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Intermittency and anomalous scaling for magnetic fluctuations

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The generation of magnetic fluctuations by turbulent flow of conducting fluid with a zero mean magnetic field for small magnetic Prandtl numbers is studied. The equation for the high-order correlation functions of the magnetic field is derived. It is shown that the growth rates of the higher moments of the magnetic field are larger than those of the lower moments, i.e., the spatial distribution of the magnetic field is intermittent. The problem of anomalous scaling of magnetic fluctuations is discussed as well. The turbulent fluid velocity field is assumed to be homogeneous and isotropic with a power-law energy spectrum (proportional to k^{-p}) and with a very short scale-dependent correlation time. It is found that magnetic fluctuations can be generated when the exponent $p>3/2$. In addition, the effect of compressibility (i.e., $\nabla\cdot\mathbf{u}\neq 0$) of the low-Mach-number turbulent fluid flow $\mathbf{u}$ is studied. It is shown that the threshold for the generation of magnetic fluctuations by turbulent fluid flow with $\nabla\cdot\mathbf{u}\neq 0$ is higher than that for incompressible fluid. This implies that the compressibility impairs the generation of magnetic fluctuations. [S1063-651X(97)07406-0]

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I. INTRODUCTION

The problem of the generation of magnetic fluctuations by turbulent flow of conducting fluid is of fundamental importance in view of various applications in astrophysical and laboratory plasmas (see, e.g., [1–11]). Analytical and numerical studies of intermittency and the generation of magnetic fluctuations by homogeneous, isotropic, and incompressible turbulence with a zero mean magnetic field were carried out mainly for magnetic Prandtl numbers $\mathrm{Pr}_m\geqslant 1$ (see, e.g., [12–17]). The study of magnetic fluctuations with $\mathrm{Pr}_m\ll 1$ shows that magnetic fluctuations cannot be generated by turbulent fluid flow with the Kolmogorov energy spectrum [18,19]. In addition, in numerical simulations by [20,21] the generation of magnetic fluctuations for $\mathrm{Pr}_m\ll 1$ were not observed.

However, in astrophysical plasmas the magnetic Prandtl numbers is small ($\mathrm{Pr}_m\ll 1$). Thus a mechanism for the generation of magnetic fluctuations for $\mathrm{Pr}_m\ll 1$ still remains poorly understood. On the other hand, in astrophysical applications (e.g., accretion disks, solar and stellar convection zones, and galaxies) the turbulent velocity field cannot be considered as divergence-free.

In the present paper we study the generation of magnetic fluctuations with a zero mean magnetic field for $\mathrm{Pr}_m\ll 1$. The turbulent fluid velocity field is assumed to be homogeneous and isotropic with a very short scale-dependent correlation time. We have found that magnetic fluctuations can be generated by turbulent motions of conducting fluid flow even with the Kolmogorov energy spectrum $\mathrm{Pr}_m\ll 1$. The equation for the high-order correlation functions of the magnetic field is derived. It is shown that the spatial distribution of the magnetic field is intermittent. In addition, we study the effect of compressibility (i.e., $\nabla\cdot\mathbf{u}\neq 0$) of the low-Mach-number turbulent fluid flow $\mathbf{u}$ on the generation of magnetic fluctuations.

II. GOVERNING EQUATIONS

In this section we describe dynamics of magnetic fluctuations. The induction equation in a compressible (i.e., $\nabla\cdot\mathbf{v}\neq 0$) fluid flow reads

$$\frac{\partial\mathbf{H}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{H}=(\mathbf{H}\cdot\nabla)\mathbf{v}-\mathbf{H}(\nabla\cdot\mathbf{v})+\eta\Delta\mathbf{H}, \tag{1}$$

where $\eta=c^2/4\pi\sigma$ is the magnetic diffusion, σ is the electrical conductivity, and c is the speed of light. We derive equations for the mean field and for the high-order correlation functions of the magnetic field. To this purpose we use the stochastic calculus that was applied in magnetohydrodynamics [15–17] and passive scalar transport in incompressible [15,16,22] and compressible [23–25] turbulent flows. Magnetic diffusion in this method is described by means of an average over an ensemble of random Wiener trajectories. The solution of the induction equation (1) with the initial condition $\mathbf{H}(t=t_0,\mathbf{x})=\mathbf{H}_0(\mathbf{x})$ is given by a modified Feynman-Kac formula [15,16]

$$H_i(t,\mathbf{x})=M\{G_{ij}(t,t_0)H_{0j}[\xi(t,t_0)]\}, \tag{2}$$

where the function G_{ij} is determined by the equation

$$\frac{d}{ds}G_{ij}(t_s,t_0)=N_{ik}G_{kj}, \tag{3}$$

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with the initial condition $G_{ij}=\delta_{ij}$ for $t_s=t_0$. Here $M\{\ \}$ is a mathematical expectation over the ensemble of Wiener paths $t_s=t+s$,

$$N_{ik}=\frac{\partial v_i}{\partial x_k}-\delta_{ik}\nabla\cdot\mathbf{v}, \tag{4}$$

and the Wiener path $\boldsymbol{\xi}_t\equiv\boldsymbol{\xi}(t,t_0)$ is given by

$$\boldsymbol{\xi}_t=\mathbf{x}-\int_{t_0}^{t}\mathbf{v}(t_s,\xi_s)ds+\sqrt{2\eta}\mathbf{w}(t), \tag{5}$$

where $\mathbf{w}_t$ is a Wiener process. Equation (5) describes a set of the random trajectories that pass through the point $\mathbf{x}$ at time t. The use of this technique of path integrals allows us to derive the equation for the mean magnetic field $\mathbf{B}=\langle\mathbf{H}\rangle$,

$$\frac{\partial\mathbf{B}}{\partial t}=\nabla\times(\mathbf{V}_{\rm eff}\times\mathbf{B})+\boldsymbol{\nabla}_i(\eta_{im}\boldsymbol{\nabla}_m\mathbf{B}+B_i\boldsymbol{\nabla}_k\langle\tau u_k\mathbf{u}\rangle-2B_k\langle\tau u_i\boldsymbol{\nabla}_k\mathbf{u}\rangle) \tag{6}$$

(see details in the Appendix), where

$$\eta_{pm}=\eta\delta_{pm}+\langle\tau u_p u_m\rangle, \tag{7}$$

$$\mathbf{V}_{\rm eff}=\mathbf{V}-\langle\tau\mathbf{u}(\boldsymbol{\nabla}\cdot\mathbf{u})\rangle, \tag{8}$$

τ is the scale-dependent momentum relaxation time, $\mathbf{V}=\langle\mathbf{v}\rangle$ is the mean velocity field, and the angular brackets mean statistical averaging. We introduce a tensor α_{mn} and a vector $\mathbf{q}$,

$$\alpha_{mn}=(\varepsilon_{mji}\langle\tau u_i\nabla_n u_j\rangle+\varepsilon_{nji}\langle\tau u_i\nabla_m u_j\rangle)/2, \tag{9}$$

$$q_m=\langle\tau u_m(\nabla\cdot\mathbf{u})\rangle-\nabla_n\eta_{nm}/2, \tag{10}$$

which satisfy the identity

$$\langle\tau u_i\nabla_n u_k\rangle=(\varepsilon_{ikm}\alpha_{mn}+\delta_{kn}q_i-\delta_{in}q_k+\nabla_n\eta_{ki})/2,$$

where δ_{mn} is the Kronecker tensor and ε_{ikm} is the Levi-Civita tensor. The tensor α_{mn} describes the α effect. In isotropic turbulence $\langle u_i u_j\rangle=\langle\mathbf{u}^2\rangle\delta_{ij}/3$ and Eq. (9) reduces to the well-known formula for the α effect (see, e.g., [1–4]), $\alpha_{ij}=-\langle\tau\mathbf{u}(\nabla\times\mathbf{u})\rangle\delta_{ij}/3$. Equation (10) for the vector $\mathbf{q}$ in the case of incompressible isotropic turbulence coincides with the well-known formula for the velocity that describes turbulent diamagnetism (see, e.g., [26,4]). Using Eqs. (9) and (10), we rewrite Eq. (6) for the mean magnetic field as

$$\frac{\partial\mathbf{B}}{\partial t}=\nabla\times(\mathbf{U}_{\rm eff}\times\mathbf{B}+\hat{\alpha}\mathbf{B}-\hat{\eta}\nabla\times\mathbf{B}), \tag{11}$$

where $\hat{\eta}=(\eta_{pp}\delta_{ij}-\eta_{ij})/2$, $\hat{\alpha}=\alpha_{ij}$, and $\mathbf{U}_{\rm eff}=\mathbf{V}-\nabla_p\langle\tau u_p\mathbf{u}\rangle/2$. Equation (11) is the induction equation for the mean magnetic field in a low-Mach-number compressible turbulent fluid flow. This equation coincides in form with that for incompressible turbulent fluid flow (see, e.g., [1–4]). Therefore, compressibility does not change the form of the induction equation for the mean magnetic field. This fact demonstrates a difference between the problems of evolution of the scalar field (the number density of particles) and vector (magnetic) field advected by a low-Mach-number compressible turbulent fluid flow. In particular, the compressibility results in the appearance of a new term in the equation for the mean number density of particles. The new term describes an additional nondiffusive turbulent flux of particles due to compressibility $(\nabla\cdot\mathbf{u}\neq0)$ of fluid [23,24]. This causes the formation of the large-scale inhomogeneous spatial distributions of the mean number density of particles.

The technique of path integrals allows us also to derive the equation for the second-order correlation function $h_{ij}=\langle h_i(t,\mathbf{x})h_j(t,\mathbf{y})\rangle$:

$$\frac{\partial h_{ij}}{\partial t}=[\hat{L}_{ik}(\mathbf{x})\delta_{js}+\hat{L}_{sj}(\mathbf{y})\delta_{ik}+\hat{M}_{ijks}]h_{ks}+I_{ij} \tag{12}$$

(for details see the Appendix), where the turbulent component of the magnetic field is $\mathbf{h}(t,\mathbf{x})=\mathbf{H}(t,\mathbf{x})-\mathbf{B}(t,\mathbf{x})$ and

$$\hat{L}_{ij}=\varepsilon_{iks}\frac{\partial}{\partial x_k}\left[\varepsilon_{smj}U_m+\alpha_{sj}-\hat{\eta}_{sm}\varepsilon_{mpj}\frac{\partial}{\varepsilon x_p}\right], \tag{13}$$

$$\frac{1}{2}\hat{M}_{ijks}=\delta_{ik}\delta_{js}f_{mn}\frac{\partial^2}{\partial x_m\partial y_n}-\delta_{ik}\frac{\partial f_{mj}}{\partial y_s}\frac{\partial}{\partial x_m}-\delta_{js}\frac{\partial f_{in}}{\partial x_k}\frac{\partial}{\partial y_n}+\frac{\partial^2 f_{ij}}{\partial x_k\partial y_s}+\delta_{ik}\delta_{js}\frac{\partial f_{mp}}{\partial y_p}\frac{\partial}{\partial x_m}+\delta_{ik}\delta_{js}\frac{\partial f_{pn}}{\partial x_p}\frac{\partial}{\partial y_n}-\delta_{ik}\frac{\partial^2 f_{pj}}{\partial x_p\partial y_s}-\delta_{js}\frac{\partial^2 f_{ip}}{\partial x_k\partial y_p}+\delta_{ik}\delta_{js}\frac{\partial^2 f_{pl}}{\partial x_p\partial y_l}, \tag{14}$$

$$I_{ij}=\hat{M}_{ijks}B_k(\mathbf{x})B_s(\mathbf{y}), \tag{15}$$

$f_{mn}=\langle\tau u_m(\mathbf{x})u_n(\mathbf{y})\rangle$, and $U_m=(\mathbf{U}_{\rm eff})_m$. We seek a solution for the second moment of the magnetic field in the form

$$\langle h_m(\mathbf{x})h_n(\mathbf{x}+\mathbf{r})\rangle=W(r)\delta_{mn}+\frac{rW'}{2}\left(\delta_{mn}-\frac{r_m r_n}{r^2}\right). \tag{16}$$

This form of the second moment corresponds to the condition $\nabla\cdot\mathbf{h}=0$ and an assumption of the homogeneous and isotropic magnetic fluctuations. Multiplying Eq. (12) by $r_i r_j/r^2$ and using the identities (A14)–(A19) (see the Appendix) yields the equation for the correlation function $W(r,t)=\langle h_\tau(\mathbf{x},t)h_\tau(\mathbf{y},t)\rangle$:

$$\frac{\partial W}{\partial t}=\frac{1}{m}W''+\mu W'-\frac{\kappa}{m}W, \tag{17}$$

where h_τ is the projection of the magnetic field $\mathbf{h}$ on the direction $\mathbf{r}=\mathbf{x}-\mathbf{y}$ and

$$\frac{1}{m}=\frac{2}{\rm Rm}+\frac{2}{3}[1-F-(rF_c)'],$$

$$\mu=\frac{4}{mr}+\left(\frac{1}{m}\right)',\qquad \kappa=\frac{2m}{r}(f'+2f_c'),$$

$$f=F+rF'/3,\qquad f_c=F_c+rF_c'/3,$$

${\rm Rm}=u_0 l_0/\eta\gg1$ is the magnetic Reynolds number, u_0 is the characteristic velocity in the maximum scale l_0 of turbulent motions, and $F'=dF/dr$. Here we consider a homogeneous, isotropic, and reflectionally invariant (with zero mean helicity) compressible turbulent fluid velocity field. In this case the correlation function $\langle \tau u_m u_n \rangle$ is given by

$$\langle \tau u_m(\mathbf{x}) u_n(\mathbf{x}+\mathbf{r}) \rangle = \eta_T \bigg[[F(r)+F_c(r)]\delta_{mn} + \frac{rF'}{2}\left(\delta_{mn} - \frac{r_m r_n}{r^2}\right) + rF_c' \frac{r_m r_n}{r^2} \bigg] \tag{18}$$

(for details see [23]), where $F'=dF/dr$, $\eta_T = u_0^2 \tau_0/3$ is the turbulent magnetic diffusion, $\tau_0 = l_0/u_0$, and $F(0) = 1 - F_c(0)$. The function $F_c(r)$ describes the compressible (potential) component, whereas $F(r)$ corresponds to the vortical part of the turbulence. Equation (17) is written in dimensionless variables: Coordinates and time are measured in the units l_0 and τ_0, the velocity is measured in the units u_0, and the magnetic field is measured in the units B_0. By means of Eq. (17) we will study the generation of magnetic fluctuations (see Sec. III).

The technique of path integrals that is described in the Appendix also allows us to derive the equation for the high-order correlation function $\Phi_s^{\{\alpha_j\}}(t,\mathbf{x}^{\{j\}}) = \langle \Pi_{j=1}^s h_{\alpha_j}(t,\mathbf{x}^{(j)}) \rangle$:

$$\frac{\partial \Phi_s^{\{\alpha_j\}}}{\partial t} + \sum_{j=1}^{s} \hat{L}_{\alpha_k}^{\alpha_j} \prod_{p,q=1;p\neq j}^{s} \delta_{\alpha_p}^{\alpha_q} \Phi_s^{\{\alpha_p\}} = \sum_{j=1}^{s} \sum_{i=1}^{s-1} \hat{M}_{\alpha_k \alpha_p}^{\alpha_j \alpha_q} \prod_{m,n=1;n\neq j,q}^{s} \delta_{\alpha_n}^{\alpha_m} \Phi_s^{\{\alpha_p\}}, \tag{19}$$

where $\{\alpha_j\} = \alpha_1\alpha_2, \ldots, \alpha_s$, $\mathbf{x}^{\{j\}} = (\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(s)})$, and in Eqs. (13) and (14) we change $\hat{L}_{ik} \to \hat{L}_{\alpha_k}^{\alpha_i}$ and $\hat{M}_{jqkp} \to \hat{M}_{\alpha_k \alpha_p}^{\alpha_j \alpha_q}$. Now we consider the case of a zero mean magnetic field $\mathbf{B}=\mathbf{0}$. When the mean field $\mathbf{B}\neq\mathbf{0}$ a source $I^{\{\alpha_j\}}$ (which depends on $\mathbf{B}$ and on the structure functions of the lower orders) appears in Eq. (19). We seek a solution to Eq. (19) in the form

$$\Phi_s^{\{\alpha_j\}} = \prod_{i,k}^{s} \Phi_2^{\alpha_j \alpha_k}(\mathbf{x}^{(i)} - \mathbf{x}^{(k)}) \exp(\gamma_s t), \tag{20}$$

where $i \neq j$. Substitution of Eq. (20) into Eq. (19) yields

$$\gamma_s = \frac{s(s-1)}{2} \gamma_2 . \tag{21}$$

Equation (21) implies that if the second-order correlation function of the magnetic field grows ($\gamma_2 > 0$), then all high-order correlation functions grow. It is shown in Sec. III that under certain conditions $\gamma_2 > 0$. Note that the higher moments grow faster than the lower moments of the magnetic field. Therefore, the spatial distribution of the magnetic fluctuations is intermittent (i.e., $\gamma_s > s\gamma_2/2$). This is in agreement with a dynamo theorem [27,15,16].

We use in the present paper the δ-correlated-in-time random process to describe a turbulent velocity field:

$$\langle u_m(t,\mathbf{x}) u_n(t',\mathbf{x}+\mathbf{r}) \rangle = \delta\left(\frac{t-t'}{2\tau(\mathbf{r})}\right) \langle u_m(\mathbf{x}) u_n(\mathbf{x}+\mathbf{r}) \rangle, \tag{22}$$

where $\tau(\mathbf{r})$ is the scale-dependent momentum relaxation time. It follows from Eq. (22) that

$$\int_0^\infty \langle u_m(t,\mathbf{x}) u_n(t',\mathbf{x}+\mathbf{r}) \rangle dt' = \langle \tau u_m(\mathbf{x}) u_n(\mathbf{x}+\mathbf{r}) \rangle .$$

Using the δ-correlated-in-time random process allows us to provide the analytical calculations and to obtain closed results for the growth rate of the high-order correlation functions of the magnetic field, the threshold of the generation of the magnetic fluctuations, and their anomalous scaling. The results remain valid also for the velocity field with a finite correlation time if the high-order correlation functions of the magnetic field vary slowly in comparison to the correlation time of the turbulent velocity field (see, e.g., [28,15,16]). We also take into account the dependence of the momentum relaxation time on the scale of the turbulent velocity field: $\tau(\mathbf{k}) = \tau_0 (k/k_0)^{1-p}$, where p is the exponent in the spectrum of kinetic turbulent energy, k is the wave number, and $k_0 = l_0^{-1}$. The equations derived for the high-order correlation functions are valid as long as the momentum relaxation time of the velocity field is small in comparison to the characteristic time of variations of the magnetic fluctuations.

III. GROWTH OF MAGNETIC FLUCTUATIONS

Let us study the evolution of magnetic fluctuations in a low-Mach-number compressible turbulent fluid flow. A mechanism of the generation of magnetic fluctuations with a zero mean magnetic field was proposed by Zeldovich and co-workers (see, e.g., [16,29]) and comprises stretching, twisting, and folding of the original loop of a magnetic field. These nontrivial motions are three dimensional and result in an amplification of the magnetic field. The magnetic diffusion leads to the reconnection of the field at an X point.

The generation of magnetic fluctuations can be described by Eq. (17) for the second moment of the magnetic field. We seek a solution of Eq. (17) in the form

$$W(t,r) = \frac{\Psi(r)\sqrt{m}}{r^2} \exp(\gamma t), \tag{23}$$

where the unknown function $\Psi(r)$ is determined by

$$\frac{1}{m(r)} \frac{d^2\Psi}{dr^2} - [\gamma + U(r)]\Psi = 0 \tag{24}$$

and

$$U(r) = \frac{1}{4m(r)} (\chi^2 + 2\chi' + 4\kappa), \quad \chi(r) = \frac{4}{r} + m\left(\frac{1}{m}\right)' .$$

We consider the case of small magnetic Prandtl numbers $\mathrm{Pr}_m = \nu/\eta \ll 1$, which is typical for many astrophysical and geophysical applications, where ν is the kinematic viscosity. The latter allows us to consider magnetic fluctuations only in the inertial range of the turbulent velocity field. We choose the following model of turbulence. Incompressible $F(r)$ and compressible $F_c(r)$ components in the inertial range of turbulence $r \ll 1$ are given by

$$F(r)=(1-\varepsilon)(1-r^{q-1}), \quad F_c(r)=\varepsilon(1-\beta r^{q-1}),$$

where $r_d < r \ll 1$, q is the exponent in the spectrum of the function $\langle \tau u_m u_n \rangle$, $r_d = \mathrm{Re}^{-1/(3-p)}$, p is the exponent in the spectrum of kinetic turbulent energy, and $\mathrm{Re}=u_0 l_0/\nu \gg 1$ is the Reynolds number. Thus the functions $m(r)$, $\chi(r)$, and $\kappa(r)$ for $r \ll 1$ are given by

$$\frac{1}{m(r)} = \frac{2}{\mathrm{Rm}}[1+\beta_m \mathrm{Rm} r^{q-1}], \quad \kappa(r) = -2m\kappa_0 r^{q-3}, \tag{25}$$

$$\chi(r) = \frac{4}{r}\left(1+\frac{(q-1)/4}{1+(r^{q-1}\beta_m \mathrm{Rm})^{-1}}\right), \tag{26}$$

where

$$\beta_m = \frac{1}{3}(1-\varepsilon)(1+q\sigma), \quad \sigma = \frac{\varepsilon\beta}{1-\varepsilon},$$

$$\kappa_0 = \frac{1}{3}(1-\varepsilon)(1+2\sigma)(2+q)(q-1).$$

Note that the exponent p in the spectrum of kinetic turbulent energy differs from that of the function $\langle \tau u_m u_n \rangle$ due to the scale dependence of the momentum relaxation time τ of the turbulent velocity field $\mathbf{u}$. The relation between p and q can be found as follows. The spectrum function of the turbulent magnetic diffusion is defined as

$$\eta(k) = \tau(k) E(k), \tag{27}$$

where k is the wave number, $E(k) \propto k^{-p}$ is the spectrum function of the turbulent velocity field $\mathbf{u}$, $\tau(k)$ is the scale-dependent momentum relaxation time, and $\langle \mathbf{u}^2 \rangle = \int_0^\infty E(k)dk$. Consider a Kolmogorov-type turbulence. For this turbulence the energy flux over the spectrum is constant, i.e.,

$$\frac{E(k)k}{\tau(k)} = \mathrm{const}. \tag{28}$$

Equations (27) and (28) yield $\eta(k) \propto E^2(k) k \propto k^{1-2p}$. For homogeneous and isotropic fields the scale-dependent turbulent magnetic diffusion is given by

$$\eta(r) \equiv \langle \tau \mathbf{u}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}+\mathbf{r}) \rangle = \int_0^\infty \frac{\sin(kr)}{kr}\, \eta(k) dk.$$

Thus we obtain that $q=2p-1$. Note that if $p>3/2$, the exponent $q>2$. We will see in this section that the magnetic fluctuations can be generated if $q>2$.

The solution of Eq. (24) can be obtained using an asymptotic analysis (see, e.g., [15,17,23,25]). This analysis is based on the separation of scales. In particular, the solution of the Schrödinger equation (24) with a variable mass has different regions where the forms of the potential $U(r)$, mass $m(r)$, and therefore eigenfunctions $\Psi(r)$ are different. Solutions in these different regions can be matched at their boundaries. Note that the most important part of the solution is localized in small scales (i.e., $r \ll 1$). The results obtained by this asymptotic analysis are presented below. The solution of the Schrödinger equation (24) has the discrete [for $U(r)<0$] and continuous [for $U(r)>0$] spectra. The discrete spectrum describes the self-excitation of magnetic fluctuations ($\gamma_2>0$), while the continuous spectrum corresponds to the dissipation of magnetic energy ($\gamma_2<0$). The solution of Eq. (24) has three characteristic regions.

In region I, i.e., for $0 \leq r \leq \mathrm{Rm}^{-1/(q-1)}$, the mass $m(r)$, the potential $U(r)$, and the function $W(r)$ are given by

$$\frac{1}{m(r)} \sim \frac{2}{\mathrm{Rm}}[1+\beta_m \mathrm{Rm} r^{q-1}],$$

$$U(r) \sim \frac{1}{m}\left(\frac{2}{r^2} - \kappa_0 \mathrm{Rm} r^{q-3}\right),$$

$$W(r) = A_1 m^{1/2} r^{-3/2} J_\lambda[2\lambda \sqrt{\kappa_0 \mathrm{Rm}}\, r^{3/2\lambda}/3],$$

where $\lambda = 3/(q-1)$, J_λ is the Bessel function of the first kind, and the coefficient A_1 is given by

$$A_1 = \sqrt{\frac{2}{\mathrm{Rm}}}\,\Gamma\left(\frac{q+2}{q-1}\right)\left(\frac{3}{\lambda\sqrt{\kappa_0 \mathrm{Rm}}}\right)^\lambda.$$

The correlation function $W(r)$ for $r \ll \mathrm{Rm}^{-1/(q-1)}$ is given by $W(r) = 1-\beta_0 \mathrm{Rm} r^{q-1}$, where $\beta_0 = \beta_m + \kappa_0(q-1)/(q+2)$.

In region II, i.e., for $\mathrm{Rm}^{-1/(q-1)} \ll r \ll 1$,

$$\frac{1}{m(r)} \sim 2\beta_m r^{q-1}[1+(\beta_m \mathrm{Rm} r^{q-1})^{-1}],$$

$$U(r) \sim -\frac{1+4b^2}{4mr^2}, \quad b^2 = \left(\frac{q^2-4}{4}\right)\left(\frac{3+\sigma(4-q)}{1+q\sigma}\right), \tag{29}$$

$$W(r) = A_2 m^{1/2} r^{-3/2} \cos(b \ln r + \varphi_0). \tag{30}$$

In region III ($r \gg 1$),

$$\frac{1}{m(r)} \sim 2/3, \quad U(r) \sim 4/3r^2,$$

$$W(r) = A_3 r^{-2}(\sqrt{|\gamma|}+r^{-1})\exp(-\sqrt{|\gamma|}r).$$

Matching functions $W(r)$ and $W'(r)$ at the boundaries of these regions yields the constant A_k, φ_0, and the growth (or damping) rate $\gamma_2 \equiv \gamma$ of the magnetic fluctuations. The latter is given by

$$\gamma \simeq \frac{4b^2+(4-q)^2}{4(q-1)} \ln\left(\frac{\mathrm{Rm}}{\mathrm{Rm}^{(\mathrm{cr})}}\right),$$

where the critical magnetic Reynolds number $\mathrm{Rm}^{(\mathrm{cr})}$ is given by

$$\mathrm{Rm}^{(\mathrm{cr})} \simeq \exp\left[\frac{q-1}{b}\left(\pi k + \arctan\frac{4-q}{2b} + \arctan S\right)\right], \tag{31}$$

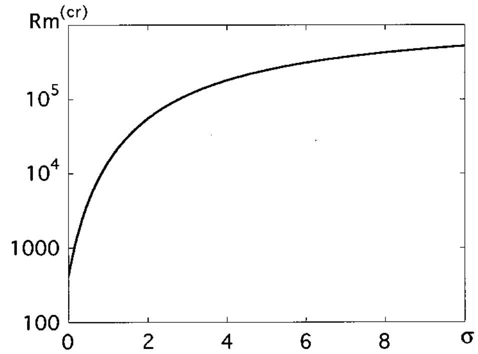

FIG. 1. Dependence of the critical magnetic Reynolds number versus the parameter compressibility σ for $p=5/3$ (Kolmogorov turbulence) and $\beta=1$.

$$S=\frac{\sqrt{\kappa_0}}{b}\frac{J'_\lambda[2\lambda\sqrt{\kappa_0}/3]}{J_\lambda[2\lambda\sqrt{\kappa_0}/3]},$$

and $k=1,2,3,\ldots$. This analysis shows that the characteristic scales of localization of the magnetic fluctuations is of order

$$l_f \sim l_0 \mathrm{Rm}^{-1/2(p-1)} \exp\left[\frac{1}{b}(\arctan S+\pi n+\pi/2)\right], \quad (32)$$

where $n \leqslant k$.

The critical magnetic Reynolds number as a function of the parameter compressibility σ for $p=5/3$ (Kolmogorov turbulence) and $\beta=1$ is presented in Fig. 1. It is seen in Fig. 1 that the threshold for the generation of magnetic fluctuations (i.e., the critical magnetic Reynolds number) by turbulent fluid flow with $\nabla\cdot\mathbf{u}\neq 0$ is higher than that for the case of incompressible fluid. For incompressible fluid $\sigma=0$ the critical magnetic Reynolds number $\mathrm{Rm}^{(\mathrm{cr})}=412$, while for compressible fluid flow $\sigma=0.1$ the value $\mathrm{Rm}^{(\mathrm{cr})}=740$. For a larger parameter of compressibility the critical magnetic Reynolds number increases sharply. The latter implies that the compressibility impairs the generation of magnetic fluctuations.

Now we discuss the effect of compressibility on the generation of magnetic fluctuations. In compressible ideal conducting fluid flow the vector $\mathbf{H}/\rho$ is frozen into the motion of the fluid. In incompressible flow, at any time the mass of the fluid flowing into a small volume exactly equals a mass outflow from this volume. On the other hand, in compressible flow $(\nabla\cdot\mathbf{u}\neq 0)$ a mass of fluid flowing into a small volume does not equal a mass outflow from the volume at any instance. Therefore, at times smaller than a characteristic time of the turbulent velocity field the fluid density ρ and the magnetic field $\mathbf{H}$ increase (or decrease) when $\nabla\cdot\mathbf{u}<0$ (or $\nabla\cdot\mathbf{u}>0$). Note that the increase and decrease of the magnetic field in a small control volume are separated in time and small molecular magnetic diffusion breaks a reversibility in time. The latter may cause an additional increase $\langle\mathbf{h}^2\rangle$ caused by the compressibility of the fluid flow.

On the other hand, the compressibility affects the turbulent magnetic diffusion as well since it increases the turbulent magnetic diffusion in small scales. When $q>2$ the increase of the turbulent magnetic diffusion is stronger than that the increase of the growth of the magnetic fluctuations due to the compressibility of the fluid flow (see below). Note also that the scale of the localization of the magnetic fluctuations increases when the degree of the compressibility σ increases [see Eq. (32)]. On the other hand, the scale-dependent turbulent magnetic diffusion increases with the increase of scale. Thus the compressibility impairs the generation of magnetic fluctuations for $q>2$ (see below).

The generation of magnetic fluctuations depends on the form of solution for the correlation function $W(r)$ in the second region $\mathrm{Rm}^{-1/(q-1)} \ll r \ll 1$ (i.e., in the inertial range of the turbulence). The form of the solution depends on the value $m(r)U(r) \propto -(b^2+1/4)$ (see, e.g., [30]). When $b^2>0$ magnetic fluctuations are excited. It follows from Eq. (29) that $b_c^2-b_{\mathrm{inc}}^2=-\sigma(q^2-4)(q-1)$, where $b_{\mathrm{inc}}=b(\sigma=0)$ is the value of the parameter b for incompressible fluid flow and $b_c=b(\sigma>0)$ is the value of the parameter b for compressible fluid flow. From this we see that for $q>2$ (i.e., $b^2>0$) we obtain $0<b_c^2<b_{\mathrm{inc}}^2$. The latter implies that the compressibility impairs the generation of magnetic fluctuations for $q>2$.

Now we discuss the effect of the exponent of the energy spectrum of the turbulent velocity field on the generation of magnetic fluctuations. The solution (30) for the correlation function $W(r)$ is valid for $b^2>0$ (i.e., for $q>2$). In this case $(q>2)$ magnetic fluctuations can be excited. Since $q=2p-1$ the necessary condition for the excitation of magnetic fluctuations is $p>3/2$. On the other hand, when $p<3/2$ (i.e., for $q<2$) the solution for $W(r)$ is given by

$$W(r)=m^{1/2}r^{-3/2}(A_2 r^{-b}+A_5 r^b). \quad (33)$$

In this case magnetic fluctuations are not excited.

Note that magnetic fluctuations in a δ-correlated-in-time incompressible turbulent fluid flow were studied in [18] for $\mathrm{Pr}_m \ll 1$. In the latter model the correlation time is assumed to be independent of the scale of turbulent motions; therefore $q=p$ and the necessary condition for the excitation of magnetic fluctuations is $p>2$. The Kolmogorov turbulence with $p=5/3$ and for $\mathrm{Pr}_m \ll 1$ cannot generate magnetic fluctuations in the model with a scale-independent correlation time of the turbulent velocity field [18]. On the other hand, in the model with a scale-dependent correlation time of the turbulent velocity field considered in the present study the magnetic fluctuations are excited for $p>3/2$.

Remarkably, a spectrum of magnetic fluctuations proportional to $k^{-3/2}$ (the Kraichnan spectrum) occur when the magnetic energy equals the hydrodynamic energy $M(k)=E(k)\propto k^{-3/2}$ (see [31]). Here $M(k)$ and $E(k)$ are spectral functions of the magnetic and hydrodynamic energy, respectively. We have shown here that if the energy spectrum of hydrodynamic turbulent motions is steeper than the $k^{-3/2}$ spectrum, the magnetic fluctuations can be generated.

Note that the condition of the validity of the assumption of the very short correlated velocity field is as follows. The characteristic time of variations of the magnetic fluctuations $\tau_B \sim \gamma_s^{-1}$ is very large in comparison to the momentum relaxation time τ of the velocity field in the scale l_f. The scale l_f determines the conditions of generation and localization of the magnetic fluctuations. This allows us to estimate the maximum number s of the high-order correlation function of the magnetic field,

$$s < \mathrm{Rm}^{1/2} \Big/ \left| \ln\left(\frac{\mathrm{Rm}}{\mathrm{Rm}^{(\mathrm{cr})}}\right) \right|^{1/2}, \tag{34}$$

when the theory is valid. For the Kolmogorov turbulence ($p=5/3$) and $\mathrm{Rm}=10^6$ (e.g., the convection zone of the sun) we obtain $s<10^3$.

IV. ANOMALOUS SCALING OF MAGNETIC FLUCTUATIONS

The problems of anomalous scalings for vector (magnetic) and scalar (particles number density or temperature) fields passively advected by a turbulent fluid flow have been a subject of active research in recent years (see, e.g., [32–38]). The anomalous scaling means the deviation of the scaling exponents of the correlation function of a vector (scalar) field from their values obtained by the dimensional analysis. For incompressible turbulent flow the anomalous scalings for the scalar field can occur only for a fourth-order correlation function (see, e.g., [33–35]), while for the vector (magnetic) field the anomalous scalings appear already in the second moment [37].

The anomalous scalings of the magnetic fluctuations were considered in [37] using the model of the δ-correlated-in-time turbulent velocity field. The correlation time in this model is independent of the scale of turbulent motions. Note that in this model the generation of magnetic fluctuations is possible only for $q=p>2$. Taking into account the dependence of the correlation time on the scale of turbulent motions, we have shown (see Sec. III) that the magnetic fluctuations can be generated for $3/2<p<3$.

Now we discuss the anomalous scaling for the model with the scale-dependent correlation time of turbulent motions. Consider the case when the magnetic Reynolds number $\mathrm{Rm}<\mathrm{Rm}^{(\mathrm{cr})}$ and the magnetic fluctuations are caused by an external source. The condition $\mathrm{Rm}<\mathrm{Rm}^{(\mathrm{cr})}$ implies that there is no self-excitation (i.e., exponential growth) of the magnetic fluctuations (see Sec. III). The solution for the correlation function $W(r)$ in the inertial range is given by $W(r)\sim r^{-(2p+1)/2}\cos(b\ln r+\varphi_0)$ [see Eq. (30)]. This corresponds to the anomalous scaling of the magnetic fluctuations. The normal scaling for the second moment of the magnetic fluctuations is given by $W(r)\sim r^{4-2p}$. The normal scaling for the second moment of magnetic fluctuations occurs when the flux of magnetic fluctuations over the spectrum is constant, i.e.,

$$\frac{\langle \mathbf{B}^2\rangle_k}{\tau_\eta(k)} = \mathrm{const}, \tag{35}$$

where $\langle \mathbf{B}^2\rangle_k=\int_k^\infty M(k)dk$, $M(k)$ is the spectrum function of magnetic fluctuations, $\tau_\eta(k)=(\eta_k k^2)^{-1}$, and $\eta_k\equiv\langle \tau \mathbf{u}^2\rangle_k =\int_k^\infty \eta(k)dk$. Dimensional analysis shows that under condition (35) the spectrum of magnetic fluctuations (or vector field) is given by

$$M(k)\propto \frac{1}{k^5 E^2(k)} \propto k^{2p-5}. \tag{36}$$

When $p=5/3$ the spectrum of magnetic fluctuations $M(k) \propto k^{-5/3}$. Note that the spectrum (36) is similar to the Obukhov-Corrsin spectrum for the passive scalar (see, e.g., [39–41]).

Equation (36) yields the second moment of magnetic fluctuations $W(r)\propto r^{4-2p}\propto r^{3-q}$, where $q=2p-1$. Note that the general solution of Eq. (17) with an external source I includes the solution (30) [or (33)] and the solution $W(r) \propto r^{4-2p}$ (i.e., it includes solutions describing the anomalous and normal scalings). The obtained anomalous scaling $W(r)\sim r^{-(2p+1)/2}\cos(b\ln r+\varphi_0)$ can be presented as the real part of the power-law function r^ζ with the complex exponent $\zeta=-1/2-p-ib$. This anomalous scaling is significantly different from the normal scaling $W(r)\propto r^{4-2p}$ and corresponds to the deviation from the condition (35) of the constant flux of magnetic fluctuations over the spectrum. The obtained anomalous scaling is valid when the exponent of the energy spectrum of the turbulent velocity field is $3/2<p<3$. When $1<p<3/2$ the anomalous exponent in a low-Mach-number compressible turbulent flow is real, i.e., $\zeta=-1/2-p+|b(\sigma,q)|$. In the case of incompressible turbulent flow ($\sigma=0$) this result coincides with that obtained in [37].

V. DISCUSSION

In this study we have shown that the magnetic fluctuations can be generated for small magnetic Prandtl numbers, which occur in many astrophysical and geophysical applications. We have studied the effect of compressibility of the low-Mach-number turbulent fluid flow. The compressibility impairs the generation of magnetic fluctuations (i.e., the threshold for the generation of magnetic fluctuations by turbulent fluid flow with $\mathrm{div}\mathbf{u}\neq 0$ is higher than that for the case of divergence-free fluid flow). The reason is that the compressibility results in an increase of the scale of localization of the magnetic fluctuations. In larger scales the scale-dependent turbulent magnetic diffusion is larger. However, the contribution of the compressibility to the generation of the magnetic fluctuations proportional to $(\boldsymbol{\nabla}\cdot\mathbf{u})^2$ is smaller in the larger scales. This results in an increase of the threshold for the generation of magnetic fluctuations in a compressible fluid flow.

The model of very short scale-dependent correlation time of the turbulent velocity field is considered. It is shown that the magnetic fluctuations can be generated even in the Kolmogorov turbulence (i.e., for $p=5/3$). The anomalous scaling for the second moment of the magnetic field is found for the case $3/2<p<3$. This anomalous scaling r^ζ has the complex exponent $\zeta=-1/2-p-ib$. When $1<p<3/2$ the anomalous exponent in a low-Mach-number compressible turbulent flow is real: $\zeta=-1/2-p+|b(\sigma,q)|$. In the case of incompressible turbulent flow ($\sigma=0$) this result coincides with that obtained in [37]. The equation for the high-order correlation functions of the magnetic field is derived. It is shown that the growth rates of the higher moments of the magnetic field are higher than those of the lower moments, i.e., the spatial distribution of the magnetic field is intermittent.

The presented analysis explains why in the numerical simulations by [20,21] the magnetic fluctuations with a zero mean magnetic field were not generated. The parameters in the numerical simulation [21] with a zero mean field ($\mathbf{B}=\mathbf{0}$) are $\mathrm{Rm}\leqslant 200$ and $\sigma=0.01$. We have shown that even for incompressible turbulent fluid flow ($\sigma=0$) the threshold of the excitation of the magnetic fluctuations $\mathrm{Rm}^{(\mathrm{cr})}=412$ in the case of $\mathrm{Pr}_m\leqslant 1$. Thus the magnetic fluctuations cannot be generated for the parameters used in [21]. Note that the value $\mathrm{Rm}^{(\mathrm{cr})}=412$ also cannot be achieved in laboratory experiments. On the other hand, in astrophysical conditions $\mathrm{Rm}\gg\mathrm{Rm}^{(\mathrm{cr})}$ and $\mathrm{Pr}_m\ll 1$; therefore the magnetic fluctuations can be generated in the astrophysical conditions. This is in agreement with observations of the magnetic fields in the Sun (see, e.g., [21]).

Note that the use of the δ-correlated-in-time random process to describe a turbulent velocity field is certainly an approximation. However, we study in the present paper two specific problems: (a) conditions for the excitations of the magnetic fluctuations (i.e., the threshold of the generation and growth rate of the magnetic fluctuations in the vicinity of the threshold) and (b) the anomalous scaling behavior, which is determined by the ''zero mode'' of the equations for high-order correlation functions of the magnetic field (''zero mode'' is a mode with zero growth rate).

In the vicinity of the threshold the characteristic time of variations of the high-order correlation functions of the magnetic field is much larger than the momentum relaxation time $\tau(\mathbf{r})$. The latter allows us to use the δ-correlated-in-time random process to describe a turbulent velocity field.

Note that the problem of the generation and dynamics of the magnetic fluctuations in the turbulent velocity field with a finite correlation time is very important and is a subject for future investigations. A significant difference between the results obtained with a very short correlation time and those obtained with a Navier-Stokes turbulent velocity field probably will be observed in the case of a strong deviation from the threshold of excitation of the magnetic fluctuations. In this case the problem will be strongly nonlinear and may be solved only numerically (if large magnetic Reynolds numbers will be achieved in numerical simulations with small magnetic Prandtl numbers).

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APPENDIX: DERIVATION OF THE EQUATIONS FOR THE MEAN MAGNETIC FIELD AND FOR THE HIGH-ORDER CORRELATION FUNCTION OF THE MAGNETIC FIELD

We derive here the equation for the mean magnetic field and second moment of the magnetic field. We use here the stochastic calculus [15–17,22–25].

If the total field H_i is specified at time t, then we can determine the total field $H_i(t+\Delta t)$ at time $t+\Delta t$ by means of substitutions $t\to t+\Delta t$ and $t_0\to t$ in Eq. (2). The result is given by

$$H_i(t+\Delta t,\mathbf{x})=M\{G_{ij}(t+\Delta t,t)H_{0j}[t,\boldsymbol{\xi}(t+\Delta t,t)]\}, \tag{A1}$$

where

$$\boldsymbol{\xi}(t+\Delta t,t)\equiv\boldsymbol{\xi}_{\Delta t}=\mathbf{x}-\int_0^{\Delta t}\mathbf{v}(t_\sigma,\boldsymbol{\xi}_\sigma)d\sigma+\sqrt{2\eta}\mathbf{w}(\Delta t), \tag{A2}$$

$t_\sigma=t+\Delta t-\sigma$, and $\boldsymbol{\xi}(t_2,t_1)\equiv\boldsymbol{\xi}_{t_2-t_1}$, i.e., $\boldsymbol{\xi}_\sigma=\boldsymbol{\xi}(t+\Delta t,t_\sigma)$.

In order to find the function $G_{ij}(t+\Delta t,t)$ we solve Eq. (3) by iterations. The result is given by

$$G_{ij}(t+\Delta t,t)\approx\delta_{ij}+\int_0^{\Delta t}N_{ij}(t_\sigma,\boldsymbol{\xi}_\sigma)d\sigma+\int_0^{\Delta t}N_{ik}(t_s,\boldsymbol{\xi}_s)ds\int_0^{s}N_{kj}(t_\sigma,\boldsymbol{\xi}_\sigma)d\sigma, \tag{A3}$$

where we keep terms greater than or equal to $O((\Delta t)^2)$. Expanding the function $H_i(t,\boldsymbol{\xi}_{\Delta t})$ in a Taylor series in the vicinity of the point $\mathbf{x}$ yields

$$H_i(t,\boldsymbol{\xi}_{\Delta t})\simeq H_i(t,\mathbf{x})+\frac{\partial H_i}{\partial x_m}(\boldsymbol{\xi}_{\Delta t}-\mathbf{x})_m+\frac{1}{2}\frac{\partial^2 H_i}{\partial x_m\partial x_n}\times(\boldsymbol{\xi}_{\Delta t}-\mathbf{x})_m(\boldsymbol{\xi}_{\Delta t}-\mathbf{x})_n+\cdots. \tag{A4}$$

Using the equation for the Wiener trajectory (A2) we obtain

$$[\boldsymbol{\xi}(t_2,t_1)-\mathbf{x}]_m=-\int_0^{t_2-t_1}v_m(t_s,\boldsymbol{\xi}_s)ds+\sqrt{2\eta}w_m(t_2-t_1), \tag{A5}$$

where $\boldsymbol{\xi}(t_2,t_2-s)\equiv\boldsymbol{\xi}_s$. Expanding the velocity $v_m(t_s,\boldsymbol{\xi}_s)$ in a Taylor series in the vicinity of the point $\mathbf{x}$ and using Eq. (A5) yields

$$v_m(t_s,\boldsymbol{\xi}_s)\simeq v_m(t_s,\mathbf{x})-v_l\frac{\partial v_m}{\partial x_l}s+\sqrt{2\eta}\frac{\partial v_m}{\partial x_l}w_l(s)+\cdots. \tag{A6}$$

Here we assume that velocity $\mathbf{v}$ remains constant (time independent) at small time intervals $(0,\Delta t),(\Delta t,2\Delta t),\ldots$ and changes every small time Δt and that the velocity is statistically independent at different time intervals. Substituting Eq. (A6) into Eq. (A5) and calculating the integrals in Eq. (A5) accurate up to $\sim(t_2-t_1)^2$ terms yields

$$[\boldsymbol{\xi}(t_2,t_1)-\mathbf{x}]_m\simeq-(t_2-t_1)v_m+\frac{1}{2}(t_2-t_1)^2v_l\frac{\partial v_m}{\partial x_l}-\sqrt{2\eta}\frac{\partial v_m}{\partial x_l}\int_0^{t_2-t_1}w_l ds+\sqrt{2\eta}w_m(t_2-t_1)+\cdots. \tag{A7}$$

The combination of Eqs. (A7) and (A4) yields the formula for the field $H_i(t,\boldsymbol{\xi}_{\Delta t})$,

$$H_i(t,\boldsymbol{\xi}_{\Delta t}) \simeq H_i(t,\mathbf{x}) + \frac{\partial H_i}{\partial x_m}\bigg(-v_m\Delta t + \frac{1}{2} v_l \frac{\partial v_m}{\partial x_l}(\Delta t)^2 + \sqrt{2\eta} w_m - \sqrt{2\eta}\frac{\partial v_m}{\partial x_l}\int_0^{\Delta t} w_l ds \bigg) + \frac{1}{2}\frac{\partial^2 H_i}{\partial x_m \partial x_l}[v_m v_l (\Delta t)^2 + 2\eta w_m w_l - \sqrt{2\eta}\Delta t(v_m w_l + v_l w_m)]\}, \tag{A8}$$

where we keep terms greater than or equal to $O((\Delta t)^2)$. Similar calculations for $N_{ij}(t,\boldsymbol{\xi}_{\Delta t})$, accurate up to $\sim O(\Delta t)$, yield

$$N_{ij}(t,\boldsymbol{\xi}_{\Delta t}) = N_{ij}(t,\mathbf{x}) + \frac{\partial N_{ij}}{\partial x_m}(-v_m\Delta t + \sqrt{2\eta} w_m) + \eta\frac{\partial^2 N_{ij}}{\partial x_m \partial x_l} w_m w_l . \tag{A9}$$

Therefore,

$$\int_0^s N_{ij}(t_\sigma,\xi_\sigma)d\sigma \simeq N_{ij}(t,\mathbf{x})s - \frac{1}{2} v_q \frac{\partial N_{ij}}{\partial x_q}s^2 + \sqrt{2\eta}\frac{\partial N_{ij}}{\partial x_q}\int_0^s w_q d\sigma + \eta\frac{\partial^2 N_{ij}}{\partial x_m \partial x_l}\int_0^s w_m w_l d\sigma + \cdots , \tag{A10}$$

where we also keep terms greater than or equal to $O((\Delta t)^2)$. Using Eqs. (A10) and (A9), we calculate the function $G_{ij}(t+\Delta t,t)$ accurate up to $\sim(\Delta t)^2$ terms,

$$G_{ij}(t+\Delta t,t) \simeq \delta_{ij} + N_{ij}(t,\mathbf{x})\Delta t + \frac{1}{2}\bigg(-v_q \frac{\partial N_{ij}}{\partial x_q} + N_{ik}N_{kj}\bigg)\times(\Delta t)^2 + \sqrt{2\eta}\frac{\partial N_{ij}}{\partial x_q}\int_0^{\Delta t} w_q d\sigma + \cdots . \tag{A11}$$

Using the identity

$$\frac{\partial}{\partial x_k}\bigg(v_i \frac{\partial v_k}{\partial x_j}\bigg) = \frac{\partial v_i}{\partial x_k}\frac{\partial v_k}{\partial x_j} + v_i \frac{\partial^2 v_k}{\partial x_k \partial x_j},$$

Eq. (A11) can be written as

$$G_{ij}(t+\Delta t,t) \simeq \delta_{ij} + N_{ij}(t,\mathbf{x})\Delta t + M_{ij}(\Delta t)^2 + \sqrt{2\eta}\frac{\partial N_{ij}}{\partial x_q}\int_0^{\Delta t} w_q d\sigma + \cdots , \tag{A12}$$

where

$$M_{ij} = \frac{1}{2}\bigg[\delta_{ij}\frac{\partial}{\partial x_k}(bv_k) - \frac{\partial}{\partial x_j}(bv_i) + \frac{\partial}{\partial x_k}\bigg(v_i \frac{\partial v_k}{\partial x_j} - v_k \frac{\partial v_i}{\partial x_j}\bigg)\bigg].$$

The substitution of Eqs. (A12) and (A8) into Eq. (A1) allows us to determine $H_i(t+\Delta t,\mathbf{x})$,

$$H_i(t+\Delta t,\mathbf{x}) \simeq H_i(t,\mathbf{x}) + M\bigg(q_i(\mathbf{x})\Delta t + p_i(\mathbf{x})(\Delta t)^2 + \sqrt{2\eta} Q_{in}(\mathbf{x})\int_0^{\Delta t} w_n d\sigma \bigg), \tag{A13}$$

where

$$q_i = H_m \frac{\partial v_i}{\partial x_m} - v_m \frac{\partial H_i}{\partial x_m} - bH_i + \eta w_m w_n \frac{\partial^2 H_i}{\partial x_m \partial x_n}(\Delta t)^{-1},$$

$$p_i = \frac{1}{2} H_n \bigg[\frac{\partial}{\partial x_m}\bigg(v_i \frac{\partial v_m}{\partial x_n} - v_m \frac{\partial v_i}{\partial x_n}\bigg) - \frac{\partial}{\partial x_n}(bv_i) + \delta_{in}\frac{\partial}{\partial x_m}(bv_m)\bigg] + \frac{\partial H_n}{\partial x_m}\bigg(bv_m\delta_{in} - v_m \frac{\partial v_i}{\partial x_n} + \frac{1}{2} v_k \frac{\partial v_m}{\partial x_k}\delta_{in}\bigg) + \frac{1}{2} v_m v_n \frac{\partial^2 H_i}{\partial x_m \partial x_n},$$

$$Q_{iw} = H_j \frac{\partial N_{ij}}{\partial x_n} - \frac{\partial H_i}{\partial x_m}\frac{\partial v_m}{\partial x_n}.$$

Averaging Eq. (A13) over the ensembles of the turbulent velocities, we obtain the equation for the mean field $\mathbf{B}(t+\Delta t,\mathbf{x}) = \langle \mathbf{H}(t+\Delta t,\mathbf{x})\rangle$. Calculating the limit $[\mathbf{B}(t+\Delta t,\mathbf{x}) - \mathbf{B}(t,\mathbf{x})]/\Delta t$ for $\Delta t \to 0$ yields Eq. (6) for the mean magnetic field. Now we calculate the correlation function $h_{ij} = \langle h_i(t+\Delta t,\mathbf{x}) h_j(t+\Delta t,\mathbf{y})\rangle$ by means of Eq. (A13), where the turbulent component of the magnetic field $\mathbf{h}(t,\mathbf{x}) = \mathbf{H}(t,\mathbf{x}) - \mathbf{B}(t,\mathbf{x})$. Now we determine $[h_{ij}(t+\Delta t,\mathbf{x}) - h_{ij}(t,\mathbf{x})]/\Delta t$ for $\Delta t \to 0$. The result is given by Eq. (12). We seek a solution for the second moment of the magnetic field in the form given by Eq. (16). Multiplying Eq. (12) by $r_i r_j / r^2$ and using the identities

$$3\frac{r_i r_j}{r^2} f_{mn}\frac{\partial^2 h_{ij}}{\partial r_m \partial r_n} = W''(F + F_c + rF_c') + \frac{4W'}{r}(F + F_c + rF'/2), \tag{A14}$$

$$3\frac{r_i r_j}{r^2}\frac{\partial f_{mj}}{\partial r_s}\frac{\partial h_{is}}{\partial r_m} = W'\bigg[\frac{3}{2}F' + 2F_c' + r^2\bigg(\frac{F_c'}{r}\bigg)'\bigg], \tag{A15}$$

$$3\frac{r_i r_j}{r^2}\frac{\partial f_{ml}}{\partial r_l}\frac{\partial h_{ij}}{\partial r_m} = rW'\bigg(F_c'' + \frac{4F_c'}{r}\bigg), \tag{A16}$$

$$3\frac{r_i r_j}{r^2}\frac{\partial^2 f_{il}}{\partial r_k \partial r_l} h_{kj} = W(rF_c''' + 5F_c''), \tag{A17}$$

$$3\frac{r_i r_j}{r^2}\frac{\partial^2 f_{pl}}{\partial r_p \partial r_l} h_{ij} = W(rF_c''' + 7F_c'' + 8F_c'/r), \tag{A18}$$

$$3\frac{r_i r_j}{r^2}\frac{\partial^2 f_{ij}}{\partial r_m \partial r_n}h_{mn}=W(rF_c'''+5F_c''+F''+4F'/r)+W'(rF_c''+2F'), \quad (A19)$$

we obtain Eq. (17) for the correlation function $W(t,r)$.

Now we derive the equation for the high-order correlation function $\Phi_s^{\{\alpha_j\}}(t+\Delta t,\mathbf{x}^{\{j\}})=\langle \Pi_{j=1}^{s} h_{\alpha_j}(t+\Delta t,\mathbf{x}^{(j)})\rangle$ using Eq. (A13):

$$\Phi_s^{\{\alpha_j\}}=\left\langle \prod_{j=1}^{s} h_{\alpha_j}(t,\mathbf{x}^{(j)})\right\rangle + M\Bigg\{\sum_{j=1}^{s}\left\langle \Delta t q_{\alpha_j}(\mathbf{x}^{(j)})\prod_{k=1;k\neq j}^{s} h_{\alpha_k}(t,\mathbf{x}^{(k)})\right\rangle + \sum_{j=1}^{s}\left\langle (\Delta t)^2 p_{\alpha_j}(\mathbf{x}^{(j)})\prod_{k=1;k\neq j}^{s} h_{\alpha_k}(t,\mathbf{x}^{(k)})\right\rangle + \sum_{j=1}^{s}\left\langle (\Delta t)^2 q_{\alpha_j}(\mathbf{x}^{(j)}) q_{\alpha_k}(\mathbf{x}^{(k)}) \times \prod_{n=1;n\neq j,k}^{s} h_{\alpha_n}(t,\mathbf{x}^{(n)})\right\rangle\Bigg\}, \quad (A20)$$

where $\{\alpha_j\}=\alpha_1,\alpha_2,\ldots,\alpha_s$ and $\mathbf{x}^{\{j\}}=(\mathbf{x}^{(1)},\mathbf{x}^{(2)},\ldots,\mathbf{x}^{(s)})$. Now we change $\hat{L}_{ik}\to\hat{L}_{\alpha_k}^{\alpha_j}$ and $\hat{M}_{jqkp}\to\hat{M}_{\alpha_k\alpha_p}^{\alpha_j\alpha_q}$ in Eqs. (13) and (14). Use Eq. (A20), we find $[\Phi_s^{\{\alpha_j\}}(t+\Delta t,\mathbf{x}^{\{j\}})-\Phi_s^{\{\alpha_j\}}(t,\mathbf{x}^{\{j\}})]/\Delta t$ for $\Delta t\to 0$. The result is given by Eq. (19).

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