Magnetic fluctuations with a zero mean field in a random fluid flow with a finite correlation time and a small magnetic diffusion

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Magnetic fluctuations with a zero mean field in a random flow with a finite correlation time and a small yet finite magnetic diffusion are studied. Equation for the second-order correlation function of a magnetic field is derived. This equation comprises spatial derivatives of high orders due to a nonlocal nature of magnetic field transport in a random velocity field with a finite correlation time. For a random Gaussian velocity field with a small correlation time the equation for the second-order correlation function of the magnetic field is a third-order partial differential equation. For this velocity field and a small magnetic diffusion with large magnetic Prandtl numbers the growth rate of the second moment of magnetic field is estimated. The finite correlation time of a turbulent velocity field causes an increase of the growth rate of magnetic fluctuations. It is demonstrated that the results obtained for the cases of a small yet finite magnetic diffusion and a zero magnetic diffusion are different.

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I. INTRODUCTION

In recent time magnetic fluctuations are a subject of intensive study (see, e.g., [1–10]). There are two types of magnetic fluctuations: the fluctuations with a zero and a nonzero mean magnetic field. These two types of magnetic fluctuations have different mechanisms of generation and different properties. Magnetic fluctuations with a zero mean magnetic field in a random velocity field are generated by the stretch-twist-fold mechanism (see, e.g., [1,2]). On the other hand, magnetic fluctuations with a nonzero mean magnetic field are generated by a tangling of the mean magnetic field by a random velocity field (see, e.g., [11–14]).

In the present paper we considered only magnetic fluctuations with a zero mean magnetic field that were observed, e.g., in the ionosphere of Venus (see, e.g., [15,16]), in the quiet sun (see, e.g., [14]) and probably in galaxies (see, e.g., [17]). In spite of that the dynamics of a mean magnetic field at least in kinematic (linear) stage is well studied (see, e.g., [11–14,17]), a generation of magnetic fluctuations with a zero mean magnetic field even in kinematic stage still remains a subject of numerous discussions. Most studies starting with a seminal paper by Kazantsev [18] were performed in the δ correlated in time approximation for a random velocity field (see, e.g., [1,2,8,9], and references therein). The use of δ correlated in time approximation for a random velocity field is a great mathematical convenience.

However, a real velocity field in astrophysical and geophysical applications cannot be considered as the δ correlated in time velocity field. As follows from the analysis in [19,20] a finite correlation time of the velocity field does not essentially change a form of the mean-field equations and the growth rates of the mean fields. In particular, there is a wide range of scales in which the mean-field equations are the second-order partial differential equations (in spatial derivatives). However, the effect of a finite correlation time of the velocity field on magnetic fluctuations is poorly understood. It is not clear how conditions for the generation of magnetic fluctuations are changed in a random velocity field with a finite correlation time.

In this study we took into account a finite correlation time of a random velocity field and a small yet finite magnetic diffusion caused by an electrical conductivity of fluid. We derived an equation for the second-order correlation function of magnetic field in a random velocity field with a finite correlation time using a method described in [19–21]. The derived equation comprises spatial derivatives of high orders. For a random Gaussian velocity field with a small correlation time the equation for the second-order correlation function of the magnetic field is a third-order partial differential equation. We calculated the growth rate of the second moment of magnetic field for this velocity field and a small magnetic diffusion with large magnetic Prandtl numbers. In the limit of extremely small correlation time of a random velocity field we recovered the results obtained in the δ correlated in time approximation for a random velocity field.

Recently, the finite correlation time effects of a random velocity field in the kinematic dynamo in the case of a zero magnetic diffusion have been studied in [10]. We will show that the results obtained for the cases of a zero magnetic diffusion and of a small yet finite magnetic diffusion are different.

II. GOVERNING EQUATIONS

We study magnetic fluctuations with a zero mean magnetic field. A mechanism of the generation of magnetic fluctuations with a zero mean magnetic field was proposed by Zeldovich (see, e.g., [1,2]) and comprises stretching, twisting, and folding of the original loop of a magnetic field. These nontrivial motions are three dimensional and result in an amplification of the magnetic field. The magnetic field $\mathbf{b}(t,\mathbf{r})$ is determined by the induction equation

$$\frac{\partial \mathbf{b}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{b}=(\mathbf{b}\cdot\nabla)\mathbf{v}-\mathbf{b}(\nabla\cdot\mathbf{v})+D_m\Delta\mathbf{b}, \tag{1}$$

where D_m is the magnetic diffusion caused by an electrical conductivity of a fluid, $\mathbf{v}$ is a random velocity field. The goal of the present paper is to derive an equation for the second-order correlation function of the magnetic field in a random velocity field with a finite correlation time.

Now we discuss a method of derivation of the equation for the second-order correlation function of the magnetic field (for details, see Appendix A). We use an exact solution of Eq. (1) in the form of a functional integral for an arbitrary velocity field taking into account a small yet finite molecular magnetic diffusion. The molecular magnetic diffusion can be described by a random Brownian motions of a particle. The functional integral implies an averaging over a random Brownian motions of a particle. The form of the exact solution used in the present paper allows us to separate the averaging over both, a random Brownian motions of a particle and a random velocity field. This method allows us to derive an equation for the second-order correlation function $\Phi_{ij}(t,\mathbf{x},\mathbf{y})=\langle b_i(t,\mathbf{x})b_j(t,\mathbf{y})\rangle$ of the magnetic field,

$$\Phi_{ij}(t,\mathbf{r})=P_{ijpl}(\tau,\mathbf{r},i\nabla)\Phi_{pl}(s,\mathbf{r}), \tag{2}$$

$$P_{ijpl}(\tau,\mathbf{r},i\nabla)=M_{\boldsymbol{\xi}}\{\langle G_{ip}(\mathbf{x})G_{jl}(\mathbf{y})\exp(\tilde{\boldsymbol{\xi}}\cdot\nabla)\rangle\}, \tag{3}$$

(see Appendix A), where $\tau=t-s$, $G_{ij}(\mathbf{x})\equiv G_{ij}(t,s,\boldsymbol{\xi}(\mathbf{x}))$ is determined by equation $dG_{ij}(t,s,\boldsymbol{\xi})/ds=N_{ik}G_{kj}(t,s,\boldsymbol{\xi})$ with the initial condition $G_{ij}(t=s)=\delta_{ij}$, and the tensor G_{ij} can be considered as the Jacobian for magnetic field transport. Here $N_{ik}=\partial v_i/\partial x_j-\delta_{ij}(\nabla\cdot\mathbf{v})$, $M_{\boldsymbol{\xi}}\{\cdot\}$ denotes the mathematical expectation over the Wiener paths $\boldsymbol{\xi}(\mathbf{x})=\mathbf{x}-\int_0^{t-s}\mathbf{v}(t-\sigma,\boldsymbol{\xi})d\sigma+(2D_m)^{1/2}\mathbf{w}(t-s)$, and $\tilde{\boldsymbol{\xi}}=\boldsymbol{\xi}(\mathbf{y})-\boldsymbol{\xi}(\mathbf{x})-\mathbf{r}$, $\mathbf{r}=\mathbf{y}-\mathbf{x}$, $\nabla=\partial/\partial\mathbf{r}$, the angular brackets $\langle\cdot\rangle$ denote the ensemble average over the random velocity field, and the molecular magnetic diffusion D_m is described by a Wiener process $\mathbf{w}(t)$. Another equivalent approach that includes a weak molecular diffusion in a Lagrangian map, with a Green's function was considered in [22,23].

Equation (2) for the second moment of a magnetic field comprises spatial derivatives of high orders due to a nonlocal nature of turbulent transport of magnetic field in a random velocity field with a finite correlation time (for details, see Appendix A).

III. THE RANDOM GAUSSIAN VELOCITY FIELD WITH A SMALL CORRELATION TIME

Now we use the model of the random Gaussian velocity field with a small yet finite correlation time. We seek a solution for the second moment of the magnetic field in the form

$$\Phi_{ij}(t,\mathbf{r})\equiv\langle b_i(t,\mathbf{x})b_j(t,\mathbf{y})\rangle=W(t,r)\,\delta_{ij}+(rW'/2)P_{ij}(\mathbf{r}), \tag{4}$$

where $W(t,r)=\langle\tilde{b}(t,\mathbf{x})\tilde{b}(t,\mathbf{y})\rangle$, $\tilde{b}=\mathbf{b}\cdot\mathbf{r}$, $\mathbf{r}=\mathbf{y}-\mathbf{x}$, $P_{ij}(\mathbf{r})=\delta_{ij}-r_{ij}$, $r_{ij}=r_ir_j/r^2$ and $W'=\partial W(t,r)/\partial r$. This form of the second moment corresponds to the condition $\nabla\cdot\mathbf{b}=0$ and an assumption of the homogeneous and isotropic magnetic fluctuations. We considered a homogeneous, isotropic, and incompressible random velocity field (see below). The equation for the correlation function $W(t,r)$ is given by

$$\frac{\partial W(t,r)}{\partial t}=(1/3)\sigma_\xi r^3W'''+m^{-1}(r)W''+\mu(r)W'+\kappa W, \tag{5}$$

(for details, see Appendix B), where in the leading order of asymptotic expansion $\kappa=(20/3)(1+\sigma_\xi/4)$, and

$$1/m(r)=2/\mathrm{Pr}+(2/3)r^2(1+8\sigma_\xi),$$

$$\mu(r)=\frac{4}{m(r)r}+\left(\frac{1}{m(r)}\right)'-27\sigma_\xi r,$$

$\mathrm{Pr}=\nu/D_m$ is the magnetic Prandtl number, ν is the kinematic viscosity, $\sigma_\xi=(2/3)\mathrm{St}^2$, $\mathrm{St}=\tau u_d/l_d$ is the Strouhal number. Equation (5) is written in dimensionless form: the distance r is measured in the units of the inner scale of turbulence $l_d=l_0\,\mathrm{Re}^{-3/4}$, the time t is measured in the units $\tau_d=\tau_0\,\mathrm{Re}^{-1/2}$, where τ_d is the turnover time of eddies in the inner scale l_d and the velocity v is measured in the units $u_d=l_d/\tau_d$, $\mathrm{Re}=u_0l_0/\nu\gg1$ is the Reynolds number, u_0 is the characteristic turbulent velocity in the maximum scale of turbulent motions l_0 and $\tau_0=l_0/u_0$. In this study we consider the case of large magnetic Prandtl numbers. For the derivation of Eq. (5) we used a homogeneous, isotropic, and incompressible random velocity field and the correlation function $f_{ij}(t,\mathbf{r})=\langle v_i(t,\mathbf{x})v_j(t,\mathbf{y})\rangle$ for the velocity field is given by

$$f_{ij}=(1/3)[F(r)\delta_{ij}+(rF'/2)P_{ij}(\mathbf{r})]. \tag{6}$$

We assumed that in dissipative range $(0\leqslant r\leqslant1)$ of a turbulent velocity field the function $F(r)$ is given by $F(r)=1-r^2$.

Now we analyze a solution of Eq. (5). In a molecular magnetic diffusion region of scales whereby $r\ll\mathrm{Pr}^{-1/2}$, all terms proportional to r^2 may be neglected. Then the solution of Eq. (5) is given by $W(t,r)=(1-\alpha\,\mathrm{Pr}\,r^2)\exp(\gamma t)$, where γ are the eigenvalues to be found, $\alpha=(\kappa-\gamma)/20$ and $\kappa>\gamma$. In a turbulent magnetic diffusion region of scales, $\mathrm{Pr}^{-1/2}\ll r\ll1$, the molecular magnetic diffusion term proportional to $1/\mathrm{Pr}$ is negligible. Thus, the solution of Eq. (5) in this region is $W(t,r)=A_1r^{-\lambda}\exp(\gamma t)$, where λ is determined by an equation

$$\sigma_\xi\lambda^3-(3+13\sigma_\xi)\lambda^2+(15-1226\sigma_\xi)\lambda+(9/2)\gamma-30-5\sigma_\xi=0. \tag{7}$$

For a small parameter σ_ξ we obtain $\lambda\approx5/2-424\sigma_\xi\pm ia_0$, where $a_0^2=3(5-2\gamma_0)/4$, $\gamma=\gamma_0+\sigma_\xi\gamma_1$, and $\gamma_1\approx348$. For $r\gg1$ the solution for $W(t,r)$ decays rapidly with r. The value γ_0 can be calculated by matching the correlation function $W(t,r)$ and its first and second derivatives at the boundaries of the above regions, i.e., at the points $r=\mathrm{Pr}^{-1/2}$ and $r=1$. In particular, the matching yields $a_0 \approx 2\pi k/\ln \mathrm{Pr}$, where the parameter $k=1,2,3,\ldots$ determines modes with different numbers of zero points $(W=0)$ in the correlation function $W(r)$. In particular, the mode with $k=1$ has only one zero point in the correlation function $W(r)$. Thus, the growth rate γ of magnetic fluctuations is given by

$$\gamma \approx \frac{5}{2} - \frac{2}{3}\left(\frac{2\pi k}{\ln \mathrm{Pr}}\right)^2 + 348\sigma_\xi . \tag{8}$$

The correlation function $W(t,r)$ has global maximum at $r=0$. This implies that the real part of λ is positive. Thus, $\tau < 0.1\ (l_d/u_d)$. It follows from Eq. (8) that the finite correlation time of a turbulent velocity field causes an increase of the growth rate of magnetic fluctuations. The latter is important in view of applications in astrophysics and planetary physics because the real velocity field has a finite correlation time. Note that the considered case corresponds to the fast dynamo because the growth rate tends to the nonzero constant at very large magnetic Reynolds numbers.

IV. DISCUSSION

In the present paper we studied an effect of a finite correlation time of a turbulent velocity field on dynamics of magnetic fluctuations with a zero mean magnetic field in the case of a small yet finite magnetic diffusion. The finite correlation time results in an increase of the growth rate of magnetic fluctuations. However, the developed theory is limited by an assumption about small correlation time, i.e., $\tau < 0.1\ (l_d/u_d)$. The latter estimate is quite realistic, e.g., for galactic turbulence (see Ref. [17]). We showed also that for an arbitrary correlation time of a turbulent velocity field the equation for the second moment of the turbulent magnetic field comprises higher-order spatial derivatives.

In this study we took into account a small yet finite magnetic diffusion caused by an electrical conductivity of a fluid. The obtained results are different from that derived for a zero magnetic diffusion (see [10]). In particular, the finite correlation time of a turbulent velocity field reduces the growth rate of magnetic fluctuations in the case of a zero magnetic diffusion (see [10]). A difference between two cases with a zero magnetic diffusion and a small yet finite magnetic diffusion can be demonstrated even for the δ correlated in time random velocity field. For instance, for large magnetic Prandtl numbers the growth rate of the second moment of a turbulent magnetic field is given by

$$\gamma = \frac{5(1+\sigma/3)}{2(1+3\sigma)} - \frac{2(1+3\sigma)}{3(1+\sigma)}\left(\frac{2\pi k}{\ln \mathrm{Pr}}\right)^2, \tag{9}$$

where $\sigma = \langle(\boldsymbol{\nabla}\cdot\mathbf{v})^2\rangle/\langle(\boldsymbol{\nabla}\times\mathbf{v})^2\rangle$ is the degree of compressibility of fluid velocity field. Equation (9) is obtained using Eqs. (29) and (30) of Ref. [8] and implies that the compressibility of fluid velocity field causes a reduction of the growth rate of the second moment of a turbulent magnetic field. On the other hand, in the case of a zero magnetic diffusion the growth rate of the second moment of magnetic fluctuations generated by the δ correlated in time random velocity field is given by

$$\gamma = \frac{10(1+2\sigma)}{3(1+\sigma)} \tag{10}$$

(see [10]), and the compressibility results in an increase of the growth rate of the second moment of a turbulent magnetic field. This implies that a transition from the case of a zero magnetic diffusion to that of a small yet finite magnetic diffusion is singular. The limit of zero magnetic diffusion is singular because the growth rate γ of magnetic fluctuations is discontinuous at zero magnetic diffusion, i.e., it is different from the limit of magnetic diffusion tending to zero. This stresses a danger for an application of the results obtained for a zero magnetic diffusion to astrophysics and planetary physics where the magnetic diffusion caused by an electrical conductivity of fluid is small yet finite.

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APPENDIX A: DERIVATION OF EQ. (2)

When $D_m \neq 0$ the magnetic field $\mathbf{b}(t,\mathbf{x})$ is given by

$$b_i(t,\mathbf{x}) = M_\xi\{G_{ij}(t,\boldsymbol{\xi})\exp(\boldsymbol{\xi}^*\cdot\boldsymbol{\nabla})b_j(s,\mathbf{x})\}, \tag{A1}$$

where $\boldsymbol{\xi}^* = \boldsymbol{\xi} - \mathbf{x}$. In order to derive Eq. (A1) we use an exact solution of Eq. (1) with an initial condition $\mathbf{b}(t=s,\mathbf{x}) = \mathbf{b}(s,\mathbf{x})$ in the form of the Feynman-Kac formula:

$$b_i(t,\mathbf{x}) = M_\xi\{G_{ij}(t,s,\boldsymbol{\xi}(t,s))b_j(s,\boldsymbol{\xi}(t,s))\}, \tag{A2}$$

where $dG_{ij}(t,s,\boldsymbol{\xi})/ds = N_{ik}G_{kj}(t,s,\boldsymbol{\xi})$, $N_{ik} = \partial v_i/\partial x_j - \delta_{ij}(\boldsymbol{\nabla}\cdot\mathbf{v})$, and $M_\xi\{\cdot\}$ denotes the mathematical expectation over the Wiener paths $\boldsymbol{\xi}(t,s) = \mathbf{x} - \int_0^{t-s}\mathbf{v}[t-\sigma,\boldsymbol{\xi}(t,\sigma)]d\sigma + (2D_m)^{1/2}\mathbf{w}(t-s)$. Now we assume that

$$\mathbf{b}(t,\boldsymbol{\xi}) = \int \exp(i\boldsymbol{\xi}\cdot\mathbf{q})\mathbf{b}(s,\mathbf{q})d\mathbf{q}. \tag{A3}$$

Substituting Eq. (A3) into Eq. (A2) we obtain

$$b_i(s,\mathbf{x}) = \int M_\xi\{G_{ij}(t,s,\boldsymbol{\xi}(t,s))\exp[i\boldsymbol{\xi}^*\cdot\mathbf{q}]b_j(s,\mathbf{q})\} \times \exp(i\mathbf{q}\cdot\mathbf{x})d\mathbf{q}. \tag{A4}$$

In Eq. (A4) we expand the function $\exp[i\boldsymbol{\xi}^*\cdot\mathbf{q}]$ in Taylor series at $\mathbf{q}=0$, i.e., $\exp[i\boldsymbol{\xi}^*\cdot\mathbf{q}] = \Sigma_{k=0}^{\infty}(1/k!)(i\boldsymbol{\xi}^*\cdot\mathbf{q})^k$. Using the identity $(i\mathbf{q})^k\exp[i\mathbf{x}\cdot\mathbf{q}] = \boldsymbol{\nabla}^k\exp[i\mathbf{x}\cdot\mathbf{q}]$ and Eq. (A4) we get

$$b_i(t,\mathbf{x}) = M_{\xi}\left\{ G_{ij}(t,s,\boldsymbol{\xi}) \left[\sum_{k=0}^{\infty} (1/k!)(\boldsymbol{\xi}^* \cdot \boldsymbol{\nabla})^k \right] \times \int b_j(s,\mathbf{q}) \exp(i\mathbf{q}\cdot\mathbf{x}) d\mathbf{q} \right\}. \tag{A5}$$

After the inverse Fourier transformation in Eq. (A5) we obtain Eq. (A1). Equation (A3) can be formally considered as an inverse Fourier transformation of the function $b_i(t,\boldsymbol{\xi})$. However, $\boldsymbol{\xi}$ is the Wiener path that is not a usual spatial variable. Therefore, it is desirable to derive Eq. (A1) by a more rigorous method as it is done below.

To this end we use an exact solution of the Cauchy problem for Eq. (1) with an initial condition $\mathbf{b}(t=s,\mathbf{x}) = \mathbf{b}(s,\mathbf{x})$ in the form

$$b_i(t,\mathbf{x}) = M_{\zeta}\{ J(t,s,\boldsymbol{\zeta}) \tilde{G}_{ij}(t,s,\boldsymbol{\zeta}) b_j(s,\boldsymbol{\zeta}(t,s))\}, \tag{A6}$$

where the matrix $\tilde{G}_{ij}$ is determined by the equation $d\tilde{G}_{ij}(t,s,\boldsymbol{\zeta})/ds = N_{ik}\tilde{G}_{kj}(t,s,\boldsymbol{\zeta})$ with the initial condition $\tilde{G}_{ij}(t=s) = \delta_{ij}$, and the function $J(t,s,\boldsymbol{\zeta})$ is given by

$$J(t,s,\boldsymbol{\zeta}) = \exp\left[-(2D_m)^{-1/2} \int_0^{t-s} \mathbf{v}(t-\eta, \boldsymbol{\zeta}(t,\eta)) \cdot d\mathbf{w}(\eta) - (4D_m)^{-1} \int_0^{t-s} \mathbf{v}^2(t-\eta, \boldsymbol{\zeta}(t,\eta)) d\eta \right], \tag{A7}$$

$\mathbf{w}(t)$ is a Wiener process, and $M_{\zeta}\{\cdot\}$ denotes the mathematical expectation over the paths $\boldsymbol{\zeta}(t,s) = \mathbf{x} + (2D_m)^{1/2}[\mathbf{w}(t) - \mathbf{w}(s)]$. The solution (A6) was first found in [19] for a magnetic field in an incompressible fluid flow. Equation (A6) generalizes the solution obtained in [19] for a magnetic field in a compressible random velocity field. The first integral $\int_0^{t-s} \mathbf{v}(t-\eta, \boldsymbol{\zeta}(t,\eta)) \cdot d\mathbf{w}(\eta)$ in Eq. (A7) is the Ito stochastic integral (see, e.g., [24]).

The difference between the solutions (A6) and (A2) is as follows. The function $b_j(s,\boldsymbol{\xi}(t,s))$ in Eq. (A2) explicitly depends on the random velocity field $\mathbf{v}$ via the Wiener path $\boldsymbol{\xi}$, while the function $b_j(s,\boldsymbol{\zeta}(t,s))$ in Eq. (A6) is independent of the velocity $\mathbf{v}$. Trajectories in the Feynman-Kac formula (A2) are determined by both, a random velocity field and magnetic diffusion. On the other hand, trajectories in Eq. (A6) are determined only by magnetic diffusion. Due to the Markovian property of the Wiener process the solution (A6) can be rewritten in the form

$$b_i(t,\mathbf{x}) = E\{S_{ij}(t,s,\mathbf{x},\mathbf{X}') b_j(s,\mathbf{X}')\} = \int Q_{ij}(t,s,\mathbf{x},\mathbf{x}') b_j(s,\mathbf{x}') d\mathbf{x}', \tag{A8}$$

where

$$Q_{ij}(t,s,\mathbf{x},\mathbf{x}') = [4\pi D_m(t-s)]^{3/2} \exp\left(-\frac{(\mathbf{x}'-\mathbf{x})^2}{4D_m(t-s)} \right) \times S_{ij}(t,s,\mathbf{x},\mathbf{x}'), \tag{A9}$$

$S_{ij}(t,s,\mathbf{x},\mathbf{x}') = M_{\boldsymbol{\mu}}\{J(t,s,\boldsymbol{\mu})\tilde{G}_{ij}(t,s,\boldsymbol{\mu})\}$ and $M_{\boldsymbol{\mu}}\{\cdot\}$ means the path integral taken over the set of trajectories $\boldsymbol{\mu}$ that connect points $(t,\mathbf{x})$ and $(s,\mathbf{x}')$. The mathematical expectation $E\{\cdot\}$ in Eq. (A8) denotes the averaging over the set of random points $\mathbf{X}'$ that have a Gaussian statistics (see, e.g., [25]). We used here the following property of the averaging over the Wiener process $E\{M_{\boldsymbol{\mu}}\{\cdot\}\} = M_{\zeta}\{\cdot\}$. We considered a random velocity field with a finite renewal time. In the intervals $\ldots(-\tau,0],(0,\tau],(\tau,2\tau],\ldots$ the velocity fields are assumed to be statistically independent and have the same statistics. This implies that the velocity field loses memory at the prescribed instants $t = n\tau$, where $n = 0, \pm 1, \pm 2, \ldots$. This velocity field cannot be considered as a stationary velocity field for small times $\sim\tau$, however, it behaves similar to a stationary field for $t \gg \tau$. Note that the fields $b_j(s,\mathbf{x}')$ and $Q_{ij}(t,s,\mathbf{x},\mathbf{x}')$ are statistically independent because the field $b_j(s,\mathbf{x}')$ is determined in the time interval $(-\infty,s]$, whereas the function $Q_{ij}(t,s,\mathbf{x},\mathbf{x}')$ is defined on the interval $(s,t]$. Due to a renewal, the velocity field as well as its functionals $b_j(s,\mathbf{x}')$ and $Q_{ij}(t,s,\mathbf{x},\mathbf{x}')$ in these two time intervals are statistically independent. Now we make a change of variables $(\mathbf{x},\mathbf{x}') \to (\mathbf{x},\mathbf{x}' = \mathbf{z}+\mathbf{x})$ in Eq. (A8), i.e., $\tilde{Q}_{ij}(t,s,\mathbf{x},\mathbf{x}') = \tilde{Q}_{ij}(t,s,\mathbf{x},\mathbf{z}+\mathbf{x}) = Q_{ij}(t,s,\mathbf{x},\mathbf{z})$. The Fourier transformation in Eq. (A8) yields

$$b_i(t,\mathbf{x}) = \int \int Q_{ij}(t,s,\mathbf{x},\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{z}) d\mathbf{k} \times \int b_j(s,\mathbf{q}) \exp[i\mathbf{q}\cdot(\mathbf{z}+\mathbf{x})] d\mathbf{q} d\mathbf{z}.$$

Since $\delta(\mathbf{k}+\mathbf{q}) = (2\pi)^{-3} \int \exp[i(\mathbf{k}+\mathbf{q})\cdot\mathbf{z}] d\mathbf{z}$, we obtain that

$$b_i(t,\mathbf{x}) = (2\pi)^3 \int Q_{ij}(t,s,\mathbf{x},-\mathbf{q}) b_j(s,\mathbf{q}) \exp(i\mathbf{q}\cdot\mathbf{x}) d\mathbf{q}. \tag{A10}$$

In Eq. (A10) the function $Q_{ij}(t,s,\mathbf{x},-\mathbf{q})$ is given by

$$Q_{ij}(t,s,\mathbf{x},-\mathbf{q}) = (2\pi)^{-3} \int Q_{ij}(t,s,\mathbf{x},\mathbf{z}) \exp(i\mathbf{q}\cdot\mathbf{z}) d\mathbf{z}. \tag{A11}$$

Substituting $\tilde{Q}_{ij}(t,s,\mathbf{x},\mathbf{x}') = Q_{ij}(t,s,\mathbf{x},\mathbf{z})$ in Eq. (A8) and taking into account that $\mathbf{x}' = \mathbf{z}+\mathbf{x}$ we obtain

$$b_i(t,\mathbf{x}) = \int Q_{ij}(t,s,\mathbf{x},\mathbf{z}) b_j(s,\mathbf{z}+\mathbf{x}) d\mathbf{z}. \tag{A12}$$

Equation (A11) can be rewritten in the form

$$(2\pi)^3 Q_{ij}(t,s,\mathbf{x},-\mathbf{q}) \exp(i\mathbf{q}\cdot\mathbf{x}) = \int Q_{ij}(t,s,\mathbf{x},\mathbf{z}) \exp[i\mathbf{q}\cdot(\mathbf{z}+\mathbf{x})] d\mathbf{z}. \tag{A13}$$

The right hand sides of Eqs. (A12) and (A13) coincide when $\mathbf{b}(s,\mathbf{z}+\mathbf{x}) = \mathbf{e} \exp[i\mathbf{q}\cdot(\mathbf{z}+\mathbf{x})]$, where $\mathbf{e}$ is a unit vector. Thus, a particular solution (A12) of Eq. (1) with the initial condition $\mathbf{b}(s,\mathbf{x}') = \mathbf{e}\exp(i\mathbf{q}\cdot\mathbf{x}')$ coincides in form with the integral (A13). On the other hand, a solution of Eq. (1) is given by Eq. (A6). Substituting the initial condition $\mathbf{b}(s,\boldsymbol{\zeta}) = \mathbf{e}\exp(i\mathbf{q}\cdot\boldsymbol{\zeta}) = \mathbf{e}\exp[i\mathbf{q}\cdot(\mathbf{x}+(2D_m)^{1/2}\mathbf{w})]$ into Eq. (A6) we obtain

$$b_i(t,\mathbf{x}) = M_{\boldsymbol{\zeta}}\{J(t,s,\boldsymbol{\zeta})\tilde{G}_{ij}(t,s,\boldsymbol{\zeta})e_j \times\exp[i\mathbf{q}\cdot(\mathbf{x}+(2D_m)^{1/2}\mathbf{w})]\}. \quad (A14)$$

Comparing Eqs. (A12)–(A14) we get

$$Q_{ij}(t,s,\mathbf{x},-\mathbf{q}) = (2\pi)^{-3}M_{\boldsymbol{\zeta}}\{J(t,s,\boldsymbol{\zeta})\tilde{G}_{ij}(t,s,\boldsymbol{\zeta}) \times\exp[i(2D_m)^{1/2}\mathbf{q}\cdot\mathbf{w}]\}. \quad (A15)$$

Now we rewrite Eq. (A15) using Feynman-Kac formula (A2). The result is given by

$$Q_{ij}(t,s,\mathbf{x},-\mathbf{q}) = (2\pi)^{-3}M_{\boldsymbol{\xi}}\{G_{ij}(t,s,\boldsymbol{\xi}(t,s))\exp[i\mathbf{q}\cdot\boldsymbol{\xi}^*]\}, \quad (A16)$$

where $\boldsymbol{\xi}^* = \boldsymbol{\xi} - \mathbf{x}$. Substituting Eq. (A16) into Eq. (A10) we obtain

$$b_i(t,\mathbf{x}) = \int M_{\boldsymbol{\xi}}\{G_{ij}(t,s,\boldsymbol{\xi})\exp[i\mathbf{q}\cdot\boldsymbol{\xi}^*]b_j(s,\mathbf{q})\} \times\exp(i\mathbf{q}\cdot\mathbf{x})d\mathbf{q}. \quad (A17)$$

The Fourier transformation in Eq. (A17) yields Eq. (A1). The above derivation proves that the assumption (A3) is correct for a Wiener path $\boldsymbol{\xi}$. In order to derive an equation for the second-order correlation function $\Phi_{ij}(t,\mathbf{x},\mathbf{y}) = \langle b_i(t,\mathbf{x})b_j(t,\mathbf{y})\rangle$ we use Eq. (A17), where the angular brackets $\langle\cdot\rangle$ denote the ensemble average over the random velocity field. After the Fourier transformation we obtain

$$\Phi_{ij}(t,\mathbf{x},\mathbf{y}) = (2\pi)^{-6}\int\int P_{ijpl}(\tau,\mathbf{x},\mathbf{y},\mathbf{k}_1,\mathbf{k}_2)\exp[i(\mathbf{k}_1\cdot\mathbf{x} + \mathbf{k}_2\cdot\mathbf{y})]\left[\int\int\Phi_{pl}(s,\mathbf{x}',\mathbf{y}')\exp[-i(\mathbf{k}_1\cdot\mathbf{x}' + \mathbf{k}_2\cdot\mathbf{y}')]d\mathbf{x}'d\mathbf{y}'\right]d\mathbf{k}_1 d\mathbf{k}_2, \quad (A18)$$

where

$$P_{ijpl}(\tau,\mathbf{x},\mathbf{y},\mathbf{k}_1,\mathbf{k}_2) = M_{\boldsymbol{\xi}}\{\langle G_{ip}(\mathbf{x})G_{jl}(\mathbf{y})\exp[i(\mathbf{k}_1\cdot\boldsymbol{\xi}^*(\mathbf{x}) + \mathbf{k}_2\cdot\boldsymbol{\xi}^*(\mathbf{y}))]\rangle\}, \quad (A19)$$

$G_{ij}(\mathbf{x}) \equiv G_{ij}(\tau,\boldsymbol{\xi}(\mathbf{x}))$ and $\tau = t - s$. For a homogeneous and isotropic random flow Eq. (A18) reads

$$\Phi_{ij}(t,\mathbf{r}) = \int\int P_{ijpl}(\tau,-\mathbf{q},\mathbf{r})\exp[i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')] \times\Phi_{pl}(s,\mathbf{r}')d\mathbf{r}'d\mathbf{q}, \quad (A20)$$

where $\mathbf{r} = \mathbf{y} - \mathbf{x}$,

$$P_{ijpl}(\tau,-\mathbf{q},\mathbf{r}) = M_{\boldsymbol{\xi}}\{\langle G_{ip}(\mathbf{x})G_{jl}(\mathbf{y})\exp(i\mathbf{q}\cdot\tilde{\boldsymbol{\xi}})\rangle\} \quad (A21)$$

and $\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi}^*(\mathbf{y}) - \boldsymbol{\xi}^*(\mathbf{x})$. The Fourier transformation of Eq. (A20) yields Eq. (2).

APPENDIX B: DERIVATION OF EQ. (5)

Now we use the model of the random velocity field with a small correlation time. We expand the functions $\boldsymbol{\xi}^*$ and $G_{ij}(\tau,\boldsymbol{\xi})$ in Taylor series of small time τ. Then an expression for the function $P_{ijpl}(\tau,\mathbf{r},i\boldsymbol{\nabla})$ reads

$$P_{ijpl}(\tau,\mathbf{r},i\boldsymbol{\nabla}) = \delta_{ip}\delta_{jl} + \tau B_{ijpl} + \tau U_{ijplm}\boldsymbol{\nabla}_m + \tau D_{ijplmn}\boldsymbol{\nabla}_m\boldsymbol{\nabla}_n + \cdots, \quad (B1)$$

where

$$D_{ijplmn} = (1/2\tau)M_{\boldsymbol{\xi}}\{\langle\tilde{\xi}_m\tilde{\xi}_n G_{ip}(\mathbf{x})G_{jl}(\mathbf{y})\rangle\}, \quad (B2)$$

$$U_{ijplm}(r) = \tau^{-1}[\delta_{jl}M_{\boldsymbol{\xi}}\{\langle g_{ip}(\mathbf{x})\xi_m^*(\mathbf{y})\rangle\} + \delta_{ip}M_{\boldsymbol{\xi}}\{\langle g_{jl}(\mathbf{x})\xi_m^*(\mathbf{y})\rangle\} - (1/2)M_{\boldsymbol{\xi}}\{\langle g_{ip}(\mathbf{x})g_{jl}(\mathbf{y})\tilde{\xi}_m\rangle\}], \quad (B3)$$

$$B_{ijpl}(r) = \tau^{-1}M_{\boldsymbol{\xi}}\{\langle g_{ip}(\mathbf{x})g_{jl}(\mathbf{y})\rangle\}, \quad (B4)$$

and $G_{ij} = \delta_{ij} + g_{ij}$ and $M_{\boldsymbol{\xi}}\{\langle g_{ij}\rangle\} = 0$. Thus an equation for the second-order correlation function for a magnetic field in a random velocity field with a small yet finite correlation time reads

$$\partial\Phi_{ij}/\partial t = [B_{ijpl} + U_{ijplm}\boldsymbol{\nabla}_m + D_{ijplmn}\boldsymbol{\nabla}_m\boldsymbol{\nabla}_n]\Phi_{pl}(t,\mathbf{r}). \quad (B5)$$

Now we consider a random velocity field with a Gaussian statistics. This assumption allows us to calculate the tensors D_{ijplmn}, U_{ijplm}, and B_{ijpl}. We omit the lengthy algebra and present the final results

$$D_{ijplqn} = D_{ijplqn}^{(1)} + D_{ijplqn}^{(2)} + D_{ijplqn}^{(3)} + 2D_m\delta_{qn}\delta_{ip}\delta_{jl}, \quad (B6)$$

$$D_{ijplmn}^{(1)} = 2\tau\{\tilde{f}_{mn} + \mathrm{St}^2[(\boldsymbol{\nabla}_s f_{kn})(\boldsymbol{\nabla}_k f_{ms}) - \tilde{f}_{sk}(\boldsymbol{\nabla}_s\boldsymbol{\nabla}_k f_{mn})]\}\delta_{ip}\delta_{jl}, \quad (B7)$$

$$D_{ijplmn}^{(2)} = (1/2)\tau\mathrm{St}^2[(\boldsymbol{\nabla}_k f_{im})(\boldsymbol{\nabla}_p f_{nk}) - (\boldsymbol{\nabla}_k f_{mn})(\boldsymbol{\nabla}_p f_{ik}) + 2\tilde{f}_{ms}(\boldsymbol{\nabla}_s\boldsymbol{\nabla}_p f_{in})]\delta_{jl}, \quad (B8)$$

$$D_{ijplmn}^{(3)} = (1/2)\tau\mathrm{St}^2[(\boldsymbol{\nabla}_p f_{im})(\boldsymbol{\nabla}_l f_{jn}) - \tilde{f}_{mn}(\boldsymbol{\nabla}_p\boldsymbol{\nabla}_l f_{ij})], \quad (B9)$$

$$B_{ijpl} = -2\tau\{(\boldsymbol{\nabla}_p\boldsymbol{\nabla}_l f_{ij}) - \mathrm{St}^2[(\boldsymbol{\nabla}_k\boldsymbol{\nabla}_s f_{ij})(\boldsymbol{\nabla}_p\boldsymbol{\nabla}_l f_{ks}) + 2(\boldsymbol{\nabla}_p\boldsymbol{\nabla}_m f_{is})(\boldsymbol{\nabla}_l\boldsymbol{\nabla}_s f_{jm})]\}, \quad (B10)$$

$$U_{ijplm}=4\tau\{(\nabla_p f_{im})\delta_{jl}+\mathrm{St}^2[((\nabla_k \tilde{f}_{is})(\nabla_p\nabla_s f_{km})+(\nabla_p f_{sk})\times(\nabla_k\nabla_s \tilde{f}_{im})-(\nabla_s f_{km})(\nabla_k\nabla_p \tilde{f}_{is}))\delta_{jl}+2((\nabla_k \tilde{f}_{jm})\times(\nabla_p\nabla_l \tilde{f}_{ik})+(\nabla_l f_{km})(\nabla_k\nabla_p \tilde{f}_{ij}))]\}, \quad (B11)$$

where $\mathrm{St}=\tau u_d/l_d$ is the Strouhal number, $\tilde{f}_{mn}=f_{mn}(0)-f_{mn}(\mathbf{r})$, and we changed $\tau\to 2\tau$ in order to compare the obtained results with those for the δ correlated in time approximation for a random velocity field. Here the small terms of the order of $\sim O(\mathrm{St}^4)$ are being neglected. In Eqs. (B6)–(B11) we took into account a commutative symmetry in every pair of the following indexes: (i,j); (p,l) and (m,n). The latter is due to a symmetry of the following tensors: r_{ij}, Φ_{pl}, and $\nabla_m\nabla_n$. In Eqs. (B6)–(B11) we assumed also that the form of the tensor $\tilde{f}_{mn}$ is given by $\tilde{f}_{mn}=C_{mnps}r_p r_s$, where C_{mnps} is an arbitrary constant tensor. This satisfies for the model of the velocity field (6) with $F(r)=1-r^2$.

Now we seek a solution for the second moment of the magnetic field in the form of Eq. (4). Multiplying Eq. (B5) by r_{ij} and using Eq. (4) we obtain the equation for the correlation function $W(t,r)=\langle b_{\mathbf{r}}(t,\mathbf{x})b_{\mathbf{r}}(t,\mathbf{y})\rangle$. This equation is given by Eq. (5). For the derivation of Eq. (5) we used the following identities:

$$\hat{D}W\equiv r_{ij}D_{ijplmn}\nabla_m\nabla_n\Phi_{pl}=(2\tau/3)[r^2W''+8rW'+(\sigma_\xi/4)\times(2r^3W'''+31r^2W''+12rW')+(3/\mathrm{Pr})(W''+4W/r)], \quad (B12)$$

$$\hat{B}W\equiv r_{ij}B_{ijpl}\Phi_{pl}=(4\tau/3)[2rW'+5W+(\sigma_\xi/4)(rW'+5W)], \quad (B13)$$

$$\hat{U}W\equiv r_{ij}U_{ijplm}\nabla_m\Phi_{pl}=(2\tau/3)\{-6rW'+(\sigma_\xi/4)[r^2W''+(29/2)rW']\}. \quad (B14)$$

Equations (B12)–(B14) are derived by means of Eqs. (6), (B6)–(B11), and we also used the following identities:

$$\nabla_n\Phi_{pl}=(1/2)[(W''-W'/r)P_{pl}r_m+(W'/r)(4\delta_{pl}r_m-\delta_{pm}r_l-\delta_{lm}r_p), \quad (B15)$$

$$\nabla_m\nabla_n\Phi_{pl}=(1/2)[(rW''')P_{pl}r_{mn}+(W''-W'/r)(P_{mn}P_{pl}+4P_{pl}r_{mn}-P_{pm}r_{ln}-P_{lm}r_{pn}-P_{pn}r_{lm}-P_{ln}r_{pm}+2r_{plmn})+(W'/r)(4\delta_{pl}\delta_{mn}-\delta_{pm}\delta_{ln}-\delta_{lm}\delta_{pn})]. \quad (B16)$$

The corresponding derivatives for f_{pl} coincide with Eqs. (B15) and (B16) after the change $W(r)\to(1/3)F(r)$. Note that for $F(r)=1-r^2$ the following identities are valid: $F''-F'/r=0$ and $F'''=0$. Turbulent magnetic diffusion is determined by function $\hat{D}W=r_{ij}D_{ijplmn}\nabla_m\nabla_n\Phi_{pl}(t,\mathbf{r})$. The latter depends on the field of Lagrangian trajectories $\boldsymbol{\xi}$ [see Eqs. (B2) and (B5)]. Due to a finite correlation time of a random velocity field $\langle(\nabla\cdot\boldsymbol{\xi})^2\rangle\neq 0$ even if the velocity field is incompressible. Indeed, $\langle(\nabla\cdot\boldsymbol{\xi})^2\rangle\approx(4/9)\mathrm{St}^4=\sigma_\xi^2$. Thus the parameter σ_ξ describes the compressibility of the field of Lagrangian trajectories. The latter results in a change of the dynamics of magnetic fluctuations. Thus, the equation for the correlation function $W(t,r)$ is given by Eq. (5).

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