

# Electromotive force and large-scale magnetic dynamo in a turbulent flow with a mean shear

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An effect of sheared large-scale motions on a mean electromotive force in a nonrotating turbulent flow of a conducting fluid is studied. It is demonstrated that in a homogeneous divergence-free turbulent flow the  $\alpha$  effect does not exist, however a mean magnetic field can be generated even in a nonrotating turbulence with an imposed mean velocity shear due to a “shear-current” effect. A mean velocity shear results in an anisotropy of turbulent magnetic diffusion. A contribution to the electromotive force related to the symmetric parts of the gradient tensor of the mean magnetic field (the  $\kappa$  effect) is found in nonrotating turbulent flows with a mean shear. The  $\kappa$  effect and turbulent magnetic diffusion reduce the growth rate of the mean magnetic field. It is shown that a mean magnetic field can be generated when the exponent of the energy spectrum of the background turbulence (without the mean velocity shear) is less than 2. The shear-current effect was studied using two different methods: the  $\tau$  approximation (the Orszag third-order closure procedure) and the stochastic calculus (the path integral representation of the solution of the induction equation, Feynman-Kac formula, and Cameron-Martin-Girsanov theorem). Astrophysical applications of the obtained results are discussed.

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## I. INTRODUCTION

Generation of magnetic fields by a turbulent flow of a conducting fluid is a fundamental problem which has a number of applications in solar physics and astrophysics, geophysics and planetary physics (see, e.g., Refs. [1–5]). It is known that small-scale magnetic fluctuations with a zero mean magnetic field can be generated in a homogeneous nonhelical and nonrotating turbulence by a stretch-twist-fold mechanism (see, e.g., Refs. [6–11]). On the other hand, in a homogeneous divergence-free turbulent flow the helicity and the  $\alpha$  effect vanish.

However, the mean magnetic field can be generated in a rotating homogeneous turbulent flow due to the combined action of the  $\mathbf{\Omega} \times \mathbf{J}$  effect and a nonuniform (differential) rotation [12–16], where  $\mathbf{\Omega}$  is the angular velocity and  $\mathbf{J}$  is the mean electric current. The evolution of the mean magnetic field  $\bar{\mathbf{B}}$  is determined by equation

$$\frac{\partial \bar{\mathbf{B}}}{\partial t} = \nabla \times (\bar{\mathbf{U}} \times \bar{\mathbf{B}} + \mathcal{E} - D_m \nabla \times \bar{\mathbf{B}}), \quad (1)$$

where  $\bar{\mathbf{U}}$  is the mean velocity,  $D_m$  is the magnetic diffusion due to the electrical conductivity of fluid,  $\mathcal{E} = \langle \mathbf{u} \times \mathbf{b} \rangle$  is the mean electromotive force. The general form of the mean electromotive force was suggested in Ref. [17] using the symmetry arguments:

$$\begin{aligned} \mathcal{E}_i = & \alpha_{ij} \bar{B}_j - \beta_{ij} (\nabla \times \bar{\mathbf{B}})_j + (\mathbf{V}^{(\text{eff})} \times \bar{\mathbf{B}})_i - [\boldsymbol{\delta} \times (\nabla \times \bar{\mathbf{B}})]_i \\ & - \kappa_{ijk} (\partial \hat{\mathbf{B}})_{jk}, \end{aligned} \quad (2)$$

where  $(\partial \hat{\mathbf{B}})_{ij} = (\nabla_i \bar{B}_j + \nabla_j \bar{B}_i)/2$ ,  $\mathbf{u}$  and  $\mathbf{b}$  are fluctuations of the velocity and magnetic field, respectively, angular brackets denote averaging over an ensemble of turbulent fluctuations, the tensors  $\alpha_{ij}$  and  $\beta_{ij}$  describe the  $\alpha$  effect and turbulent magnetic diffusion, respectively,  $\mathbf{V}^{(\text{eff})}$  is the effective diamagnetic (or paramagnetic) velocity,  $\kappa_{ijk}$  and  $\boldsymbol{\delta}$  describe a nontrivial behavior of the mean magnetic field in an anisotropic turbulence. The  $\mathbf{\Omega} \times \mathbf{J}$  effect, e.g., is associated with the  $\boldsymbol{\delta}$  term in the mean electromotive force.

In the present paper, we suggested a mechanism of generation of a mean magnetic field by a *nonrotating homogeneous* and nonhelical turbulence with an imposed mean velocity shear. This mechanism is associated with a “shear-current” effect determined by the  $\boldsymbol{\delta}$  term in the mean electromotive force. On the other hand, the turbulent magnetic diffusion and the  $\kappa$  effect can reduce the growth rate of the mean magnetic field. The  $\kappa$  effect arises in an anisotropic turbulence caused by the mean velocity shear. Our analysis of the mean-field magnetic dynamo showed that the generation of a mean magnetic field can occur when  $q < 2$ , where  $q$  is the exponent of the energy spectrum of the background turbulence (without a mean velocity shear). In particular, in Kolmogorov background turbulence with  $q = 5/3$  a mean magnetic field can be generated. The shear-current effect was studied using two different methods: the  $\tau$  approximation (the Orszag third-order closure procedure [18], see Sec. IV) and the stochastic calculus (the path integral representation of the solution of the induction equation, Feynman-Kac formula, and Cameron-Martin-Girsanov theorem, see Appendixes A and B). We also calculated the mean electromotive force for an arbitrary weakly inhomogeneous turbulence with an imposed mean velocity shear. The inhomogeneity of turbulence and mean velocity shear cause the  $\alpha$  effect and the effective drift velocity of the mean magnetic field.

The  $\boldsymbol{\delta}$  term in the electromotive force which is responsible for the shear-current effect was also independently found in Ref. [19] in a problem of a screw dynamo using the

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modified second-order correlation approximation. Note also that for homogeneous and nonhelical flows another mechanism for the magnetic dynamo associated with a “negative turbulent magnetic diffusivity” was recently discussed in Refs. [20–22].

This paper is organized as follows. In Sec. II, the general form of the mean electromotive force which includes the shear-current effect is obtained using simple symmetry reasoning, and the mechanism for the shear-current effect is also discussed. In Sec. III, the governing equations for turbulent velocity and magnetic fields are formulated, which then are used for studying an effect of a mean velocity shear on a turbulence (Sec. IV) and on a cross helicity (Sec. V). This allows us to determine the mean electromotive force in a turbulent flow of a conducting fluid with an imposed mean velocity shear (Sec. VI). The implications of the results for the mean electromotive force to the mean-field magnetic dynamo in a nonrotating homogeneous turbulence are performed in Sec. VII. Conclusions and astrophysical applications of the obtained results are discussed in Sec. VIII. In Appendixes A and B the shear-current effect is studied using another approach, i.e., stochastic calculus. In Appendix C, we derived identities used for the derivation of equations for the second moment of the velocity field and the cross-helicity tensor.

## II. THE QUALITATIVE DESCRIPTION

The mean electromotive force can be written in the form

$$\mathcal{E}_i = a_{ij}\bar{B}_j + b_{ijk}\bar{B}_{j,k} + O(\nabla^2\bar{B}_i). \quad (3)$$

Following Ref. [17] we use an identity  $\bar{B}_{j,k} \equiv \nabla_k\bar{B}_j = (\partial\hat{B})_{jk} - \varepsilon_{jkl}(\nabla \times \bar{\mathbf{B}})_l/2$  which allows us to rewrite Eq. (3) for the mean electromotive force in the form of Eq. (2), where  $\varepsilon_{ijk}$  is the fully antisymmetric Levi-Civita tensor, and

$$\alpha_{ij} = (a_{ij} + a_{ji})/2, \quad (4)$$

$$\beta_{ij} = (\varepsilon_{ikp}b_{jkp} + \varepsilon_{jkp}b_{ikp})/4, \quad (5)$$

$$V_k^{(\text{eff})} = \varepsilon_{kji}a_{ij}/2, \quad (6)$$

$$\delta_i = (b_{jji} - b_{jij})/4, \quad (7)$$

$$\kappa_{ijk} = -(b_{ijk} + b_{ikj})/2. \quad (8)$$

In a homogeneous and nonhelical turbulence the tensor  $a_{ij}$  vanishes, which implies that  $\alpha_{ij} = 0$  and  $V_k^{(\text{eff})} = 0$ . Below we consider this case.

The general form of the mean electromotive force in a turbulent flow with a mean velocity shear can be obtained even from simple symmetry reasoning. Indeed, the electromotive force  $\mathcal{E}$  is a true vector, whereas the mean magnetic field  $\bar{\mathbf{B}}$  is a pseudovector. Therefore, the tensor  $b_{ijk}$  is a pseudotensor [see Eq. (3)]. For homogeneous, isotropic, and nonhelical turbulence the tensor  $b_{ijk} = \beta_T \varepsilon_{ijk}$ , where  $\beta_T$  is the turbulent magnetic diffusion coefficient. In a turbulent flow with an imposed mean velocity shear, the tensor  $b_{ijk}$

depends on the true tensor  $\nabla_j\bar{U}_i$ . Note that the tensor  $\nabla_j\bar{U}_i$  can be written as a sum of the symmetric and antisymmetric parts, i.e.,  $\nabla_j\bar{U}_i = (\partial\hat{U})_{ij} - (1/2)\varepsilon_{ijk}\bar{W}_k$ , where  $(\partial\hat{U})_{ij} = (\nabla_i\bar{U}_j + \nabla_j\bar{U}_i)/2$  is the true tensor and  $\bar{\mathbf{W}} = \nabla \times \bar{\mathbf{U}}$  is the mean vorticity (pseudovector). Hereafter we take into account the effect which is linear in  $\nabla_j\bar{U}_i$ . Therefore, the pseudotensor  $b_{ijk}$  has the following general form:

$$b_{ijk} = \beta_T \varepsilon_{ijk} + l_0^2 [a_1 \varepsilon_{ijm} (\partial\hat{U})_{mk} + a_2 \varepsilon_{ikm} (\partial\hat{U})_{mj} + a_3 \varepsilon_{jkm} (\partial\hat{U})_{mi} + a_4 \delta_{ij} \bar{W}_k + a_5 \delta_{ik} \bar{W}_j], \quad (9)$$

where  $a_k$  are the unknown coefficients,  $l_0$  is the maximum scale of turbulent motions, and the term  $\propto \delta_{jk} \bar{W}_i$  vanishes since  $\nabla \cdot \bar{\mathbf{B}} = 0$  [see Eq. (3)]. Using Eqs. (4)–(9) we determine the turbulent coefficients defining the mean electromotive force for a homogeneous and nonhelical turbulence:

$$\beta_{ij} = \beta_T \delta_{ij} - 2\beta_0 l_0^2 (\partial\hat{U})_{ij}, \quad (10)$$

$$\delta = l_0^2 \delta_0 \bar{\mathbf{W}}, \quad (11)$$

$$\kappa_{ijk} = l_0^2 [\kappa_1 \delta_{ij} \bar{W}_k + \kappa_2 \varepsilon_{ijm} (\partial\hat{U})_{mk}], \quad (12)$$

where

$$\beta_0 = (a_1 - a_2 - 2a_3)/4, \quad \delta_0 = (a_4 - a_5)/2, \quad (13)$$

$$\kappa_1 = -(a_4 + a_5), \quad \kappa_2 = -(a_1 + a_2), \quad (14)$$

and  $\beta_T = u_0 l_0/3$  is the coefficient of isotropic part of turbulent magnetic diffusion, while the second term in Eq. (10) determines anisotropic part of turbulent magnetic diffusion caused by the mean velocity shear. Here  $u_0$  is the characteristic turbulent velocity in the maximum scale of turbulent motions. The  $\kappa$  effect (determined by the tensor  $\kappa_{ijk}$ ) describes a contribution to the electromotive force related to the symmetric parts of the gradient tensor of the mean magnetic field and arises in an anisotropic turbulence caused by the mean velocity shear. Since the tensor  $\kappa_{ijk}$  is multiplied by the symmetric tensor  $(\partial\hat{B})_{jk}$  in the mean electromotive force, this allows us to rewrite the tensor  $\kappa_{ijk}$  determined by Eq. (12) in a more simple but not in a symmetric form. We will show in this paper that the  $\delta$  term in Eqs. (2) and (11) for the mean electromotive force describes the shear-current effect which can cause the mean-field magnetic dynamo in a homogeneous nonrotating turbulence with an imposed mean velocity shear.

Consider a homogeneous divergence-free turbulence with a mean velocity shear, e.g.,  $\bar{\mathbf{U}} = (0, Sx, 0)$  and  $\bar{\mathbf{W}} = (0, 0, S)$ . The mean magnetic field is determined by equation

$$\frac{\partial \bar{\mathbf{B}}}{\partial t} = \nabla \times [\bar{\mathbf{U}} \times \bar{\mathbf{B}} - \hat{\beta}(\nabla \times \bar{\mathbf{B}}) - \delta \times (\nabla \times \bar{\mathbf{B}}) - \hat{\kappa}(\partial\hat{B})], \quad (15)$$

where  $\hat{\beta} \equiv \beta_{ij}$  and  $\hat{\kappa} \equiv \kappa_{ijk}$ . Now for simplicity we use the mean magnetic field in the form  $\bar{\mathbf{B}} = (\bar{B}_x(z), \bar{B}_y(z), 0)$ . Then Eq. (15) reads

$$\frac{\partial \bar{B}_x}{\partial t} = -Sl_0^2 \sigma_0 \bar{B}_y'' + \beta_T \bar{B}_x'', \quad (16)$$

$$\frac{\partial \bar{B}_y}{\partial t} = S\bar{B}_x + \beta_T \bar{B}_y'', \quad (17)$$

where

$$\sigma_0 = \delta_0 - \beta_0 - \kappa_1/2 - \kappa_2/4, \quad (18)$$

and  $\bar{B}'' = \partial^2 \bar{B} / \partial z^2$ . In Eq. (17) we took into account that  $l_0^2 \bar{B}_x'' \ll \bar{B}_x$ , i.e., the characteristic spatial scale  $L_B$  of the mean magnetic field variations is much larger than the maximum scale of turbulent motions  $l_0$ . This assumption corresponds to the mean-field approach. The first term ( $\propto S\bar{B}_x$ ) on the right hand side of Eq. (17) plays a role of the differential rotation. Indeed,  $\nabla \times (\bar{\mathbf{U}} \times \bar{\mathbf{B}}) = (\bar{\mathbf{B}} \cdot \nabla) \bar{\mathbf{U}} - (\bar{\mathbf{U}} \cdot \nabla) \bar{\mathbf{B}} = S\bar{B}_x \mathbf{e}_y$ , and for the chosen configuration of the mean magnetic field,  $(\bar{\mathbf{U}} \cdot \nabla) \bar{\mathbf{B}} = 0$ .

We seek for a solution of Eqs. (16) and (17) in the form  $\propto \exp(\gamma t + iKz)$ , where  $\gamma$  is given by

$$\gamma = Sl_0 K \sqrt{\sigma_0 - \beta_T K^2}, \quad (19)$$

where  $\sigma_0 = (a_2 + a_3 + 2a_4)/2$ . The magnetic dynamo instability can be excited when  $\sigma_0 > 0$ . In this paper we will find unknown coefficients  $a_k$ , which will allow us to determine the conditions for the generation of the mean magnetic field due to the magnetic dynamo instability caused by the shear-current effect.

In order to elucidate the physics of the shear-current effect, let us compare the  $\alpha$  term in the electromotive force which is responsible for the generation of the mean magnetic field, i.e.,

$$\mathcal{E}_i^\alpha \equiv \alpha \bar{B}_i \alpha - (\boldsymbol{\Omega} \cdot \boldsymbol{\Lambda}) \bar{B}_i \quad (20)$$

(see, e.g., Refs. [3,16]), with the  $\delta$  term in the electromotive force caused by the shear-current effect, i.e.,

$$\mathcal{E}_i^\delta \equiv -(\boldsymbol{\delta} \times (\nabla \times \bar{\mathbf{B}}))_i \alpha - (\bar{\mathbf{W}} \cdot \nabla) \bar{B}_i, \quad (21)$$

where  $\boldsymbol{\Lambda} = \nabla \langle u^2 \rangle / \langle u^2 \rangle$  determines the inhomogeneity of turbulence. Here for simplicity we considered an isotropic  $\alpha$  tensor, i.e.,  $\alpha_{ij} = \alpha \delta_{ij}$ . There is an analogy between the  $\alpha$  term and the  $\delta$  term in the electromotive force. In particular, the mean vorticity  $\bar{\mathbf{W}}$  plays a role of rotation  $\boldsymbol{\Omega}$  and an inhomogeneity of the mean magnetic field plays a role of the inhomogeneity of turbulence in the  $\alpha \boldsymbol{\Omega}$  dynamo (see below). During the generation of the mean magnetic field in both cases, the mean electric current along the original mean magnetic field arises. The  $\alpha$  effect is related to the hydrodynamic helicity  $\propto (\boldsymbol{\Omega} \cdot \boldsymbol{\Lambda})$  in an inhomogeneous turbulence. The deformation of the magnetic field lines is caused by upward

and downward rotating turbulent eddies. Since the turbulence is inhomogeneous (which breaks a symmetry between the upward and downward eddies), their total effect on the mean magnetic field does not vanish and it creates the mean electric current along the original mean magnetic field.

In a turbulent flow with an imposed mean velocity shear, the inhomogeneity of the original mean magnetic field breaks a symmetry between the influence of upward and downward turbulent eddies on the mean magnetic field. The deformation of the magnetic field lines is caused by upward and downward turbulent eddies which causes the mean electric current along the mean magnetic field and produces the magnetic dynamo. The magnetic dynamo instability due to the shear-current effect is determined by a system of Eqs. (16) and (17), and there is a coupling between the components of the mean magnetic field. In particular, the field  $\bar{B}_y$  generates the field  $\bar{B}_x$  due to the shear-current effect [see Eq. (16)]. This is similar to the  $\alpha$  effect. On the other hand, the field  $\bar{B}_x$  generates the field  $\bar{B}_y$  due to the pure shear effect [see Eq. (17)], such as the differential rotation in  $\alpha \boldsymbol{\Omega}$  dynamo.

In the following sections, we will describe the above magnetic dynamo effect quantitatively using two different methods: the  $\tau$  approximation (the Orszag third-order closure procedure [18]) and the stochastic calculus (the path integral representation of the solution of the induction equation, Feynman-Kac formula, and Cameron-Martin-Girsanov theorem).

### III. THE GOVERNING EQUATIONS

Our goal is to study an effect of sheared large-scale motions on a mean electromotive force in nonrotating turbulent flows of a conducting fluid. The momentum equation for the fluid velocity  $\mathbf{v}$  and the induction equation for the magnetic field  $\mathbf{h}$  read

$$\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\frac{\nabla P}{\rho} + \mathbf{F}_m(\mathbf{h}) + \nu \Delta \mathbf{v} + \mathbf{F}^{(st)}, \quad (22)$$

$$\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{h} = (\mathbf{h} \cdot \nabla) \mathbf{v} + D_m \Delta \mathbf{h}, \quad (23)$$

where  $\nabla \cdot \mathbf{v} = 0$  and  $\nabla \cdot \mathbf{h} = 0$ ,  $\nu$  is the kinematic viscosity,  $\mathbf{F}_m(\mathbf{h}) = -(1/\mu\rho)[\mathbf{h} \times (\nabla \times \mathbf{h})]$  is the magnetic force,  $\mu$  is the magnetic permeability of the fluid,  $\mathbf{F}^{(st)}$  is the random external stirring force,  $P$  and  $\rho$  are the pressure and density of fluid, respectively.

We will use a mean-field approach whereby the velocity, pressure, and magnetic field are separated into the mean and fluctuating parts:  $\mathbf{v} = \bar{\mathbf{U}} + \mathbf{u}$ ,  $P = \bar{P} + p$ , and  $\mathbf{h} = \bar{\mathbf{B}} + \mathbf{b}$ , the fluctuating parts have zero mean values, and  $\bar{\mathbf{U}} = \langle \mathbf{v} \rangle$ ,  $\bar{P} = \langle P \rangle$ ,  $\bar{\mathbf{B}} = \langle \mathbf{h} \rangle$ . Averaging Eqs. (22) and (23) over an ensemble of fluctuations we obtain the mean-field equations. In particular, the evolution of the mean magnetic field  $\bar{\mathbf{B}}$  is determined by Eq. (1), where  $\boldsymbol{\mathcal{E}} = \langle \mathbf{u} \times \mathbf{b} \rangle$  is the mean electromotive force. To determine the mean electromotive force we use equations for fluctuations  $\mathbf{u}(t, \mathbf{r})$  and  $\mathbf{b}(t, \mathbf{r})$  which are

obtained by subtracting equations for the mean fields from the corresponding equations (22) and (23) for the total fields:

$$\frac{\partial \mathbf{u}}{\partial t} = -(\bar{\mathbf{U}} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \bar{\mathbf{U}} - \frac{\nabla p}{\rho} + \mathbf{F}_m(\mathbf{b}, \bar{\mathbf{B}}) + \mathbf{F}^{(\text{st})} + \mathbf{U}_N, \quad (24)$$

$$\frac{\partial \mathbf{b}}{\partial t} = -(\bar{\mathbf{U}} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \bar{\mathbf{U}} - (\mathbf{u} \cdot \nabla) \bar{\mathbf{B}} + (\bar{\mathbf{B}} \cdot \nabla) \mathbf{u} + \mathbf{B}_N, \quad (25)$$

where

$$\mathbf{F}_m(\mathbf{b}, \bar{\mathbf{B}}) = -\frac{1}{\mu\rho} [\mathbf{b} \times (\nabla \times \bar{\mathbf{B}}) + \bar{\mathbf{B}} \times (\nabla \times \mathbf{b})],$$

$$\mathbf{U}_N = \langle (\mathbf{u} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{u} + [ \langle \mathbf{b} \times (\nabla \times \mathbf{b}) - \mathbf{b} (\nabla \times \mathbf{b}) \rangle ] / (\mu\rho) + \nu \Delta \mathbf{u},$$

$$\mathbf{B}_N = \nabla \times (\mathbf{u} \times \mathbf{b} - \langle \mathbf{u} \times \mathbf{b} \rangle) + D_m \Delta \mathbf{b}.$$

We consider a turbulent flow with large hydrodynamic ( $Re = l_0 u_0 / \nu \gg 1$ ) and magnetic ( $Rm = l_0 u_0 / D_m \gg 1$ ) Reynolds numbers, where  $u_0$  is the characteristic velocity in the maximum scale  $l_0$  of turbulent motions. In the following sections, we will use Eqs. (24) and (25) to study an effect of a mean velocity shear on a turbulence (Sec. IV) and on a cross helicity (Sec. V) in order to determine the mean electromotive force.

#### IV. EFFECT OF A MEAN VELOCITY SHEAR ON A TURBULENCE

In this section, we study an effect of a mean velocity shear on a turbulence using Eq. (24). We neglect an effect of the mean magnetic field on the turbulence, i.e., we neglect the magnetic force  $\mathbf{F}_m(\mathbf{b}, \bar{\mathbf{B}})$  in Eq. (24). This is valid when  $\bar{B}^2 / \mu \ll \rho \langle u^2 \rangle / 2$ , i.e., we do not consider the quenching effects (see, e.g., Ref. [23–28]). We use a two-scale approach, i.e., a correlation function is written as follows:

$$\begin{aligned} \langle u_i(\mathbf{x}) u_j(\mathbf{y}) \rangle &= \int \langle u_i(\mathbf{k}_1) u_j(\mathbf{k}_2) \rangle \\ &\quad \times \exp[i(\mathbf{k}_1 \cdot \mathbf{x} + \mathbf{k}_2 \cdot \mathbf{y})] d\mathbf{k}_1 d\mathbf{k}_2 \\ &= \int f_{ij}(\mathbf{k}, \mathbf{R}) \exp(i\mathbf{k} \cdot \mathbf{r}) d\mathbf{k}, \end{aligned}$$

$$f_{ij}(\mathbf{k}, \mathbf{R}) = \int \langle u_i(\mathbf{k} + \mathbf{K}/2) u_j(-\mathbf{k} + \mathbf{K}/2) \rangle \exp(i\mathbf{K} \cdot \mathbf{R}) d\mathbf{K}$$

(see, e.g., Refs. [29,30]), where  $\mathbf{R}$  and  $\mathbf{K}$  correspond to the large scales, and  $\mathbf{r}$  and  $\mathbf{k}$  to the small scales, i.e.,  $\mathbf{R} = (\mathbf{x} + \mathbf{y})/2$ ,  $\mathbf{r} = \mathbf{x} - \mathbf{y}$ ,  $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2$ ,  $\mathbf{k} = (\mathbf{k}_1 - \mathbf{k}_2)/2$ . We assume that there exists a separation of scales, i.e., the maximum scale of turbulent motions  $l_0$  is much smaller than the characteristic scales of inhomogeneities of the mean fields.

Now we calculate

$$\begin{aligned} \frac{\partial f_{ij}(\mathbf{k}_1, \mathbf{k}_2)}{\partial t} &\equiv \left\langle P_{in}(\mathbf{k}_1) \frac{\partial u_n(\mathbf{k}_1)}{\partial t} u_j(\mathbf{k}_2) \right\rangle \\ &\quad + \left\langle u_i(\mathbf{k}_1) P_{jn}(\mathbf{k}_2) \frac{\partial u_n(\mathbf{k}_2)}{\partial t} \right\rangle, \quad (26) \end{aligned}$$

where we multiplied equation of motion (24) rewritten in  $\mathbf{k}$  space by  $P_{ij}(\mathbf{k}) = \delta_{ij} - k_{ij}$  in order to exclude the pressure term from the equation of motion,  $\delta_{ij}$  is the Kronecker tensor, and  $k_{ij} = k_i k_j / k^2$ . Thus, the equation for  $f_{ij}(\mathbf{k}, \mathbf{R})$  is given by

$$\frac{\partial f_{ij}}{\partial t} = \hat{I}_{ijmn}(\bar{\mathbf{U}}) f_{mn} + F_{ij} + f_{ij}^{(N)}, \quad (27)$$

where

$$\begin{aligned} \hat{I}_{ijmn}(\bar{\mathbf{U}}) &= \left( 2k_{iq} \delta_{mp} \delta_{jn} + 2k_{jq} \delta_{im} \delta_{pn} - \delta_{im} \delta_{jq} \delta_{np} \right. \\ &\quad \left. - \delta_{iq} \delta_{jn} \delta_{mp} + \delta_{im} \delta_{jn} k_q \frac{\partial}{\partial k_p} \right) \nabla_p \bar{U}_q, \quad (28) \end{aligned}$$

and  $f_{ij}^{(N)}(\mathbf{k}, \mathbf{R})$  is the third moment appearing due to the nonlinear term,  $\nabla = \partial / \partial \mathbf{R}$ ,

$$F_{ij}(\mathbf{k}, \mathbf{R}) = \langle \tilde{F}_i(\mathbf{k}, \mathbf{R}) u_j(-\mathbf{k}, \mathbf{R}) \rangle + \langle u_i(\mathbf{k}, \mathbf{R}) \tilde{F}_j(-\mathbf{k}, \mathbf{R}) \rangle,$$

and

$$\tilde{\mathbf{F}}(\mathbf{k}, \mathbf{R}, t) = -\mathbf{k} \times [\mathbf{k} \times \mathbf{F}^{(\text{st})}(\mathbf{k}, \mathbf{R})] / k^2.$$

Equation (27) is written in a frame moving with a local velocity  $\bar{\mathbf{U}}$  of the mean flows. In Eqs. (27) and (28) we neglected small terms which are of the order of  $O(|\nabla^3 \bar{\mathbf{U}}|)$ . Note that Eqs. (27) and (28) do not contain terms proportional to  $O(|\nabla^2 \bar{\mathbf{U}}|)$ . To get Eqs. (27) and (28) we used an identity derived in Appendix C.

Equation (27) for the second moment  $f_{ij}(\mathbf{k}, \mathbf{R})$  contains the third moment  $f_{ij}^{(N)}(\mathbf{k}, \mathbf{R})$  and a problem of closing the equations for the higher moments arises. Various approximate methods have been proposed for the solution of problems of this type (see, e.g., Refs. [18,31,32]). The simplest procedure is the  $\tau$  approximation (the Orszag third-order closure procedure [18]). For magnetohydrodynamic turbulence this approximation was used in Ref. [33] (see also Refs. [26,34–36]). In the simplest variant, it allows us to express the deviations of the third moment  $f_{ij}^{(N)}(\mathbf{k}, \mathbf{R}) - f_{ij}^{(N0)}(\mathbf{k}, \mathbf{R})$  in terms of that for the second moment  $f_{ij}(\mathbf{k}, \mathbf{R}) - f_{ij}^{(0)}(\mathbf{k}, \mathbf{R})$ :

$$f_{ij}^{(N)} - f_{ij}^{(N0)} = -\frac{f_{ij} - f_{ij}^{(0)}}{\tau(k)}, \quad (29)$$

where the superscript (0) corresponds to the background turbulence (it is a turbulence with zero gradients of the mean fluid velocity,  $\nabla_i \bar{U}_j = 0$ ), and  $\tau(k)$  is the correlation time of the turbulent velocity field. Here we assumed that the time  $\tau(k)$  is independent of gradients of the mean fluid velocity

because in the framework of the mean-field approach we may only consider a weak shear:  $\tau_0 |\nabla \bar{U}| \ll 1$ , where  $\tau_0 = l_0 / u_0$ .

The  $\tau$  approximation is in general similar to eddy damped quasnormal markovian (EDQNM) approximation. However some principal difference exists between these two approaches (see Refs. [18,32]). The EDQNM closures do not relax to equilibrium, and this procedure does not describe properly the motions in the equilibrium state in contrast to the  $\tau$  approximation. Within the EDQNM theory, there is no dynamically determined relaxation time, and no slightly perturbed steady state can be approached [18]. In the  $\tau$  approximation, the relaxation time for small departures from equilibrium is determined by the random motions in the equilibrium state, but not by the departure from equilibrium [18]. As follows from the analysis in Ref. [18] the  $\tau$  approximation describes the relaxation to equilibrium state (the background turbulence) much more accurately than the EDQNM approach.

Note that we applied the  $\tau$  approximation (29) only to study the deviations from the background turbulence which are caused by the spatial derivatives of the mean velocity. The background turbulence is assumed to be known. Here we use the following model of the background nonhelical, isotropic, and weakly inhomogeneous turbulence:

$$f_{ij}^{(0)}(\mathbf{k}, \mathbf{R}) = \frac{1}{8\pi k^2} \left( P_{ij}(\mathbf{k}) + \frac{i}{2k^2} (k_i \nabla_j - k_j \nabla_i) \right) u_0^2 E(k, \mathbf{R}), \quad (30)$$

where  $\tau(k) = 2\tau_0 \bar{\tau}(k)$ ,  $E(k) = -d\bar{\tau}(k)/dk$ ,  $\bar{\tau}(k) = (k/k_0)^{1-q}$ ,  $1 < q < 3$  is the exponent of the kinetic energy spectrum (e.g.,  $q = 5/3$  for the Kolmogorov spectrum),  $k_0 = 1/l_0$ .

We assume that the characteristic time of variation of the second moment  $f_{ij}(\mathbf{k}, \mathbf{R})$  is substantially larger than the correlation time  $\tau(k)$  for all turbulence scales. Thus in a steady state, Eq. (27) reads

$$[\delta_{im} \delta_{jn} - \tau \hat{I}_{ijmn}(\bar{\mathbf{U}})] [f_{mn}(\mathbf{k}, \mathbf{R}) - f_{mn}^{(0)}] = \tau \hat{I}_{ijmn}(\bar{\mathbf{U}}) f_{mn}^{(0)}, \quad (31)$$

where we used Eq. (29). The term  $F_{ij}$  in Eq. (27) determines the background turbulence. The solution of Eq. (31) yields the second moment  $f_{ij}(\mathbf{k}, \mathbf{R})$ :

$$f_{ij}(\mathbf{k}, \mathbf{R}) \approx f_{ij}^{(0)} + \tau \hat{I}_{ijmn}(\bar{\mathbf{U}}) f_{mn}^{(0)}(\mathbf{k}, \mathbf{R}), \quad (32)$$

where we neglected terms which are of the order of  $O(|\tau \nabla \bar{U}|^2) \ll 1$ . The first term in Eq. (32) is independent of the mean velocity shear and it describes the background turbulence. The second term in Eq. (32) determines an effect of the mean velocity shear on turbulence.

## V. EFFECT OF A MEAN VELOCITY SHEAR ON THE CROSS HELICITY

In order to determine the mean electromotive force  $\mathcal{E}_i(\mathbf{r} = \mathbf{0}, \mathbf{R}) \equiv \varepsilon_{ilm} \int g_{mn}(\mathbf{k}, \mathbf{R}) d\mathbf{k}$ , we derive equation for the cross-helicity tensor:

$$g_{ij}(\mathbf{k}, \mathbf{R}) = \int \langle b_i(\mathbf{k} + \mathbf{K}/2) u_j(-\mathbf{k} + \mathbf{K}/2) \rangle \exp(i\mathbf{K} \cdot \mathbf{R}) d\mathbf{K},$$

using Eqs. (24) and (25), i.e., we calculate

$$\frac{\partial g_{ij}(\mathbf{k}_1, \mathbf{k}_2)}{\partial t} \equiv \left\langle b_i(\mathbf{k}_1) P_{jn}(\mathbf{k}_2) \frac{\partial u_n(\mathbf{k}_2)}{\partial t} \right\rangle + \left\langle \frac{\partial b_i(\mathbf{k}_1)}{\partial t} u_j(\mathbf{k}_2) \right\rangle. \quad (33)$$

This yields an equation for  $g_{ij}(\mathbf{k}, \mathbf{R})$ ,

$$\frac{\partial g_{ij}}{\partial t} = \hat{J}_{ijmn}(\bar{\mathbf{U}}) g_{mn} + \hat{M}_{ijmn}(\bar{\mathbf{B}}) f_{mn} + g_{ij}^{(N)}, \quad (34)$$

which describes an effect of a mean velocity shear on the cross helicity, where

$$\begin{aligned} \hat{J}_{ijmn}(\bar{\mathbf{U}}) &= 2k_{jq} \delta_{im} \nabla_n \bar{U}_q - \delta_{im} \nabla_n \bar{U}_j + \delta_{jn} \nabla_m \bar{U}_i \\ &\quad + \delta_{im} \delta_{jn} (\nabla_q \bar{U}_p) k_p \frac{\partial}{\partial k_q}, \end{aligned} \quad (35)$$

$$\begin{aligned} \hat{M}_{ijmn}(\bar{\mathbf{B}}) &= \delta_{im} \delta_{jn} \bar{B}_p \left( ik_p + \frac{1}{2} \nabla_p \right) - \delta_{jn} \bar{B}_{i,m} \\ &\quad - \frac{1}{2} \delta_{im} \delta_{jn} \bar{B}_{p,q} k_p \frac{\partial}{\partial k_q}, \end{aligned} \quad (36)$$

and  $g_{ij}^{(N)}(\mathbf{k}, \mathbf{R})$  is the third moment appearing due to the nonlinear terms,  $\bar{B}_{i,j} = \partial \bar{B}_i / \partial R_j$ . Equation (34) is written in a frame moving with a local velocity  $\bar{\mathbf{U}}$  of the mean flows. To get Eqs. (34)–(36) we used an identity derived in Appendix C.

Now we use the  $\tau$  approximation which allows us to express the third moments  $g_{ij}^{(N)}(\mathbf{k}, \mathbf{R})$  in terms of the second moments  $g_{ij}(\mathbf{k}, \mathbf{R})$ :

$$g_{ij}^{(N)} = -\frac{g_{ij}}{\tilde{\tau}(k)}, \quad (37)$$

where  $\tilde{\tau}(k)$  is the characteristic relaxation time, and we took into account that the cross-helicity tensor  $g_{ij}$  for  $\bar{\mathbf{B}} = \mathbf{0}$  is zero, i.e.,  $g_{ij}(\bar{\mathbf{B}} = \mathbf{0}) = \mathbf{0}$ . Note that we applied the  $\tau$  approximation (37) only to study the deviations from the original turbulence (i.e., the turbulence with  $\bar{\mathbf{B}} = \mathbf{0}$ ). These deviations are caused by a weak mean magnetic field. We considered the case when the original turbulence does not have magnetic fluctuations. Now we assume that the characteristic time of variation of the second moment  $g_{ij}(\mathbf{k}, \mathbf{R})$  is substantially larger than the correlation time  $\tau(k) \sim \tilde{\tau}(k)$  for all turbulence

scales. This allows us to get an equation for the cross-helicity tensor  $g_{ij}(\mathbf{k}, \mathbf{R})$  in a steady state:

$$g_{mn}(\mathbf{k}, \mathbf{R})[\delta_{im}\delta_{jn} - \tau\hat{J}_{ijmn}(\bar{\mathbf{U}})] = \tau\hat{M}_{ijmn}(\bar{\mathbf{B}})f_{mn}, \quad (38)$$

where  $f_{mn}(\mathbf{k}, \mathbf{R})$  in Eq. (38) is determined by Eq. (32). The solution of Eq. (38) yields

$$g_{ij}(\mathbf{k}, \mathbf{R}) = \tau\hat{M}_{ijmn}(\bar{\mathbf{B}})f_{mn}^{(0)} + \tau^2[\hat{M}_{ijmn}(\bar{\mathbf{B}})\hat{I}_{mnlk}(\bar{\mathbf{U}}) + \hat{J}_{ijmn}(\bar{\mathbf{U}})\hat{M}_{mnlk}(\bar{\mathbf{B}})]f_{lk}^{(0)}, \quad (39)$$

where we neglected terms which are of the order of  $O(|\tau\bar{\mathbf{V}}\bar{\mathbf{U}}|^2) \ll 1$ . The first term in Eq. (39) is independent of the mean velocity shear and it describes the isotropic turbulent magnetic diffusion. The second term in Eq. (39) determines an effect of the mean velocity shear on turbulence which causes a modification of the mean electromotive force, i.e., this term describes an indirect modification of the mean electromotive force. The last term in Eq. (39) determines a direct modification of the mean electromotive force by the mean velocity shear.

## VI. THE MEAN ELECTROMOTIVE FORCE

Using Eqs. (32) and (39) we determine the mean electromotive force for an inhomogeneous turbulence with a mean velocity shear:

$$\mathcal{E}_i(\mathbf{r}=\mathbf{0}, \mathbf{R}) \equiv \varepsilon_{imn} \int g_{mn}(\mathbf{k}, \mathbf{R}) d\mathbf{k} = a_{ij}\bar{B}_j + b_{ijk}\bar{B}_{j,k}, \quad (40)$$

where

$$a_{ij} = -\frac{l_0^2}{18} \left\{ 6\Lambda_m \left[ \frac{4q-9}{5} \varepsilon_{imn} (\partial\hat{U})_{nj} + \varepsilon_{ijn} (\partial\hat{U})_{mn} \right] + 3\varepsilon_{ijm} (\bar{\mathbf{W}} \times \boldsymbol{\Lambda})_m + \delta_{ij} (\bar{\mathbf{W}} \cdot \boldsymbol{\Lambda}) - \bar{W}_i \Lambda_j \right\}, \quad (41)$$

$$b_{ijk} = \beta_T \varepsilon_{ijk} - \frac{l_0^2}{45} [2(8q-11)\varepsilon_{ijm} (\partial\hat{U})_{mk} + 18\varepsilon_{ikm} (\partial\hat{U})_{mj} + (4q-17)\delta_{ij}\bar{W}_k + (4q-7)\delta_{ik}\bar{W}_j]. \quad (42)$$

Here  $\boldsymbol{\Lambda} = \nabla \langle u^2 \rangle / \langle u^2 \rangle$  and  $l_0 = \tau_0 u_0$ . Equations (41) and (42) allow us to obtain the turbulent coefficients defining the mean electromotive force:

$$\alpha_{ij} = -\frac{l_0^2}{18} \left( (\bar{\mathbf{W}} \cdot \boldsymbol{\Lambda}) \delta_{ij} - \frac{1}{2} (\bar{W}_i \Lambda_j + \bar{W}_j \Lambda_i) + \frac{3}{5} (4q-9) \times \Lambda_m [\varepsilon_{imn} (\partial\hat{U})_{jn} + \varepsilon_{jmn} (\partial\hat{U})_{in}] \right), \quad (43)$$

$$\mathbf{V}^{(\text{eff})} = \frac{l_0^2}{30} \left( (4q+1) (\partial\hat{U})_{ij} \Lambda_j + \frac{25}{6} (\bar{\mathbf{W}} \times \boldsymbol{\Lambda}) \right), \quad (44)$$

and the tensor of turbulent magnetic diffusion  $\beta_{ij}$ , the  $\delta$  effect, and the tensor  $\kappa_{ijk}$  are determined by Eqs. (10)–(12) with

$$\beta_0 = 2(5-2q)/45, \quad \delta_0 = 1/9, \quad (45)$$

$$\kappa_1 = -8(3-q)/45, \quad \kappa_2 = 4(4q-1)/45, \quad (46)$$

where we used Eqs. (4)–(8). It is seen from Eqs. (43) and (44) that the  $\alpha$  effect described by the tensor  $\alpha_{ij}$  and the effective drift velocity  $\mathbf{V}^{(\text{eff})}$  of the mean magnetic field require an inhomogeneity of turbulence (i.e.,  $\boldsymbol{\Lambda} \neq \mathbf{0}$ ). The  $\kappa$  effect determined by the tensor  $\kappa_{ijk}$  arises in an anisotropic turbulence caused by the mean velocity shear. We will show in the following section that the  $\delta$  term in the equation for the mean electromotive force can cause the mean-field magnetic dynamo in a homogeneous nonrotating turbulence with an imposed mean velocity shear.

## VII. THE MEAN-FIELD MAGNETIC DYNAMO IN A HOMOGENEOUS TURBULENCE WITH A MEAN SHEAR

Consider a homogeneous divergence-free turbulence with a mean velocity shear, e.g.,  $\bar{\mathbf{U}} = (0, Sx, 0)$ . In this case  $\bar{\mathbf{W}} = (0, 0, S)$ ,  $\boldsymbol{\Lambda} = \mathbf{0}$ , the  $\alpha$  effect and the effective drift velocity  $\mathbf{V}^{(\text{eff})}$  of the mean magnetic field vanish. The mean magnetic field is determined by Eq. (15), where  $\boldsymbol{\delta} = \delta_0 l_0^2 \bar{\mathbf{W}}$  describes the shear-current effect and  $\beta_{ij} = \beta_T \delta_{ij} - 2\beta_0 l_0^2 (\partial\hat{U})_{ij}$  corresponds to the turbulent magnetic diffusion with an anisotropic part  $\propto \beta_0$ . The tensor  $\kappa_{ijk} = l_0^2 [\kappa_1 \delta_{ij} \bar{W}_k + \kappa_2 \varepsilon_{ijm} (\partial\hat{U})_{mk}]$  describes a contribution to the electromotive force related to the symmetric parts of the gradient tensor of the mean magnetic field and arises in an anisotropic turbulence caused by the mean velocity shear. Since the tensor  $\kappa_{ijk}$  is multiplied by the symmetric tensor  $(\partial\hat{B})_{jk}$  in the mean electromotive force, this allows us to rewrite the tensor  $\kappa_{ijk}$  in a more simple but not in a symmetric form. For simplicity we use the mean magnetic field in the form  $\bar{\mathbf{B}} = (\bar{B}_x(z), \bar{B}_y(z), 0)$ . Then Eq. (15) reduces to Eqs. (16) and (17), where the parameters  $\delta_0$ ,  $\beta_0$ ,  $\kappa_1$ , and  $\kappa_2$  are determined by Eqs. (45) and (46).

We seek for a solution of Eqs. (16) and (17) in the form  $\propto \exp(\gamma t + iKz)$ . Thus the growth rate of the mean magnetic field is given by

$$\gamma = Sl_0 K \sqrt{\delta_0 - \beta_0 - \kappa_0} - \beta_T K^2, \quad (47)$$

where  $\kappa_0 = (2\kappa_1 + \kappa_2)/4 = (8q-13)/45$ . It follows from Eq. (47) that the shear-current effect ( $\propto \delta_0$ ) causes the generation of the mean magnetic field, whereas the anisotropic ( $\propto \beta_0$ ) and isotropic ( $\propto \beta_T$ ) turbulent magnetic diffusions and the  $\kappa$  effect ( $\propto \kappa_0$ ) reduce the growth rate of the mean magnetic field. Note that the maximum growth rate of the mean magnetic field,  $\gamma_{\text{max}} = S^2 l_0^2 (\delta_0 - \beta_0 - \kappa_0) / 4\beta_T$ , is attained at  $K = K_m = Sl_0 \sqrt{\delta_0 - \beta_0 - \kappa_0} / 2\beta_T$ . Using expressions for  $\delta_0$ ,  $\beta_0$ , and  $\kappa_0$  we rewrite the growth rate of the mean magnetic field in the form

$$\gamma = \frac{2}{3} \left( \frac{2-q}{5} \right)^{1/2} S l_0 K - \beta_T K^2. \quad (48)$$

Therefore, the generation of the mean magnetic field is possible when the exponent  $q$  of the energy spectrum of the background homogeneous turbulence (without imposed mean velocity shear) is less than 2. Thus, in the Kolmogorov background turbulence with  $q=5/3$  the mean magnetic field can be generated due to the shear-current effect. The sufficient condition  $\gamma > 0$  for the dynamo instability reads  $L_B/l_0 > \pi\sqrt{5}/(\tau_0 S\sqrt{2-q})$ , where  $L_B \equiv 2\pi/K$ .

The magnetic dynamo instability due to the shear-current effect is different from the magnetic instability suggested in Ref. [22]. The latter instability is caused by the negative turbulent magnetic diffusivity and is determined by Eq. (4.2) for  $\bar{B}_x$  in Ref. [22]. This equation is decoupled from that for the field  $\bar{B}_y$ , i.e., there is no real coupling between the components of the mean magnetic field  $\bar{B}_x$  and  $\bar{B}_y$ . In contrast to this, the magnetic dynamo instability due to the shear-current effect is determined by a system of equations (16) and (17) for the components  $\bar{B}_x$  and  $\bar{B}_y$ . This implies that there is a coupling between these components of the mean magnetic field. In particular, the field  $\bar{B}_y$  generates the field  $\bar{B}_x$  due to the  $\delta$  term (shear-current effect), see the first term in Eq. (16). This is similar to the  $\alpha$  effect. On the other hand, the field  $\bar{B}_x$  generates the field  $\bar{B}_y$  due to the pure shear effect [see the first term in Eq. (17)], such as the differential rotation in  $\alpha\Omega$  dynamo. In this sense the instability due to the shear-current effect is a pure magnetic dynamo instability.

However, the above mechanism is different from that for  $\alpha\Omega$  dynamo. Indeed, the dynamo mechanism due to the shear-current effect acts even in homogeneous small-scale turbulence, while the  $\alpha$  effect vanishes for homogeneous turbulence. The difference between these magnetic dynamo mechanisms can be seen in the form of the growth rate of the mean magnetic field. Indeed, the generation of the mean magnetic field is caused by a coupling between the shear-current effect [described by the first term in Eq. (16), which is proportional to the second-order spatial derivative of the mean magnetic field] and the pure shear effect [described by the first term in Eq. (17) which is proportional to the mean magnetic field]. Then the first term in the expression for growth rate in Eq. (47) (which is responsible for the generation of the mean magnetic field due to the shear-current effect) is proportional to the wave number  $K$ .

On the other hand, the  $\alpha\Omega$  dynamo is caused by a coupling of the  $\alpha$  effect (the corresponding term in the mean-field equation is proportional to the first-order spatial derivative of the mean magnetic field) and the differential rotation (the corresponding term is proportional to the mean magnetic field). Then the term in the expression for growth rate of the instability (which is responsible for the generation of the mean magnetic field in the  $\alpha\Omega$  dynamo) is proportional to  $K^{1/2}$ .

Note that the properties of the magnetic dynamo caused by the shear-current effect are also different from that for the  $\Omega \times \mathbf{J}$  effect. In particular, the mean magnetic field can be

generated due to the  $\Omega \times \mathbf{J}$  effect for an arbitrary exponent  $q$  of the energy spectrum of the background homogeneous turbulence (see Ref. [16]). The  $\Omega \times \mathbf{J}$  effect is caused by the term  $\delta \times (\nabla \times \bar{\mathbf{B}})$  in the mean electromotive force. In a slow rotating ( $\Omega \tau_0 \ll 1$ ) and homogeneous turbulence  $\delta = -(2/9)l_0^2 \Omega$  (for details, see Ref. [16]). Note also that the  $\Omega \times \mathbf{J}$  effect cannot generate the mean magnetic field without a differential rotation.

## VIII. CONCLUSIONS

In the present paper, we discussed a mechanism of a generation of a mean magnetic field by a nonrotating and non-helical homogeneous turbulence with an imposed mean velocity shear. This mechanism is associated with a shear-current effect. We showed that when the exponent of the energy spectrum of the background turbulence (without the mean velocity shear) is less than 2, a mean magnetic field can be generated. We calculated the mean electromotive force for an arbitrary weakly inhomogeneous turbulence ( $\Lambda l_0 \ll 1$ ) with an imposed mean velocity shear. Inhomogeneity of turbulence and mean velocity shear cause the  $\alpha$  effect and the effective drift velocity of the mean magnetic field. The shear-current effect was studied using two different methods: the  $\tau$  approximation (the Orszag third-order closure procedure) and the stochastic calculus (the path integral representation of the solution of the induction equation, Feynman-Kac formula, and Cameron-Martin-Girsanov theorem, see Appendixes A and B).

The obtained results may be important in astrophysics, e.g., in extragalactic clusters and in interstellar clouds. The extragalactic clusters are nonrotating objects which have a homogeneous turbulence in the center of a extragalactic cluster. Sheared motions are created between interacting clusters. The observed magnetic fields cannot be explained by a small-scale turbulent magnetic dynamo (see, e.g., Ref. [5]). It is plausible to suggest that the shear-current effect can produce a mean magnetic field in the extragalactic clusters. The sheared motions can be also formed between interacting interstellar clouds. The latter can result in a generation of a mean magnetic field.

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## APPENDIX A: INVESTIGATION OF THE SHEAR-CURRENT EFFECT USING STOCHASTIC CALCULUS

In this appendix, we study the shear-current effect using stochastic calculus for a random velocity field with a finite correlation time. In order to derive an equation for the mean magnetic field we use an exact solution of the induction equation (23) in the form of a functional integral for an arbitrary velocity field taking into account a small yet finite magnetic diffusion caused by the electrical conductivity of

fluid. This magnetic diffusion  $D_m$  can be described by a random Brownian motion of a particle. The functional integral implies an averaging over a random Brownian motion of a particle. The form of the exact solution used in the present paper allows us to separate the averaging over both, a random Brownian motion of a particle and a random velocity field. This method yields the solution of the induction equation (23) with an initial condition  $\mathbf{h}(t=s, \mathbf{x}) = \mathbf{h}(s, \mathbf{x})$  in the form

$$h_i(t, \mathbf{x}) = M_{\xi} \{ G_{ij}(t, s, \xi) \exp(\xi^* \cdot \nabla) h_j(s, \mathbf{x}) \} \quad (\text{A1})$$

(see Appendix B and Ref. [11]), where  $\xi^* = \xi - \mathbf{x}$ ,  $G_{ij}(t, s, \xi)$  is determined by equation  $dG_{ij}(t, s, \xi)/ds = N_{ik} G_{kj}(t, s, \xi)$  with the initial condition  $G_{ij}(t=s) = \delta_{ij}$ . Here  $N_{ij} = \partial v_i / \partial x_j$ ,  $M_{\xi} \{ \cdot \}$  denotes the mathematical expectation over the Wiener paths  $\xi = \mathbf{x} - \int_0^{t-s} \mathbf{v}(t-\sigma, \xi) d\sigma + (2D_m)^{1/2} \mathbf{w}(t-s)$ , and the magnetic diffusion  $D_m$  is described by a Wiener process  $\mathbf{w}(t)$ .

Consider a random velocity field with a finite constant renewal time. Assume that in the intervals  $\dots (-\tau, 0]; (0, \tau]; (\tau, 2\tau]; \dots$  the velocity fields are statistically independent and have the same statistics. This implies that the velocity field loses memory at the prescribed instants  $t = k\tau$ , where  $k=0, \pm 1, \pm 2, \dots$ . This velocity field cannot be considered as a stationary velocity field for small times  $\sim \tau$ , however, it behaves like a stationary field for  $t \gg \tau$ .

In Eq. (A1) we specify instants  $t = (m+1)\tau$  and  $s = m\tau$ . Note that the fields  $h_j(m\tau, \mathbf{x})$  and  $G_{ij}((m+1)\tau, m\tau, \xi)$  are statistically independent because the field  $h_j(m\tau, \mathbf{x})$  is determined in the time interval  $(-\infty, m\tau]$ , whereas the function  $G_{ij}((m+1)\tau, m\tau, \xi)$  is defined on the interval  $(m\tau, (m+1)\tau]$ . Due to a renewal, the velocity field as well as its functionals  $h_j(m\tau, \mathbf{x})$  and  $G_{ij}((m+1)\tau, m\tau, \xi)$  in these two time intervals are statistically independent. Averaging Eq. (A1) over the random velocity field yields the equation for the mean magnetic field,

$$\bar{B}_i((m+1)\tau, \mathbf{x}) = M_{\xi} \{ \langle G_{ij}(t, s, \xi) \exp(\xi^* \cdot \nabla) \rangle \} \bar{B}_j(m\tau, \mathbf{x}), \quad (\text{A2})$$

where the operator  $\exp(\xi^* \cdot \nabla)$  is determined by

$$\exp(\xi^* \cdot \nabla) = 1 + \xi^* \cdot \nabla + \frac{1}{2!} (\xi^* \cdot \nabla)^2 + \dots + \frac{1}{m!} (\xi^* \cdot \nabla)^m + \dots, \quad (\text{A3})$$

and the angular brackets  $\langle \cdot \rangle$  denote the ensemble average over the random velocity field. Note that  $|(\xi^* \cdot \nabla) \bar{\mathbf{B}}| / |\bar{\mathbf{B}}| \sim l_0 / L_B \ll 1$ . Thus in the framework of the mean-field approach we can neglect in Eqs. (A2) and (A3) the terms  $\sim O[(\xi^* \cdot \nabla)^3 \bar{\mathbf{B}}]$ . Now we use the identity

$$\bar{B}_i(t + \tau, \mathbf{x}) = \exp\left(\tau \frac{\partial}{\partial t}\right) \bar{B}_i(t, \mathbf{x}), \quad (\text{A4})$$

which follows from the Taylor expansion

$$f(t + \tau) = \sum_{m=1}^{\infty} \left( \tau \frac{\partial}{\partial t} \right)^m f(t) = \exp\left(\tau \frac{\partial}{\partial t}\right) \frac{f(t)}{m!}.$$

Therefore, Eqs. (A2)–(A4) yield

$$\begin{aligned} \exp\left(\tau \frac{\partial}{\partial t}\right) \bar{B}_i(t, \mathbf{x}) &= (\bar{G}_{ij} + \bar{G}_{ij} \bar{\xi}_m \nabla_m + A_{ijm} \nabla_m \\ &\quad + C_{ijmn} \nabla_m \nabla_n) \bar{B}_j \equiv \exp(\tau \hat{L}) \bar{\mathbf{B}}, \end{aligned} \quad (\text{A5})$$

where  $\bar{G}_{ij} = M_{\xi} \{ \langle G_{ij} \rangle \} = \delta_{ij} + \bar{U}_{i,j} \tau + O[(\nabla \bar{U})^2]$ ,  $\bar{\xi}_i = M_{\xi} \{ \langle \xi_i^* \rangle \} = -\bar{U}_i \tau + O[(\nabla \bar{U})^2]$ ,  $\bar{U}_{i,j} = \partial \bar{U}_i / \partial x_j$ ,  $A_{ijm} = M_{\xi} \{ \langle G_{ij} \xi_m^* \rangle \}$ ,  $C_{ijmn} = M_{\xi} \{ \langle G_{ij} \xi_m^* \xi_n^* \rangle \}$ , and we introduced the operator  $\hat{L}$ , which allows us to reduce the integral equation (A2) to a partial differential equation. Indeed, Eq. (A5), which is rewritten in the form

$$\exp\left[\tau \left( \hat{L} - \frac{\partial}{\partial t} \right)\right] \bar{\mathbf{B}} = \bar{\mathbf{B}}, \quad (\text{A6})$$

reduces to

$$\frac{\partial \bar{\mathbf{B}}}{\partial t} = \hat{L} \bar{\mathbf{B}}. \quad (\text{A7})$$

The Taylor expansion of the function  $\exp(\tau \hat{L})$  reads

$$\exp(\tau \hat{L}) = \hat{E} + \tau \hat{L} + (\tau \hat{L})^2 / 2 + \dots, \quad (\text{A8})$$

where  $\hat{E}$  is the unit operator. Thus, Eqs. (A5) and (A8) yield

$$\begin{aligned} \hat{L} \equiv L_{ij} &= \frac{1}{\tau} (\bar{G}_{ij} - \delta_{ij} + \bar{G}_{ij} \bar{\xi}_m \nabla_m + A_{ijm} \nabla_m) + D_{ijmn} \nabla_m \nabla_n \\ &\quad + O(\nabla^3), \end{aligned} \quad (\text{A9})$$

where  $D_{ijmn} = (C_{ijmn} - A_{ikm} A_{kjn}) / 2\tau$ . Now we consider homogeneous and nonhelical background turbulence, then  $A_{ijk} = 0$  and the equation for the mean magnetic field is given by

$$\frac{\partial \bar{B}_i}{\partial t} = [\nabla \times (\bar{\mathbf{U}} \times \bar{\mathbf{B}})]_i + D_{ijmn} \nabla_m \nabla_n \bar{B}_j, \quad (\text{A10})$$

where

$$D_{ijmn} = \frac{1}{2\tau} M_{\xi} \{ \langle G_{ij} \xi_m^* \xi_n^* \rangle \}. \quad (\text{A11})$$

For a turbulent flow with an imposed mean velocity gradient, the turbulence is anisotropic. Let us determine the tensor  $D_{ijmn}$  in this case. Solution of the equation  $dG_{ij}/ds = N_{ik} G_{kj}$  with the initial condition  $G_{ij}(t=s) = \delta_{ij}$  is given by



$$G_{ij}(t+\tau, t) = \delta_{ij} + \int_0^\tau N_{ij}(t_\sigma, \xi) d\sigma \\ + \int_0^\tau N_{ik}(t_s, \xi) ds \int_0^s N_{kj}(t_\sigma, \xi) d\sigma + \dots, \quad (\text{A12})$$

which was solved by iterations, where  $t_\sigma = t + \tau - \sigma$ . Since the velocity field is separated into the mean and fluctuating parts:  $\mathbf{v} = \bar{\mathbf{U}} + \mathbf{u}$ , the tensor  $G_{ij}$  can be presented in the form

$$G_{ij} = g_{ij} + G_{ij}^{(L)}, \quad (\text{A13})$$

$$g_{ij} = \int_0^\tau \frac{\partial u_i(t_\sigma, \xi)}{\partial x_k} g_{kj}(t_\sigma, \xi) d\sigma, \quad (\text{A14})$$

$$G_{ij}^{(L)}(\bar{\mathbf{U}}) = \bar{U}_{i,k} \int_0^\tau g_{kj}(t_\sigma, \xi) d\sigma + \int_0^\tau \frac{\partial u_i(t_\sigma, \xi)}{\partial x_k} G_{kj}^{(L)}(\bar{\mathbf{U}}) d\sigma. \quad (\text{A15})$$

Using Eqs. (A12)–(A15) we obtain

$$G_{ij} = g_{ij} + g_{ik} \bar{U}_{k,p} \int_0^\tau g_{pj}(t_\sigma, \xi) d\sigma + O((\nabla \bar{U})^2). \quad (\text{A16})$$

For the derivation of Eq. (A16) we used the identity

$$(\hat{E} - \hat{X})^{-1} = \hat{E} + \hat{X} + \hat{X}\hat{X} + \hat{X}\hat{X}\hat{X} + \dots, \quad (\text{A17})$$

where  $\hat{X}$  is an arbitrary operator. Similarly, the trajectory  $\xi_i^*$  can be written in the form

$$\xi_i^* = \bar{\xi}_i - \bar{U}_{i,k} \int_0^\tau \bar{\xi}_k(t_\sigma, \xi) d\sigma + O[(\nabla \bar{U})^2], \quad (\text{A18})$$

where  $\bar{\xi}_i = -\int_0^\tau u_i(t_\sigma, \xi) d\sigma + (2D_m)^{1/2} w_i(\tau)$ . Using Eqs. (A16) and (A18) we determine the tensor  $D_{ijmn}$ ,

$$D_{ijmn} = [\langle g_{ij} \bar{\xi}_m \bar{\xi}_n \rangle - (\bar{U}_{m,p} \delta_{nk} + \bar{U}_{n,p} \delta_{mk}) K_{ijkp} + F_{ijmn}] / 2\tau, \quad (\text{A19})$$

where  $F_{ijmn} = \bar{U}_{k,p} \langle g_{ik} \bar{\xi}_m \bar{\xi}_n \int_0^\tau g_{pj} d\sigma \rangle$  and  $K_{ijmn} = \langle g_{ij} \bar{\xi}_m \int_0^\tau \bar{\xi}_n d\sigma \rangle$ . In this section hereafter the angular brackets denote both averagings: the averaging over a random velocity field and the averaging over the Wiener trajectories. Now we determine the tensor  $\langle g_{ij} \bar{\xi}_m \bar{\xi}_n \rangle$ , which describes turbulent magnetic diffusion. We take into account that in a homogeneous and nonhelical turbulence without mean shear ( $\bar{U}_{i,j} = 0$ ), the mean-field equation reads

$$\frac{\partial \bar{B}_i}{\partial t} = -[\nabla \times (\beta_T \nabla \times \bar{\mathbf{B}})]_i = -\beta_T \varepsilon_{imk} \varepsilon_{knj} \nabla_m \nabla_n \bar{B}_j. \quad (\text{A20})$$

In this case  $b_{ijk} = \beta_T \varepsilon_{ijk}$ . Therefore,

$$\langle g_{ij} \bar{\xi}_m \bar{\xi}_n \rangle = -2\tau \beta_T \varepsilon_{imk} \varepsilon_{knj} \equiv 2\tau \beta_T (\delta_{ij} \delta_{mn} - \delta_{in} \delta_{mj}), \quad (\text{A21})$$

where

$$\beta_T = \frac{1}{3} \int_0^\infty \langle u_p(0, \xi) u_p(\sigma, \xi) \rangle d\sigma, \quad (\text{A22})$$

and we used an identity

$$\left\langle \int_0^\tau u_i(\mu, \xi) d\mu \int_0^\tau u_j(\sigma, \xi) d\sigma \right\rangle \\ = 2\tau \int_0^\infty \langle u_i(0, \xi) u_j(\sigma, \xi) \rangle d\sigma. \quad (\text{A23})$$

The integration of Eq. (A21) over  $\tau$  yields the tensor  $K_{ijmn}$ :

$$K_{ijmn} = \tau^2 \beta_T (\delta_{ij} \delta_{mn} - \delta_{in} \delta_{mj}). \quad (\text{A24})$$

Now we construct the tensor  $F_{ijmn}$ . The general form of this tensor reads

$$F_{ijmn} = 2\tau^2 \beta_T (C_1 \bar{U}_{m,j} \delta_{in} + C_2 \bar{U}_{j,m} \delta_{in} + C_3 \bar{U}_{i,j} \delta_{mn} \\ + C_4 \bar{U}_{j,i} \delta_{mn}), \quad (\text{A25})$$

where we took into account that the tensor  $F_{ijmn}$  in Eq. (A10) is multiplied by a tensor  $\nabla_m \nabla_n \bar{B}_j$  which is symmetric with respect to indices  $(m, n)$ . Since  $\nabla \cdot \bar{\mathbf{B}} = 0$ , the tensor  $F_{ijmn}$  does not contain the terms with the tensors  $\delta_{jm}$  and  $\delta_{jn}$ , and  $F_{ijmn}$  satisfies the condition  $F_{ijmn} \nabla_i \nabla_m \nabla_n \bar{B}_j = 0$ . The latter equation yields  $C_1 = -C_3$  and  $C_2 = -C_4$ . Equation (A25) do not have the term  $\propto \varepsilon_{imk} \varepsilon_{jpm} (\partial \hat{U})_{pk}$  because we considered a nonhelical turbulence. This is the reason that Eqs. (42) and (A27) does not contain the term  $\propto \varepsilon_{jkm} (\partial \hat{U})_{mi}$  [cf. with the general form of the tensor  $b_{ijk}$  determined by Eq. (9)]. Therefore, Eqs. (A19), (A21), (A24), and (A25) yield the tensor  $D_{ijmn}$ :

$$D_{ijmn} = \beta_T \{ \delta_{ij} \delta_{mn} + \tau [C_1 (\bar{U}_{m,j} \delta_{in} - \bar{U}_{i,j} \delta_{mn}) \\ + C_2 (\bar{U}_{j,m} \delta_{in} - \bar{U}_{j,i} \delta_{mn}) - (\partial \hat{U})_{mn} \delta_{ij}] \}. \quad (\text{A26})$$

Now we use the identities

$$(\bar{U}_{m,j} \delta_{in} - \bar{U}_{i,j} \delta_{mn}) \nabla_m \nabla_n \bar{B}_j = [\nabla \times (\nabla \times (\bar{\mathbf{B}} \cdot \nabla) \bar{\mathbf{U}})]_i,$$

$$(\bar{U}_{j,m} \delta_{in} - \bar{U}_{j,i} \delta_{mn}) \nabla_m \nabla_n \bar{B}_j = [\nabla \times (\nabla \times (\bar{B}_j \nabla \bar{U}_j))]_i,$$

$$(\partial \hat{U})_{mn} \nabla_m \nabla_n \bar{B}_i = (\nabla \times \mathbf{Q})_i,$$

where  $Q_i = \varepsilon_{ijp} (\partial \hat{U})_{pn} \nabla_n \bar{B}_j$ . The latter identity is derived using the following identity:  $\varepsilon_{ims} \varepsilon_{sjp} (\partial \hat{U})_{pn} = (\partial \hat{U})_{mn} \delta_{ij} - (\partial \hat{U})_{in} \delta_{mj}$ . Note that we neglect the second- and higher-order spatial derivatives of the mean velocity. We also neglected the cross-effect terms which describe an interaction

between molecular and turbulent effects. Thus, Eq. (A26) and the above identities allow us to determine the tensor  $b_{ijk}$ :

$$b_{ijk} = \beta_T \varepsilon_{ijk} + \beta_T \tau [(C_1 + C_2) \varepsilon_{ikm} (\partial \hat{U})_{mj} + \frac{1}{2} (C_2 - C_1) \delta_{ij} \bar{W}_k - \varepsilon_{ijm} (\partial \hat{U})_{mk}]. \quad (\text{A27})$$

Using Eqs. (4)–(8) and (A27) we determine the turbulent coefficients defining the mean electromotive force. They are given by Eqs. (10)–(12) with

$$\beta_0 = -\frac{1}{12} (C_1 + C_2 + 1), \quad \delta_0 = \frac{1}{12} (C_2 - C_1), \quad (\text{A28})$$

$$\kappa_1 = -\frac{1}{6} (C_2 - C_1), \quad \kappa_2 = -\frac{1}{3} (C_1 + C_2 - 1), \quad (\text{A29})$$

where  $l_0 = \sqrt{3\beta_T\tau}$ . For simplicity we use the mean magnetic field in the form  $\bar{\mathbf{B}} = (\bar{B}_x(z), \bar{B}_y(z), 0)$ . Then Eq. (15) reduces to Eqs. (16) and (17), where the parameters  $\delta_0$ ,  $\beta_0$ ,  $\kappa_1$ , and  $\kappa_2$  are determined by Eqs. (A28) and (A29). We seek for solution of Eqs. (16) and (17) in the form  $\propto \exp(\gamma t + iKz)$ . Thus the growth rate of the mean magnetic field due to the shear-current effect is given by

$$\gamma \approx Sl_0 K \sqrt{C_2/3} - \beta_T K^2, \quad (\text{A30})$$

where we used that  $\sigma_0 = C_2/3$ . Therefore, the magnetic dynamo instability can be excited when  $C_2 > 0$ .

This approach does not allow us to take into account the effect of mean velocity shear on turbulence. The method used in this appendix only describes the effect of shear on the cross-helicity tensor. This is one of the reasons that the results obtained by this method are quantitatively different from that obtained by the  $\tau$  approximation. However, the form of the electromotive force and a possibility for the large-scale magnetic dynamo in a homogeneous turbulence due to the shear-current effect are clearly demonstrated by two different approaches.

## APPENDIX B: DERIVATION OF EQ. (A1)

In order to derive Eq. (A1) we use an exact solution of Eq. (23) with an initial condition  $\mathbf{h}(t=s, \mathbf{x}) = \mathbf{h}(s, \mathbf{x})$  in the form of the Feynman-Kac formula:

$$h_i(t, \mathbf{x}) = M_{\xi} \{ G_{ij}(t, s, \xi(t, s)) h_j(s, \xi(t, s)) \}, \quad (\text{B1})$$

where the Wiener paths  $\xi(t, s) = \mathbf{x} - \int_0^{t-s} \mathbf{v}[t - \sigma, \xi(t, \sigma)] d\sigma + (2D_m)^{1/2} \mathbf{w}(t-s)$ . Now we assume that

$$h_i(t, \xi) = \int \exp(i\xi \cdot \mathbf{q}) h_i(s, \mathbf{q}) d\mathbf{q}. \quad (\text{B2})$$

Substituting Eq. (B2) into Eq. (B1) we obtain

$$h_i(s, \mathbf{x}) = \int M_{\xi} \{ G_{ij}(t, s, \xi(t, s)) \exp[i\xi^* \cdot \mathbf{q}] h_j(s, \mathbf{q}) \} \times \exp(i\mathbf{q} \cdot \mathbf{x}) d\mathbf{q}. \quad (\text{B3})$$

In Eq. (B3) we expand the function  $\exp[i\xi^* \cdot \mathbf{q}]$  in Taylor series at  $\mathbf{q} = 0$ , i.e.,  $\exp[i\xi^* \cdot \mathbf{q}] = \sum_{k=0}^{\infty} (1/k!) (i\xi^* \cdot \mathbf{q})^k$ . Using the identity  $(i\mathbf{q})^k \exp[i\mathbf{x} \cdot \mathbf{q}] = \nabla^k \exp[i\mathbf{x} \cdot \mathbf{q}]$  and Eq. (B3) we get

$$h_i(t, \mathbf{x}) = M_{\xi} \left\{ G_{ij}(t, s, \xi) \left[ \sum_{k=0}^{\infty} (1/k!) (\xi^* \cdot \nabla)^k \right] \times \int h_j(s, \mathbf{q}) \exp(i\mathbf{q} \cdot \mathbf{x}) d\mathbf{q} \right\}. \quad (\text{B4})$$

After the inverse Fourier transformation in Eq. (B4) we obtain Eq. (A1). Equation (B2) can be formally considered as an inverse Fourier transformation of the function  $h_i(t, \xi)$ . However,  $\xi$  is the Wiener path which is not a usual spatial variable. Therefore, it is desirable to derive Eq. (A1) by a more rigorous method as it is done below.

To this end we use an exact solution of the Cauchy problem for Eq. (23) with an initial condition  $\mathbf{h}(t=s, \mathbf{x}) = \mathbf{h}(s, \mathbf{x})$  in the form

$$h_i(t, \mathbf{x}) = M_{\xi} \{ J(t, s, \xi) \tilde{G}_{ij}(t, s, \xi) h_j(s, \xi(t, s)) \}, \quad (\text{B5})$$

where the matrix  $\tilde{G}_{ij}$  is determined by the equation  $d\tilde{G}_{ij}(t, s, \xi)/ds = N_{ik} \tilde{G}_{kj}(t, s, \xi)$  with the initial condition  $\tilde{G}_{ij}(t=s) = \delta_{ij}$ , and the function  $J(t, s, \xi)$  is given by

$$J(t, s, \xi) = \exp \left( -\frac{1}{\sqrt{2D_m}} \int_0^{t-s} v_i(t-\eta, \xi(t, \eta)) dw_i(\eta) - \frac{1}{4D_m} \int_0^{t-s} v^2(t-\eta, \xi(t, \eta)) d\eta \right), \quad (\text{B6})$$

$\mathbf{w}(t)$  is a Wiener process, and  $M_{\xi} \{ \cdot \}$  denotes the mathematical expectation over the paths  $\xi(t, s) = \mathbf{x} + (2D_m)^{1/2} [\mathbf{w}(t) - \mathbf{w}(s)]$ . Solution (B5) was first found in Ref. [37]. The first integral  $\int_0^{t-s} \mathbf{v}(t-\eta, \xi(t, \eta)) \cdot d\mathbf{w}(\eta)$  in Eq. (B6) is the Ito stochastic integral (see, e.g., Ref. [38]). As follows from Cameron-Martin-Girsanov theorem the transformation from Eq. (B1) to Eq. (B5) can be considered as a change of variables  $\xi \rightarrow \zeta$  in the path integral (B1) (see, e.g., Ref. [39]).

The difference between the solutions (B1) and (B5) is as follows. The function  $h_j(s, \xi(t, s))$  in Eq. (B1) explicitly depends on the random velocity field  $\mathbf{v}$  via the Wiener path  $\xi$ , while the function  $h_j(s, \zeta(t, s))$  in Eq. (B5) is independent of the velocity  $\mathbf{v}$ . Trajectories in the Feynman-Kac formula (B1) are determined by both, a random velocity field and magnetic diffusion. On the other hand, trajectories in Eq. (B5) are determined only by magnetic diffusion. Due to the Markovian property of the Wiener process, solution (B5) can be rewritten in the form

$$h_i(t, \mathbf{x}) = E \{ S_{ij}(t, s, \mathbf{x}, \mathbf{X}') h_j(s, \mathbf{X}') \} = \int Q_{ij}(t, s, \mathbf{x}, \mathbf{x}') h_j(s, \mathbf{x}') d\mathbf{x}', \quad (\text{B7})$$

(B3) where

$$Q_{ij}(t, s, \mathbf{x}, \mathbf{x}') = [4\pi D_m(t-s)]^{3/2} \exp\left(-\frac{(\mathbf{x}' - \mathbf{x})^2}{4D_m(t-s)}\right) \times S_{ij}(t, s, \mathbf{x}, \mathbf{x}'), \quad (\text{B8})$$

$S_{ij}(t, s, \mathbf{x}, \mathbf{x}') = M_{\boldsymbol{\mu}}\{J(t, s, \boldsymbol{\mu})\tilde{G}_{ij}(t, s, \boldsymbol{\mu})\}$  and  $M_{\boldsymbol{\mu}}\{\cdot\}$  means the path integral taken over the set of trajectories  $\boldsymbol{\mu}$  which connect points  $(t, \mathbf{x})$  and  $(s, \mathbf{x}')$ . The mathematical expectation  $E\{\cdot\}$  in Eq. (B7) denotes the averaging over the set of random points  $\mathbf{X}'$  which have a Gaussian statistics (see, e.g., Ref. [40]). We used here the following property of the averaging over the Wiener process  $E\{M_{\boldsymbol{\mu}}\{\cdot\}\} = M_{\xi}\{\cdot\}$ . We considered a random velocity field with a finite renewal time. Due to a renewal, the velocity field as well as its functionals  $h_j(s, \mathbf{x}')$  and  $Q_{ij}(t, s, \mathbf{x}, \mathbf{x}')$  in the two time intervals are statistically independent. Now we make a change of variables  $(\mathbf{x}, \mathbf{x}') \rightarrow (\mathbf{x}, \mathbf{x}' = \mathbf{z} + \mathbf{x})$  in Eq. (B7), i.e.,  $\tilde{Q}_{ij}(t, s, \mathbf{x}, \mathbf{x}') = \tilde{Q}_{ij}(t, s, \mathbf{x}, \mathbf{z} + \mathbf{x}) = Q_{ij}(t, s, \mathbf{x}, \mathbf{z})$ . The Fourier transformation in Eq. (B7) yields

$$h_i(t, \mathbf{x}) = \int \int Q_{ij}(t, s, \mathbf{x}, \mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{z}) d\mathbf{k} \times \int h_j(s, \mathbf{q}) \exp[i\mathbf{q} \cdot (\mathbf{z} + \mathbf{x})] d\mathbf{q} d\mathbf{z}.$$

Since  $\delta(\mathbf{k} + \mathbf{q}) = (2\pi)^{-3} \int \exp[i(\mathbf{k} + \mathbf{q}) \cdot \mathbf{z}] d\mathbf{z}$ , we obtain that

$$h_i(t, \mathbf{x}) = (2\pi)^3 \int Q_{ij}(t, s, \mathbf{x}, -\mathbf{q}) h_j(s, \mathbf{q}) \exp(i\mathbf{q} \cdot \mathbf{x}) d\mathbf{q}. \quad (\text{B9})$$

In Eq. (B9) the function  $Q_{ij}(t, s, \mathbf{x}, -\mathbf{q})$  is given by

$$Q_{ij}(t, s, \mathbf{x}, -\mathbf{q}) = (2\pi)^{-3} \int Q_{ij}(t, s, \mathbf{x}, \mathbf{z}) \exp(i\mathbf{q} \cdot \mathbf{z}) d\mathbf{z}. \quad (\text{B10})$$

Substituting  $\tilde{Q}_{ij}(t, s, \mathbf{x}, \mathbf{x}') = Q_{ij}(t, s, \mathbf{x}, \mathbf{z})$  in Eq. (B7) and taking into account that  $\mathbf{x}' = \mathbf{z} + \mathbf{x}$  we obtain

$$h_i(t, \mathbf{x}) = \int Q_{ij}(t, s, \mathbf{x}, \mathbf{z}) h_j(s, \mathbf{z} + \mathbf{x}) d\mathbf{z}. \quad (\text{B11})$$

Equation (B10) can be rewritten in the form

$$(2\pi)^3 Q_{ij}(t, s, \mathbf{x}, -\mathbf{q}) \exp(i\mathbf{q} \cdot \mathbf{x}) = \int Q_{ij}(t, s, \mathbf{x}, \mathbf{z}) \exp[i\mathbf{q} \cdot (\mathbf{z} + \mathbf{x})] d\mathbf{z}. \quad (\text{B12})$$

The right hand sides of Eqs. (B11) and (B12) coincide when  $\mathbf{h}(s, \mathbf{z} + \mathbf{x}) = \mathbf{e} \exp[i\mathbf{q} \cdot (\mathbf{z} + \mathbf{x})]$ , where  $\mathbf{e}$  is a unit vector. Thus, a particular solution (B11) of Eq. (23) with the initial condition  $\mathbf{h}(s, \mathbf{x}') = \mathbf{e} \exp(i\mathbf{q} \cdot \mathbf{x}')$  coincides in form with integral (B12). On the other hand, a solution of Eq. (23) is given by Eq. (B5). Substituting the initial condition  $\mathbf{h}(s, \boldsymbol{\zeta}) = \mathbf{e} \exp(i\mathbf{q} \cdot \boldsymbol{\zeta}) = \mathbf{e} \exp(i\mathbf{q} \cdot [\mathbf{x} + (2D_m)^{1/2} \mathbf{w}])$  into Eq. (B5) we obtain

$$h_i(t, \mathbf{x}) = M_{\boldsymbol{\zeta}}\{J(t, s, \boldsymbol{\zeta}) \tilde{G}_{ij}(t, s, \boldsymbol{\zeta}) e_j \exp[i\mathbf{q} \cdot (\mathbf{x} + (2D_m)^{1/2} \mathbf{w})]\}. \quad (\text{B13})$$

Comparing Eqs. (B11)–(B13) we get

$$Q_{ij}(t, s, \mathbf{x}, -\mathbf{q}) = (2\pi)^{-3} M_{\boldsymbol{\zeta}}\{J(t, s, \boldsymbol{\zeta}) \tilde{G}_{ij}(t, s, \boldsymbol{\zeta}) \times \exp[i(2D_m)^{1/2} \mathbf{q} \cdot \mathbf{w}]\}. \quad (\text{B14})$$

Now we rewrite Eq. (B14) using Feynman-Kac formula (B1). The result is given by

$$Q_{ij}(t, s, \mathbf{x}, -\mathbf{q}) = (2\pi)^{-3} M_{\boldsymbol{\xi}}\{G_{ij}(t, s, \boldsymbol{\xi}(t, s)) \exp[i\mathbf{q} \cdot \boldsymbol{\xi}^*]\}, \quad (\text{B15})$$

where  $\boldsymbol{\xi}^* = \boldsymbol{\xi} - \mathbf{x}$ . Substituting Eq. (B15) into Eq. (B9) we obtain

$$h_i(t, \mathbf{x}) = \int M_{\boldsymbol{\xi}}\{G_{ij}(t, s, \boldsymbol{\xi}) \exp[i\mathbf{q} \cdot \boldsymbol{\xi}^*] h_j(s, \mathbf{q})\} \times \exp(i\mathbf{q} \cdot \mathbf{x}) d\mathbf{q}. \quad (\text{B16})$$

The Fourier transformation in Eq. (B16) yields Eq. (A1). The above derivation proves that assumption (B2) is correct for a Wiener path  $\boldsymbol{\xi}$ .

### APPENDIX C: IDENTITY USED FOR DERIVATION OF EQS. (27) AND (34)

For the derivation of Eqs. (27) and (34) we used the following identity:

$$ik_i \int f_{ij}(\mathbf{k} - \frac{1}{2}\mathbf{Q}, \mathbf{K} - \mathbf{Q}) \bar{U}_p(\mathbf{Q}) \exp(i\mathbf{K} \cdot \mathbf{R}) d\mathbf{K} d\mathbf{Q} = -\frac{1}{2} \bar{U}_p \nabla_i f_{ij} + \frac{1}{2} f_{ij} \nabla_i \bar{U}_p - \frac{i}{4} (\nabla_s \bar{U}_p) \left( \nabla_i \frac{\partial f_{ij}}{\partial k_s} \right) + \frac{i}{4} \left( \frac{\partial f_{ij}}{\partial k_s} \right) (\nabla_s \nabla_i \bar{U}_p). \quad (\text{C1})$$

To derive Eq. (C1) we multiply the equation  $\nabla \cdot \mathbf{u} = 0$  [written in  $\mathbf{k}$  space for  $u_i(\mathbf{k}_1 - \mathbf{Q})$ ] by  $u_j(\mathbf{k}_2) \bar{U}_p(\mathbf{Q}) \exp(i\mathbf{K} \cdot \mathbf{R})$ , integrate over  $\mathbf{K}$  and  $\mathbf{Q}$ , and average over the ensemble of velocity fluctuations. Here  $\mathbf{k}_1 = \mathbf{k} + \mathbf{K}/2$  and  $\mathbf{k}_2 = -\mathbf{k} + \mathbf{K}/2$ . This yields

$$\int i \left( k_i + \frac{1}{2} K_i - Q_i \right) \left\langle u_i \left( \mathbf{k} + \frac{1}{2} \mathbf{K} - \mathbf{Q} \right) u_j \left( -\mathbf{k} + \frac{1}{2} \mathbf{K} \right) \right\rangle \times \bar{U}_p(\mathbf{Q}) \exp(i\mathbf{K} \cdot \mathbf{R}) d\mathbf{K} d\mathbf{Q} = 0. \quad (\text{C2})$$

Next, we introduce new variables:  $\tilde{\mathbf{k}}_1 = \mathbf{k} + \mathbf{K}/2 - \mathbf{Q}$ ,  $\tilde{\mathbf{k}}_2 = -\mathbf{k} + \mathbf{K}/2$  and  $\tilde{\mathbf{k}} = (\tilde{\mathbf{k}}_1 - \tilde{\mathbf{k}}_2)/2 = \mathbf{k} - \mathbf{Q}/2$ ,  $\tilde{\mathbf{K}} = \tilde{\mathbf{k}}_1 + \tilde{\mathbf{k}}_2 = \mathbf{K} - \mathbf{Q}$ . This allows us to rewrite Eq. (C2) in the form

$$\int i \left( k_i + \frac{1}{2} K_i - Q_i \right) f_{ij} \left( \mathbf{k} - \frac{1}{2} \mathbf{Q}, \mathbf{K} - \mathbf{Q} \right) \bar{U}_p(\mathbf{Q}) \times \exp(i\mathbf{K} \cdot \mathbf{R}) d\mathbf{K} d\mathbf{Q} = 0. \quad (\text{C3})$$

Since  $|\mathbf{Q}| \ll |\mathbf{k}|$  we use the Taylor expansion

$$f_{ij}(\mathbf{k}-\mathbf{Q}/2, \mathbf{K}-\mathbf{Q}) \approx f_{ij}(\mathbf{k}, \mathbf{K}-\mathbf{Q}) - \frac{1}{2} \frac{\partial f_{ij}(\mathbf{k}, \mathbf{K}-\mathbf{Q})}{\partial k_s} Q_s + O(Q^2). \quad (\text{C4})$$

We also use the following identities:

$$\begin{aligned} [f_{ij}(\mathbf{k}, \mathbf{R}) \bar{U}_p(\mathbf{R})]_{\mathbf{K}} &= \int f_{ij}(\mathbf{k}, \mathbf{K}-\mathbf{Q}) \bar{U}_p(\mathbf{Q}) d\mathbf{Q}, \\ \nabla_p [f_{ij}(\mathbf{k}, \mathbf{R}) \bar{U}_p(\mathbf{R})] &= \int iK_p [f_{ij}(\mathbf{k}, \mathbf{R}) \bar{U}_p(\mathbf{R})]_{\mathbf{K}} \exp(i\mathbf{K} \cdot \mathbf{R}) d\mathbf{K}. \end{aligned} \quad (\text{C5})$$

Therefore, Eqs. (C3)–(C5) yield Eq. (C1).

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