

# Magnetic fluctuations and formation of large-scale inhomogeneous magnetic structures in a turbulent convection

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Magnetic fluctuations generated by a tangling of the mean magnetic field by velocity fluctuations are studied in a developed turbulent convection with large magnetic Reynolds numbers. We show that the energy of magnetic fluctuations depends on the magnetic Reynolds number only when the mean magnetic field is smaller than  $B_{\text{eq}}/4\text{Rm}^{1/4}$ , where  $B_{\text{eq}}$  is the equipartition mean magnetic field determined by the turbulent kinetic energy and  $\text{Rm}$  is the magnetic Reynolds number. Generation of magnetic fluctuations in a turbulent convection with a nonzero mean magnetic field results in a decrease of the total turbulent pressure and may cause the formation of large-scale inhomogeneous magnetic structures even in an originally uniform mean magnetic field. This effect is caused by a negative contribution of the turbulent convection to the effective mean Lorentz force. The inhomogeneous large-scale magnetic fields are formed due to excitation of the large-scale instability. The energy for this instability is supplied by small-scale turbulent convection. The discussed effects might be useful for understanding the origin of solar nonuniform magnetic fields: e.g., sunspots.

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## I. INTRODUCTION

Magnetic fields in astrophysics are strongly nonuniform (see, e.g., [1–8]). Large-scale magnetic structures are observed in the form of sunspots, solar coronal magnetic loops, etc. There are different mechanisms for the formation of the large-scale magnetic structures: e.g., the magnetic buoyancy instability of stratified continuous magnetic fields [2,9–11], magnetic flux expulsion [12], topological magnetic pumping [13], etc.

Magnetic buoyancy applies in the literature for different situations (see [11]). The first corresponds to the magnetic buoyancy instability of stratified continuous magnetic fields (see, e.g., [2,9–11]), and the magnetic flux tube concept is not used there. The magnetic buoyancy instability of stratified continuous magnetic fields is excited when the scale of variations of the initial magnetic field is less than the density stratification length. On the other hand, buoyancy of discrete magnetic flux tubes has been discussed in a number of studies in solar physics and astrophysics (see, e.g., [11,14–18]). This phenomenon is also related to the problem of the storage of magnetic fields in the overshoot layer near the bottom of the solar convective zone (see, e.g., [19–22]).

A universal mechanism of the formation of the nonuniform distribution of magnetic flux is associated with a magnetic flux expulsion. In particular, the expulsion of magnetic flux from two-dimensional flows (a single vortex and a grid of vortices) was demonstrated in [12]. In the context of solar and stellar convection, the topological asymmetry of stationary thermal convection plays a very important role in magnetic field dynamics. In particular, topological magnetic pumping is caused by topological asymmetry of the thermal convection [13]. The fluid rises at the centers of the convective cells and falls at their peripheries. The ascending fluid

elements (contrary to the descending fluid elements) are disconnected from one another. This causes a topological magnetic pumping effect, allowing downward transport of the mean horizontal magnetic field to the bottom of a cell, but impeding its upward return [4,13,23].

Turbulence may form inhomogeneous large-scale magnetic fields due to turbulent diamagnetic and paramagnetic effects (see, e.g., [3,24–28]). Inhomogeneous velocity fluctuations lead to a transport of mean magnetic flux from regions with a high intensity of velocity fluctuations. Inhomogeneous magnetic fluctuations due to the small-scale dynamo cause a turbulent paramagnetic velocity; i.e., the magnetic flux is pushed into regions with a high intensity of magnetic fluctuations. Other effects are the effective drift velocities of the mean magnetic field caused by inhomogeneities of the fluid density [26,27] and pressure [29]. In a nonlinear stage of the magnetic field evolution, inhomogeneities of the mean magnetic field contribute to the diamagnetic or paramagnetic drift velocities depending on the level of magnetic fluctuations due to the small-scale dynamo and level of the mean magnetic field [30]. The diamagnetic velocity causes a drift of the magnetic field components from regions with a high intensity of the mean magnetic field.

The nonlinear drift velocities of the mean magnetic field in a turbulent convection have been determined in [31]. This study demonstrates that nonlinear drift velocities are caused by three kinds of inhomogeneities: i.e., inhomogeneous turbulence, the nonuniform fluid density, and the nonuniform turbulent heat flux. The nonlinear drift velocities of the mean magnetic field cause the small-scale magnetic buoyancy and magnetic pumping effects in the turbulent convection. These phenomena are different from the large-scale magnetic buoyancy and magnetic pumping effects which are due to the effect of the mean magnetic field on the large-scale density stratified fluid flow. The small-scale magnetic buoyancy and magnetic pumping can be stronger than these large-scale effects when the mean magnetic field is smaller than the equipartition field determined by the turbulent kinetic energy

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[31]. The pumping of magnetic flux in three-dimensional compressible magnetoconvection has been studied in direct numerical simulations in [32] by calculating the turbulent diamagnetic and paramagnetic velocities.

Turbulence may affect also the Lorentz force of the large-scale magnetic field (see [33–36]). This effect can also form inhomogeneous magnetic structures. In this study a theoretical approach proposed in [33–36] for a nonconvective turbulence is further developed and applied to investigate the modification of the large-scale magnetic force by turbulent convection and to elucidate a mechanism of formation of inhomogeneous magnetic structures.

This paper is organized as follows. In Sec. II we discuss the physics of the effect of turbulence on the large-scale Lorentz force. In Sec. III we formulate the governing equations, the assumptions, and the procedure of the derivations of the large-scale effective magnetic force in turbulent convection. In Sec. IV we study magnetic fluctuations and determine the modification of the large-scale effective Lorentz force by the turbulent convection. In Sec. V we discuss the formation of large-scale magnetic inhomogeneous structures in the turbulent convection due to excitation of the large-scale instability. Finally, we draw conclusions in Sec. VI. In the Appendix we perform the derivation of the large-scale effective Lorentz force in the turbulent convection.

## II. TURBULENT PRESSURE AND EFFECTIVE MEAN MAGNETIC PRESSURE

In this section we discuss the physics of the effect of turbulence on the large-scale Lorentz force. First, let us examine an isotropic turbulence. The Lorentz force of the small-scale magnetic fluctuations can be written in the form  $\mathcal{F}_i^{(m)} = \nabla_j \sigma_{ij}^{(m)}$ , where the magnetic stress tensor  $\sigma_{ij}^{(m)}$  is given by

$$\sigma_{ij}^{(m)} = -\frac{\langle \mathbf{b}^2 \rangle}{2} \delta_{ij} + \langle b_i b_j \rangle, \quad (1)$$

$\mathbf{b}$  are the magnetic fluctuations, and  $\delta_{ij}$  is the Kronecker tensor. Hereafter we omit the magnetic permeability of the fluid  $\mu$  and include  $\mu^{-1/2}$  in the definition of the magnetic field. The angular brackets in Eq. (1) denote ensemble averaging. For isotropic turbulence  $\langle b_i b_j \rangle = \delta_{ij} \langle \mathbf{b}^2 \rangle / 3$ , and the magnetic stress tensor reads

$$\sigma_{ij}^{(m)} = -\frac{\langle \mathbf{b}^2 \rangle}{6} \delta_{ij} = -\frac{W_m}{3} \delta_{ij}, \quad (2)$$

where  $W_m = \langle \mathbf{b}^2 \rangle / 2$  is the energy density of the magnetic fluctuations. The magnetic pressure  $p_m$  is related to the magnetic stress tensor:  $\sigma_{ij}^{(m)} = -p_m \delta_{ij}$ , where  $p_m = W_m / 3$ . Similarly, in an isotropic turbulence the Reynolds stresses  $\langle u_i u_j \rangle$  read  $\langle u_i u_j \rangle = \delta_{ij} \langle \mathbf{u}^2 \rangle / 3$ , and

$$\sigma_{ij}^{(v)} = -\rho_0 \langle u_i u_j \rangle = -\frac{\rho_0 \langle \mathbf{u}^2 \rangle}{3} \delta_{ij} = -\frac{2W_k}{3} \delta_{ij}, \quad (3)$$

where  $\mathbf{u}$  are the velocity fluctuations,  $W_k = \rho_0 \langle \mathbf{u}^2 \rangle / 2$  is the kinetic energy density of the velocity fluctuations, and  $\rho_0$  is

the fluid density. Equation (3) yields the hydrodynamic pressure  $p_v = 2W_k / 3$ , where  $\sigma_{ij}^{(v)} = -p_v \delta_{ij}$ . Therefore, the equation of state for the isotropic turbulence is given by

$$p_T = \frac{1}{3} W_m + \frac{2}{3} W_k \quad (4)$$

(see also [37,38]), where  $p_T$  is the total (hydrodynamic plus magnetic) turbulent pressure. Similarly, the equation of state for an anisotropic turbulence reads

$$p_T = \frac{2}{3(2 + A_N)} W_m + \frac{4 + 3A_N}{3(2 + A_N)} W_k, \quad (5)$$

where  $A_N = (2/3)[\langle \mathbf{u}_\perp^2 \rangle / \langle \mathbf{u}_z^2 \rangle - 2]$  is the degree of anisotropy of the turbulent velocity field  $\mathbf{u} = \mathbf{u}_\perp + u_z \mathbf{e}$ . For an isotropic three-dimensional turbulence  $\langle \mathbf{u}_\perp^2 \rangle = 2\langle \mathbf{u}_z^2 \rangle$  and the parameter  $A_N = 0$ , while for a two-dimensional turbulence  $\langle \mathbf{u}_z^2 \rangle = 0$  and the degree of anisotropy  $A_N \rightarrow \infty$ . Here  $\mathbf{e}$  is the vertical unit vector perpendicular to the plane of the two-dimensional turbulence.

In a two-dimensional turbulence  $A_N \rightarrow \infty$  and the total turbulent pressure  $p_T \rightarrow W_k$ . Note that the magnetic pressure in a two-dimensional turbulence vanishes. Indeed, for isotropic magnetic fluctuations in a two-dimensional turbulence  $\langle b_i b_j \rangle = (1/2) \langle \mathbf{b}^2 \rangle \delta_{ij}^{(2)}$ , and therefore,  $\sigma_{ij}^{(m)} \equiv -(1/2) \langle \mathbf{b}^2 \rangle \delta_{ij}^{(2)} + \langle b_i b_j \rangle = 0$ , where  $\delta_{ij}^{(2)} = \delta_{ij} - e_i e_j$ .

The total energy density  $W_T = W_k + W_m$  of the homogeneous turbulence with a mean magnetic field  $\mathbf{B}$  is determined by the equation

$$\frac{\partial W_T}{\partial t} = I_T - \frac{W_T}{\tau_0} + \eta_T (\nabla \times \mathbf{B})^2 \quad (6)$$

(see, e.g., [36]), where  $\tau_0$  is the correlation time of the turbulent velocity field in the maximum scale  $l_0$  of turbulent motions,  $I_T$  is the energy source of turbulence,  $\eta_T$  is the turbulent magnetic diffusion, and the mean magnetic field  $\mathbf{B}$  is given (prescribed). The second term  $W_T / \tau_0$  on the right-hand side of Eq. (6) determines the dissipation of the turbulent energy. For a given time-independent source of turbulence  $I_T$  the solution of Eq. (6) is given by

$$W_T = \tau_0 [I_T + \eta_T (\nabla \times \mathbf{B})^2] \left[ 1 - \exp\left(-\frac{t}{\tau_0}\right) \right] + \tilde{W}_T \exp\left(-\frac{t}{\tau_0}\right), \quad (7)$$

where  $\tilde{W}_T = W_T(t=0)$ . For instance, a time-independent source of the turbulence exists in the Sun. The mean nonuniform magnetic field causes an additional energy source of turbulence,  $I_N = \eta_T (\nabla \times \mathbf{B})^2$ . The ratio  $I_N / I_T$  of these two sources of turbulence is of the order of

$$\frac{I_N}{I_T} \simeq \left( \frac{l_0}{L_B} \right)^2 \frac{\mathbf{B}^2}{\rho_0 \langle \mathbf{u}^2 \rangle^{(0)}} \ll 1, \quad (8)$$

where  $L_B$  is the characteristic scale of the spatial variations of the mean magnetic field. Since  $l_0 \ll L_B$  and  $\mathbf{B}^2 \ll \rho_0 \langle \mathbf{u}^2 \rangle^{(0)}$ , we can neglect the small magnetic source  $I_N$  of the turbulence. Thus, for  $t \gg \tau_0$  the total energy density of the turbulence reaches a steady state  $W_T = \text{const} = \tau_0 I_T$ . Therefore, the

total energy density  $W_T$  of the homogeneous turbulence is conserved (the dissipation is compensated by a supply of energy); i.e.,

$$W_k + W_m = \text{const.} \quad (9)$$

A more rigorous derivation of Eq. (9) is given in the Appendix [see Eq. (A16)]. Equation (9) implies that the uniform large-scale magnetic field performs no work on the turbulence. It can only redistribute the energy between hydrodynamic and magnetic fluctuations.

Combining Eqs. (4) and (9) we can express the change of turbulent pressure,  $\delta p_T$ , in terms of the change of the magnetic energy density,  $\delta W_m$ , for an isotropic turbulence,  $\delta p_T = -(1/3)\delta W_m$  (see [33,34,36]). Therefore, the turbulent pressure is reduced when magnetic fluctuations are generated ( $\delta W_m > 0$ ). Similarly, for an anisotropic turbulence, the generation of magnetic fluctuations reduces the turbulent pressure; i.e.,

$$\delta p_T = -\frac{2 + 3A_N}{3(2 + A_N)} \delta W_m.$$

The total turbulent pressure is decreased also by the tangling of the large-scale mean magnetic field  $\mathbf{B}$  by the velocity fluctuations (see, e.g., [1–4], and references therein). The mean magnetic field generates additional small-scale magnetic fluctuations due to a tangling of the mean magnetic field by velocity fluctuations. For a small energy of the mean magnetic field,  $\mathbf{B}^2 \ll \rho_0 \langle \mathbf{u}^2 \rangle$ , the energy of magnetic fluctuations,  $\langle \mathbf{b}^2 \rangle - \langle \mathbf{b}^2 \rangle^{(0)}$ , caused by a tangling of the mean magnetic field can be written in the form

$$\langle \mathbf{b}^2 \rangle - \langle \mathbf{b}^2 \rangle^{(0)} = a_m(B, \text{Rm}) \mathbf{B}^2 + O(\mathbf{B}^4 / (\rho_0 \langle \mathbf{u}^2 \rangle)^2), \quad (10)$$

where  $\langle \mathbf{b}^2 \rangle^{(0)}$  are the magnetic fluctuations with a zero mean magnetic field generated by a small-scale dynamo. Equation (10) allows us to determine the variation of the magnetic energy  $\delta W_m$ . Therefore, the total turbulent pressure reads

$$p_T = p_T^{(0)} - q_p \frac{\mathbf{B}^2}{2}, \quad (11)$$

where  $p_T^{(0)}$  is the turbulent pressure in a flow with a zero mean magnetic field and the coefficient  $q_p \propto a_m(B, \text{Rm})$ . Here we neglect the small terms  $\sim O(\mathbf{B}^4 / (\rho_0 \langle \mathbf{u}^2 \rangle)^2)$ . The coefficient  $q_p$  is positive when magnetic fluctuations are generated, and it is negative when they are damped. The total pressure is

$$P_{\text{tot}} \equiv P_k + p_T + P_B(B) = P_k + p_T^{(0)} + (1 - q_p) \frac{\mathbf{B}^2}{2}, \quad (12)$$

where  $P_k$  is the mean fluid pressure and  $P_B(B) = \mathbf{B}^2/2$  is the magnetic pressure of the mean magnetic field. Now we examine the part of the total pressure  $P_{\text{tot}}$  that depends on the mean magnetic field  $\mathbf{B}$ ; i.e., we consider

$$P_m(B) = P_B(B) - q_p(B) \frac{\mathbf{B}^2}{2} = (1 - q_p) \frac{\mathbf{B}^2}{2} \quad (13)$$

(see [33,34,36]), where now  $P_{\text{tot}} = P + P_m(B)$  and  $P = P_k + p_T^{(0)}$ . The pressure  $P_m(B)$  is called the effective (or com-

bined) mean magnetic pressure. Note that both the hydrodynamic and magnetic fluctuations contribute to the combined mean magnetic pressure. However, the gain in the turbulent magnetic pressure  $p_m$  is not as large as the reduction of the turbulent hydrodynamic pressure  $p_v$  by the mean magnetic field  $\mathbf{B}$ . This is due to different coefficients multiplying by  $W_m$  and  $W_k$  in the equation of state (4) [see also Eq. (5)]. Therefore, this effect is caused by a negative contribution of the turbulence to the combined mean magnetic pressure.

We consider the case when  $P \gg \mathbf{B}^2/2$ , so that the total pressure  $P_{\text{tot}}$  is always positive. Only the combined mean magnetic pressure  $P_m(B)$  may be negative when  $q_p > 1$ , while the pressure  $P_B(B)$  as well as the values  $P_k, p_v, p_m$ , and  $p_T$  are always positive. When a mean magnetic field  $\mathbf{B}$  is superimposed on an isotropic turbulence, the isotropy breaks down. Nevertheless Eq. (13) remains valid, while the relationship between  $q_p$  and  $a_m$  may change.

In this section we use the conservation law (9) for the total turbulent energy only for elucidation of the principle of the effect, but we have not employed Eq. (9) to develop the theory of this effect (see for details [34–36]). In particular, the high-order closure procedure [34,36] and the renormalization procedure [35] have been used for the investigation of the nonconvective turbulence at large magnetic and hydrodynamic Reynolds numbers.

In this study we investigate the modification of the large-scale magnetic force by turbulent convection. We demonstrate that turbulent convection enhances modification of the effective magnetic force and causes a large-scale instability. This results in the formation of large-scale inhomogeneous magnetic structures.

### III. GOVERNING EQUATIONS AND THE PROCEDURE OF DERIVATION

In order to study magnetic fluctuations and the modification of the large-scale Lorentz force by turbulent convection we use a mean-field approach in which the magnetic and velocity fields and entropy are decomposed into the mean and fluctuating parts, where the fluctuating parts have zero mean values. We assume that there exists a separation of scales; i.e., the maximum scale of turbulent motions  $l_0$  is much smaller than the characteristic scale  $L_B$  of the mean magnetic field variations. We apply here an approach which is described in [30,31,39] and outlined below.

We consider a nonrotating turbulent convection with large Rayleigh numbers and large magnetic Reynolds numbers. We use the equations for fluctuations of the fluid velocity  $\mathbf{u}$ , entropy  $s'$ , and the magnetic field  $\mathbf{b}$ . The equations for velocity and entropy fluctuations are rewritten in the new variables  $\mathbf{v} = \sqrt{\rho_0} \mathbf{u}$  and  $s = \sqrt{\rho_0} s'$ . We also use the new variable  $\mathbf{H} = \mathbf{B} / \sqrt{\rho_0}$  for the mean magnetic field  $\mathbf{B}$ . On the other hand, we do not use a new variable for magnetic fluctuations  $\mathbf{b}$ . Equations for fluctuations of fluid velocity, entropy, and magnetic field are applied in the anelastic approximation, which is a combination of the Boussinesq approximation and the condition  $\text{div}(\rho_0 \mathbf{u}) = 0$ . The turbulent convection is regarded as a small deviation from a well-mixed adiabatic reference state. This implies that we consider the hydrostatic nearly isentropic basic reference state.

Using these equations for fluctuations of fluid velocity, entropy, and magnetic field written in a Fourier space we derive equations for the two-point second-order correlation functions of the velocity fluctuations  $\langle v_i v_j \rangle$ , the magnetic fluctuations  $\langle b_i b_j \rangle$ , the entropy fluctuations  $\langle s s \rangle$ , the cross helicity  $\langle b_i v_j \rangle$ , and the turbulent heat flux  $\langle s v_i \rangle$  and  $\langle s b_i \rangle$ . The equations for these correlation functions are given by Eqs. (A4)–(A9) in the Appendix. We split the tensor  $\langle b_i b_j \rangle$  of magnetic fluctuations into nonhelical,  $h_{ij}$ , and helical,  $h_{ij}^{(H)}$ , parts. The helical part  $h_{ij}^{(H)}$  depends on the magnetic helicity (see below). We also split all second-order correlation functions  $M^{(II)}$  into symmetric and antisymmetric parts with respect to the wave vector  $\mathbf{k}$ —e.g.,  $h_{ij} = h_{ij}^{(s)} + h_{ij}^{(a)}$ , where the tensor  $h_{ij}^{(s)} = [h_{ij}(\mathbf{k}) + h_{ij}(-\mathbf{k})]/2$  describes the symmetric part of the tensor and  $h_{ij}^{(a)} = [h_{ij}(\mathbf{k}) - h_{ij}(-\mathbf{k})]/2$  determines the antisymmetric part of the tensor.

The second-moment equations include the first-order spatial differential operators  $\hat{N}$  applied to the third-order moments  $M^{(III)}$ . A problem arises as to how to close the equations for the second moments—i.e., how to express the third-order terms  $\hat{N}M^{(III)}$  through the second moments  $M^{(II)}$  (see, e.g., [40–42]). We will use the spectral  $\tau$  approximation which postulates that the deviations of the third-moment terms  $\hat{N}M^{(III)}(\mathbf{k})$  from the contributions to these terms afforded by the background turbulent convection,  $\hat{N}M^{(III,0)}(\mathbf{k})$ , are expressed through similar deviations of the second moments,  $M^{(II)}(\mathbf{k}) - M^{(II,0)}(\mathbf{k})$ :

$$\hat{N}M^{(III)}(\mathbf{k}) - \hat{N}M^{(III,0)}(\mathbf{k}) = -\frac{1}{\tau(k)}[M^{(II)}(\mathbf{k}) - M^{(II,0)}(\mathbf{k})] \quad (14)$$

(see, e.g., [30,34,36,40,43]), where  $\tau(k)$  is the scale-dependent relaxation time. In the background turbulent convection the mean magnetic field is zero. The  $\tau$  approximation is applied for large hydrodynamic and magnetic Reynolds numbers and large Rayleigh numbers. In this case there is only one relaxation time  $\tau$  which can be identified with the correlation time of the turbulent velocity field. A justification of the  $\tau$  approximation for different situations has been performed in numerical simulations and theoretical studies in [44–49] (see also review [8]). The  $\tau$  approximation is also discussed in Sec. VI.

We apply the spectral  $\tau$  approximation only for the nonhelical part  $h_{ij}$  of the tensor of magnetic fluctuations. The helical part  $h_{ij}^{(H)}$  depends on the magnetic helicity, and it is determined by the dynamic equation which follows from the magnetic helicity conservation arguments (see, e.g., [50–56] and the review [8], and references therein). The characteristic time of evolution of the nonhelical part of the tensor  $h_{ij}$  is of the order of the turbulent time  $\tau_0 = l_0/u_0$ , while the relaxation time of the helical part of the tensor  $h_{ij}^{(H)}$  of magnetic fluctuations is of the order of  $\tau_0 \text{Rm}$ , where  $\text{Rm} = l_0 u_0 / \eta$  is the magnetic Reynolds number,  $u_0$  is the characteristic turbulent velocity in the maximum scale of turbulent motions  $l_0$ , and  $\eta$  is the magnetic diffusivity due to electrical conductivity of the fluid.

In this study we consider an intermediate nonlinearity which implies that the mean magnetic field is not strong enough in order to affect the correlation time of the turbulent velocity field. The theory can be expanded to the case of a very strong mean magnetic field after taking into account a dependence of the correlation time of the turbulent velocity field on the mean magnetic field.

We assume that the characteristic time of variation of the mean magnetic field  $\mathbf{B}$  is substantially larger than the correlation time  $\tau(k)$  for all turbulence scales. This allows us to get a stationary solution for the equations for the second-order moments given by Eqs. (A10)–(A14) in the Appendix. For the integration in  $\mathbf{k}$  space of the second moments we have to specify a model for the background turbulent convection (with a zero mean magnetic field,  $\mathbf{B} = 0$ ). Here we use the model of the background shear-free turbulent convection with a given heat flux [see Eqs. (A17)–(A20) in the Appendix]. In this model velocity and magnetic fluctuations are homogeneous and isotropic.

This procedure allows us to study magnetic fluctuations with a nonzero mean magnetic field and to investigate the modification of the large-scale Lorentz force by turbulent convection (see Sec. IV).

#### IV. MAGNETIC FLUCTUATIONS AND LARGE-SCALE EFFECTIVE LORENTZ FORCE

##### A. Magnetic fluctuations with a nonzero mean magnetic field

Let us study magnetic fluctuations with a nonzero mean magnetic field using the approach outlined in Sec. III. Integration in  $\mathbf{k}$  space in Eq. (A11) yields an analytical expression for the energy of magnetic fluctuations,  $\langle \mathbf{b}^2 \rangle$  [see Eq. (A21) in the Appendix]. The energy of magnetic fluctuations versus the mean magnetic field  $B/B_{\text{eq}}$  is shown in Fig. 1, where  $B_{\text{eq}}$  is the equipartition mean magnetic field determined by the turbulent kinetic energy. The asymptotic formulas for  $\langle \mathbf{b}^2 \rangle$  are given below. In particular, for a very weak mean magnetic field,  $B \ll B_{\text{eq}}/4\text{Rm}^{1/4}$ , the energy of magnetic fluctuations is given by

$$\langle \mathbf{b}^2 \rangle = \langle \mathbf{b}^2 \rangle^{(0)} + \frac{4}{3}[\langle \mathbf{v}^2 \rangle^{(0)} - \langle \mathbf{b}^2 \rangle^{(0)}]B^2 \ln \text{Rm} + \frac{8a_*}{5}\langle \mathbf{v}^2 \rangle^{(0)}B^2(2 - 3 \cos^2 \phi), \quad (15)$$

where the quantities with the superscript (0) correspond to the background turbulent convection (with a zero mean magnetic field), and  $\langle \mathbf{v}^2 \rangle^{(0)}$  and  $\langle \mathbf{b}^2 \rangle^{(0)}$  are the velocity and magnetic fluctuations in the background turbulent convection. Here the magnetic field  $B$  is measured in units of  $B_{\text{eq}}$  and  $\phi$  is the angle between the vertical unit vector  $\mathbf{e}$  and the mean magnetic field  $\mathbf{B}$ . The unit vector  $\mathbf{e}$  is directed opposite to the gravity field. The parameter  $a_*$  characterizing the turbulent convection is determined by the budget equation for the total energy, and it is given by

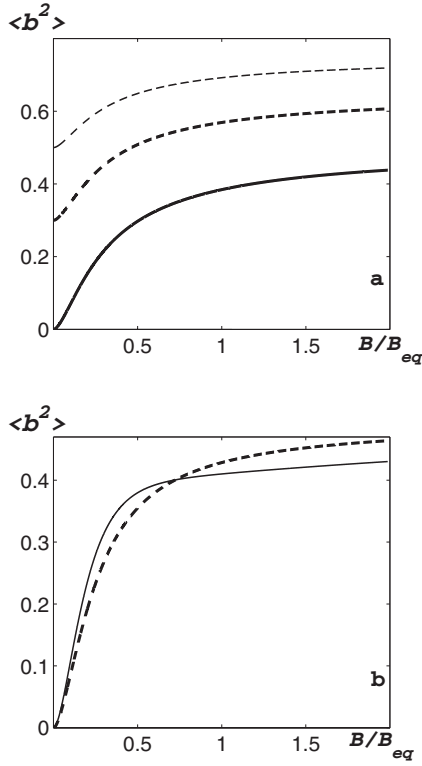


FIG. 1. (a) The energy of magnetic fluctuations  $\langle \mathbf{b}^2 \rangle$  versus the mean magnetic field  $B/B_{eq}$  for a nonconvective turbulence ( $a_*=0$ ),  $Rm=10^6$ , and different values of the parameter  $\epsilon \equiv \langle \mathbf{b}^2 \rangle^{(0)} / \langle \mathbf{u}^2 \rangle^{(0)}$ :  $\epsilon=0$  (solid line),  $\epsilon=0.3$  (dashed line), and  $\epsilon=0.5$  (thin dashed line). (b) The energy of magnetic fluctuations  $\langle \mathbf{b}^2 \rangle$  versus the horizontal (dashed line) and vertical (thin solid line) mean magnetic field for a convective turbulence ( $a_*=0.7$ ) and for  $Rm=10^6$ ,  $\epsilon=0$ .

$$a_*^{-1} = 1 + \frac{\nu_T (\nabla U)^2 + \eta_T (\nabla B)^2 / \rho_0}{g F_*},$$

where  $\nu_T$  is the turbulent viscosity,  $U$  is the mean fluid velocity, and  $F_* = \langle u_z s' \rangle^{(0)}$  is the vertical heat flux in the background turbulent convection. The energy of magnetic fluctuations for a very weak mean magnetic field,  $B \ll B_{eq} / 4Rm^{1/4}$ , depends on the magnetic Reynolds number:  $\langle \mathbf{b}^2 \rangle \propto \ln Rm$ . This is an indication of that the spectrum of magnetic fluctuations is  $k^{-1}$  in the limit of a small yet finite mean magnetic field [34–36,57] (see also discussion in [58]). When the mean magnetic field  $B_{eq} / 4Rm^{1/4} \ll B \ll B_{eq} / 4$ , the energy of magnetic fluctuations is given by

$$\begin{aligned} \langle \mathbf{b}^2 \rangle &= \langle \mathbf{b}^2 \rangle^{(0)} + \frac{16}{3} [\langle \mathbf{v}^2 \rangle^{(0)} - \langle \mathbf{b}^2 \rangle^{(0)}] B^2 |\ln(4B)| \\ &+ \frac{8a_*}{5} \langle \mathbf{v}^2 \rangle^{(0)} B^2 (2 + 3 \cos^2 \phi), \end{aligned} \quad (16)$$

and for  $B \gg B_{eq} / 4$  it is given by

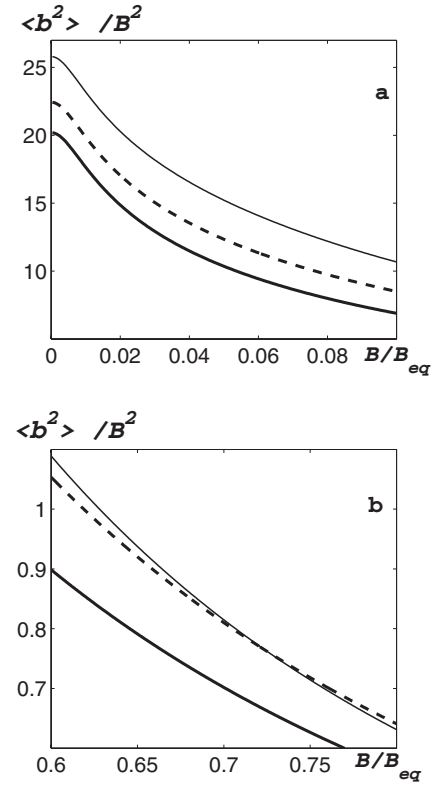


FIG. 2. The normalized energy of magnetic fluctuations  $\langle \mathbf{b}^2 \rangle / B^2$  versus the mean magnetic field  $B/B_{eq}$  for a nonconvective turbulence ( $a_*=0$ ) (solid line) and for a convective turbulence ( $a_*=0.7$ ) with the horizontal (dashed line) and vertical (thin solid line) mean magnetic field, where the cases  $B \ll B_{eq}$  (a) and  $B \sim B_{eq}$  (b) are shown. Here  $Rm=10^6$  and  $\epsilon=0$ .

$$\begin{aligned} \langle \mathbf{b}^2 \rangle &= \frac{1}{2} [\langle \mathbf{v}^2 \rangle^{(0)} + \langle \mathbf{b}^2 \rangle^{(0)}] - \frac{\pi}{24B} [\langle \mathbf{v}^2 \rangle^{(0)} - \langle \mathbf{b}^2 \rangle^{(0)}] \\ &+ \frac{\pi a_*}{40B} \langle \mathbf{v}^2 \rangle^{(0)} (1 - 3 \cos^2 \phi). \end{aligned} \quad (17)$$

The normalized energy of magnetic fluctuations  $\langle \mathbf{b}^2 \rangle / B^2$  versus the mean magnetic field is shown in Fig. 2 for a nonconvective and convective turbulence. Inspection of Figs. 1 and 2 shows that turbulent convection increases the level of magnetic fluctuations in comparison with the nonconvective turbulence. It follows from Eqs. (15)–(17) that in the case of Alfvénic equipartition,  $\langle \mathbf{u}^2 \rangle^{(0)} = \langle \mathbf{b}^2 \rangle^{(0)}$ , a deviation of the energy of magnetic fluctuations from the background level is caused by the turbulent convection.

## B. Large-scale effective Lorentz force

The effective (combined) mean magnetic force, which takes into account the effect of turbulence on magnetic force, can be written in the form  $\mathcal{F}_i^{\text{eff}} = \nabla_j \sigma_{ij}^{\text{eff}}$ , where the effective stress tensor  $\sigma_{ij}^{\text{eff}}$  reads

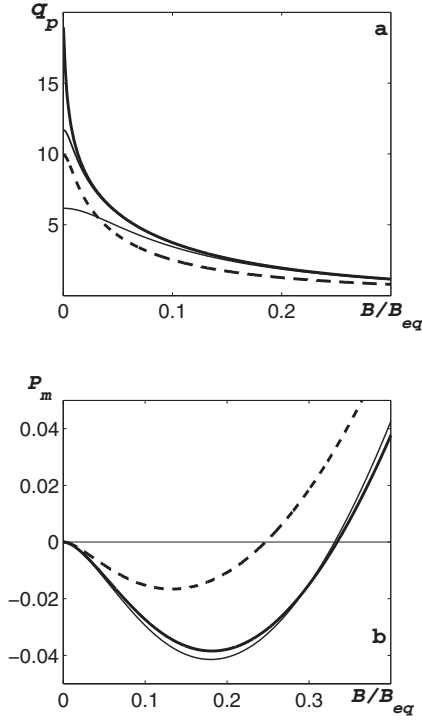


FIG. 3. (a) The nonlinear coefficient  $q_p(B)$  for different values of the magnetic Reynolds numbers  $Rm$ :  $Rm=10^3$  (thin solid line),  $Rm=10^6$  (dash-dotted line), and  $Rm=10^{10}$  (thick solid line) for a nonconvective turbulence ( $a_*=0$ ) and at  $Rm=10^6$  (dashed line) for a convective turbulence ( $a_*=0.7$ ). (b) The effective (combined) mean magnetic pressure  $P_m(B)=(1-q_p)B^2/B_{eq}^2$  at  $Rm=10^6$  for a nonconvective turbulence ( $a_*=0$ ) (thick solid line) and for a convective turbulence ( $a_*=0.7$ ) for the horizontal field (dashed line) and for the vertical field (thin solid line).

$$\sigma_{ij}^{\text{eff}} = -\frac{1}{2}\mathbf{B}^2\delta_{ij} + B_iB_j - \frac{1}{2}\langle \mathbf{b}^2 \rangle \delta_{ij} + \langle b_i b_j \rangle - \rho_0 \langle u_i u_j \rangle. \quad (18)$$

The last three terms on the right-hand side (RHS) of Eq. (18) determine the contribution of velocity and magnetic fluctuations to the effective (combined) mean magnetic force. Using Eqs. (A10) and (A11) for  $\langle u_i u_j \rangle$  and  $\langle b_i b_j \rangle$  after the integration in  $\mathbf{k}$  space we arrive at the expression for the effective stress tensor:

$$\sigma_{ij}^{\text{eff}} = -[1 - q_p(B)]\frac{B^2}{2}\delta_{ij} + [1 - q_s(B)]B_iB_j + a_*\sigma_{ij}^A(B), \quad (19)$$

where the analytical expressions for the nonlinear coefficients  $q_p(B)$  and  $q_s(B)$  are given by Eqs. (A23) and (A24) in the Appendix, and the tensor  $\sigma_{ij}^A(B)$  is the anisotropic contribution caused by turbulent convection to the effective stress tensor [which is given by Eq. (A22) in the Appendix].

The nonlinear coefficients  $q_p(B)$  and  $q_s(B)$  in Eq. (19) for the effective stress tensor are shown in Figs. 3(a) and 4(a) for different values of the magnetic Reynolds numbers. The nonlinear coefficients  $q_p(B)$  and  $q_s(B)$  increase with the mag-

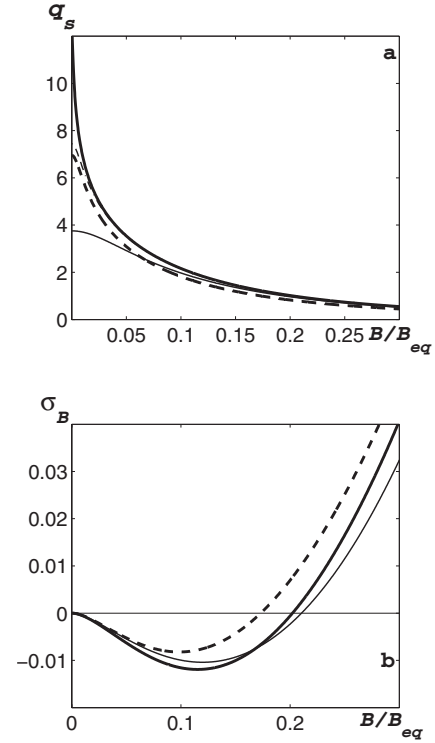


FIG. 4. (a) The nonlinear coefficient  $q_s(B)$  for different values of the magnetic Reynolds numbers  $Rm$ :  $Rm=10^3$  (thin solid line),  $Rm=10^6$  (thin dash-dotted line), and  $Rm=10^{10}$  (thick solid line) for a nonconvective turbulence  $a_*=0$  and at  $Rm=10^6$  (dashed line) for a convective turbulence  $a_*=0.7$ . (b) The effective (combined) mean magnetic tension  $\sigma_B(B)=(1-q_s)B^2/B_{eq}^2$  at  $Rm=10^6$  for a nonconvective turbulence ( $a_*=0$ ) (thick solid line) and for a convective turbulence ( $a_*=0.7$ ) for the horizontal field (dashed line) and for the vertical field (thin solid).

netic Reynolds numbers in the range of weak mean magnetic fields ( $B < 0.1B_{eq}$ ). On the other hand, the turbulent convection reduces these nonlinear coefficients in comparison with the case of a nonconvective turbulence. The asymptotic formulas for the nonlinear coefficients  $q_p(B)$  and  $q_s(B)$  are given below. In particular, for a very weak mean magnetic field,  $B \ll B_{eq}/4Rm^{1/4}$ , the nonlinear coefficients  $q_p(B)$  and  $q_s(B)$  are given by

$$q_p(B) = \frac{4}{5}(1 - \epsilon) \left[ \ln Rm + \frac{4}{45} \right] - \frac{8a_*}{35}(11 - 13 \cos^2 \phi), \quad (20)$$

$$q_s(B) = \frac{8}{15}(1 - \epsilon) \left[ \ln Rm + \frac{2}{15} \right] - \frac{24a_*}{35}, \quad (21)$$

where  $\epsilon \equiv \langle \mathbf{b}^2 \rangle^{(0)} / \langle \mathbf{u}^2 \rangle^{(0)}$ . For  $B_{eq}/4Rm^{1/4} \ll B \ll B_{eq}/4$  these nonlinear coefficients are given by

$$q_p(B) = \frac{16}{25}(1 - \epsilon)[5|\ln(4B)| + 1 + 32B^2] - \frac{8a_*}{35}(11 - 13 \cos^2 \phi), \quad (22)$$

$$q_s(B) = \frac{32}{15}(1 - \epsilon) \left[ |\ln(4B)| + \frac{1}{30} + 12B^2 \right] - \frac{24a_*}{35}, \quad (23)$$

and for  $B \gg B_{\text{eq}}/4$  they are given by

$$q_p(B) = \frac{1}{6B^2}(1 - \epsilon) + \frac{\pi a_*}{80B}(1 - 5 \cos^2 \phi), \quad (24)$$

$$q_s(B) = \frac{\pi}{48B^3}(1 - \epsilon) + \frac{3\pi a_*}{160B}(1 - 3 \cos^2 \phi). \quad (25)$$

The effective (combined) mean magnetic pressure  $P_m(B) = (1 - q_p)B^2/B_{\text{eq}}^2$  is shown in Fig. 3(b). Inspection of Fig. 3(b) shows that the combined mean magnetic pressure  $P_m(B) = (1 - q_p)B^2/B_{\text{eq}}^2$  vanishes at some value of the mean magnetic field  $B = B_p \sim (0.2 - 0.3)B_{\text{eq}}$ . This causes the following effect. Let us consider an isolated tube of magnetic field lines. When  $B = B_p$ , the combined mean magnetic pressure  $P_m(B) = 0$ , the fluid pressure, and fluid density inside and outside the isolated tube are the same, and therefore, this isolated tube is in equilibrium. When  $B > B_p$ , the combined mean magnetic pressure  $P_m(B) > 0$ ; the fluid pressure and fluid density inside the isolated tube are smaller than the fluid pressure and fluid density outside the isolated tube. This results in an upward floating of the isolated tube. On the other hand, when  $B < B_p$ , the combined mean magnetic pressure  $P_m(B) < 0$ , the fluid pressure, and fluid density inside the isolated tube are larger than the fluid pressure and fluid density outside the isolated tube. Therefore, this isolated magnetic tube flows down.

The effective (combined) mean magnetic tension  $\sigma_B(B) = (1 - q_s)B^2/B_{\text{eq}}^2$  is shown in Fig. 4(b). The combined mean magnetic tension  $\sigma_B(B)$  vanishes at some value of the mean magnetic field  $B = B_S \sim 0.2B_{\text{eq}}$  [see Fig. 4(b)]. When  $B > B_S$ , the combined mean magnetic tension  $\sigma_B(B) > 0$ , and Alfvénic and magnetosound waves can propagate in the isolated tubes. On the other hand, when  $B < B_S$ , the combined mean magnetic tension  $\sigma_B(B) < 0$ , and Alfvénic and magnetosound waves cannot propagate in the isolated tubes.

The anisotropic contributions  $\sigma_{ij}^A(B)$  to the effective stress tensor determine the anisotropic mean magnetic tension due to the turbulent convection. The tensor  $\sigma_{ij}^A(B)$  is characterized by the function  $\sigma_A(B) = \sigma_{ij}^A(B)e_{ij} = q_e B^2/B_{\text{eq}}^2$ . The nonlinear coefficient  $q_e(B)$  and the anisotropic mean magnetic tension  $\sigma_A(B)$  are shown in Figs. 5(a) and 5(b). In the next section we show that the anisotropic mean magnetic tension  $\sigma_A(B)$  caused by the turbulent convection strongly affects the dynamics of the horizontal mean magnetic field.

In this section we demonstrate that turbulent convection strongly modifies the large-scale magnetic force. Let us discuss a possibility for a study of the effect of turbulence on the effective (combined) mean Lorentz force in the direct numerical simulations. Consider the mean magnetic field which is directed along the  $x$  axis; i.e.,  $\mathbf{B} = B\mathbf{e}_x$ . Let us introduce the functions  $\sigma_x(B)$  and  $\sigma_y(B)$ ,

$$\sigma_x(B) = -\frac{1}{2}\langle b^2 \rangle + \langle b_x^2 \rangle - \rho_0 \langle u_x^2 \rangle, \quad (26)$$

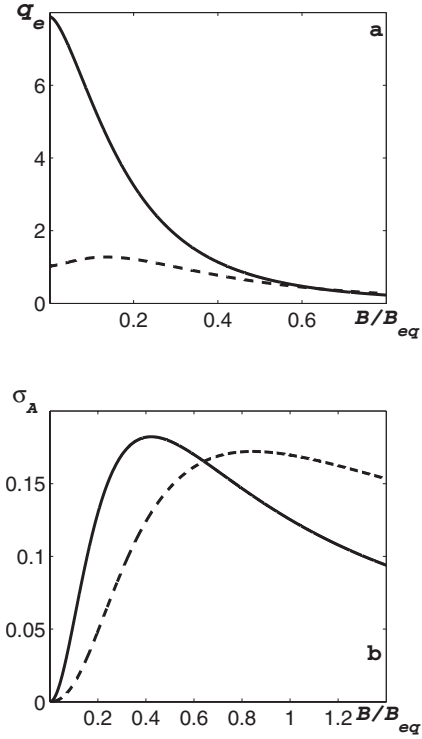


FIG. 5. (a) The nonlinear coefficient  $q_e(B)$  versus the mean magnetic field and (b) the anisotropic mean magnetic tension  $\sigma_A(B) = \sigma_{ij}^A e_{ij} = q_e B^2/B_{\text{eq}}^2$  for a convective turbulence ( $a_* = 1$ ) with the horizontal mean magnetic field (solid line) and vertical mean magnetic field (dashed line).

$$\sigma_y(B) = -\frac{1}{2}\langle b^2 \rangle + \langle b_y^2 \rangle - \rho_0 \langle u_y^2 \rangle, \quad (27)$$

which allow us to determine the coefficients  $q_p(B)$  and  $q_s(B)$  in the effective stress tensor:

$$q_p(B) = \frac{2}{B^2}[\sigma_y(B) - \sigma_y(B=0)], \quad (28)$$

$$q_s(B) = \frac{1}{2}q_p(B) - \frac{1}{B^2}[\sigma_x(B) - \sigma_x(B=0)]. \quad (29)$$

Therefore, Eqs. (26)–(29) allow us to determine the effective (combined) mean magnetic force in the direct numerical simulations.

## V. LARGE-SCALE INSTABILITY

The modification of the mean magnetic force by turbulent convection causes a large-scale instability. In this study we investigate the large-scale instability of continuous magnetic fields in small-scale turbulent convection and we do not consider buoyancy of the discrete magnetic flux tubes. In order to study the large-scale instability in a small-scale turbulent convection we use the equation of motion (with the effective magnetic force  $\mathcal{F}_i^{\text{eff}} = \nabla_j \sigma_{ij}^{\text{eff}}$  determined in Sec. IV), the induction equation, and the equation for the evolution of the mean entropy [see Eqs. (A25)–(A27) in the Appendix]. We

estimate the growth rate  $\gamma$  of this instability and the frequencies  $\omega$  of generated modes, neglecting turbulent dissipative processes for simplicity's sake. We also neglect a very small Brunt-Väisälä frequency based on the gradient of the mean entropy. We seek a solution of these equations in the form  $\propto \exp(\gamma t + i\omega t - i\mathbf{K} \cdot \mathbf{R})$ .

### A. Horizontal mean magnetic field

First, we study the large-scale instability of a horizontal mean magnetic field that is perpendicular to the gravity field. Let the  $z$  axis of a Cartesian coordinate system be directed opposite to the gravitational field, and let the  $x$  axis lie along the mean magnetic field  $\mathbf{B}$ . Consider the case  $K_x=0$ , which corresponds to the interchange mode. The dispersive relation for the instability reads

$$\hat{\sigma}^2 = \left( \frac{K_y C_A}{KL_\rho} \right)^2 (Q + iD), \quad (30)$$

where  $\hat{\sigma} = \gamma + i\omega$ ,  $L_\rho$  is the density stratification length,  $L_B$  is the characteristic scale of the mean magnetic field variations,  $C_A = B/\sqrt{\rho_0}$  is the Alfvén speed,

$$Q = \left\{ 1 - q_p(y) - yq_p'(y) + a_* \left[ y\sigma_A''(y) \left( 1 - 2\frac{L_\rho}{L_B} \right) + \frac{1}{2}\sigma_A'(y) \left( 1 - 4\frac{L_\rho}{L_B} \right) \right] \right\}_{y=B^2} \left( \frac{L_\rho}{L_B} - 1 \right),$$

$$D = a_* K_z L_\rho \left[ \sigma_A'(y) \right]_{y=B^2} \left( \frac{L_\rho}{L_B} - 1 \right),$$

$\sigma_A(y) = q_e(y)y$ ,  $\sigma_A'(y) = d\sigma_A(y)/dy$ ,  $B$  is measured in units of the equipartition field  $B_{\text{eq}} = \sqrt{\rho_0 u_0}$ , and  $K = \sqrt{K_z^2 + K_y^2}$ .

When  $Q \geq 0$  the growth rate of perturbations with frequency

$$\omega = \frac{K_y}{\sqrt{2K}} \left( \frac{C_A}{L_\rho} \right) D (\sqrt{Q^2 + D^2} + Q)^{-1/2} \quad (31)$$

is given by

$$\gamma = \frac{K_y}{\sqrt{2K}} \left( \frac{C_A}{L_\rho} \right) (\sqrt{Q^2 + D^2} + Q)^{1/2}. \quad (32)$$

When  $Q < 0$  the growth rate of perturbations with frequency

$$\omega = -\text{sgn}(D) \frac{K_y}{\sqrt{2K}} \left( \frac{C_A}{L_\rho} \right) (\sqrt{Q^2 + D^2} + |Q|)^{1/2} \quad (33)$$

is given by

$$\gamma = \frac{K_y}{\sqrt{2K}} \left( \frac{C_A}{L_\rho} \right) |D| (\sqrt{Q^2 + D^2} + |Q|)^{-1/2}. \quad (34)$$

Therefore, in small-scale turbulent convection this large-scale instability causes excitation of oscillatory modes with growing amplitude. In a nonconvective turbulence ( $a_*=0$ ) this large-scale instability is aperiodic.

The growth rate  $\gamma$  of the large-scale instability and frequency  $\omega$  of the generated modes for the horizontal mean

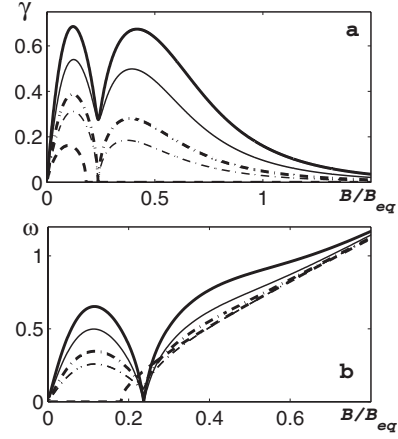


FIG. 6. The growth rate (a) of the large-scale instability and frequency of the generated modes (b) for the horizontal mean magnetic field with  $L_\rho/L_B=0.2$  versus  $B/B_{\text{eq}}$  for different values  $K_z$ :  $K_z L_\rho=3$  (dash-dotted line) and  $K_z L_\rho=5$  (solid line) for turbulent convection with  $a_*=1$  (thick curves) and  $a_*=0.3$  (thin curves) and for a nonconvective turbulence  $a_*=0$  (dashed line). Here  $\epsilon=0$  and  $\text{Rm}=10^6$ .

magnetic field versus  $B/B_{\text{eq}}$  for different values  $K_z$  and  $L_\rho/L_B$  for a nonconvective and convective turbulence are shown in Figs. 6 and 7. Here  $\gamma$  and  $\omega$  are measured in units of  $t_*^{-1}$ , where  $t_* = (L_\rho/u_0)(K/K_y)$ . In turbulent convection there are two ranges for the large-scale instability of the horizontal mean magnetic field (when  $Q > 0$  and  $Q < 0$ ), while in a nonconvective turbulence ( $a_*=0$ ) there is only one range for the instability. The first range for the instability is related to the negative contribution of turbulence to the effective magnetic pressure for the case of  $L_\rho < L_B$ , while the second range is mainly caused by the anisotropic contribution  $\sigma_A(B)$  due to turbulent convection.

In the absence of turbulence (small Reynolds numbers) or turbulent convection (small Rayleigh numbers) the coeffi-

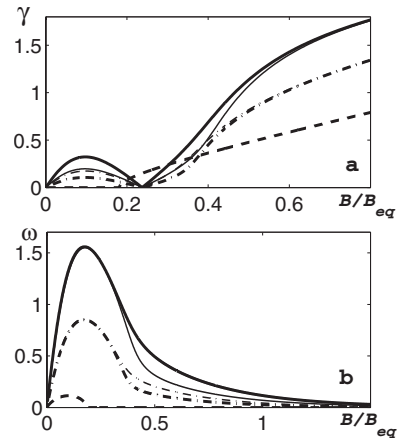


FIG. 7. The growth rate (a) of the large-scale instability and frequency of the generated modes (b) for the horizontal mean magnetic field with  $L_\rho/L_B=10$  versus  $B/B_{\text{eq}}$  for different values  $K_z$ :  $K_z L_\rho=3$  (dash-dotted line) and  $K_z L_\rho=5$  (solid line) for turbulent convection with  $a_*=1$  (thick curves) and  $a_*=0.3$  (thin curves) and for a nonconvective turbulence  $a_*=0$  (dashed line). Here  $\epsilon=0$  and  $\text{Rm}=10^6$ .



coefficients  $Q=L_\rho/L_B-1$ ,  $D=0$ , and the criterion for the large-scale instability,  $L_\rho>L_B$ , coincides with that of the Parker's magnetic buoyancy instability [2,9,10]. In this case,  $\omega=0$ ; i.e., the oscillatory modes with growing amplitude are not excited. On the other hand, in a developed turbulent convection the effective (combined) magnetic pressure becomes negative and the Parker's magnetic buoyancy instability cannot be excited. However, the instability due to the modification of the mean magnetic force by small-scale turbulent convection can be excited even when  $L_\rho<L_B$ —i.e., even in a uniform mean magnetic field.

The instability mechanism due to the modification of the mean magnetic force consists in the following. An isolated tube of magnetic field lines moving upwards turns out to be lighter than the surrounding plasma. This is due to the fact that the decrease of the magnetic field in the isolated tube caused by its expansion is accompanied by an increase of the effective magnetic pressure inside the tube. Since the effective (combined) magnetic pressure is negative, this leads to a decrease of the density inside the tube. The arising buoyant force results in the upwards floating of the isolated tube; i.e., it causes excitation of the large-scale instability (see also discussion in [36]).

### B. Vertical uniform mean magnetic field

Consider a vertical uniform mean magnetic field; i.e., the magnetic field is directed along  $z$  axis. The growth rate of perturbations is given by

$$\gamma = C_A K_z \left[ q_s(y) - 1 - 2y q'_s(y) \frac{K_\perp^2}{K^2} \right]_{y=B^2}^{1/2}, \quad (35)$$

where  $\mathbf{K}_\perp$  is the component of the wave vector that is perpendicular to the  $z$  axis and  $K = \sqrt{K_z^2 + \mathbf{K}_\perp^2}$ . It follows from Eq. (35) that the large-scale instability of the vertical uniform mean magnetic field is caused by modification of the mean magnetic tension by small-scale turbulent convection. When  $q_s > 1$  (i.e., when  $B < 0.2B_{\text{eq}}$ ) the instability occurs for an arbitrary value  $K_\perp$ , while for  $q_s \leq 1$  the necessary condition for the instability reads  $K_\perp > K\sqrt{\chi}$ , where

$$\chi = \left[ \frac{1 - q_s(y)}{2y |q'_s(y)|} \right]_{y=B^2},$$

and we take into account that  $q'_s(y) < 0$ . The growth rate of the instability is reduced by the turbulent dissipation  $\propto \nu_T K^2$ . The maximum growth rate  $\gamma_{\text{max}}$  at a fixed value of the wave number  $K$  (i.e., at a fixed value of the turbulent dissipation) is attained at  $K_z = K_m = K [(1-\chi)/2]^{1/2}$ , and it is given by

$$\gamma_{\text{max}} = K C_A (1 - \chi) \left[ \frac{y |q'_s(y)|}{2} \right]_{y=B^2}^{1/2}, \quad (36)$$

where  $\chi < 1$ . The maximum growth rate  $\gamma_{\text{max}}$  of the instability for the vertical uniform mean magnetic field versus  $B/B_{\text{eq}}$  is plotted in Fig. 8. Here  $\gamma_{\text{max}}$  is measured in units of  $u_0 K$ . The value of  $\gamma_{\text{max}}$  is larger for a nonconvective turbulence, but the range for the instability is wider for the turbulent convection (see Fig. 8).

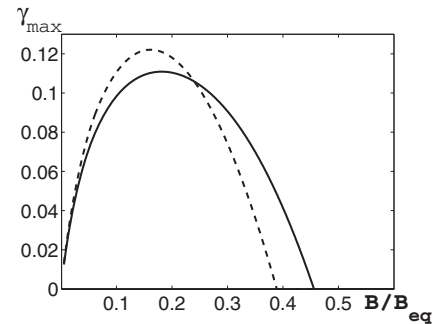


FIG. 8. The maximum growth rate of the large-scale instability for the vertical mean magnetic field versus  $B/B_{\text{eq}}$  for turbulent convection with  $a_* = 1$  (solid line) and for a nonconvective turbulence  $a_* = 0$  (dashed line). Here  $\epsilon = 0$  and  $\text{Rm} = 10^6$ .

## VI. DISCUSSION

In the present study we investigate magnetic fluctuations generated by a tangling of the mean magnetic field in a developed turbulent convection. When the mean magnetic field  $B \ll B_{\text{eq}}/4\text{Rm}^{1/4}$ , the energy of magnetic fluctuations depends on the magnetic Reynolds number. We study the modification of the large-scale magnetic force by turbulent convection. We show that the generation of magnetic fluctuations in a turbulent convection results in a decrease of the total turbulent pressure and may cause the formation of large-scale magnetic structures even in an originally uniform mean magnetic field. This phenomenon is due to a negative contribution of the turbulent convection to the effective mean magnetic force.

The large-scale instability causes the formation of inhomogeneous magnetic structures. The energy for these processes is supplied by the small-scale turbulent convection, and this effect can develop even in an initially uniform magnetic field. In contrast, the Parker's magnetic buoyancy instability is excited when the density stratification scale is larger than the characteristic scale of the mean magnetic field variations (see [2,9,10]). The free energy in the Parker's magnetic buoyancy instability is drawn from the gravitational field. The characteristic time of the large-scale instability is of the order of the Alfvén time based on the large-scale magnetic field.

We study an initial stage of formation of large-scale magnetic structures for horizontal and vertical mean magnetic fields relative to the vertical direction of the gravity field. In turbulent convection there are two ranges for the large-scale instability of the horizontal mean magnetic field. The first range for the instability is related to the negative contribution of turbulence to the effective magnetic pressure for the case of  $L_\rho < L_B$ , while the second range for the instability is mainly caused by the anisotropic contribution of the turbulent convection to the effective magnetic force. The large-scale instability of the vertical uniform mean magnetic field is caused by modification of the mean magnetic tension by small-scale turbulent convection. The discussed effects in the present study might be useful for understanding of the origin of sunspot formation.

Since in the present study we neglect the very small Brunt-Väisälä frequency based on the gradient of the mean

entropy, we do not investigate the large-scale dynamics of the mean entropy. This problem was addressed in [59] whereby the modification of the mean magnetic force by the turbulent convection was not taken into account.

In order to study magnetic fluctuations and the modification of the large-scale Lorentz force by turbulent convection we apply the spectral  $\tau$  approximation (see Sec. III). The  $\tau$  approach is a universal tool in turbulent transport that allows one to obtain closed results and compare them with the results of laboratory experiments, observations, and numerical simulations. The  $\tau$  approximation reproduces many well-known phenomena found by other methods in the turbulent transport of particles and magnetic fields, in turbulent convection, and in stably stratified turbulent flows (see below).

In turbulent transport, the  $\tau$  approximation yields correct formulas for turbulent diffusion, turbulent thermal diffusion, and turbulent barodiffusion (see, e.g., [60,61]). The phenomenon of turbulent thermal diffusion (a nondiffusive streaming of particles in the direction of the mean heat flux) has been predicted using stochastic calculus (the path integral approach) and the  $\tau$  approximation. This phenomenon has been already detected in laboratory experiments in oscillating grid turbulence [62] and in a multiple-fan turbulence generator [63] in stably and unstably stratified fluid flows. The experimental results obtained in [62,62] are in a good agreement with the theoretical studies performed by means of different approaches (see [60,64]).

The  $\tau$  approximation reproduces the well-known  $k^{-7/3}$  spectrum of anisotropic velocity fluctuations in a sheared turbulence (see [65]). This spectrum was found previously in analytical, numerical, and laboratory studies and was observed in atmospheric turbulence (see, e.g., [66]). In the turbulent boundary layer problems, the  $\tau$  approximation yields correct expressions for turbulent viscosity, turbulent thermal conductivity, and the classical heat flux. This approach also describes the counterwind heat flux and the Deardorff's heat flux in convective boundary layers (see [65]). These phenomena have been studied previously using different approaches (see, e.g., [41,42,67]).

The theory of turbulent convection [65] based on the  $\tau$  approximation explains the recently discovered hysteresis phenomenon in laboratory turbulent convection [68]. The results obtained using the  $\tau$  approximation allow us also to explain the most pronounced features of typical semiorganized coherent structures observed in the atmospheric convective boundary layers (“cloud cells” and “cloud streets”) [69]. The theory [65] based on the  $\tau$  approximation predicts realistic values of the following parameters: the aspect ratios of structures, the ratios of the minimum size of semiorganized structures to the maximum scale of turbulent motions, and the characteristic lifetime of semiorganized structures. The theory [65] also predicts excitation of convective-shear waves propagating perpendicular to the convective rolls (“cloud streets”). These waves have been observed in atmospheric convective boundary layers with cloud streets [69].

A theory [70] for stably stratified atmospheric turbulent flows based on the  $\tau$  approximation and the budget equations for the key second moments, turbulent kinetic and potential energies, and vertical turbulent fluxes of momentum and buoyancy is in good agreement with data from atmospheric

and laboratory experiments, direct numerical simulations, and large-eddy simulations (see the detailed comparison in Sec. V of [70]).

A detailed verification of the  $\tau$  approximation in direct numerical simulations of turbulent transport of passive scalars has been recently performed in [46]. In particular, the results on turbulent transport of passive scalars obtained using direct numerical simulations of homogeneous isotropic turbulence have been compared with those obtained using a closure model based on the  $\tau$  approximation. The numerical and analytical results are in good agreement.

In magnetohydrodynamics, the  $\tau$  approximation reproduces many well-known phenomena found by different methods: e.g., the  $\tau$  approximation yields correct formulas for the  $\alpha$  effect [3,28,71,72], the turbulent diamagnetic and paramagnetic velocities [24–28], the turbulent magnetic diffusion [3,25,28,30,73], the  $\mathbf{\Omega} \times \mathbf{J}$  and  $\kappa$  effects [3,28], and the shear-current effect [30,39,74].

Generation of the large-scale magnetic field in a nonhelical turbulence with an imposed mean velocity shear has been recently investigated in [75] using direct numerical simulations. The results of these numerical simulations are in good agreement with the theoretical predictions based on the  $\tau$  approximation (see [30,39,74]) and with the numerical solutions of the nonlinear dynamo equations performed in [76,77] (see the detailed comparison in [39]).

The validity of the  $\tau$  approximation has been tested in the context of dynamo theory in direct numerical simulations in [47]. The  $\alpha$  effect in mean-field dynamo theory becomes proportional to a relaxation time scale multiplied by the difference between kinetic and current helicities. It is shown in [47] that the value of the relaxation time is positive and, in units of the turnover time at the forcing wave number, it is of the order of unity. Kinetic and current helicities are shown in [47] to be dominated by large-scale properties of the flow. Recent studies in [48] of the nonlinear  $\alpha$  effect showed that in the limit of small magnetic and hydrodynamic Reynolds numbers, both the second-order correlation approximation (or first-order smoothing approximation) and the  $\tau$  approximation give identical results. This is also supported by simulations [49] of isotropically forced helical turbulence whereby the contributions to kinetic and magnetic  $\alpha$  effects are computed. The study performed in [49] provides an extra piece of evidence that the  $\tau$  approximation is a usable formalism for describing simulation data and for predicting the behavior in situations that are not yet accessible to direct numerical simulations.

#### ACKNOWLEDGMENTS

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#### APPENDIX: VELOCITY AND MAGNETIC FLUCTUATIONS IN TURBULENT CONVECTION

In order to study the velocity and magnetic fluctuations with a nonzero mean magnetic field and to derive the effec-

tive stress tensor in the turbulent convection, we use a mean-field approach in which the magnetic and velocity fields and entropy are decomposed into the mean and fluctuating parts, where the fluctuating parts have zero mean values. The equations for fluctuations of the fluid velocity, entropy, and the magnetic field are given by

$$\frac{1}{\sqrt{\rho_0}} \frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t} = -\nabla \left( \frac{p}{\rho_0} \right) - \frac{\mathbf{g}}{\sqrt{\rho_0}} s + \frac{1}{\sqrt{\rho_0}} \left[ (\mathbf{b} \cdot \nabla) \mathbf{H} + (\mathbf{H} \cdot \nabla) \mathbf{b} + \frac{\Lambda_p}{2} [2\mathbf{e}(\mathbf{b} \cdot \mathbf{H}) - (\mathbf{b} \cdot \mathbf{e})\mathbf{H}] \right] + \mathbf{v}^N, \quad (\text{A1})$$

$$\frac{\partial \mathbf{b}(\mathbf{x}, t)}{\partial t} = (\mathbf{H} \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{H} + \frac{\Lambda_p}{2} [\mathbf{v}(\mathbf{H} \cdot \mathbf{e}) - \mathbf{H}(\mathbf{v} \cdot \mathbf{e})] + \mathbf{b}^N, \quad (\text{A2})$$

$$\frac{\partial s(\mathbf{x}, t)}{\partial t} = -\frac{\Omega_b^2}{g} (\mathbf{v} \cdot \mathbf{e}) + s^N, \quad (\text{A3})$$

where we used new variables  $(\mathbf{v}, s, \mathbf{H})$  for fluctuating fields  $\mathbf{v} = \sqrt{\rho_0} \mathbf{u}$  and  $s = \sqrt{\rho_0} s'$  and also for the mean field  $\mathbf{H} = \mathbf{B} / \sqrt{\rho_0}$ . Here  $\mathbf{B}$  is the mean magnetic field,  $\rho_0$  is the fluid density,  $\mathbf{e}$  is the vertical unit vector directed opposite to the gravity field,  $\Omega_b^2 = -\mathbf{g} \cdot \nabla S$  is the Brunt-Väisälä frequency,  $S$  is the mean entropy,  $\mathbf{g}$  is the acceleration of gravity,  $\mathbf{u}$ ,  $\mathbf{b}$ , and  $s'$  are fluctuations of velocity, magnetic field, and entropy (we have not used new variables for magnetic fluctuations),  $\mathbf{v}^N$ ,  $\mathbf{b}^N$ , and  $s^N$  are nonlinear terms which include the molecular viscous and diffusion terms,  $p = p' + \sqrt{\rho_0} (\mathbf{H} \cdot \mathbf{b})$  are the fluctuations of total pressure, and  $p'$  are the fluctuations of fluid pressure.

Equations (A1)–(A3) for fluctuations of fluid velocity, entropy, and magnetic field are written in the anelastic approximation, which is a combination of the Boussinesq approximation and the condition  $\text{div}(\rho_0 \mathbf{u}) = 0$ . The equation  $\text{div} \mathbf{u} = \Lambda_p (\mathbf{u} \cdot \mathbf{e})$  in the new variables reads  $\text{div} \mathbf{v} = (\Lambda_p / 2) (\mathbf{v} \cdot \mathbf{e})$ , where  $\nabla \rho_0 / \rho_0 = -\Lambda_p \mathbf{e}$ . The quantities with the subscript 0 correspond to the hydrostatic nearly isentropic basic reference state; i.e.,  $\nabla P_0 = \rho_0 \mathbf{g}$  and  $\mathbf{g} \cdot [(\tilde{\gamma} P_0)^{-1} \nabla P_0 - \rho_0^{-1} \nabla \rho_0] \approx 0$ , where  $\tilde{\gamma}$  is the specific heat ratio and  $P_0$  is the fluid pressure in the basic reference state. The turbulent convection is regarded as a small deviation from a well-mixed adiabatic reference state.

Using Eqs. (A1)–(A3) and performing the procedure described in Sec. III we derive equations for the two-point second-order correlation functions of the velocity fluctuations  $f_{ij} = \langle v_i v_j \rangle$ , the magnetic fluctuations  $h_{ij} = \langle b_i b_j \rangle$ , the entropy fluctuations  $\Theta = \langle s s \rangle$ , the cross-helicity  $g_{ij} = \langle b_i v_j \rangle$ , and the turbulent heat flux  $F_i = \langle s v_i \rangle$  and  $G_i = \langle s b_i \rangle$ . The equations for these correlation functions are given by

$$\frac{\partial f_{ij}(\mathbf{k})}{\partial t} = i(\mathbf{k} \cdot \mathbf{H}) \Phi_{ij} + \hat{\mathcal{N}} f_{ij}, \quad (\text{A4})$$

$$\frac{\partial h_{ij}(\mathbf{k})}{\partial t} = -i(\mathbf{k} \cdot \mathbf{H}) \Phi_{ij} + \hat{\mathcal{N}} h_{ij}, \quad (\text{A5})$$

$$\frac{\partial g_{ij}(\mathbf{k})}{\partial t} = i(\mathbf{k} \cdot \mathbf{H}) [f_{ij}(\mathbf{k}) - h_{ij}(\mathbf{k})] + g e_n P_{jn}(k) G_i(-\mathbf{k}) + \hat{\mathcal{N}} g_{ij}, \quad (\text{A6})$$

$$\frac{\partial F_i(\mathbf{k})}{\partial t} = -i(\mathbf{k} \cdot \mathbf{H}) G_i(\mathbf{k}) + g e_n P_{in}(k) \Theta(\mathbf{k}) + \hat{\mathcal{N}} F_i, \quad (\text{A7})$$

$$\frac{\partial G_i(\mathbf{k})}{\partial t} = -i(\mathbf{k} \cdot \mathbf{H}) F_i(\mathbf{k}) + \hat{\mathcal{N}} G_i, \quad (\text{A8})$$

$$\frac{\partial \Theta(\mathbf{k})}{\partial t} = -\frac{\Omega_b^2}{g} F_z(\mathbf{k}) + \hat{\mathcal{N}} \Theta \quad (\text{A9})$$

(see for details [31]), where  $\Phi_{ij}(\mathbf{k}) = g_{ij}(\mathbf{k}) - g_{ji}(-\mathbf{k})$ ,  $\hat{\mathcal{N}} f_{ij} = g e_n [P_{in}(k) F_j(\mathbf{k}) + P_{jn}(k) F_i(-\mathbf{k})] + \hat{\mathcal{N}} \tilde{f}_{ij}$ , and  $\hat{\mathcal{N}} \tilde{f}_{ij}$ ,  $\hat{\mathcal{N}} h_{ij}$ ,  $\hat{\mathcal{N}} g_{ij}$ ,  $\hat{\mathcal{N}} F_i$ ,  $\hat{\mathcal{N}} G_i$ , and  $\hat{\mathcal{N}} \Theta$  are the third-order moment terms appearing due to the nonlinear terms. The terms  $\sim F_i$  in the tensor  $\hat{\mathcal{N}} f_{ij}$  can be considered as a stirring force for the turbulent convection. Note that a stirring force in the Navier-Stokes turbulence is an external parameter.

We split the tensor of magnetic fluctuations into nonhelical,  $h_{ij}$ , and helical,  $h_{ij}^{(H)}$ , parts. The helical part  $h_{ij}^{(H)}$  depends on the magnetic helicity, and it is determined by the dynamic equation which follows from the magnetic helicity conservation arguments. We also split all second-order correlation functions into symmetric and antisymmetric parts with respect to the wave vector  $\mathbf{k}$ —e.g.,  $f_{ij} = f_{ij}^{(s)} + f_{ij}^{(a)}$ , where the tensor  $f_{ij}^{(s)} = [f_{ij}(\mathbf{k}) + f_{ij}(-\mathbf{k})] / 2$  describes the symmetric part of the tensor and  $f_{ij}^{(a)} = [f_{ij}(\mathbf{k}) - f_{ij}(-\mathbf{k})] / 2$  determines the antisymmetric part of the tensor. We use the spectral  $\tau$  approximation [see Eq. (14) in Sec. III]. We assume also that the characteristic time of variation of the mean magnetic field  $\mathbf{B}$  is substantially larger than the correlation time  $\tau(k)$  for all turbulence scales. This allows us to get a stationary solution for the equations for the second-order moments:

$$f_{ij}^{(s)}(\mathbf{k}) \approx \frac{1}{1 + 2\psi} [(1 + \psi) f_{ij}^{(0s)}(\mathbf{k}) + \psi h_{ij}^{(0s)}(\mathbf{k}) - 2\psi \tau g e_n P_{in}(k) F_j^{(s)}(\mathbf{k})], \quad (\text{A10})$$

$$h_{ij}^{(s)}(\mathbf{k}) \approx \frac{1}{1 + 2\psi} [\psi f_{ij}^{(0s)}(\mathbf{k}) + (1 + \psi) h_{ij}^{(0s)}(\mathbf{k}) + \psi \tau g e_n P_{in}(k) F_j^{(s)}(\mathbf{k})], \quad (\text{A11})$$

$$g_{ij}^{(a)}(\mathbf{k}) \approx \frac{i\tau(\mathbf{k} \cdot \mathbf{H})}{1 + 2\psi} [f_{ij}^{(0s)}(\mathbf{k}) - h_{ij}^{(0s)}(\mathbf{k}) + \tau g e_n P_{in}(k) F_j^{(s)}(\mathbf{k})], \quad (\text{A12})$$

$$F_i^{(s)}(\mathbf{k}) \approx \frac{F_i^{(0s)}(\mathbf{k})}{1 + \psi/2}, \quad (\text{A13})$$

$$G_i^{(a)}(\mathbf{k}) \approx -i\tau(\mathbf{k} \cdot \mathbf{H})F_i^{(s)}(\mathbf{k}) \quad (\text{A14})$$

(for details see [31]), where  $\psi(\mathbf{k})=2(\boldsymbol{\tau}\mathbf{k} \cdot \mathbf{H})^2$  and we neglected terms  $\sim O(\Omega_b^2)$ . In Eqs. (A10)–(A14) we neglected also the large-scale spatial derivatives. The correlation functions  $f_{ij}^{(a)}$ ,  $h_{ij}^{(a)}$ ,  $g_{ij}^{(s)}$ ,  $F_i^{(a)}$ , and  $G_i^{(s)}$  vanish because they are proportional to the first-order spatial derivatives. Equations (A10) and (A11) yield

$$\begin{aligned} f_{ij}^{(s)}(\mathbf{k}) + h_{ij}^{(s)}(\mathbf{k}) \\ = f_{ij}^{(0s)}(\mathbf{k}) + h_{ij}^{(0s)}(\mathbf{k}) - \frac{\psi}{1+2\psi} \tau g e_n P_{in}(k) F_j^{(s)}(\mathbf{k}). \end{aligned} \quad (\text{A15})$$

Therefore, when the mean heat flux  $F_i^{(0s)}$  in the background turbulence is zero (i.e., for the nonconvective turbulence), we obtain

$$f_{ij}^{(s)}(\mathbf{k}) + h_{ij}^{(s)}(\mathbf{k}) = f_{ij}^{(0s)}(\mathbf{k}) + h_{ij}^{(0s)}(\mathbf{k}). \quad (\text{A16})$$

This is in agreement with the fact that a uniform mean magnetic field performs no work on the turbulence (without mean heat flux). It can only redistribute the energy between hydrodynamic fluctuations and magnetic fluctuation. A change of the total energy of fluctuations is caused by a nonuniform mean magnetic field. For the integration in  $\mathbf{k}$  space of these second moments we have to specify a model for the background turbulent convection (with zero mean magnetic field,  $\mathbf{B}=0$ ). Here we use the following model of the background turbulent convection [denoted with the superscript (0)]:

$$f_{ij}^{(0)}(\mathbf{k}) = \rho_0 \langle \mathbf{u}^2 \rangle^{(0)} W(k) P_{ij}(\mathbf{k}), \quad (\text{A17})$$

$$h_{ij}^{(0)}(\mathbf{k}) = \langle \mathbf{b}^2 \rangle^{(0)} W(k) P_{ij}(\mathbf{k}), \quad (\text{A18})$$

$$F_i^{(0)}(\mathbf{k}) = 3\rho_0 \langle u_i s' \rangle^{(0)} W(k) e_j P_{ij}(\mathbf{k}), \quad (\text{A19})$$

$$\Theta^{(0)}(\mathbf{k}) = 2\rho_0 \langle s'^2 \rangle^{(0)} W(k), \quad (\text{A20})$$

where  $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ ,  $W(k) = E(k) / 8\pi k^2$ ,  $\tau(k) = 2\tau_0 \bar{\tau}(k)$ ,  $E(k) = -d\bar{\tau}(k)/dk$ ,  $\bar{\tau}(k) = (k/k_0)^{1-q}$ ,  $1 < q < 3$  is the exponent of the kinetic energy spectrum (e.g.,  $q=5/3$  for Kolmogorov spectrum),  $k_0 = 1/l_0$ , and  $\tau_0 = l_0/u_0$ . Note also that  $g_{ij}^{(0)}(\mathbf{k}) = 0$  and  $G_i^{(0)}(\mathbf{k}) = 0$ .

This procedure allows us to study magnetic fluctuations with a nonzero mean magnetic field and to derive the effective stress tensor in the turbulent convection (see Sec. III). In particular, integration in  $\mathbf{k}$  space in Eq. (A11) yields the energy of magnetic fluctuations,

$$\begin{aligned} \langle \mathbf{b}^2 \rangle = \langle \mathbf{b}^2 \rangle^{(0)} + \frac{1}{12} (\langle \mathbf{v}^2 \rangle^{(0)} - \langle \mathbf{b}^2 \rangle^{(0)}) [6 - 3A_1^{(0)}(4B) - A_2^{(0)}(4B)] \\ + \frac{a_*}{6} \langle \mathbf{v}^2 \rangle^{(0)} [2\Psi\{A_1\} + (1 + 3\cos^2\phi)\Psi\{A_2\}], \end{aligned} \quad (\text{A21})$$

where  $\Psi\{X\} = X^{(1)}(2B) - X^{(1)}(4B)$ ,  $\phi$  is the angle between the vertical unit vector  $\mathbf{e}$  and the mean magnetic field  $\mathbf{B}$ , the functions  $A_n^{(0)}(y)$  are given by Eqs. (A33), (A35), and (A36),

and the functions  $A_n^{(1)}(y)$  are given by Eq. (A34). The magnetic stress tensor is given by Eqs. (19), where the anisotropic contribution  $\sigma_{ij}^A$  to the magnetic stress tensor is determined by

$$\begin{aligned} \sigma_{ij}^A = \frac{1}{2} [e_{ij} \Psi\{2C_1^{(1)} + A_1^{(1)} + A_2^{(1)}\} \\ + \cos\phi (e_i \beta_j + e_j \beta_i) \Psi\{2C_3^{(1)} - A_2^{(1)}\}], \end{aligned} \quad (\text{A22})$$

and the nonlinear coefficients  $q_p(B)$  and  $q_s(B)$  are given by

$$\begin{aligned} q_p(B) = \frac{1}{12B^2} [(1-\epsilon)[A_1^{(0)}(0) - A_1^{(0)}(4B) - A_2^{(0)}(4B)] \\ + 2a_* [\Psi\{6C_1^{(1)} - 2A_1^{(1)} - A_2^{(1)}\} \\ + \cos^2\phi \Psi\{6C_3^{(1)} + A_2^{(1)}\}], \end{aligned} \quad (\text{A23})$$

$$\begin{aligned} q_s(B) = -\frac{1}{12B^2} [(1-\epsilon)A_2^{(0)}(4B) \\ + 6a_* [\Psi\{C_3^{(1)}\} + \cos^2\phi \Psi\{C_2^{(1)}\}]]. \end{aligned} \quad (\text{A24})$$

The asymptotic formulas for these coefficients are given by Eqs. (20)–(25).

In order to study the large-scale instability we use the equation of motion, the induction equation, and the equation for the mean entropy:

$$\begin{aligned} \frac{DU_i}{Dt} = -\nabla_i \left( \frac{\tilde{P}_{\text{tot}}}{\rho_0} \right) + \frac{1}{\rho_0} \left[ [1 - q_p(B)] \frac{\mathbf{B}^2}{2} \Lambda_\rho e_i + (\mathbf{B} \cdot \nabla) \right. \\ \left. \times \{ [1 - q_s(B)] B_i \} + \nabla_j [2\rho_0 \nu_T(B) (\partial U)_{ij}] + \nabla_j \sigma_{ij}^A \right] \\ - \mathbf{g} S - \nu_T(B) \Lambda_\rho e_i \text{div } \mathbf{U}, \end{aligned} \quad (\text{A25})$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times [\mathbf{U} \times \mathbf{B} - \eta_T(B) (\nabla \times \mathbf{B})], \quad (\text{A26})$$

$$\frac{DS}{Dt} = -\nabla \cdot \hat{\mathbf{F}}^{(s)}, \quad (\text{A27})$$

where  $\eta_T(B)$  is the turbulent magnetic diffusion,  $D/Dt = \partial/\partial t + (\mathbf{U} \cdot \nabla)$ , and  $\hat{\mathbf{F}}^{(s)} = -\kappa_{ij}^{(T)}(B) \nabla S$  is the turbulent heat flux,  $\kappa_{ij}^{(T)}$  is the tensor for the nonlinear turbulent thermal diffusivity, and

$$\tilde{P}_{\text{tot}} = P_k + [1 - q_p(B)] \frac{\mathbf{B}^2}{2} - \nu_T(B) \rho_0 \text{div } \mathbf{U},$$

$P_k$  is the mean fluid pressure,  $2(\partial U)_{ij} = \nabla_i U_j + \nabla_j U_i$ , and  $\nu_T(B)$  is the turbulent viscosity. The turbulent viscosity  $\nu_T(B)$  in Eq. (A25) is given by

$$\begin{aligned} \nu_T(B) = \nu_T & \left[ 2A_1^{(1)} + (1 + \epsilon)A_2^{(1)} - 2(1 - \epsilon)H\{A_1\} \right. \\ & - \frac{1}{6}[(1 - 29\epsilon)C_1^{(1)} - 4(7 - 8\epsilon)H\{C_1\}] \\ & \left. + (1 - 3\epsilon)G\{C_1\} - 2(1 - \epsilon)Q\{C_1\} \right]_{y=4B}. \end{aligned} \quad (\text{A28})$$

The explicit form of the functions  $H\{X\}$ ,  $G\{X\}$ , and  $Q\{X\}$  is given in [30]. The asymptotic formula for  $\nu_T(B)$  for a weak mean magnetic field,  $B \ll B_{\text{eq}}/4$ , is given by  $\nu_T(B) = \nu_T(1 + 2\epsilon)$ , and for  $B \gg B_{\text{eq}}/4$  it is given by  $\nu_T(B) = (\nu_T/4B)(1 + \epsilon)$ . The turbulent heat flux  $\hat{F}_i^{(s)}(\mathbf{B}) = -\kappa_{ij}^{(T)}(\mathbf{B})\nabla_j S$  and the tensor for the nonlinear turbulent thermal diffusivity,

$$\kappa_{ij}^{(T)}(\mathbf{B}) = \frac{\kappa_*^{(T)}}{4} [(2A_1^{(0)}(2B) + A_2^{(0)}(2B))\delta_{ij} - A_2^{(0)}(2B)\beta_{ij}], \quad (\text{A29})$$

where  $\kappa_*^{(T)} = u_0 l_0/3$  and  $\beta_{ij} = B_i B_j / B^2$ . The asymptotic formula for  $\kappa_{ij}^{(T)}(\mathbf{B})$  for  $B \ll B_{\text{eq}}/2\text{Rm}^{1/4}$  reads

$$\kappa_{ij}^{(T)}(\mathbf{B}) = \frac{\kappa_*^{(T)}}{20} [2(10 - \beta^2 \ln \text{Rm})\delta_{ij} + \beta^2 \ln \text{Rm}\beta_{ij}], \quad (\text{A30})$$

where  $\beta = \sqrt{8}B/B_{\text{eq}}$ . When  $B_{\text{eq}}/2\text{Rm}^{1/4} \ll B \ll B_{\text{eq}}/2$ , the function  $\kappa_{ij}^{(T)}(\mathbf{B})$  is

$$\kappa_{ij}^{(T)}(\mathbf{B}) = \frac{\kappa_*^{(T)}}{5} [(5 - 2\beta^2 |\ln \beta|)\delta_{ij} + \beta^2 |\ln \beta| \beta_{ij}], \quad (\text{A31})$$

and when  $B \gg B_{\text{eq}}/2$ , it is

$$\kappa_{ij}^{(T)}(\mathbf{B}) = \kappa_*^{(T)} \frac{\sqrt{2}\pi}{4\beta} (\delta_{ij} + \beta_{ij}). \quad (\text{A32})$$

In order to integrate in Eqs. (A10)–(A14) over the angles in  $\mathbf{k}$  space we used the following identity:

$$\bar{K}_{ij} = \int \frac{k_{ij} \sin \theta}{1 + a \cos^2 \theta} d\theta d\varphi = \bar{A}_1 \delta_{ij} + \bar{A}_2 \beta_{ij},$$

$$\begin{aligned} \bar{K}_{ijmn} &= \int \frac{k_{ijmn} \sin \theta}{1 + a \cos^2 \theta} d\theta d\varphi \\ &= \bar{C}_1 (\delta_{ij} \delta_{mn} + \delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) \\ &\quad + \bar{C}_2 \beta_{ijmn} + \bar{C}_3 (\delta_{ij} \beta_{mn} + \delta_{im} \beta_{jn} \\ &\quad + \delta_{in} \beta_{jm} + \delta_{jm} \beta_{in} + \delta_{jn} \beta_{im} + \delta_{mn} \beta_{ij}), \end{aligned}$$

$$\bar{H}_{ijmn}(a) = \int \frac{k_{ijmn} \sin \theta}{(1 + a \cos^2 \theta)^2} d\theta d\varphi = \bar{K}_{ijmn}(a) + a \frac{\partial}{\partial a} \bar{K}_{ijmn}(a),$$

$$\bar{G}_{ijmn}(a) = \int \frac{k_{ijmn} \sin \theta}{(1 + a \cos^2 \theta)^3} d\theta d\varphi = \bar{H}_{ijmn}(a) + \frac{a}{2} \frac{\partial}{\partial a} \bar{H}_{ijmn}(a),$$

where  $a = \beta^2 / \bar{\tau}(k)$ , and

$$\bar{A}_1 = \frac{2\pi}{a} \left[ (a+1) \frac{\arctan(\sqrt{a})}{\sqrt{a}} - 1 \right],$$

$$\bar{A}_2 = -\frac{2\pi}{a} \left[ (a+3) \frac{\arctan(\sqrt{a})}{\sqrt{a}} - 3 \right]$$

$$\bar{C}_1 = \frac{\pi}{2a^2} \left[ (a+1)^2 \frac{\arctan(\sqrt{a})}{\sqrt{a}} - \frac{5a}{3} - 1 \right],$$

$$\bar{C}_2 = \bar{A}_2 - 7\bar{A}_1 + 35\bar{C}_1, \quad \bar{C}_3 = \bar{A}_1 - 5\bar{C}_1$$

(for details, see [30]). The functions  $A_n^{(m)}(\beta)$  are given by

$$A_n^{(0)}(\beta) = \frac{3\beta^2}{\pi} \int_{\beta}^{\beta \text{Rm}^{1/4}} \frac{\bar{A}_n(X^2)}{X^3} dX, \quad (\text{A33})$$

$$A_n^{(1)}(\beta) = \frac{3\beta^4}{\pi} \int_{\beta}^{\beta \text{Rm}^{1/4}} \frac{\bar{A}_n(X^2)}{X^5} dX, \quad (\text{A34})$$

and similarly for  $C_n^{(m)}(\beta)$ , where  $X^2 = \beta^2 (k/k_0)^{2/3} = a$ . The explicit form of the functions  $A_n^{(m)}(\beta)$  and  $C_n^{(m)}(\beta)$  for  $m=1,2$  are given in [30], and the functions  $A_1^{(0)}(\beta)$  and  $A_2^{(0)}(\beta)$  are given by

$$\begin{aligned} A_1^{(0)}(\beta) &= \frac{1}{5} \left[ 2 + 2 \frac{\arctan \beta}{\beta^3} (3 + 5\beta^2) - \frac{6}{\beta^2} - \beta^2 \ln \text{Rm} \right. \\ &\quad \left. - 2\beta^2 \ln \left( \frac{1 + \beta^2}{1 + \beta^2 \sqrt{\text{Rm}}} \right) \right], \end{aligned} \quad (\text{A35})$$

$$\begin{aligned} A_2^{(0)}(\beta) &= \frac{2}{5} \left[ 2 - \frac{\arctan \beta}{\beta^3} (9 + 5\beta^2) + \frac{9}{\beta^2} - \beta^2 \ln \text{Rm} \right. \\ &\quad \left. - 2\beta^2 \ln \left( \frac{1 + \beta^2}{1 + \beta^2 \sqrt{\text{Rm}}} \right) \right], \end{aligned} \quad (\text{A36})$$

where  $\beta = \sqrt{8}B/B_{\text{eq}}$ . For  $B \ll B_{\text{eq}}/4\text{Rm}^{1/4}$  these functions are given by

$$A_1^{(0)}(\beta) \sim 2 - \frac{1}{5} \beta^2 \ln \text{Rm},$$

$$A_2^{(0)}(\beta) \sim -\frac{2}{5} \beta^2 \left[ \ln \text{Rm} + \frac{2}{15} \right].$$

For  $B_{\text{eq}}/4\text{Rm}^{1/4} \ll B \ll B_{\text{eq}}/4$  these functions are given by

$$A_1^{(0)}(\beta) \sim 2 + \frac{2}{5} \beta^2 \left[ 2 \ln \beta - \frac{16}{15} + \frac{4}{7} \beta^2 \right],$$

$$A_2^{(0)}(\beta) \sim \frac{2}{5} \beta^2 \left[ 4 \ln \beta - \frac{2}{15} - 3\beta^2 \right],$$

and for  $B \gg B_{\text{eq}}/4$  they are given by

$$A_1^{(0)}(\beta) \sim \frac{\pi}{\beta} - \frac{3}{\beta^2}, \quad A_2^{(0)}(\beta) \sim -\frac{\pi}{\beta} + \frac{6}{\beta^2}.$$

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