Strange behavior of a passive scalar in a linear velocity field

Tov Elperin, Nathan Kleeorin, and Igor Rogachevskii
The Pearlstone Center for Aeronautical Engineering Studies, Department of Mechanical Engineering, Ben-Gurion University of the Negev, Beer-Sheva 84105, P.O. Box 653, Israel

Dmitry Sokoloff
Department of Physics, Moscow State University, 117234 Moscow, Russia

(Received 13 December 1999; published 26 March 2001)

Damping (or growth) rates of a typical realization, mean-field and high-order correlation functions of a passive scalar (e.g., a number density of particles) advected by a linear velocity fields are estimated. It is shown that all statistical moments higher than the first moment and a typical realization of a passive scalar without an external pumping decay for both laminar and random incompressible linear velocity fields. Strong compressibility of a laminar linear velocity field can result in a growth of a typical realization and the high-order moments of a passive scalar. It is demonstrated that for a laminar compressible linear velocity field the flux of particles from the infinity does not vanish and the total number of particles is not conserved. For a random compressible linear velocity field a typical realization decays whereas the high-order moments of a passive scalar can grow. Comparison of the obtained results with those for dynamics of a passive scalar advected by a homogeneous isotropic and compressible turbulent flow with a given longitudinal two-point correlation function $F=1-r^2$ is performed (where r is the distance between two points measured in the units of the maximum scale of turbulent motions).

DOI: 10.1103/PhysRevE.63.046305 PACS number(s): 47.27.Qb, 47.40.−x

I. INTRODUCTION

Fluctuations of a passive scalar advected by a turbulent velocity field were studied in a number of publications (see, e.g., Refs. [1–7], and references therein) starting from the seminal paper by Kraichnan [8] where the equation for the second-order correlation function of a passive scalar in the δ-correlated time approximation for a random incompressible velocity field was derived. Recently a model of linear velocity field was used to study a turbulent transport of a passive scalar (see, e.g., Refs. [5,9], and references therein). Employment of this simple model makes it possible to perform calculations in a closed form for a passive scalar. Note that a model of a linear velocity field can be viewed as (i) an expansion in Taylor series of the velocity field in a local frame of reference moving with a fluid element and (ii) a real flow field in an infinite space (e.g., similar to the Hubble flow in cosmology).

In this study we analyzed some peculiar aspects of a transport of a passive scalar in a linear velocity field. In particular, we studied the Cauchy problem for a passive scalar in both laminar and random linear velocity fields. It is shown that a spatial distribution of a passive scalar evolves either into a thin infinite ''pancake'' or into an infinite ''rope'' structure. The properties of a passive scalar are similar for laminar and random incompressible linear velocity fields. A compressibility of a fluid flow results in a nonzero flux of particles from the infinity. For strong compressibility a spatial distribution of a passive scalar evolves into a ball (or ellipsoid) with very small radius which is of the order of a molecular diffusion scale. We found that higher statistical moments can grow exponentially in spite of a decay of a typical realization of a passive scalar. Such a strange behavior of a passive scalar demonstrates that transport of a passive scalar in linear velocity field cannot be considered as a general and universal phenomenon. Comparison of the obtained results with those for dynamics of a passive scalar (number density of particles and particles mass concentration) advected by a homogeneous, isotropic, and compressible turbulent flow with a given longitudinal two-point correlation function is performed. The dynamics of magnetic field in a linear velocity field was studied in Refs. [10–13]. We demonstrated that a dynamics of the passive scalar field in a linear velocity field is similar to the dynamics of a magnetic field.

II. GOVERNING EQUATIONS

The evolution of a passive scalar [e.g., the number density $n(t,\mathbf{r})$ of small particles] in a compressible fluid flow is determined by the equation

$$\partial n/\partial t+\boldsymbol{\nabla}\cdot(n\mathbf{v})=D\Delta n \tag{1}$$

with an initial condition $n(t=0,\mathbf{r})=n_0(\mathbf{r})$, where D is the coefficient of molecular diffusion, $\mathbf{v}$ is a velocity field. We consider a kinematic problem, i.e., we study the dynamics of a passive scalar in a prescribed linear velocity field $v_i=A_{ij}x_j$. Any arbitrary matrix can be represented as a linear combination of a matrix with a zero trace and a unit matrix, i.e., $A_{ij}=C_{ij}+\delta_{ij}(\mathrm{Tr}\,\mathbf{A}/3)$, where $\mathrm{Tr}\,\mathbf{C}=C_{ii}=0$. Direct calculation yields $b\equiv\boldsymbol{\nabla}\cdot\mathbf{v}=A_{ij}(\partial x_i/\partial x_j)=A_{ij}\delta_{ij}=\mathrm{Tr}\,\mathbf{A}$. Therefore, the linear compressible velocity field can be represented as $v_i=(C_{ij}+b\delta_{ij}/3)x_j$. Note that we do not consider this field as the first term in an expansion in Taylor series of the real velocity field. We also consider here only a smooth linear velocity field (e.g., it is not a δ-correlated time velocity field). Since we study the kinematic problem we do not discuss in this section the boundary conditions for the velocity field at infinity. In a laminar steady velocity field C_{ij} and b are time independent while in a random linear velocity field $C_{ij}=C_{ij}(t)$ and $b=b(t)$. The solution of Eq. (1) is given by

$$n(t,\mathbf{r})=\int n_f(t,\mathbf{k}_0)\exp[i\mathbf{k}(t)\cdot\mathbf{r}]d\mathbf{k}_0, \tag{2}$$

where the wave vector $\mathbf{k}(t)$ is determined by the equation

$$d\mathbf{k}/dt=-(\mathbf{C}^T+b/3)\mathbf{k}, \tag{3}$$

and $\mathbf{k}_0=\mathbf{k}(t=0)$ and $\mathbf{C}^T$ is the matrix transpose to $\mathbf{C}$ (in general it is nondiagonal and time-dependent matrix). Substituting Eqs. (2) and (3) into Eq. (1) we obtain

$$dn_f/dt=-[b+Dk^2(t)]n_f, \tag{4}$$

where $n_f(t=0,\mathbf{k}_0)$ is a Fourier transformation of the initial condition $n_0(\mathbf{r})$, i.e.,

$$n_0(\mathbf{r})=\int n_f(t=0,\mathbf{k}_0)\exp(i\mathbf{k}_0\cdot\mathbf{r})d\mathbf{k}_0 \tag{5}$$

[see Eqs. (2)]. The field $n(t,\mathbf{r})$ we calculate at the vicinity $r\to0$ (i.e., for $\mathbf{v}\to0$). Solution of Eq. (1) in the form given by Eqs. (2)–(4) was found in Ref. [10] for a magnetic field advected by a linear velocity field.

III. INCOMPRESSIBLE LINEAR VELOCITY FIELD

First, we consider an incompressible $(b=0)$ linear velocity field. The solution of Eq. (4) for $b=0$ reads

$$n_f(t,\mathbf{k}_0)=n_f(t=0,\mathbf{k}_0)\exp\left[-D\int_0^t k^2(t')dt'\right], \tag{6}$$

where $\mathbf{k}(t)$ is given by

$$\mathbf{k}(t)=\exp[-\mathbf{C}^T t]\mathbf{k}_0 \tag{7}$$

[see Eq. (3)]. For an arbitrary matrix $\mathbf{C}$ the matrix $\mathbf{T}=\exp(-\mathbf{C}^T t)$ can be calculated using the Jordan representation (see, e.g., Refs. [15,16]). A Jordan form of a general 3×3 matrix $\mathbf{C}$ with $\mathrm{Tr}\,\mathbf{C}=0$ and the corresponding matricies $\mathbf{T}$ are given by

$$\mathbf{C}^{(1)}=\begin{pmatrix}-c_1-c_2 & 0 & 0\\ 0 & c_2 & 0\\ 0 & 0 & c_1\end{pmatrix},$$

$$\mathbf{T}^{(1)}=\begin{pmatrix}\exp[(c_1+c_2)t] & 0 & 0\\ 0 & \exp(-c_2t) & 0\\ 0 & 0 & \exp(-c_1t)\end{pmatrix},$$

$$\mathbf{C}^{(2)}=\begin{pmatrix}-2c_1 & 0 & 0\\ 0 & c_1 & 0\\ 0 & 1 & c_1\end{pmatrix},$$

$$\mathbf{T}^{(2)}=\begin{pmatrix}\exp(2c_1t) & 0 & 0\\ 0 & \exp(-c_1t) & -t\exp(-c_1t)\\ 0 & 0 & \exp(-c_1t)\end{pmatrix},$$

$$\mathbf{C}^{(3)}=\begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix},\quad \mathbf{T}^{(3)}=\begin{pmatrix}1 & -t & t^2/2\\ 0 & 1 & -t\\ 0 & 0 & 1\end{pmatrix}$$

(see Ref. [10]), where the matrix $\mathbf{C}^{(1)}$ corresponds to a case with all different eigenvectors, the matrix $\mathbf{C}^{(2)}$ describes a case with two equal eigenvalues and two independent eigenvectors, and the matrix $\mathbf{C}^{(3)}$ corresponds to a case with three equal eigenvalues and one eigenvector. In all cases except for the pure rotation $(c_1=c_2^*)$ at least of one element of the matrix $\mathbf{T}$ grows with time. In all cases (except for two cases which are described by the matrix $\mathbf{C}^{(2)}$ with $c_1=0$ or $\mathbf{C}^{(3)}$) the growth of matrix $\mathbf{T}$ is exponential. The latter causes a superexponential decay of the Fourier components of the passive scalar (see below).

Indeed, let us consider a general case which is described by the matricies $\mathbf{C}^{(1)}$ and $\mathbf{T}^{(1)}$. We assume that, e.g., $c_1>0$ and $c_2>0$. In this case the wave vector $\mathbf{k}$ decays in the directions $\mathbf{e}_2$ and $\mathbf{e}_3$, and it increases in the direction $\mathbf{e}_1$ (where $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are the eigenvectors of the matrix $\mathbf{C}$). Equation (6) implies that the main contribution to the integral $\int_0^t k^2(t')dt'$ is due to the growing component $k_1(t)$. From the instant t which is determined by

$$D\int_0^t k^2(t')dt'\leqslant1, \tag{8}$$

there is a superexponential decay of the Fourier components of the passive scalar. For times which are smaller than the time t the Fourier components of the passive scalar can be considered as a constant. Equation (8) determines the range of integration in $\mathbf{k}$ space for the integral $\int n_f(t,\mathbf{k}_0)\exp[i\mathbf{k}(t)\cdot\mathbf{r}]d\mathbf{k}_0$ in Eq. (2). The evaluation of this integral yields

$$n(t,\mathbf{r}=0)\sim n_f^* V_k(t), \tag{9}$$

where n_f^* is the characteristic value of the Fourier component of the initial passive scalar field and $V_k(t)$ is the volume in $\mathbf{k}$ space. This volume can be estimated as follows. Substituting Eq. (7) into Eq. (8) and integrating yields

$$D\left[\frac{k_{01}^2}{2(c_1+c_2)}\{\exp[2(c_1+c_2)t]-1\}+\frac{k_{02}^2}{2c_2}+\frac{k_{03}^2}{2c_1}\right]\sim1, \tag{10}$$

where we neglected terms $\sim k_{03}^2\exp(-c_1t)$ and $\sim k_{02}^2\exp(-c_2t)$. On the other hand, the initial components k_{01}, k_{02}, and k_{03} satisfy the inequality

$$k_{01}^2+k_{02}^2+k_{03}^2\leqslant\overline{k_0^2}, \tag{11}$$

where $\overline{k_0}$ is the maximum value of the wave number in the initial distribution of the passive scalar. It follows from Eqs.

(10) and (11) that $k_{02}\sim k_{03}\leqslant\overline{k_0}$ and $k_{01}\leqslant[(c_1+c_2)/D]^{1/2}\exp[-(c_1+c_2)t]$. Here we take into account that $\exp[2(c_1+c_2)t]\gg 1$. Therefore the volume

$$V_k(t)\sim k_{01}k_{02}k_{03}\propto\overline{k_0^2}[(c_1+c_2)/D]^{1/2}\exp[-(c_1+c_2)t]. \tag{12}$$

Equations (9) and (12) yield

$$n(t,\mathbf{r}=0)\propto\exp[-(c_1+c_2)t]. \tag{13}$$

Consider now the spatial structure of the solution $n(t,\mathbf{r})$ for the passive scalar. Equation for the matrix $\mathbf{T}^{(1)}$ implies that the wave vector $\mathbf{k}(t)$ grows in the direction $\mathbf{e}_1$ and it decays in the directions $\mathbf{e}_2$ and $\mathbf{e}_3$. Therefore, the spatial scale of the passive scalar field in the direction $\mathbf{e}_1$ decreases up to the molecular diffusion scale $l^{(1)}\sim[D/(c_1+c_2)]^{1/2}$. On the other hand, the spatial scales $l^{(2)}$ and $l^{(3)}$ of the passive scalar field in the directions $\mathbf{e}_2$ and $\mathbf{e}_3$ increase, i.e., $l^{(2)}\sim\exp(c_2t)$ and $l^{(3)}\sim\exp(c_1t)$. Thus, e.g., if the initial spatial distribution of the passive scalar has the form of a ball, it evolves into a thin ''pancake.'' The thickness of the ''pancake'' $\sim l^{(1)}$, i.e., it is determined by the molecular diffusion. Now we calculate the total number of particles $N^{(\mathrm{tot})}=\int n\,d\mathbf{r}\sim n(t,\mathbf{r}=0)V_r=\mathrm{const}$, where $V_r\sim V_k^{-1}$. We also estimate the higher moments of the passive scalar field $\int n^p d\mathbf{r}$:

$$\int n^p d\mathbf{r}\propto n^p(t,\mathbf{r}=0)V_r\propto\exp[-(p-1)(c_1+c_2)t]. \tag{14}$$

Equation (14) shows that the moments of the passive scalar field $\int n^p d\mathbf{r}$ decay when $p>1$ and they grow when $p<1$. Equation (14) for $p=1$ implies the conservation of the total number of particles.

In the case $c_1>0$, $c_2<0$, and $c_1+c_2>0$ the wave vector $\mathbf{k}(t)$ grows in the directions $\mathbf{e}_1$ and $\mathbf{e}_2$, and it decays in the direction $\mathbf{e}_3$. This results in that the passive scalar evolves into a long ''rope.'' A similar kind of solution for a passive scalar was found in [14] and for a passive vector (magnetic field) in Refs. [11,12].

Now we consider a random linear velocity field with a constant renewal time τ. In this case $\mathbf{C}$ is the random matrix. The solution of Eq. (3) is given by

$$\mathbf{k}(t)=\hat{T}\exp\left[-\int_0^t\mathbf{C}^T(t')\,dt'\right]\mathbf{k}_0 = \lim_{m\to\infty}\Pi_{i=0}^{m-1}[\mathbf{I}-\mathbf{C}^T(t_{m-i})\Delta t], \tag{15}$$

where $\hat{T}\exp[-\int_0^t\mathbf{C}^T(t')\,dt']$ implies the Volterra multiplicative integral or T exponent (see, e.g., Ref. [15]). Note that Eq. (15) cannot be applied in a straightforward manner to a δ-correlated time linear velocity field. We specify $t=m\tau$ and take into account that $\mathbf{T}_t=\Pi_{p=1}^m\mathbf{T}^{(p)}$, where $\mathbf{T}^{(p)}=\hat{T}\exp[-\int_{(p-1)\tau}^{p\tau}\mathbf{C}^T(t')\,dt']$ are the random independent matrices which have the same statistical distributions and $\det\mathbf{T}^{(p)}=1$. The properties of a product of random matricies are described by the Furstenberg theorem (see, e.g., Refs. [10,17–19]). This theorem implies the existence of a random independent set $(\mathbf{E}_1,\mathbf{E}_2,\mathbf{E}_3)$ in which the diagonal elements $T_{\alpha\alpha}$ of the product of random matrices grow exponentially, i.e.,

$$T_{\alpha\alpha}\propto\exp(\gamma_\alpha t+\eta_\alpha t^{1/2}+\cdots), \tag{16}$$

where η_α are Gaussian random variables, and the Lyapunov exponents γ_α satisfy the equation $\gamma_1+\gamma_2+\gamma_3=0$, and $\gamma_1>0$. The random independent set $(\mathbf{E}_1,\mathbf{E}_2,\mathbf{E}_3)$ is determined as follows. For a given realization of the matrix $\mathbf{T}^{(p)}$ there is a random number m for which the product $\Pi_{p=1}^m\mathbf{T}^{(p)}$ is diagonal in the random independent set $(\mathbf{E}_1,\mathbf{E}_2,\mathbf{E}_3)$ with the accuracy $O[\exp(-am)]$, where $a>0$. The independent set $(\mathbf{E}_1,\mathbf{E}_2,\mathbf{E}_3)$ varies from a realization to a realization. We can apply the Furstenberg theorem to the matrix $\mathbf{T}_t$. Therefore Eqs. (13) and (14) after the change $c_i\to\gamma_i$ are valid for the random linear velocity field at $t\gg\tau$, i.e.,

$$n(t,\mathbf{r}=0)\propto\exp[-(\gamma_1+\gamma_2)t],$$

$$\int n^p d\mathbf{r}\propto\exp[-(p-1)(\gamma_1+\gamma_2)t].$$

Thus when $p>1$ the higher moments $\int n^p d\mathbf{r}$ decay. Therefore, the dynamics of the passive scalar in both random and laminar incompressible linear velocity fields are similar.

IV. COMPRESSIBLE LINEAR VELOCITY FIELD

Now we consider a compressible linear velocity field. In this case there is a growth of a Fourier component of the passive scalar when $b<0$. Indeed, the solution of Eq. (4) reads

$$n_f(t,\mathbf{k}_0)=n_0\exp\left[-bt-D\int_0^t k^2(t')\,dt'\right], \tag{17}$$

where the function $\mathbf{k}(t)$ is given by

$$\mathbf{k}(t)=\exp(|b|t/3)\exp[-\mathbf{C}^Tt]\mathbf{k}_0.$$

Equation (8) in this case becomes

$$bt+D\int_0^t k^2(t')\,dt'\leqslant 1. \tag{18}$$

Now we consider the case when the passive scalar evolves into a ''pancake,'' i.e., when $b<0$, $c_1>|b|/3$, and $c_2>|b|/3$. The volume $V_k(t)$ is given by

$$V_k(t)\propto\overline{k_0^2}\left[\left(c_1+c_2+\frac{|b|}{3}\right)\left(\frac{|b|t+1}{D}\right)\right]^{1/2}\times\exp\left[-\left(c_1+c_2+\frac{|b|}{3}\right)t\right]. \tag{19}$$

Therefore, the evolution of a passive scalar is determined by

$$n(t,\mathbf{r}=0)\propto\exp(|b|t)V_k(t)\propto(|b|t+1)^{1/2}\times\exp\left[\left(\frac{2|b|}{3}-c_1-c_2\right)t\right]. \tag{20}$$

The higher moments of the passive scalar field $\int n^p d\mathbf{r}$ are given by

$$\int n^p d\mathbf{r}\propto(|b|t+1)^{(p-1)/2}\times\exp\left[\left(\frac{2p+1}{3}|B|-(p-1)(c_1+c_2)\right)t\right]. \tag{21}$$

Equation (21) implies that for $p=1$ the integral $\int n d\mathbf{r}$ $\propto\exp(|b|t)$. Thus the total number of particles is not conserved for a compressible linear velocity field. The reason for this unphysical behavior is that the obtained spatial distributions of particles (in the form of ''ropes'' or ''pancake'') imply the nonzero inflow or outflow of particles from any finite control volume.

Now we consider the case when the passive scalar evolves into a ''rope,'' i.e., when $b<0$, $c_1>|b|/3$, and $c_2<0$ and $c_1+c_2-b/3>0$. The volume $V_k(t)$ is given by

$$V_k(t)\propto\left(\frac{1+|b|t}{D}\right)\exp\left[-\left(c_1+\frac{2|b|}{3}\right)t\right], \tag{22}$$

and the function $n(t,\mathbf{r}=0)\propto\exp(|b|t)V_k(t)$ is determined by

$$n(t,\mathbf{r}=0)\propto\left(\frac{1+|b|t}{D}\right)\exp\left[-\left(c_1-\frac{|b|}{3}\right)t\right]. \tag{23}$$

Therefore the total number of particles $\int n d\mathbf{r}\sim n(t,\mathbf{r}=0)/V_k(t)\propto\exp(|b|t)$ is not conserved as well as in the previous case. When $|b|\gg|c_i|$ (in this case the passive scalar distribution evolves into a ''ball'') the volume

$$V_k(t)\propto\left(\frac{|b|t}{D}\right)^{3/2}\exp(-|b|t), \tag{24}$$

and the function $n(t,\mathbf{r}=0)\propto\exp(|b|t)V_k(t)\propto(|b|t/D)^{3/2}$, and the total number of particles $\int n d\mathbf{r}\sim n(t,\mathbf{r}=0)/V_k(t)\propto\exp(|b|t)$.

Now we consider a random linear compressible velocity field: $v_i=[C_{ij}(t)+(1/3)b(t)\delta_{ij}]x_j$, where the matrix C_{ij} and b are independent random processes. The solution of Eq. (3) for the wave vector is given by

$$\mathbf{k}(t)=\exp(-A(t)/3)\hat{T}\exp\left[-\int_0^t\mathbf{C}^T(t')dt'\right]\mathbf{k}_0, \tag{25}$$

where $A(t)=\int_0^t b(t')dt'$. We assume that $b(t)$ is a Gaussian random process with $\langle b\rangle=0$ and $\langle b(t)b(t+\tau)\rangle=B(\tau)$. In this case $\langle A(t)\rangle=0$ and $\langle A(t)A(t)\rangle=\int_0^t\int_0^t B(t''-t')dt'dt''$ $=2\int_0^t B(\tau)(t-\tau)d\tau\sim 2t\tau_0 B(0)$, where $\tau_0=\int_0^\infty B(\tau)d\tau/B(0)$, and $t\gg\tau_0$. For $t\gg\tau$ we use the Furstenberg theorem, which yields

$$T_{\alpha\alpha}\propto\exp[\gamma_\alpha t+(\eta_\alpha+\beta_0\beta/3)t^{1/2}+\cdots], \tag{26}$$

where $\beta_0=[2\tau_0 B(0)]^{1/2}$ and β is the Gaussian random process with zero mean value and a unit dispersion. Therefore the compressibility does not affect essentially the realizations of a passive scalar field. Indeed, the volume V_k decreases, i.e.,

$$V_k(t)\propto\exp[-\gamma_1 t-(\eta_1+\beta_0\beta/3)t^{1/2}]. \tag{27}$$

Since $n(t,\mathbf{r}=0)\propto n_\gamma V_k$ and $n_\gamma\propto\exp[\beta_0\beta t^{1/2}]$ we obtain that

$$n(t,\mathbf{r}=0)\propto\exp[-\gamma_1 t-(\eta_1-2\beta_0\beta/3)t^{1/2}]. \tag{28}$$

Therefore for large t the passive scalar field $n(t,\mathbf{r}=0)$ decreases. The spatial distribution of a passive scalar evolves into either a ''pancake'' or a ''rope.'' On the other hand, this field is homogeneous along a ''rope'' or inside a ''pancake.'' Therefore, the passive scalar field $n(t,\mathbf{r})$ decreases also with time. Now we calculate the higher-order statistical moments

$$\langle n^p\rangle\geqslant\exp[(4\beta_0^2 p/9-\gamma_1)pt]. \tag{29}$$

The latter equation shows that the higher-order statistical moments can grow in time in spite of the decay of a typical realization of the passive scalar field. Equation (29) implies a log-normal statistics for the passive scalar field.

V. DISCUSSION

In this study we analyzed some peculiar aspects of a transport of passive scalar in a linear laminar and random velocity fields. We demonstrated that all statistical moments higher than the first moment and a typical realization of a passive scalar decay for both laminar and random incompressible linear velocity fields. Strong compressibility of a laminar linear velocity field can result in a growth of a typical realization and of the high-order moments of a passive scalar. For strong compressibility a passive scalar distribution evolves into a ball (or ellipsoid) with very small radius which is of the order of a molecular diffusion scale.

Now we will compare obtained results with those for a dynamics of a passive scalar [number density of particles (see Sec. V A) and particles mass concentration (see Sec. V B)] in a homogeneous, isotropic and compressible turbulent flow with a given longitudinal two-point correlation function $F=1-r^2$. We also will compare the obtained results with those for the dynamics of a passive vector (magnetic field) advected by linear velocity field (see Sec. V C).

A. Dynamics of particles number density in a homogeneous, isotropic, and compressible turbulent flow

Consider first a passive scalar (i.e., the number density of particles) advected by a δ-correlated time random velocity field. An equation for the second-order correlation function $\Phi = \langle \Theta(\mathbf{x}) \Theta(\mathbf{y}) \rangle$ of particles concentration reads

$$\frac{\partial \Phi}{\partial t} = -2[D\delta_{pm} + D_{mn}(0) - D_{mn}(\mathbf{r})] \frac{\partial^2 \Phi}{\partial x_m \partial y_n} + 2\langle \tau b(\mathbf{x}) b(\mathbf{y}) \rangle \Phi - 4 \langle \tau v_m(\mathbf{x}) b(\mathbf{y}) \rangle \frac{\partial \Phi}{\partial y_m} + I \tag{30}$$

(see Ref. [7]), where $\Theta = n - N$, $\mathbf{r} = \mathbf{y} - \mathbf{x}$, and $I = 2\langle \tau b(\mathbf{x}) b(\mathbf{y}) \rangle N^2$, and $D_{pm}(\mathbf{r}) = \langle \tau v_p(\mathbf{x}) v_m(\mathbf{y}) \rangle$, and $N = \langle n \rangle$ is the mean number density of particles. The correlation function of a compressible homogeneous and isotropic random velocity field is given by

$$\langle v_m(\mathbf{x}) v_n(\mathbf{y}) \rangle = (u_0^2/d)[(F + F_c)\delta_{mn} + [rF'/(d-1)]P_{mn} + rF_c' r_{mn}] \tag{31}$$

(see Ref. [20]), where $P_{mn}(r) = \delta_{mn} - r_{mn}$, $r_{mn} = r_m r_n / r^2$, $F' = dF/dr$, $F(0) = 1 - F_c(0)$, d is the dimensionality of space, u_0 is the characteristic velocity in the maximum scale of turbulent motions l_0. The function $F_c(r)$ describes the potential component whereas $F(r)$ corresponds to the vortical part of the turbulent velocity of particles. Consider the case with $I = 0$. We seek a solution to Eq. (30) in the form

$$\Phi(t,r) = \Psi(r) r^{(1-d)/2} \exp\left[-\int_0^r \chi(x)\, dx \right] \exp(\gamma t), \tag{32}$$

where the function χ is defined below. Consider three-dimensional velocity field ($d = 3$). Substituting Eq. (32) into Eq. (30) yields an equation for the unknown function $\Psi(r)$:

$$\Psi''/m(r) - [\gamma + U(r)]\Psi = 0, \tag{33}$$

where $U(r) = m^{-1}(2\chi/r + \chi^2 + \chi') - \kappa(r)$, $m^{-1}(r) = (2/\mathrm{Pe}) + (2/3)[1 - F - (rF_c)']$, $\chi(r) = -m(r)(6F_c' + F' + 2rF_c'')/3$, $\kappa(r) = -2[8F_c'/r + 7F_c'' + rF_c''']/3$, and distance r is measured in units of l_0, time t is measured in units of $\tau_0 = l_0/u_0$, and $\mathrm{Pe} = l_0 u_0 / D \gg 1$ is the Peclet number. We consider a random velocity field with $F(r) = (1 - r^2)/(1 + \sigma)$, $F_c(r) = \sigma(1 - r^2)/(1 + \sigma)$ for $0 \leqslant r \leqslant 1$, and $F(r) = F_c(r) = 0$ for $r > 1$, where $\sigma = \langle (\boldsymbol{\nabla} \cdot \mathbf{v})^2 \rangle / \langle (\boldsymbol{\nabla} \times \mathbf{v})^2 \rangle$ is the degree of compressibility. When $\sigma = 0$ the velocity field is incompressible.

A solution of Eq. (33) can be obtained using an asymptotic analysis (see, e.g., Refs. [1,2,7,20]). This analysis is based on the separation of scales. In particular, the solution of the Schrödinger equation (33) with a variable mass has two regions where the form of the potential $U(r)$, mass $m(r)$ and, therefore, eigenfunctions $\Psi(r)$ are different. The functions Φ and Φ' in these different regions can be matched at their boundary. Note that the most important part of the solution is localized in small scales (i.e., $r \ll 1$). The results obtained by this asymptotic analysis are presented below. In region I, i.e., for $0 \leqslant r \leqslant 1$ the function $\chi = (2\delta/3)m(r)r$ and $\kappa = 20\sigma/(1+\sigma)$ and $1/m = 2(1 + X^2)/\mathrm{Pe}$, where $\delta = (8\sigma + 1)/(1 + 3\sigma)$, $X = (\beta_m \mathrm{Pe})^{1/2} r$ and $\beta_m = (1 + 3\sigma)/3(1 + \sigma)$. The potential $U(r)$ and the functions $\Psi(r)$ and $\Phi(r)$ in this region are given by

$$U = 2\beta_m \left[\delta(\delta + 1) - \frac{\delta(\delta - 2)}{1 + X^2} \right] - \kappa,$$

$$\Psi = (1 + X^2)^{1/2} L(X), \quad \Phi(r) = (1 + X^2)^{\lambda} L(X)/X, \tag{34}$$

where $L(X) = \mathrm{Re}\{A_1 P_\zeta^\mu(iX) + A_2 Q_\zeta^\mu(iX)\}$ is a real part of the complex function, $P_\zeta^\mu(Z)$ and $Q_\zeta^\mu(Z)$ are the Legendre functions with imaginary argument $Z = iX$, $\lambda = \sigma(\sigma - 3)/2(1 + 3\sigma)^2$, $\mu = \delta - 1 = 5\sigma/(1 + 3\sigma)$, $\zeta = -1/2 + \sqrt{\nu_I}$, and

$$\nu_I = (\mu - 3/2)^2 - |\gamma|/2\beta_m . \tag{35}$$

The correlation function has a global maximum at $r = 0$ and therefore it satisfies the conditions $\Phi'(r = 0) = 0$, and $\Phi''(r = 0) < 0$, and $\Phi(r = 0) > |\Phi(r > 0)|$. Condition $\Phi(r = 0) = 1$ implies that $L(X = 0) = 0$ and $L'(X = 0) = 1$. The function Φ for $X \ll 1$ [i.e., for $0 \leqslant r \ll \mathrm{Pe}^{-1/2}$] is given by $\Phi \sim 1 - [(\kappa - \gamma)/12\beta_m][X^2 + O(X^4)]$. The function Φ for $X \gg 1$ [i.e., for $\mathrm{Pe}^{-1/2} \ll r \leqslant 1$] is given by

$$\Phi = X^{-\alpha}(\tilde{A}_1 X^{\zeta_I} + \tilde{A}_2 X^{-\zeta_I}) \tag{36}$$

for $\nu_I > 0$, and

$$\Phi = X^{-\alpha}(\hat{A}_1 + \hat{A}_2 \ln X) \tag{37}$$

for $\nu_I = 0$, where $\zeta_I = \sqrt{|\nu_I|}$ and $\alpha = (25\sigma^2 + 24\sigma + 3)/2(1 + 3\sigma)^2$. Note that in the model of the velocity field with $F \propto 1 - r^2$ the solution $\Phi = BX^{-\alpha}\cos(\zeta_I \ln X + \varphi_1)$ in the range $X \gg 1$ for $\nu_I < 0$ does not exist. However, it can exist in the model of the velocity field with $F \propto \exp(-r^2)$ (see Ref. [20]). The solution of Eq. (33) for $1 < r < L_0$ is given by

$$\Phi = A_3 r^{-1} \sin[(L_0 - r)\sqrt{3|\gamma|/2}], \tag{38}$$

and when $r \geqslant L_0$ the correlation function $\Phi = 0$.

For the mode with $\nu_I = 0$ the damping rate of passive scalar fluctuations is given by

$$\gamma = -\frac{(\sigma - 3)^2}{6(1 + \sigma)(1 + 3\sigma)}, \tag{39}$$

where we used Eq. (35). Now we consider the mode with $\nu_I > 0$. Matching the functions Φ and Φ' at the boundary between two regions (i.e., at $r = 1$) yields the equation for ζ_I. In particular, the matching of the functions Φ and Φ' determined by Eqs. (36) and (38) yields

$$\zeta_I^2=\{\alpha-1-\sqrt{3|\gamma|/2}\cot[(L_0-1)\sqrt{3|\gamma|/2}]\}^2. \quad (40)$$

Using Eqs. (35) and (40) we can obtain the damping rate of passive scalar fluctuations for the mode with $\nu_I>0$. Equation (39) for the damping rate of fluctuations of the number density of particles is different from that obtained in a linear velocity field [compare with Eq. (29) for $p=2$].

B. Dynamics of particle mass concentration in a homogeneous, isotropic, and compressible turbulent flow

Now we consider the dynamics of mass concentration $A=m_p n/\rho$, where m_p is the mass of particles, ρ is the density of fluid. An equation for the evolution of the mass concentration A in a compressible turbulent fluid flow reads

$$\partial A/\partial t+(\mathbf{v}\cdot\boldsymbol{\nabla})A=D\Delta A. \quad (41)$$

This equation follows from Eq. (1) and the continuity equation for a fluid:

$$\partial\rho/\partial t+\boldsymbol{\nabla}\cdot(\rho\mathbf{v})=0.$$

The dynamics of mass concentration advected by a homogeneous, isotropic and compressible turbulent three-dimensional velocity field with a given longitudinal two-point correlation function $F=1-r^2$ was first studied in Ref. [20]. It was shown that fluctuations of the mass concentration can be only damped. The damping rate of the two-point correlation function Φ for fluctuations of the mass concentration is given by

$$\gamma=-\frac{(1+3\sigma)}{6(1+\sigma)}\left[\left(\frac{\sigma-3}{1+3\sigma}\right)^2+\left(\frac{4\pi(k+1/2)}{\ln(\mathrm{Pe})}\right)^2\right] \quad (42)$$

(for details see Ref. [20]), where $0\leqslant\sigma\leqslant3$. When the degree of compressibility of the fluid flow $\sigma\to3$ the damping rate of fluctuations of the mass concentration can be strongly reduced for large Peclet numbers. In Ref. [20] this effect was interpreted as a strong depletion of the scale-dependent turbulent diffusion caused by compressibility of fluid flow. Later, a similar behavior was observed in the study of the passive scalar (mass concentration) advected by a linear random velocity field (see Ref. [9]). In Ref. [9] this effect was interpreted as an inverse cascade which is caused by the compressibility of a linear velocity field. It was found in Ref. [9] that when $d<4$ and $\langle(\boldsymbol{\nabla}\cdot\mathbf{v})^2\rangle/S^2>d/4$ the cascade of the passive scalar A is inverse; otherwise it is direct, where $S^2=\langle(\boldsymbol{\nabla}_k v_m)^2\rangle$. Using Eq. (31) we get $\langle(\boldsymbol{\nabla}\cdot\mathbf{v})^2\rangle=-d(d+2)(F_c'/r)_{r\to0}$ and $S^2=-d(d+2)[(F'+F_c')/r]_{r\to0}$, i.e., $\langle(\boldsymbol{\nabla}\cdot\mathbf{v})^2\rangle/S^2=\sigma/(1+\sigma)$. This implies that the condition for the inverse cascade of the mass concentration is $\sigma>d/(4-d)$, where $d<4$. Thus, the condition for the inverse cascade of mass concentration advected by a linear random velocity field obtained in Ref. [9] is $\sigma>3$ (for three-dimensional velocity field), and $\sigma>1$ (for two-dimensional velocity field). On the other hand, the condition for the similar phenomena for mass concentration advected by a homogeneous, isotropic, and compressible turbulent flow is $\sigma\to3$. This demonstrates the difference between the dynamics of particle mass concentration in a random linear velocity field and in a homogeneous and isotropic three-dimensional turbulent flow.

Now we study the dynamics of mass concentration advected by a homogeneous, isotropic, and compressible turbulent two-dimensional velocity field with a given longitudinal two-point correlation function $F=1-r^2$. An equation for the second-order correlation function $\Phi_0=\langle\theta(\mathbf{x})\theta(\mathbf{y})\rangle$ of mass concentration in two-dimensional turbulent velocity field reads

$$\partial\Phi_0/\partial t=m_0^{-1}\Phi_0''+(\Phi_0'/r)(m_0^{-1}+rF'-rF_c'), \quad (43)$$

where $\theta=A-\langle A\rangle$, $m_0^{-1}(r)=(2/\mathrm{Pe})+1-F-(rF_c)'$. We seek a solution to Eq. (43) in the form $\Phi_0(t,r)=\Psi_0(r)r^{-1/2}\exp[-\int_0^r\chi_0(x)dx]\exp(\gamma t)$, where the unknown function $\Psi_0(r)$ is determined by the equation

$$\Psi_0''/m_0(r)-[\gamma+U_0(r)]\Psi_0=0, \quad (44)$$

$U_0(r)=m_0^{-1}(\chi_0/r+\chi_0^2+\chi_0'-1/4r^2)$, and $\chi_0(r)=m_0(r)(F'-F_c')/2$. The asymptotic analysis shows that the correlation function Φ_0 for $0\leqslant r\ll\mathrm{Pe}^{-1/2}$ is given by $\Phi_0=J_0(\sqrt{h}Z)\sim1-(|\gamma|/12\beta_m)Z^2$, where $Z=[(3/2)\beta_m\mathrm{Pe}]^{1/2}r$, $J_0(Z)$ is the Bessel function of the first kind, $h=2\alpha+|\gamma|/3\beta_m$ and $\alpha=(1-\sigma)/(1+3\sigma)$. The function Φ_0 for $Z\gg1$ [i.e., for $\mathrm{Pe}^{-1/2}\ll r\leqslant1$] is given by $\Phi=B_1Z^\alpha\sin(\xi_I\ln Z)$, where $\xi_I=|\gamma|/3\beta_m-\alpha^2$, and $\alpha\leqslant0$, i.e., $\sigma\geqslant1$. For $r>1$ the correlation function is $\Phi_0=B_2J_0(\sqrt{|\gamma|}r)+B_3Y_0(\sqrt{|\gamma|}r)$, where $Y_0(Z)$ is the Bessel function of the second kind.

Matching the functions Φ_0 and Φ_0' at the boundary between two regions (i.e., at $r=1$) yields $\xi_I=2\pi k/\ln(\mathrm{Pe})$, where $k=1,2,\ldots$. Therefore, the damping rate of the two-point correlation function Φ_0 for fluctuations of the mass concentration in two-dimensional turbulent velocity field is given by

$$\gamma=-\frac{(1+3\sigma)}{(1+\sigma)}\left[\left(\frac{\sigma-1}{1+3\sigma}\right)^2+\left(\frac{2\pi k}{\ln(\mathrm{Pe})}\right)^2\right], \quad (45)$$

where $\sigma\geqslant1$. When the degree of compressibility of the fluid flow $\sigma\to1$ the damping rate of fluctuations of the mass concentration can be strongly reduced for large Peclet numbers. The latter implies the inverse cascade of mass concentration. On the other hand, the condition for the inverse cascade of the mass concentration advected by two-dimensional turbulent linear velocity field obtained in Ref. [9] is $\sigma>1$. This also demonstrates the difference between the dynamics of particles mass concentration in a random linear velocity field and in a homogeneous and isotropic two-dimensional turbulent flow.

The reason for the difference between the dynamics of a passive scalar (number density of particles and particles mass concentration) in a random linear velocity field and the dynamics of a passive scalar in a homogeneous and isotropic turbulent flow with a given longitudinal two-point correlation function $F=1-r^2$ is as follows. A linear velocity field does not have a correlation radius, whereas the velocity field with the correlation function $F=1-r^2$ has a unit correlation radius. A linear velocity field can be considered as an expansion in Taylor series of the velocity field in a local frame attached to the fluid element in the form $v_i - v_i^0 = C_{ij} r_j$. However, this expansion is local, while the dynamics of particles in a linear velocity field is not local. Indeed, all particles are carried out from any small volume during a finite time. On the other hand, particles from other locations flow into a given local volume. This outflow and inflow determine the nonlocal dynamics of particles. In addition, a linear velocity field is not homogeneous since at $\mathbf{r}\to\infty$ the velocity $v_i\to\infty$. If we consider matching of the solution for a passive scalar in a linear velocity field $v_i - v_i^0 = C_{ij} r_j$ (which is valid for small r) with a solution for a passive scalar which vanish at the infinity, the final results can depend on the matching procedure.

C. Dynamics of a magnetic field in a linear velocity field

Now, we compare the obtained results for a passive scalar with those for the dynamics of magnetic field in a linear velocity field [10–13]. The magnetic field $\mathbf{H}$ is determined by the induction equation

$$\partial H_i/\partial t + (\mathbf{v}\cdot\boldsymbol{\nabla})H_i = (\mathbf{H}\cdot\boldsymbol{\nabla})v_i + \eta \Delta H_i, \qquad (46)$$

where η is the magnetic diffusion. A solution of Eq. (46) is similar to the solution which is determined by Eqs. (2) and (3) (see Ref. [10]). The analysis of the dynamics of magnetic field in a linear velocity field demonstrates a behavior which is similar to the dynamics of the passive scalar field: typical realization of magnetic field decays and the higher-order statistical moments $\int |\mathbf{H}|^p d\mathbf{r}$ can grow if $p<p_*$, where a threshold p_* depends on c_j. The behavior of both magnetic field and passive scalar field in a random and a laminar velocity field is similar. The difference between the dynamics of a magnetic field and a passive scalar field is that, e.g., the threshold p_* for a passive scalar in the incompressible linear velocity field is independent of c_j, i.e., $p_*=1$. This difference is caused by that the magnetic field is a divergence-free vector field and opposite-directed magnetic field lines annihilate when they approach each other.

ACKNOWLEDGMENTS

This work was partially supported by The German-Israeli Project Cooperation (DIP) administered by the Federal Ministry of Education and Research (BMBF). D.S. is grateful for a special fund for visiting senior scientists of the Faculty of Engineering of the Ben-Gurion University of the Negev and to RFBF for financial support under Grant No. 97-05-64797.

[1] Ya. B. Zeldovich, S. A. Molchanov, A. A. Ruzmaikin, and D. D. Sokoloff, Sov. Sci. Rev. Sect. C **7**, 1 (1988), and references therein.

[2] Ya. B. Zeldovich, A. A. Ruzmaikin, and D. D. Sokoloff, *The Almighty Chance* (Word Scientific, London, 1990), and references therein.

[3] K. Gawedzki and A. Kupiainen, Phys. Rev. Lett. **75**, 3834 (1995).

[4] B. I. Shraiman and E. D. Siggia, C. R. Acad. Sci. **321**, 279 (1995).

[5] M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. E **51**, 5609 (1995).

[6] T. Elperin, N. Kleeorin, and I. Rogachevskii, Phys. Rev. Lett. **76**, 224 (1996); Phys. Rev. E **55**, 2713 (1997).

[7] T. Elperin, N. Kleeorin, and I. Rogachevskii, Phys. Rev. Lett. **77**, 5373 (1996); **80**, 69 (1998); Phys. Rev. E **58**, 3113 (1998).

[8] R. H. Kraichnan, Phys. Fluids **11**, 945 (1968).

[9] M. Chertkov, I. Kolokolov, and M. Vergassola, Phys. Rev. Lett. **80**, 512 (1998).

[10] Ya. B. Zeldovich, A. A. Ruzmaikin, S. A. Molchanov, and D. D. Sokoloff, J. Fluid Mech. **144**, 1 (1984).

[11] H. K. Moffatt, J. Fluid Mech. **17**, 225 (1963).

[12] A. Clarke, Phys. Fluids **7**, 1299 (1964).

[13] V. V. Gvaramadze, J. G. Lominadze, A. A. Ruzmaikin, A. M. Shukurov, and D. D. Sokoloff, Astrophys. Space Sci. **140**, 165 (1988).

[14] J. R. A. Pearson, J. Fluid Mech. **5**, 274 (1959).

[15] F. R. Gantmacher, *The Theory of Matrices* (Chelsea, New York, 1974).

[16] P. Hartman, *Ordinary Differential Equations* (Wiley, New York, 1964).

[17] H. Furstenberg, Trans. Am. Math. Soc. **108**, 377 (1963).

[18] V. N. Tutubalin, Prob. Theory Appl. **17**, 266 (1972).

[19] A. Crisanti, G. Paladin, and A. Vulpiani, *Products of Random Matrices in Statistical Physics* (Springer-Verlag, Berlin, 1993).

[20] T. Elperin, N. Kleeorin, and I. Rogachevskii, Phys. Rev. E **52**, 2617 (1995).