Isotropic and anisotropic spectra of passive scalar fluctuations in turbulent fluid flow

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Isotropic and anisotropic spectra of passive scalar fluctuations in a turbulent fluid flow with a power law $\propto k^{-\beta}$ spectrum are analyzed. The isotropic spectra occur in flows with zero mean external gradient of passive scalar concentration and passive scalar fluctuations can be caused by an external source. On the other hand, in the presence of nonzero mean external gradient of concentration, passive scalar fluctuations are anisotropic and can be excited by ''tangling'' of the mean external gradient of the passive scalar by turbulent fluid flow. The analysis is based on the renormalization procedure in the spirit of Moffatt [J. Fluid Mech. **106**, 27 (1981); Rep. Prog. Phys. **46**, 621 (1983)]. It is shown that the anisotropic k^{-3} spectrum of passive scalar fluctuations is universal, i.e., independent of exponent β in a turbulent velocity spectrum. In the particular case of the Kolmogorov spectrum ($\beta=5/3$) of turbulent velocity field the derived general spectra recover the known spectra of passive scalar fluctuations $\propto k^{-5/3}$ and $\propto k^{-17/3}$. In addition, the ultimate Prandtl number for large Reynolds numbers is estimated ($\mathrm{Pr}^{\mathrm{lim}}\approx 0.792$) and is found to be in fairly good agreement with experimental results.

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I. INTRODUCTION

Passive scalar fluctuations in incompressible and compressible turbulent fluid flow were studied quite extensively (see, e.g., [1–7], and references therein) due to the great importance of the turbulent mixing problem. However, some of the aspects of this phenomenon, e.g., the problem of spectra of the passive scalar fluctuations, are still not completely understood. Isotropic and anisotropic spectra of the passive scalar fluctuations in different cases were analyzed in numerous studies (see, e.g., [3,8–14]) by means of different methods. The isotropic spectra occur in flows with zero mean external gradient of passive scalar concentration. In this case passive scalar fluctuations can be caused by an external source. On the other hand, in the presence of nonzero mean external gradient of concentration, passive scalar fluctuations can be excited by ''tangling'' of the mean external gradient of the passive scalar by turbulent fluid flow. In this case the spectra of passive scalar fluctuations are called anisotropic.

Some of the results on spectra of passive scalar fluctuations are a subject of discussion and controversy. Namely, the existence of the anisotropic k^{-3} spectrum of passive scalar fluctuations was repeatedly discussed in the literature (see, e.g., [2,15–17]). Although this spectrum was observed in experiments (see, e.g., [17–19]) its origin still remains poorly understood.

In this study we analyze the isotropic and anisotropic spectra of passive scalar fluctuations in a turbulent fluid flow with a power law $\propto k^{-\beta}$ spectrum using the renormalization procedure in the spirit of Moffatt [20]. Notably, this approach allows us to derive all the known spectra of passive scalar fluctuations (in scales from the maximum scale of turbulent motions to the viscous scale), including the k^{-3} spectrum.

In order to elucidate the problem, in this section we present a short review of the isotropic and anisotropic spectra of passive scalar fluctuations obtained with simple dimensional arguments and approximate estimations. First, consider isotropic homogeneous and incompressible turbulent fluid flow with zero mean external gradient of a number density of passive scalar particles. The external source of passive scalar fluctuations is localized in region of scales $l<l_0$, where l_0 is the energy containing scale of hydrodynamic turbulence. The equation for a fluctuating component q of passive scalar concentration in incompressible turbulent fluid flow $\mathbf{u}$ reads

$$\frac{\partial q}{\partial t}+(\mathbf{u}\cdot\boldsymbol{\nabla})q=D\Delta q+\epsilon, \tag{1}$$

where D is the coefficient of molecular diffusion and ϵ is the external source of the passive scalar fluctuations. As was found by Obukhov [8] and Corrsin [9] the spectrum of passive scalar fluctuations in a range $k_0<k<\min(k_\nu,k_D)$ is given by

$$\Gamma(k)\propto k^{-5/3}, \tag{2}$$

where

$$\langle q^2\rangle=\int_0^\infty \Gamma(k)\,dk; \tag{3}$$

$k_0=l_0^{-1}$, $k_D=l_D^{-1}$, l_D is the length scale in which molecular diffusion is dominant, $k_\nu=l_\nu^{-1}$, and l_ν is the ''viscous'' length scale at which molecular dissipation becomes dominant. For instance, for Kolmogorov turbulence $l_\nu \propto \mathrm{Re}^{-3/4}l_0$, where $\mathrm{Re}=u_0l_0/\nu_0$ is the Reynolds number, ν_0

is the kinematic viscosity, and u_0 is the characteristic velocity in scale l_0. The $k^{-5/3}$ spectrum of passive scalar fluctuations is obtained by means of dimensional analysis if one assumes that

$$\frac{\langle q^2\rangle_k}{\tau(k)}=\text{const},$$

$$E(k)\propto k^{-5/3}, \tag{4}$$

where

$$\tau(k)=[ku(k)]^{-1}, \quad u(k)=\sqrt{\langle \mathbf{u}^2\rangle_k}, \tag{5}$$

$$\langle q^2\rangle_k=\int^k \Gamma(k')dk', \quad \int^k \equiv \int_k^\infty,$$

$$\langle \mathbf{u}^2\rangle_k=\int^k E(k')dk',$$

and $E(k)$ is the spectrum function of turbulent velocity field $\mathbf{u}$. Conditions (4) and (5) imply the following estimate of magnitude of terms in Eq. (1): $(\mathbf{u}\cdot\boldsymbol{\nabla})q\sim\epsilon$ and a condition $\langle \epsilon q\rangle=\text{const}$. The $k^{-5/3}$ spectrum of passive scalar fluctuations depends on the exponent of the spectrum of turbulent velocity field. This spectrum exists in the region $k_0<k<\min(k_\nu,k_D)$ and is independent of the molecular Prandtl number $\mathrm{Pr}_0=\nu_0/D$.

When $\mathrm{Pr}_0\gg 1$ in the region $k_\nu<k\ll k_D$ the spectrum of passive scalar fluctuations is given by

$$\Gamma(k)\propto k^{-1}$$

(see [11,13]). This spectrum can be obtained if one assumes that

$$\frac{\langle q^2\rangle_k}{\tau(k)}=\text{const}, \tag{6}$$

$$\tau(k)=\text{const}. \tag{7}$$

Condition (7) means that in the interval $k_\nu<k\ll k_D$ there is only one characteristic time of a random velocity field.

Until now we considered the isotropic spectra of passive scalar fluctuations, i.e., passive scalar fluctuations with zero mean external gradient of passive scalar. On the other hand, when the external mean gradient of the passive scalar $\boldsymbol{\nabla} N_0 \neq \mathbf{0}$, the spectra of passive scalar fluctuations are anisotropic. Now let us discuss these anisotropic spectra. In this case passive scalar fluctuations can be excited by "tangling" of the mean external gradient of the passive scalar by turbulent fluid flow. When $\boldsymbol{\nabla} N_0\neq\mathbf{0}$ the equation for the fluctuating component q of the passive scalar concentration reads

$$\frac{\partial q}{\partial t}+(\mathbf{u}\cdot\boldsymbol{\nabla})q-D\Delta q=-(\mathbf{u}\cdot\boldsymbol{\nabla})N_0. \tag{8}$$

First consider the case $\mathrm{Pr}_0\ll 1$ and examine the range $k_D\ll k\ll k_\nu$. An estimate

$$D\Delta q\sim(\mathbf{u}\cdot\boldsymbol{\nabla})N_0 \tag{9}$$

in Eq. (8) and an assumption of the Kolmogorov spectrum of hydrodynamic turbulence $E(k)\propto k^{-5/3}$ yield the spectrum of the passive scalar fluctuations in this case:

$$\Gamma(k)\propto k^{-17/3}(\boldsymbol{\nabla} N_0)^2. \tag{10}$$

Note that the latter spectrum of the passive scalar fluctuations is anisotropic, i.e.,

$$\Phi(\mathbf{k})\propto k^{-17/3}\sin^2\theta(\boldsymbol{\nabla} N_0)^2,$$

where

$$\Gamma(k)=\int \Phi(\mathbf{k})k^2\sin\theta d\theta d\varphi,$$

and θ is the angle between a wave vector $\mathbf{k}$ and $\boldsymbol{\nabla} N_0$.

The $k^{-17/3}$ spectrum is also valid for isotropic passive scalar fluctuations $(\boldsymbol{\nabla} N_0=0)$. In the case of the isotropic passive scalar fluctuations this spectrum was derived in [12]. An estimate

$$D\Delta q\sim(\mathbf{u}\cdot\boldsymbol{\nabla})q \tag{11}$$

in Eq. (1) is valid in this case. If we assume that in the range $k_D\ll k\ll k_\nu$ the spectrum of passive scalar fluctuations is steep, then the main contribution in the $(\mathbf{u}\cdot\boldsymbol{\nabla})q$ term is from fluctuations of passive scalar $q(k')$ of the inertial range $k_0\ll k'\ll k_D$ and from fluctuations of velocity field $\mathbf{u}(k'')$ of the interval $k_D\ll k''\ll k_\nu$. This means that in the $(\mathbf{u}\cdot\boldsymbol{\nabla})q$ term the fluctuations q and $\mathbf{u}$ are not correlated. The latter yields the $k^{-17/3}$ spectrum of isotropic passive scalar fluctuations:

$$\Gamma(k)\propto k^{-17/3}|\boldsymbol{\nabla} q|^2,$$

where an assumption of the Kolmogorov spectrum of hydrodynamic turbulence $E(k)\propto k^{-5/3}$ was used.

Now we consider the spectrum of the anisotropic passive scalar fluctuations in the interval $k_0<k<\min(k_\nu,k_D)$. The spectrum in this interval can be obtained by means of Eq. (9) whereby the coefficient of the molecular diffusion D is replaced by a scale dependent coefficient of the turbulent diffusion $\eta(k)$:

$$(\mathbf{u}\cdot\boldsymbol{\nabla})N_0\sim\eta(k)\Delta q, \tag{12}$$

and

$$\eta(k)\sim\frac{u(k)}{k}. \tag{13}$$

Equations (12) and (13) yield the spectrum of the passive scalar fluctuations in the range $k_0<k<\min(k_\nu,k_D)$:

$$\Gamma(k)\propto k^{-3}(\boldsymbol{\nabla} N_0)^2$$

(see [10]). In Sec. III we derive the isotropic and anisotropic spectra of the passive scalar fluctuations (in scales from the maximum scale of turbulent motions to the viscous scale) by means of a renormalization procedure.

II. RENORMALIZATION PROCEDURE AND TURBULENT TRANSPORT COEFFICIENTS

In this section we consider properties of passive scalar fluctuations in a turbulent fluid flow. The renormalization procedure (see, e.g., [20,21]) is employed here for investigation of the passive scalar fluctuations in turbulence with a mean gradient of passive scalar. Numerous works on turbulence are confined to a study of the large-scale properties of flows by averaging the equations over the ensemble of the fluctuations (see, e.g., [1–3]). This averaging is carried out over the fluctuations in all scales of turbulence.

On the other hand, the averaging in the renormalization procedure is performed over the fluctuations of scales from l_ν to l_* within the inertial range of turbulence $l_\nu < l_* < l_0$. In the very small scales $l < l_\nu$ the molecular dissipation is important. Therefore turbulent viscosity ν and turbulent diffusion η depend on the scale of the averaging l_*. The next stage of the renormalization procedure comprises a step-by-step increase of the scale of the averaging. This procedure allows the derivation of equations for the turbulent transport coefficients.

We perform the first step of the renormalization procedure, i.e., average the Navier-Stokes equation and the equation for the concentration of the passive scalar over the fluctuations with the scales from l_ν to l_*. The averaged equations for velocity $\mathbf{v}$ and concentration n are given by

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu\Delta\mathbf{v} + \mathbf{f}, \tag{14}$$

$$\frac{\partial n}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})n = \eta\Delta n + \epsilon, \tag{15}$$

where $\boldsymbol{\nabla}\cdot\mathbf{v}=0$, ϵ is the external source of passive scalar fluctuations, $\rho\mathbf{f}$ is the external force, p is the pressure. The turbulent coefficients ν and η depend on the scale of averaging l_*.

After this averaging, the range $l > l_*$ corresponds to "mean" fields whereas fluctuations are in the range $l < l_*$. The influence of the fluctuations on the "mean" fields is described by the turbulent coefficients ν and η.

Let us change the scale of the averaging by a small value $|\Delta\mathbf{k}| \ll k_*$ (where the wave number $k_* = l_*^{-1}$) and average Eqs. (14) and (15) over the fluctuations. Now in the region $k < k_* - |\Delta\mathbf{k}|$ the velocity $\mathbf{V}$ and concentration N are the mean fields whereas the region $k > k_* - |\Delta\mathbf{k}|$ corresponds to the turbulent fields. Since Eqs. (14) and (15) have been already averaged over the fluctuations in the scales that are smaller than l_*, it is sufficient to average these equations over fluctuations in the small interval $k_* - |\Delta\mathbf{k}| < |\mathbf{k}| < k_*$. Here $\mathbf{v} = \mathbf{V} + \mathbf{u}$, $n = N + q$, $\mathbf{V} = \langle \mathbf{v} \rangle$, $N = \langle n \rangle$, and the angular brackets denote averaging over the ensemble of fluctuations in the domain $k_* - |\Delta\mathbf{k}| < |\mathbf{k}| < k_*$. Therefore the equations for the mean fields $\mathbf{V}$ and N are given by

$$\frac{\partial V_j}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})V_j + \frac{1}{\rho}\frac{\partial p}{\partial x_j} - \nu\Delta V_j - f_j = \frac{\partial}{\partial x_j}\sigma_{ij}, \tag{16}$$

$$\frac{\partial N}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})N - \eta\Delta N = -\frac{\partial}{\partial x_j}\Psi_j, \tag{17}$$

where $\boldsymbol{\nabla}\cdot\mathbf{V}=0$, the Reynolds turbulent-stress tensor $\sigma_{ij} = -\langle u_i u_j \rangle$, and the flux of particles $\Psi_j = \langle q u_j \rangle$. The equations for mean fields comprise the second moments for the turbulent fields. To obtain a closed system of equations it is necessary to find the dependence of the second moments $\langle u_i u_j \rangle$ and $\langle q u_j \rangle$ on the "mean" fields. To achieve this goal we perform the following procedure.

(1) We derive equations for the turbulent fields $\mathbf{u} = \mathbf{v} - \mathbf{V}$ and $q = n - N$ in a frame moving with a local velocity of the mean flow $\mathbf{V}$.

(2) Define a background turbulence as the turbulence without mean gradients of both mean passive scalar concentration $\boldsymbol{\nabla} N = \mathbf{0}$ and mean velocity $\nabla_i V_j = 0$. For simplicity the background turbulence is assumed to be homogeneous and isotropic. The solutions of the derived equations $\mathbf{u}^{(0)}$ and $q^{(0)}$ correspond to the background turbulence.

(3) The goal of the present study is to analyze a deviation from the solutions $\mathbf{u}^{(0)}$ and $q^{(0)}$ due to the presence of both mean passive scalar gradient $\boldsymbol{\nabla} N$ and gradient of the mean flow $\nabla_i V_j$, and we derive equations for the fields $\mathbf{u}^{(1)} = \mathbf{u} - \mathbf{u}^{(0)}$ and $q^{(1)} = q - q^{(0)}$.

(4) We solve the integrodifferential equations for the fields $\mathbf{u}^{(1)}$ and $q^{(1)}$ by iterations. We consider here the effects that are linear in the spatial derivatives of the mean fields $\mathbf{V}$ and N.

(5) We calculate the second moments for the turbulent fields in order to find the Reynolds turbulent-stress tensor σ_{ij} and flux of particles Ψ_j,

$$\sigma_{ij} = \sigma_{ij}^{(0)} - \langle u_i^{(1)} u_j^{(0)} \rangle - \langle u_i^{(0)} u_j^{(1)} \rangle - \langle u_i^{(1)} u_j^{(1)} \rangle, \tag{18}$$

$$\Psi_j = \langle q^{(0)} u_j^{(1)} \rangle + \langle q^{(1)} u_j^{(0)} \rangle + \langle q^{(1)} u_j^{(1)} \rangle, \tag{19}$$

where $\sigma_{ij}^{(0)} = -\langle u_i^{(0)} u_j^{(0)} \rangle$ is the Reynolds turbulent-stress tensor for the background turbulence. We assume that $\langle q^{(0)} u_j^{(0)} \rangle = 0$.

Substituting Eqs. (18) and (19) into (16) and (17) yields the equations for the mean fields. The described procedure enables us to derive equations for the transport coefficients: turbulent viscosity ν and turbulent diffusion η. The details of the calculations are presented in the Appendix. The result is given by

$$\frac{d\nu}{dk} = -\frac{7}{60\nu k^2}E(k), \tag{20}$$

$$\frac{d\eta}{dk} = -\frac{\mathrm{Pr}(k)}{3\nu k^2(1+\mathrm{Pr})}E(k). \tag{21}$$

Here $\mathrm{Pr}(k) = \nu(k)/\eta(k)$ is the effective Prandtl number and $E(k)$ is the hydrodynamic energy spectra of the background turbulence:

$$E(k) = (\beta - 1)\left(\frac{u_0^2}{k_0}\right)\left(\frac{k}{k_0}\right)^{-\beta},$$

where $k_0 = l_0^{-1}$. For example, for the Kolmogorov spectrum of hydrodynamic turbulence $\beta = 5/3$ (see, e.g., [22]). After the change of variables Eqs. (20) and (21) are reduced to

$$\frac{d\nu}{d\xi}=\frac{7}{20\nu},\tag{22}$$

$$\frac{d\eta}{d\xi}=\frac{\mathrm{Pr}}{\nu(1+\mathrm{Pr})},\tag{23}$$

where $d\xi=-(E/3k^2)dk$ and

$$\xi=\int_k^\infty \frac{E(k')}{3(k')^2}dk'=\frac{u_0^2}{3k_0^2}\left(\frac{\beta-1}{\beta+1}\right)\left(\frac{k}{k_0}\right)^{-\beta-1}=\frac{E(k)}{3k(\beta+1)}.\tag{24}$$

The equation for the scale dependent Prandtl number $\mathrm{Pr}(k)$ is derived from (22) and (23):

$$\frac{d\mathrm{Pr}}{d\xi}=-\frac{\mathrm{Pr}(\mathrm{Pr}-a_1)(\mathrm{Pr}+a_2)}{\nu^2(1+\mathrm{Pr})}.\tag{25}$$

Here $a_1 \sim 0.792$ and $-a_2 \sim -0.442$ are the roots of the quadratic equation $20y^2-7y-7=0$. It follows from Eq. (25) that there is a special case for $\mathrm{Pr}(k)\equiv \mathrm{Pr}^{\mathrm{lim}}=a_1\simeq 0.792$ when the Prandtl number is constant in all scales of the turbulence. The value 0.792 corresponds to the fixed point of Eq. (25). The dependence of the turbulent viscosity ν on ξ can be determined from (22) and is given by

$$\nu^2(\xi)=\nu_0^2+\frac{7}{10}[\xi(k)-\xi_d],\tag{26}$$

where $\xi_d=\xi(k=k_\nu)$. A one-parameter set of solutions of Eqs. (22) and (23) for the turbulent transport coefficients $\nu(\mathrm{Pr})$ and $\eta(\mathrm{Pr})$ is given by

$$\mathrm{Re}(\mathrm{Pr})\equiv\frac{\nu(\mathrm{Pr})}{\nu_0}=\frac{\mathrm{Pr}}{\mathrm{Pr}_0}\left|\frac{\mathrm{Pr}_0-a_1}{\mathrm{Pr}-a_1}\right|^{\alpha_1}\left|\frac{\mathrm{Pr}_0+a_2}{\mathrm{Pr}+a_2}\right|^{\alpha_2},\tag{27}$$

$$\eta(\mathrm{Pr})=\frac{\nu(\mathrm{Pr})}{\mathrm{Pr}},\tag{28}$$

where $\mathrm{Pr}_0\neq a_1$,

$$\alpha_1=\frac{a_2(1+a_1)}{a_1+a_2}\approx 0.642,\qquad \alpha_2=\frac{a_1(1-a_2)}{a_1+a_2}\approx 0.358.$$

These turbulent coefficients depend only on the Prandtl number Pr. Equation (27) allows us to describe asymptotical behavior of the Prandtl number Pr for large Reynolds number $\mathrm{Re}(k)=\nu(k)/\nu_0\gg 1$:

$$\mathrm{Pr}(k)\approx 0.792+0.618\frac{(\mathrm{Pr}_0-a_1)(\mathrm{Pr}_0+a_2)^{0.558}}{\mathrm{Pe}^{1.56}(k)},\tag{29}$$

where $\mathrm{Pe}(k)=\mathrm{Re}(k)\mathrm{Pr}_0\gg 1$. This means that in most of the inertial range (where $\mathrm{Pe}(k)=\eta(k)/D\gg 1$) the Prandtl number Pr tends to the ultimate value of $\mathrm{Pr}^{\mathrm{lim}}\simeq 0.792$. Equation (29) for $\mathrm{Pr}(k)$ and the ultimate value of $\mathrm{Pr}^{\mathrm{lim}}$ are in a fairly good agreement with experimental results (for a review see [23]).

Notably the above renormalization procedure is essentially different from the renormalization group method (RNG) described in [24]. Indeed, in the present study we do not provide a closure for the dynamical problem of turbulence. We study the interaction between the weakly inhomogeneous mean fields and background hydrodynamic turbulence. In contrast to the RNG method, the spectrum and statistical properties of the background turbulence (with zero-mean fields) are assumed to be known. Furthermore, the background turbulence can be arbitrary. On the other hand, in the RNG method an external random stirring force with Gaussian statistics is introduced. We consider a situation with very weak gradients of mean fields, and we study small deviation from the background turbulence under the influence of small gradients of mean fields, e.g.,

$$\left|\frac{\boldsymbol{\nabla} N}{N}\right|\ll\left|\frac{\sqrt{\langle(\boldsymbol{\nabla} q)^2\rangle}}{N}\right|.$$

The spectrum of the background hydrodynamic turbulence is located within a finite region of wave numbers from $k_0\neq 0$ to k_ν, where $k_0=l_0^{-1}$, $k_\nu=l_\nu^{-1}$, l_0 is the maximum scale of turbulent motions, and l_ν is the "viscous" length scale at which molecular dissipation becomes dominant.

In the above renormalization procedure the spectrum of the background hydrodynamic turbulence is not determined and it is assumed to be known. This means that we do not study the dynamic problem of the hydrodynamic turbulence, and we consider a particular problem of weak response of homogeneous and isotropic turbulence to a weak external gradient of the mean passive scalar field. Such an approach allows us to avoid the fundamental difficulties associated with divergencies, either at high wave number or at low wave number. Common in our renormalization procedure and in the renormalization group method is a step-by-step averaging. However we do not perform a renormalization of the background hydrodynamic turbulence as was done in the RNG method (see [24]).

Now let us discuss the nonlinear terms [see Eqs. (A1)–(A6)]. We do not introduce a parameter of nonlinear interaction that is assumed to be small in the RNG method (see [24]). The nonlinear terms are taken into account by means of the renormalized turbulent diffusion $\eta(k)$ and renormalized turbulent viscosity $\nu(k)$ for fluctuations with wave numbers $k_*<k<k_\nu$. However, we neglect nonlinear terms that are associated with fluctuations in the small interval $k_*-|\Delta\mathbf{k}|<|\mathbf{k}|<k_*$. It can be done for the following reasons. In the Appendix we estimated the magnitude of these nonlinear terms in Eqs. (A1)–(A6) determined by the functionals $B_{mn}(u;u)$ and $S_n(q;u)$. We also compared these nonlinear terms with the linear terms in Eqs. (A1)–(A6) related to the mean fields $\nabla_i V_j$ and N, where $\mathbf{V}$ is the mean velocity. These linear terms are determined by the functionals $L_{mn}(V;u)$ and $H_n(N;u)$. We found that the "nonlinear" functionals $B_{mn}(u;u)$ and $S_n(q;u)$ are proportional to $(\Delta k)^2$, whereas the terms $L_{mn}(V;u)$ and $H_n(N;u)$ in Eqs. (A1)–(A6) are of order $|\Delta\mathbf{k}|$. In the Appendix we also estimated the errors committed at each stage of averaging (due to neglect of nonlinear terms) that can be accumulated. The maximum error is of order $|\Delta\mathbf{k}|^{3/2}$.

Therefore the "nonlinear" functionals $B_{mn}(u;u)\sim|\Delta\mathbf{k}|^2$ [and $S_n(q;u)\sim|\Delta\mathbf{k}|^2$] can be dropped out for small $|\Delta\mathbf{k}|\ll k_*$, where $k_*>k_0$. However, it does not mean that the nonlinear terms are dropped out in all $\mathbf{k}$ space. In the first step of the renormalization procedure the equations are averaged over fluctuations of scales from k_ν^{-1} to k_*^{-1}. Therefore the nonlinear terms contribute to the turbulent transport coefficient ν and η in all scales except for only very small region of the spectrum: $k_* - |\Delta \mathbf{k}| < |\mathbf{k}| < k_*$.

The equation for $\eta(k)$ was first derived in [20]. This equation [see Eq. (2.11) in [20]] is different from Eq. (21) in our paper. The cause of this difference is that in [20] an approximation $\omega \ll k^2 \eta(k)$ was used. In our study we do not use this assumption and we choose the frequency spectrum of the background hydrodynamic turbulence in the form of the Lorenz profile [see Eq. (34)]. Integration in ω space in Eq. (A17) yields Eq. (21), which is different from Eq. (2.11) of Ref. [20]. Note that it is possible to use another model of the frequency spectrum of the background hydrodynamic turbulence. However, the assumption $\omega \ll k^2 \eta(k)$ cannot be considered as general.

The difference in the ultimate value of $\mathrm{Pr}^{\mathrm{lim}}$ obtained in [24] is caused by the fact that we do not renormalize the background turbulence, which is assumed to be known. In [24] the ultimate value of $\mathrm{Pr}^{\mathrm{lim}}$ is 0.718. Note that recently the state of the art in the renormalization theories and the unified mathematical formulation for passive scalar transport problems were discussed in [25,26].

III. RENORMALIZATION PROCEDURE AND SPECTRA OF PASSIVE SCALAR FLUCTUATIONS

Now let us derive the isotropic ($\boldsymbol{\nabla} N_0 = \mathbf{0}$) and anisotropic ($\boldsymbol{\nabla} N_0 \neq \mathbf{0}$) spectra of the passive scalar fluctuations. Here $\boldsymbol{\nabla} N_0$ is the external gradient of concentration. When $\boldsymbol{\nabla} N_0 \neq \mathbf{0}$ passive scalar fluctuations are excited by the "tangling" of the mean gradient of the passive scalar concentration by turbulent fluid flow. On the other hand, when $\boldsymbol{\nabla} N_0 = \mathbf{0}$ passive scalar fluctuations are caused by an external source ϵ. The equation for the fluctuating component of the concentration q reads (see the Appendix)

$$\frac{\partial q}{\partial t} - \eta \Delta q = -(\mathbf{u}^{(0)} \cdot \boldsymbol{\nabla}) N_0 + \epsilon. \tag{30}$$

In Eq. (30) we take into account the terms that are responsible for the generation of the passive scalar fluctuations by the "tangling" of the mean gradient of the concentration of particles $\boldsymbol{\nabla} N_0$ with hydrodynamic fluctuations $\mathbf{u}^{(0)}$. The $(\mathbf{u} \cdot \boldsymbol{\nabla}) q$ term in Eq. (30) is taken into account by means of the renormalized turbulent diffusion η [see the Appendix after Eq. (A6)]. Now we rewrite Eq. (30) in Fourier space and calculate the second moment $\langle q(\hat{k}_1) q(\hat{k}_2) \rangle$. The result is given by

$$\langle q(\hat{k}_1) q(\hat{k}_2) \rangle = -G_\eta(\hat{k}) G_\eta^*(\hat{k}) \int f_{mn}(\hat{k}, \hat{K} - \hat{Q} - \hat{Q}') Q_m Q_n' N_0(\hat{Q}) N_0(\hat{Q}') \, d\hat{Q} \, d\hat{Q}' + G_\eta(\hat{k}) G_\eta^*(\hat{k}) I(\hat{k}, \hat{K}), \tag{31}$$

where $\hat{k} = (\hat{k}_1 - \hat{k}_2)/2$ is the small-scale variable, and $\hat{K} = \hat{k}_1 + \hat{k}_2$ is the large-scale variable, and we neglect here terms $\sim O(\hat{K}^3)$,

$$f_{mn}(\hat{k}, \hat{K}) = \left\langle u_m^{(0)}\left(\hat{k} + \frac{\hat{K}}{2}\right) u_n^{(0)}\left(-\hat{k} + \frac{\hat{K}}{2}\right) \right\rangle,$$

$$I(\hat{k}, \hat{K}) = \left\langle \epsilon\left(\hat{k} + \frac{\hat{K}}{2}\right) \epsilon\left(-\hat{k} + \frac{\hat{K}}{2}\right) \right\rangle,$$

$$\hat{k} = \begin{pmatrix} \mathbf{k} \\ \omega \end{pmatrix}, \quad G_\eta = \frac{1}{-i\omega + \eta(k) k^2}.$$

Inverse Fourier transform of Eq. (31) in variable $\hat{K}$ yields

$$\Phi(\hat{k}, \mathbf{R}) = G_\eta(\hat{k}) G_\eta^*(\hat{k}) f_{mn}(\hat{k}, \mathbf{R}) \boldsymbol{\nabla}_m N_0 \boldsymbol{\nabla}_n N_0 + G_\eta(\hat{k}) G_\eta^*(\hat{k}) I(\hat{k}, \mathbf{R}). \tag{32}$$

Equation (32) allows us to derive spectra of passive scalar fluctuations. First we consider anisotropic spectra.

A. Anisotropic spectra of passive scalar fluctuations

The anisotropic $\mathbf{k}^{-3}$ *spectrum*. Consider a case $I = 0$ and a range of wave numbers $k_0 < k < \min(k_D, k_\nu)$. For the homogeneous and isotropic background turbulence,

$$f_{mn}(\hat{k}, \hat{k}') = \frac{E(k) T(k,\omega)}{8\pi k^2} \left(\delta_{mn} - \frac{k_m k_n}{k^2} \right) \delta(\hat{k} + \hat{k}'), \tag{33}$$

where the frequency component $T(k,\omega)$ is chosen as the Lorenz profile:

$$T(k,\omega) = \left(\frac{\nu(k) k^2}{\pi} \right) \frac{1}{\omega^2 + \nu^2 k^4} \equiv \left(\frac{\nu(k) k^2}{\pi} \right) G_\nu G_\nu^*, \tag{34}$$

$$\int_{-\infty}^{\infty} T(k,\omega) \, d\omega = 1.$$

Note that the time dependence of the correlation function $W(k,\tau) = \langle u(k,t) u(k,t+\tau) \rangle$ corresponds to a distribution: $W(k,\tau) = E(k) \exp[-\nu(k) k^2 \tau]$. Now we calculate the integral

$$\int \Phi(\hat{k}, \mathbf{R}) \, d\omega = \frac{E}{8\pi k^6 \nu^2(k)} \left(\frac{\mathrm{Pr}^2(k)}{1 + \mathrm{Pr}(k)} \right) \sin^2\theta (\boldsymbol{\nabla} N_0)^2, \tag{35}$$

where θ is the angle between vectors $\mathbf{k}$ and $\boldsymbol{\nabla} N_0$. We used here formula (A19). Integrating (35) over the angles in $\mathbf{k}$ space, and using Eq. (26), definition

$$\langle q^2(\mathbf{R})\rangle \equiv \int\int\int \Phi(\hat{k},\mathbf{R})\sin\theta d\theta d\varphi k^2 dk d\omega$$

$$\equiv \int \Gamma(k,\mathbf{R})dk, \tag{36}$$

and identity (24) $E=3k\xi(\beta+1)$, we find that

$$\Gamma(k,\mathbf{R})=\frac{10(\beta+1)}{7}F(k)\left(\frac{\mathrm{Pr}^2(k)}{1+\mathrm{Pr}(k)}\right)k^{-3}(\boldsymbol{\nabla}N_0)^2, \tag{37}$$

where

$$F(k)=1+\frac{1}{\mathrm{Re}^2(k)}\left(\frac{7}{30}\mathrm{Re}_0^{\alpha}-1\right), \quad \alpha=\frac{5-3\beta}{3-\beta},$$

and $\mathrm{Re}_0=\mathrm{Re}(k=k_0)$. In most of the inertial range [where $\mathrm{Pe}(k)=\eta(k)/D\gg 1$] the Prandtl number Pr tends to the ultimate value of $\mathrm{Pr}^{\mathrm{lim}}\simeq 0.792$ [see Eq. (29)]. Therefore in this case the Prandtl number Pr is independent of k and the spectrum of the passive scalar fluctuations in the presence of the mean gradient of the concentration of particles $\boldsymbol{\nabla}N_0$ is determined by Eq. (37) with $\mathrm{Pr}=0.792$. Notably, this spectrum is independent of the exponent in the spectrum of the turbulent velocity field. In this sense the anisotropic k^{-3} spectrum of the passive scalar fluctuations is universal.

The $\mathbf{k}^{-17/3}$ *spectrum*. Consider the case $I=0$ and the range of wave numbers $k_D<k<k_\nu$ (i.e., $\mathrm{Pr}\ll 1$). In this range the diffusion of the passive scalar is determined by molecular transport, i.e., $\eta(k)=D$. On the other hand, the kinematic viscosity $\nu(k)$ is determined by the turbulent transport. Integration of Eq. (32) over ω space yields

$$\Phi(\mathbf{k},\mathbf{R})=\frac{E(k)}{8\pi k^6 D^2}\sin^2\theta(\boldsymbol{\nabla}N_0)^2. \tag{38}$$

Then, integrating (38) over the angles in $\mathbf{k}$ space we find the anisotropic spectrum of passive scalar fluctuations in the range of wave numbers $k_D<k<k_\nu$:

$$\Gamma(k,\mathbf{R})=\frac{E(k)}{3D^2k^4}(\boldsymbol{\nabla}N_0)^2=\frac{(\beta-1)}{3}\frac{\mathrm{Pe}^2}{k_0^3}\left(\frac{k}{k_0}\right)^{-\beta-4}(\boldsymbol{\nabla}N_0)^2.$$

When $\beta=5/3$ the spectrum is given by

$$\Gamma(k,\mathbf{R})=\frac{2}{9}\frac{\mathrm{Pe}^2}{k_0^3}\left(\frac{k}{k_0}\right)^{-17/3}(\boldsymbol{\nabla}N_0)^2.$$

Note that the anisotropic $k^{-17/3}$ spectrum depends on exponent in the spectrum of hydrodynamic turbulence.

B. Isotropic spectra of passive scalar fluctuations

Consider a case $\boldsymbol{\nabla}N_0=0$ and $I\neq 0$. Equation (30) in $\mathbf{k}$ space is given by

$$q(\hat{k})=G_\eta(\hat{k})\epsilon(\hat{k}). \tag{39}$$

By means of Eq. (39) we calculate the second moment,

$$\chi_*(\hat{k})=\langle q(\hat{k})\epsilon(-\hat{k})\rangle+\langle q(-\hat{k})\epsilon(\hat{k})\rangle.$$

The result is given by

$$\chi_*(\hat{k})=[G_\eta(\hat{k})+G_\eta^*(\hat{k})]I(\hat{k}). \tag{40}$$

We assume that the source of passive scalar fluctuations is homogeneous and isotropic and frequency independent (i.e., it is a white noise). Therefore $I(\hat{k})=I(k)$. Integrating (40) over ω space and over the angles in $\mathbf{k}$ space we find

$$\chi_*(k)=8\pi^2 I(k). \tag{41}$$

Now we assume that the flux of the passive scalar over the spectrum is constant, i.e.,

$$\chi(k)=\int_k^{k_D}\chi_*(k')(k')^2dk'=\mathrm{const}\equiv\chi_0. \tag{42}$$

Solution of Eq. (42) for $k\ll k_D$ is given by

$$\chi_*(k)=\frac{\chi_0(3-\beta)}{\ln\mathrm{Pe}}k^{-3}, \tag{43}$$

where we take into account that $k_D=k_0\mathrm{Pe}^{1/(3-\beta)}$. Combining (43) and (41) yields

$$I(k)=\frac{\chi_0(3-\beta)}{8\pi^2\ln\mathrm{Pe}}k^{-3}. \tag{44}$$

Substituting (44) into Eq. (32), using (36), (A18), and integrating over the angles in $\mathbf{k}$ space and over ω space yields

$$\Gamma(k)=\frac{\chi_0(3-\beta)}{2\ln\mathrm{Pe}}\left(\frac{1}{\eta(k)k^3}\right). \tag{45}$$

The $\mathbf{k}^{-5/3}$ *spectrum*. Consider a range of wave numbers $k_0<k<\min(k_D,k_\nu)$. We rewrite Eq. (45) in the form

$$\Gamma(k)=\frac{\chi_0(3-\beta)}{2\ln\mathrm{Pe}}\left(\frac{\mathrm{Pr}(k)}{\nu(k)k^3}\right).$$

It was found in this section that in range $k_0<k<\min(k_D,k_\nu)$ for large Reynolds number $\mathrm{Re}(k)=\nu(k)/\nu_0\gg 1$, and the Prandtl number $\mathrm{Pr}(k)=\mathrm{const}=a_1\approx 0.792$ [see Eq. (29)]. Using the identity $E(k)=3k\xi(\beta+1)$ and Eq. (26) we obtain

$$\Gamma(k)=\frac{\chi_0 a_1(3-\beta)}{2\ln\mathrm{Pe}}\left(\frac{30(\beta+1)}{7}\right)^{1/2}\frac{k^{-5/2}}{\sqrt{E(k)}}. \tag{46}$$

When $\beta=5/3$ Eq. (46) yields

$$\Gamma(k)\approx 2.18\frac{\chi_0\tau_0}{k_0\ln\mathrm{Pe}}\left(\frac{k}{k_0}\right)^{-5/3}.$$

The isotropic $\mathbf{k}^{-17/3}$ *spectrum*. Consider a range of wave numbers $k_D<k<k_\nu$. The diffusion coefficient in this range is determined by molecular transport, i.e., $\eta(k)=D$. Solution of Eq. (8) with $\boldsymbol{\nabla}N_0=\mathbf{0}$ is given by

$$q(\hat{k})=-iG_\eta(\hat{k})\int u_m(\hat{k}-\hat{k}')k'_m q(\hat{k}')d\hat{k}'. \tag{47}$$

Comparing Eqs. (47) and (39) we obtain $\epsilon(\hat{k})$:

$$\epsilon(\hat{k}) = -i \int u_m(\hat{k}-\hat{k}')k'_m q(\hat{k}')d\hat{k}'.$$

Therefore,

$$I(\hat{k},\hat{K}) \equiv \left\langle \epsilon\left(\hat{k}+\frac{\hat{K}}{2}\right)\epsilon\left(-\hat{k}+\frac{\hat{K}}{2}\right)\right\rangle = \frac{1}{3}\langle u_m u_m\rangle\langle|\nabla q|^2\rangle, \tag{48}$$

where

$$\langle|\nabla q|^2\rangle = -\int k'_m k''_m q(\hat{k}')q(\hat{k}'')d\hat{k}'d\hat{k}''$$

and we assume that in the range $k_D \ll k \ll k_\nu$ the spectrum of passive scalar fluctuations is steep. Therefore the main contribution in $I(\hat{k},\hat{K})$ is from fluctuations of the passive scalar $q(k')$ of the inertial range $k_0 \ll k' \ll k_D$ and from fluctuations of velocity field $\mathbf{u}(k'')$ in the interval $k_D \ll k'' \ll k_\nu$. This means that $k \gg k', k''$; i.e., the fluctuations of q and $\mathbf{u}$ are not correlated. Therefore

$$\left\langle q\left(\hat{k}+\frac{\hat{K}}{2}\right)q\left(-\hat{k}+\frac{\hat{K}}{2}\right)\right\rangle = \frac{1}{3}G_D(\hat{k})G_D^*(\hat{k})\langle u_m u_m\rangle\langle|\nabla q|^2\rangle.$$

Integration over ω space and over the angles in $\mathbf{k}$ space yields a spectrum of isotropic passive scalar fluctuations in a range of wave numbers $k_D < k < k_\nu$:

$$\Gamma(k) = \frac{(\beta-1)}{3}\frac{\mathrm{Pe}^2}{k_0^3}\langle|\nabla q|^2\rangle\left(\frac{k}{k_0}\right)^{-\beta-4}. \tag{49}$$

When $\beta = 5/3$ Eq. (49) yields

$$\Gamma(k) = \frac{2}{9}\frac{\mathrm{Pe}^2}{k_0^3}\langle|\nabla q|^2\rangle\left(\frac{k}{k_0}\right)^{-17/3}.$$

Note that the scalar transport problem can be formally separated from the dynamical problem of turbulence. Indeed, we study the passive scalar fluctuations in a prescribed turbulent velocity field. Therefore, the dependence $\nu(k)$ as well as the rest of the hydrodynamic characteristics are the external parameters of this problem and must be specified independently. In this sense the hydrodynamic problem and that of passive scalar transport are independent. However, since we also determine the spectra of passive scalar fluctuations we need the explicit dependence of the Prandtl number $\mathrm{Pr}(k)$, e.g., $\nu(k)$. This is the reason that we derived the approximate equation for $\nu(k)$ using the simple renormalization procedure. Certainly this procedure does not allow us to determine the spectra of hydrodynamic turbulence β, which is assumed to be known.

Notably, we did not derive the spectrum different from the spectrum of the Corrsin-Obukhov theory. Indeed, the k^{-3} spectrum in the anisotropic case was derived in our study for a different problem. In contrast to Corrsin-Obukhov theory, we considered a completely different mechanism of excitation of passive scalar fluctuations. While Corrsin-Obukhov assumed the isotropic external source of fluctuations, we analyzed the case when passive scalar fluctuations are generated by "tangling" of an external gradient of mean passive scalar field ∇N_0 by a turbulent velocity field.

TABLE I. Isotropic and anisotropic spectra of passive scalar fluctuations. The case $\beta = 5/3$ corresponds to the Kolmogorov spectrum for turbulent velocity field.

	Isotropic spectra of the passive scalar	Anisotropic spectra of the passive scalar
$k_0 < k < \min(k_D, k_\nu)$		
$\beta > 0$:	$k^{-(5-\beta/2)}$	$k^{-3}(\nabla N_0)^2$
$\beta = \frac{5}{3}$:	$k^{-5/3}$	$k^{-3}(\nabla N_0)^2$
$k_D < k < k_\nu$		
$\beta > 0$:	$k^{-\beta-4}$	$k^{-\beta-4}(\nabla N_0)^2$
$\beta = \frac{5}{3}$:	$k^{-17/3}$	$k^{-17/3}(\nabla N_0)^2$

The mechanism of "tangling" of an external mean field by hydrodynamic fluctuations is not new. It is well known from the theory of magnetic fluctuations (passive vector field) that the result of "tangling" of mean magnetic field is a generation of anisotropic magnetic fluctuations (see, e.g., [21,27,28]). The spectrum of these anisotropic magnetic fluctuations is different from that of isotropic magnetic fluctuations excited by the Zeldovich mechanism in homogeneous and isotropic hydrodynamic turbulence (see, e.g., [4,27,29]).

A similar situation occurs in the passive scalar transport. The difference between the latter case and magnetic (passive vector field) case is that incompressible homogeneous and isotropic turbulent velocity field cannot cause self-excitation (exponential growth) of passive scalar fluctuations (in contrast to the exponential growth of the magnetic fluctuations excited by the Zeldovich mechanism). Isotropic passive scalar fluctuations can be excited by an external isotropic source as adopted in Corrsin-Obukhov theory.

Naturally, these two completely different mechanisms of generation of passive scalar fluctuations (external isotropic source and "tangling" of an external gradient of mean passive scalar field ∇N_0 by a turbulent velocity field) result in different spectra: the $k^{-5/3}$ spectrum in the Corrsin-Obukhov theory and the k^{-3} spectrum in the anisotropic case.

If two these sources (external isotropic source and "tangling" of an external gradient of mean passive scalar field) exist simultaneously the passive scalar fluctuations can be regarded as a mixture of isotropic (with the $k^{-5/3}$ spectrum) and anisotropic (with the k^{-3} spectrum). It is conceivable to suggest that in the case when there exists only one source of passive scalar fluctuations, i.e., an external gradient of mean passive scalar field, the resulting fluctuation spectrum will be a mixture of two parts, isotropic ($\sim k^{-5/3}$) and anisotropic ($\sim k^{-3}$). The latter suggestion can be verified experimentally by measurements of correlation function of concentration field in $\mathbf{r}$ space.

IV. DISCUSSION

In this study we considered the isotropic and anisotropic spectra of passive scalar fluctuations in a turbulent fluid flow with a power law $\propto k^{-\beta}$ spectrum by means of a renormalization procedure. The results are presented in Table I. It is

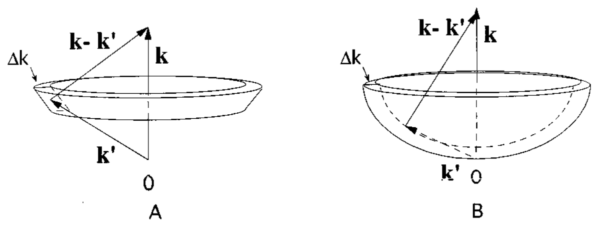

FIG. 1. Domain of integration in k space for the functionals B_{mn} (a) and L_{mn} (b).

seen that the anisotropic k^{-3} spectrum is universal. It is independent of the exponent in the spectrum of hydrodynamic turbulence. Note that the existence of the k^{-3} spectrum found in [10] was the subject of controversy (see, e.g., [2,15,17]). In the present study this spectrum has been derived by the renormalization procedure as well as the known $k^{-5/3}$ and $k^{-17/3}$ spectra. The $k^{-5/3}$ and $k^{-17/3}$ spectra of passive scalar fluctuations exist only in a turbulent fluid flow with the Kolmogorov spectrum ($\beta=5/3$). The interval of wave numbers $k_\nu<k<k_D$ (i.e., $\mathrm{Pr}_0\gg1$) cannot be studied by the renormalization procedure. In this region the k^{-1} spectrum exists [11,13].

APPENDIX: EQUATIONS FOR THE TURBULENT TRANSPORT COEFFICIENTS

Let us derive equations for the turbulent fields. The total pressure can be excluded from the equation of motion by taking the "curl" of this equation. So we repeat twice the vector multiplication of Eqs. (14) and (16) written in k and ω space by $\mathbf{k}$. Then we subtract Eq. (16) from Eq. (14) and Eq. (17) from Eq. (15), respectively, change to a frame moving with a local velocity of the mean flows $\mathbf{V}$, and transform Eqs. (17) and (15) to k and ω space. The result is given by

$$u_j(\hat{k})+\frac{i}{2}P_{jmn}G_\nu B_{mn}(u;u)=G_\nu f_j-iG_\nu\bar{P}_{jmn}L_{mn}(V;u), \tag{A1}$$

$$q(\hat{k})+iG_\eta k_n S_n(q;u)=-iG_\eta k_n H_n(N;u)+\epsilon G_\eta, \tag{A2}$$

where

$$L_{mn}(a;b)=\int a_m(\hat{k}')b_n(\hat{k}-\hat{k}')d\hat{k}',$$

$$B_{mn}(a;b)=L_{mn}(a;b)-\langle L_{mn}(a;b)\rangle,$$

$$H_n(a;b)=\int a(\hat{k}')b_n(\hat{k}-\hat{k}')d\hat{k}',$$

$$S_n(a;b)=H_n(a;b)-\langle H_n(a;b)\rangle,$$

$$P_{jmn}=\Delta_{jm}k_n+\Delta_{jn}k_m, \quad \bar{P}_{jmn}=\Delta_{jm}k_n,$$

$$\Delta_{jm}=\delta_{jm}-\frac{k_jk_m}{k^2},$$

$$\hat{k}=\begin{pmatrix}\mathbf{k}\\ \omega\end{pmatrix}, \quad \hat{k}'=\begin{pmatrix}\mathbf{k}'\\ \omega'\end{pmatrix}, \quad G_\gamma=\frac{1}{-i\omega+\gamma k^2}, \quad \gamma=\nu,\eta.$$

Here we use incompressibility condition $\boldsymbol{\nabla}\cdot\mathbf{u}=0$, and, therefore

$$\int u_n(\hat{k}-\hat{Q})Q_nN(\hat{Q})d^3Q=k_n\int u_n(\hat{k}-\hat{Q})N(\hat{Q})d^3Q.$$

Define a background turbulence as the turbulence with $\boldsymbol{\nabla}N=0$ and $\boldsymbol{\nabla}_iV_j=0$. The equations for $\mathbf{u}^{(0)}$ and $q^{(0)}$ for the background turbulence are given by

$$u_j^{(0)}(\hat{k})+\frac{i}{2}P_{jmn}G_\nu B_{mn}(u^{(0)};u^{(0)})=G_\nu f_j, \tag{A3}$$

$$q^{(0)}(\hat{k})+iG_\eta k_n S_n(q^{(0)};u^{(0)})=G_\eta\epsilon \tag{A4}$$

[see Eqs. (A1) and (A2)]. Note that the equations are written in a frame moving with a local "mean" flow $\mathbf{V}$. The correlation functions of the background turbulence have the most simple form (for example, they are homogeneous and isotropic) only in this frame, while these correlations in the laboratory frame are anisotropic [20,21].

The equations for the fields $\mathbf{u}^{(1)}=\mathbf{u}-\mathbf{u}^{(0)}$ and $\mathbf{h}^{(1)}=\mathbf{h}-\mathbf{h}^{(0)}$ are derived from (A1)–(A4):

$$u_j^{(1)}(\hat{k})+\frac{i}{2}P_{jmn}G_\nu[B_{mn}(u^{(0)};u^{(1)})+B_{mn}(u^{(1)};u^{(0)})+B_{mn}(u^{(1)};u^{(1)})]=-iG_\nu\bar{P}_{jmn}[L_{mn}(V;u^{(0)})+L_{mn}(V;u^{(1)})], \tag{A5}$$

$$q^{(1)}(\hat{k})+iG_\eta k_n[S_n(q^{(0)};u^{(1)})+S_n(q^{(1)};u^{(0)})+S_n(q^{(1)};u^{(1)})]=-iG_\eta k_n[H_n(N;u^{(0)})+H_n(N;u^{(1)})]. \tag{A6}$$

These equations describe the deviation from the background turbulence caused by nonzero mean field gradients $\boldsymbol{\nabla}N$ and $\boldsymbol{\nabla}_iV_j$.

Let us estimate the magnitude of the nonlinear terms in Eqs. (A1)–(A6) determined by the functionals $B_{mn}(u;u)$ and $S_n(q;u)$. The volume of the domain of integration in $\mathbf{k}'$ space is very small. Indeed, the end points of the vectors $\mathbf{k}'$, $\mathbf{k}-\mathbf{k}'$, and $\mathbf{k}$ fall within a thin spherical shell of the thickness $|\Delta\mathbf{k}|$ [see Fig. 1(a)]. Thus, for example,

$$|P_{jmn}B_{mn}(u^{(0)};u^{(1)})|\simeq\left|k_m\int u_m^{(0)}(\hat{k}')u_j^{(1)}(\hat{k}-\hat{k}')d\hat{k}'\right|\leq k_*U_*\int u_*^{(0)}u_*^{(1)}d\omega=\pi k_*^2(\Delta k)^2\int u_*^{(0)}u_*^{(1)}d\omega.$$

Here $U_*=\pi k_*(\Delta k)^2$ is the volume of the domain of integration [see Fig. 1(a)], $u_*^{(l)}=u^{(l)}(k=k_*)$. The other functionals $B_{mn}(u;u)$ and $S_n(q;u)$ can be estimated similarly and are proportional to $(\Delta k)^2$.

Now we find the linear terms in Eqs. (A1)–(A6) related to the "mean" fields $\nabla_i V_j$ and $\boldsymbol{\nabla} N$. The equation of the surface that determines the domain of integration for the functionals $L_{mn}(V;u)$ and $H_n(N;u)$ is given by $k_*^2=(\mathbf{k}-\mathbf{k}')^2$. Because $|\mathbf{k}|=k_*$, the equation of this surface reduces to $|\mathbf{k}'|=2k_*\cos\theta$, where θ is the angle between $\mathbf{k}$ and $\mathbf{k}'$. The volume of the domain of integration is $U_0=\pi k_*^2|\Delta\mathbf{k}|$ [see Fig. 1(b)]. Thus the terms $L_{mn}(V;u)$ and $H_n(N;u)$ in Eqs. (A1)–(A6) are of order $|\Delta\mathbf{k}|$. Therefore the functionals $B_{mn}(u;u)\sim|\Delta\mathbf{k}|^2$ [and $S_n(q;u)\sim|\Delta\mathbf{k}|^2$] can be dropped out for small $|\Delta\mathbf{k}|\ll k_*$, where $k_*>k_0$. Now we estimate the errors committed at each stage of averaging (due to neglect of nonlinear terms) which can be accumulated. The error at each stage of averaging is $\delta_\alpha\sim|\Delta\mathbf{k}|^2$. A number of small regions that yield the error can be estimated as $n_*\sim k_D/|\Delta\mathbf{k}|$. Now we assume that fluctuations of the background turbulence in these small regions are statistically independent. Therefore the maximum error committed at each stage of averaging (due to neglect of nonlinear terms) is of order

$$\delta=\sqrt{\sum_{\alpha=1}^{n_*}(\delta_\alpha)^2}\sim\sqrt{|\Delta\mathbf{k}|^4 n_*}\sim|\Delta\mathbf{k}|^{3/2}.$$

Here we will take into account only terms $\sim O(|\Delta\mathbf{k}|)$. Therefore we can neglect nonlinear terms that are associated with fluctuations in the small interval $k_*-|\Delta\mathbf{k}|<|\mathbf{k}|<k_*$.

However, this does not mean that the nonlinear terms are dropped out in all $\mathbf{k}$ space. At the first step of the renormalization procedure the equations are averaged over fluctuations of scales from k_ν^{-1} to k_*^{-1}. Therefore the nonlinear terms contribute to the turbulent transport coefficient ν and η in all scales except for only very small region of the spectrum: $k_*-|\Delta\mathbf{k}|<|\mathbf{k}|<k_*$.

The equations for the fields $u_j^{(0)}(\hat{k})$ and $q^{(0)}(\hat{k})$ of the background turbulence are reduced to

$$u_j^{(0)}(\hat{k})=G_\nu f_j,\quad q^{(0)}(\hat{k})=G_\eta\epsilon.$$

Therefore the background turbulence is assumed to be given.

In order to derive equations for the turbulent coefficients ν and η we have to find the dependence of the second moments $\langle u_i u_j\rangle$ and $\langle q u_j\rangle$ on the large-scale fields $\boldsymbol{\nabla}_i V_j$ and $\boldsymbol{\nabla} N$. Let us consider, for example, a correlation function $\langle u_i u_j\rangle$,

$$\langle u_i(\hat{x})u_j(\hat{y})\rangle=\int\langle u_i(\hat{k}_1)u_j(\hat{k}_2)\rangle\exp[i(\hat{k}_2\hat{x}+\hat{k}_2\hat{y})]d\hat{k}_1 d\hat{k}_2$$
$$=\int F_{ij}(\hat{K},\hat{r})\exp(i\hat{K}\hat{R})d\hat{K},$$

where

$$F_{ij}(\hat{K},\hat{r})=\int\langle u_i(\hat{k}+\hat{K}/2)u_j(-\hat{k}+\hat{K}/2)\rangle\exp(i\hat{k}\hat{r})d\hat{k},$$

$$\hat{R}=\frac{1}{2}(\hat{x}+\hat{y}),\quad \hat{r}=\hat{x}-\hat{y},\quad \hat{K}=\hat{k}_1+\hat{k}_2,$$

$$\hat{k}=\frac{1}{2}(\hat{k}_1-\hat{k}_2),\quad \hat{x}=\begin{pmatrix}\mathbf{x}\\ -t_1\end{pmatrix},$$

$\hat{R}$ and $\hat{K}$ correspond to the large scales, and $\hat{r}$ and $\hat{k}$ to the small ones (see, for example, [21,30]). The others second moments have the same form. These correlation functions are calculated at $\hat{r}=0$ and $t_1=t_2$.

At first we have to solve the system of the equations (A5) and (A6). Let us consider here the effects that are linear in the spatial derivatives of the mean fields $\mathbf{V}$ and N. We use the method of iterations. The first iteration corresponds to the solutions of Eqs. (A5) and (A6) when the right parts of these equations equal zero:

$$u_j^{(I)}(\hat{k})=-iG_\nu\bar{P}_{jmn}L_{mn}(V;u^{(0)}),$$

$$q^{(I)}(\hat{k})=-iG_\eta k_n H_n(N;u^{(0)}).$$

Note that the terms $B_{mn}(u;u)\sim(\Delta k)^2$ and $S_n(q;u)\sim(\Delta k)^2$ are dropped. These solutions in the explicit form are given by

$$u_j^{(I)}(\hat{k}_1)\simeq-iG_\nu\int\left(\delta_{ij}-\frac{k_i k_j}{k^2}\right)V_i(\hat{Q})Q_m u_m^{(0)}(\hat{k}_1-\hat{Q})d\hat{Q},\tag{A7}$$

$$q^{(I)}(\hat{k}_1)\simeq-iG_\eta\int N(\hat{Q})Q_m u_m^{(0)}(\hat{k}_1-\hat{Q})d\hat{Q},\tag{A8}$$

where $\hat{k}_1=\hat{k}+\hat{K}/2$. The second iteration produces terms $\sim O(Q^2N^2;Q^2V^2)$ which are neglected. Therefore $u_j^{(1)}\approx u_j^{(I)}$ and $q^{(1)}\approx q^{(I)}$.

The second moments describing a deviation of the turbulence from the background level can be obtained from Eqs. (A7) and (A8). For simplicity, the background turbulence is assumed to be homogeneous and isotropic, i.e.,

$$\left\langle u_m^{(0)}\left(\hat{k}+\frac{\hat{K}}{2}\right)u_n^{(0)}\left(-\hat{k}+\frac{\hat{K}}{2}\right)\right\rangle=\frac{E(k)T(k,\omega)}{8\pi k^2}\times\left(\delta_{mn}-\frac{k_m k_n}{k^2}\right)\delta(\hat{K}).\tag{A9}$$

Integration in Eqs. (A7) and (A8) over the angles in $\mathbf{k}$ space results in the following expressions for the second moments:

$$\langle u_m^{(1)}u_n^{(0)}\rangle=-\frac{1}{5}V_{mn}\int E(k)T(k,\omega)G_\nu dk d\omega,\tag{A10}$$

$$\langle q^{(1)}u_n^{(0)}\rangle=-\frac{1}{3}\frac{\partial N}{\partial R_n}\int E(k)T(k,\omega)G_\eta dk d\omega.\tag{A11}$$

Here $\langle q^{(0)}u_n^{(1)}\rangle=0$, because $\langle q^{(0)}u_n^{(0)}\rangle=0$, and $\langle u_m^{(1)}u_n^{(1)}\rangle \sim\langle q^{(1)}u_n^{(1)}\rangle\sim O(Q^2N^2;Q^2V^2)$, where

$$V_{mn}=\frac{\partial V_m}{\partial R_n}+\frac{1}{6}\,\frac{\partial V_n}{\partial R_m}.$$

Integration in (A10) and (A11) is performed over ω from $-\infty$ to ∞ and over $\mathbf{k}$ from $k_*-|\Delta\mathbf{k}|$ to k_*. The following integrals are used for the calculations of the second moments in (A10) and (A11):

$$\int (\mathbf{k}\cdot\mathbf{a})\frac{k_i}{k^2}\sin\theta d\theta d\varphi=\frac{4\pi}{3}a_i,$$

$$\int \left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\sin\theta d\theta d\varphi=\frac{8\pi}{3}\delta_{ij},$$

$$\int (\mathbf{k}\cdot\mathbf{a})(\mathbf{k}\cdot\mathbf{b})\frac{k_ik_j}{k^2}\sin\theta d\theta d\varphi=\frac{4\pi}{15}k^2 \times[(\mathbf{a}\cdot\mathbf{b})\delta_{ij}+a_ib_j+a_jb_i].$$

Substituting Eqs. (A10) and (A11) into the expressions for tensors (18) and (19) yields

$$\sigma_{mn}=\sigma_{mn}^{(0)}+\Delta\nu\left(\frac{\partial V_m}{\partial R_n}+\frac{\partial V_n}{\partial R_m}\right), \tag{A12}$$

$$\Psi_m=-(\boldsymbol{\nabla}N)\Delta\eta, \tag{A13}$$

where

$$\Delta\nu=\frac{7}{30}|\Delta k|\int E(k)T(k,\omega)G_\nu d\omega, \tag{A14}$$

$$\Delta\eta=\frac{1}{3}|\Delta k|\int E(k)T(k,\omega)G_\eta d\omega, \tag{A15}$$

and we take into account that for small $|\Delta k|$ the integral $\int F(k)dk\sim|\Delta k|F(k')$, where $k_*-\Delta k<k'<k_*$. Substituting (A12) and (A13) into Eqs. (16) and (17) yields the equations that coincide in form with Eqs. (14) and (15). This means that these equations are invariant under the procedure of the successive averaging, i.e., invariant under the renormalization of the turbulent transport coefficients. The term $\boldsymbol{\nabla}\sigma_{mn}^{(0)}$ is dropped for the homogeneous background turbulence and we take into account that $\langle q^{(0)}u_n^{(0)}\rangle=0$.

Now we divide Eqs. (A14) and (A15) by $\Delta k=-|\Delta k|$ and pass to the limit of small Δk. The minus sign arises because the procedure of the successive averaging is performed from small scales to large ones. Note that the background turbulence is confined in the region $k_0<k<k_\nu$. Therefore the equations are not renormalized for $k<k_0$. The small values of Δk imply that $|\Delta k|\ll k_0$. Then the equations for the turbulent viscosity and turbulent diffusion are reduced to

$$\frac{d\nu}{dk}=-\frac{7}{30}\int E(k)T(k,\omega)G_\nu d\omega, \tag{A16}$$

$$\frac{d\eta}{dk}=-\frac{1}{3}\int E(k)T(k,\omega)G_\eta d\omega. \tag{A17}$$

Using (34) and integrating in Eqs. (A16) and (A17) over ω space yield the equations for the turbulent coefficients (20) and (21). In the derivation we used integrals of the products of Green functions:

$$\int G_\alpha G_\alpha^* d\omega=\frac{\pi}{\alpha k^2},\qquad \int G_\alpha G_\alpha^* G_\beta d\omega=\frac{\pi}{\alpha k^4(\alpha+\beta)}, \tag{A18}$$

$$\int G_\alpha G_\alpha^* G_\beta G_\beta^* d\omega=\frac{\pi}{\alpha\beta k^6(\alpha+\beta)}. \tag{A19}$$

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