B_x + 4l^{-1} B_x],$$

$$\langle \tilde{\kappa}_{yjk} (\partial \tilde{b})_{jk} \rangle^{(sp)} = -\frac{\eta^{\text{eff}}}{64} [3(\nabla \times \mathbf{B})_y + 6\nabla_x B_x + \nabla_y B_y + 4l^{-1} B_y],$$

$$\langle \tilde{\kappa}_{zjk} (\partial \tilde{b})_{jk} \rangle^{(sp)} = -\frac{\eta^{\text{eff}}}{64} [13(\nabla \times \mathbf{B})_z - 2\nabla_y B_x - 8l^{-1} B_z], \quad (\text{B6})$$

where $\eta^{\text{eff}} = \tau^2 \sigma^2 / (4\eta + \tau\sigma)$, and we have taken into account that $k_x = k_y = 2$. The contribution to the electromotive force, $\mathcal{E}^{(2)}$, caused by the second moments $\langle \tilde{\alpha}_{ij} \tilde{b}_j \rangle^{(sp)}$, $\langle \tilde{\eta}_{ij} (\nabla \times \tilde{\mathbf{b}})_j \rangle^{(sp)}$, is given by

$$\mathcal{E}_x^{(2)} = \frac{\eta^{\text{eff}}}{64} [9(\nabla \times \mathbf{B})_x + 5\nabla_z B_y + 6\nabla_z B_z + \nabla_x B_x + 4l^{-1} B_x],$$

$$\mathcal{E}_y^{(2)} = \frac{\eta^{\text{eff}}}{64} [2(\nabla \times \mathbf{B})_y + 9\nabla_x B_z + 6\nabla_x B_x + \nabla_y B_y + 4l^{-1} B_y],$$

$$\mathcal{E}_z^{(2)} = \frac{\eta^{\text{eff}}}{32} [11(\nabla \times \mathbf{B})_z - \nabla_y B_x]. \quad (\text{B7})$$

This implies that the diagonal components of the total turbulent magnetic diffusion tensor are

$$\begin{aligned} \eta_{xx}^{\text{tot}} &= \frac{\tau\sigma}{4} \left[1 - \frac{9}{64} \text{Rm} \right], \\ \eta_{yy}^{\text{tot}} &= \frac{\tau\sigma}{4} \left[1 - \frac{1}{32} \text{Rm} \right], \\ \eta_{zz}^{\text{tot}} &= \frac{\tau\sigma}{4} \left[1 - \frac{11}{32} \text{Rm} \right], \end{aligned} \quad (\text{B8})$$

while the diagonal components of the total α tensor are given by

$$\alpha_{xx}^{\text{tot}} = \alpha_{yy}^{\text{tot}} = \frac{\tau\sigma}{2l} \left[1 + \frac{1}{32} \text{Rm} \right], \quad (\text{B9})$$

and $\alpha_{zz}^{\text{tot}} = 0$. Here $\text{Rm} = \tau\sigma / \eta \ll 1$ and we took into account Eqs. (18) and (B7).

- [1] H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University Press, New York, 1978).
 [2] E. Parker, *Cosmical Magnetic Fields* (Oxford University Press, New York, 1979).
 [3] F. Krause, and K. H. Rädler, *Mean-Field Magnetohydrodynamics and Dynamo Theory* (Pergamon, Oxford, 1980).
 [4] Ya. B. Zeldovich, A. A. Ruzmaikin, and D. D. Sokoloff, *Mag-*

netic Fields in Astrophysics (Gordon and Breach, New York, 1983).

- [5] A. Ruzmaikin, A. M. Shukurov, and D. D. Sokoloff, *Magnetic Fields of Galaxies* (Kluwer Academic, Dordrecht, 1988).
 [6] G. Rüdiger and R. Hollerbach, *The Magnetic Universe* (Wiley-VCH, Weinheim, 2004).
 [7] M. Ossendrijver, *Astron. Astrophys. Rev.* **11**, 287 (2003).

- [8] A. Brandenburg and K. Subramanian, *Phys. Rep.* **417**, 1 (2005).
- [9] I. Rogachevskii and N. Kleeorin, *Phys. Rev. E* **68**, 036301 (2003); **70**, 046310 (2004); **75**, 046305 (2007).
- [10] E. T. Vishniac and A. Brandenburg, *Astrophys. J.* **475**, 263 (1997).
- [11] D. D. Sokolov, *Astron. Rep.* **41**, 68 (1997).
- [12] N. A. Silant'ev, *Astron. Astrophys.* **364**, 339 (2000).
- [13] M. R. E. Proctor, *Mon. Not. R. Astron. Soc.* **382**, L39 (2007).
- [14] N. Kleeorin and I. Rogachevskii, *Phys. Rev. E* **77**, 036307 (2008).
- [15] R. Beck, A. Brandenburg, D. Moss, A. Shukurov, and D. Sokoloff, *Annu. Rev. Astron. Astrophys.* **34**, 155 (1996).
- [16] R. Kulsrud, *Annu. Rev. Astron. Astrophys.* **37**, 37 (1999).
- [17] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, MA, 1975).
- [18] W. D. McComb, *The Physics of Fluid Turbulence* (Clarendon, Oxford, 1990).
- [19] U. Frish, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, England, 1996).
- [20] S. Childress and A. Gilbert, *Stretch, Twist, Fold: The Fast Dynamo* (Springer-Verlag, Berlin, 1995), and references therein.
- [21] Ya. B. Zeldovich, A. A. Ruzmaikin, and D. D. Sokoloff, *The Almighty Chance* (World Scientific, Singapore, 1990), and references therein.
- [22] V. I. Arnold, *C. R. Hebd. Seances Acad. Sci.* **261**, 17 (1965).
- [23] M. Hénon, *C.R. Seances Acad. Sci., Ser. A* **262**, 312 (1966).
- [24] S. Childress, *J. Math. Phys.* **11**, 3063 (1970).
- [25] V. I. Arnold, Ya. B. Zeldovich, A. A. Ruzmaikin, and D. D. Sokoloff, *Zh. Eksp. Teor. Fiz.* **81**, 2052 (1981); *Sov. Phys. JETP* **54**, 1083 (1981).
- [26] V. I. Arnold and E. I. Korkina, *Vestn. Mosk. Univ., Ser. 1: Mat., Mekh.* **6**, 43 (1983).
- [27] B. Galanti, P. L. Sulem and A. Pouquet, *Geophys. Astrophys. Fluid Dyn.* **66**, 183 (1992).
- [28] D. J. Galloway and U. Frisch, *Geophys. Astrophys. Fluid Dyn.* **36**, 53 (1986).
- [29] T. Dombre, U. Frisch, J. M. Green, M. Hénon, A. Mehr, and A. M. Soward, *J. Fluid Mech.* **167**, 353 (1986).
- [30] D. J. Galloway and U. Frisch, *J. Fluid Mech.* **180**, 557 (1987).
- [31] O. Podvigina and A. Pouquet, *Physica D* **75**, 471 (1994).
- [32] B. Galanti, N. Kleeorin, and I. Rogachevskii, *Phys. Plasmas* **1**, 3843 (1994).
- [33] B. Galanti, N. Kleeorin, and I. Rogachevskii, *Phys. Plasmas* **2**, 4161 (1995).
- [34] K.-H. Rädler, *Astron. Nachr.* **301**, 101 (1980).
- [35] I. Rogachevskii and N. Kleeorin, *Phys. Rev. E* **56**, 417 (1997).
- [36] K.-H. Rädler, N. Kleeorin, and I. Rogachevskii, *Geophys. Astrophys. Fluid Dyn.* **97**, 249 (2003).
- [37] D. N. Tomin and D. D. Sokoloff, *Astron. Lett.* (to be published).
- [38] G. O. Roberts, *Philos. Trans. R. Soc. London, Ser. A* **A266**, 535 (1970); **A271**, 411 (1972).
- [39] V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, New York, 1980).
- [40] V. I. Arnold and B. A. Khesin, *Annu. Rev. Fluid Mech.* **24**, 145 (1992).
- [41] S. M. Tobias and F. Cattaneo, *Phys. Rev. Lett.* **101**, 125003 (2008).
- [42] F. H. Busse, *Annu. Rev. Fluid Mech.* **10**, 435 (1978).
- [43] R. Stieglitz and U. Müller, *Phys. Fluids* **13**, 561 (2001).
- [44] U. Müller and R. Stieglitz, *Nonlinear Processes Geophys.* **9**, 165 (2002).
- [45] U. Müller, R. Stieglitz and S. Horanyi, *J. Fluid Mech.* **498**, 31 (2004).
- [46] E. R. Priest, *Solar Magnetohydrodynamics* (D. Reidel, Dordrecht, 1982).