Compressibility effects in turbulent transport of the temperature field

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Compressibility effects in a turbulent transport of temperature field are investigated by applying the quasilinear approach for small Péclet numbers and the spectral τ approach for large Péclet numbers. The compressibility of a fluid flow reduces the turbulent diffusivity of the mean temperature field similarly to that for the particle number density and magnetic field. However, expressions for the turbulent diffusion coefficient for the mean temperature field in a compressible turbulence are different from those for the mean particle number density and the mean magnetic field. The combined effect of compressibility and inhomogeneity of turbulence causes an increase of the mean temperature in the regions with more intense velocity fluctuations due to a turbulent pumping. Formally, this effect is similar to a phenomenon of compressible turbophoresis found previously [J. Plasma Phys. **84**, 735840502 (2018)] for noninertial particles or gaseous admixtures. The gradient of the mean fluid pressure results in an additional turbulent pumping of the mean temperature field. The latter effect is similar to the turbulent barodiffusion of particles and gaseous admixtures. The compressibility of a fluid flow also causes a turbulent cooling of the surrounding fluid due to an additional sink term in the equation for the mean temperature field. There is no analog of this effect for particles.

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I. INTRODUCTION

The compressibility of a fluid flow affects the turbulent transport of particles, temperature, and magnetic fields (see, e.g., Refs. [1–5]), e.g., it causes qualitative changes in the properties of both mean fields and fluctuations. Large-scale effects of turbulence on particle concentrations and temperature field are described by means of the turbulent flux of particles and turbulent heat flux, respectively. For incompressible flow, the main contribution to the turbulent fluxes is determined by the turbulent diffusion of particles and the temperature field. This corresponds to the gradient turbulent transport of particles and temperature fields, e.g., the turbulent flux of particles is directed opposite to the gradient of the mean particle number density, while the turbulent heat flux is directed opposite to the gradient of the mean fluid temperature.

The compressibility of a turbulent flow results in a reduction of the turbulent diffusivity of a mean particle number density at small [6] and large [5,7] Péclet numbers. The Péclet number is the ratio of nonlinear to diffusion terms in the equation for particle number density fluctuations. A similar effect of the reduction of turbulent magnetic diffusivity by compressible turbulence exists also for the mean magnetic field at small [6,8] and large [5,7] magnetic Reynolds numbers. The conclusion about the reduction of turbulent diffusivity by the compressibility of fluid flow has been also confirmed by the test-field method in direct numerical simulations for an irrotational homogeneous deterministic flow [6]. Various aspects related to compressibility effects on turbulent transport have been studied using different analytical approaches (for a review, see Ref. [5]), e.g., the quasilinear approach [6–8], the spectral tau approach [7], the path-integral approach [9–11], the multiple-scale direct-interaction approximation [12,13], etc.

The compressibility of a turbulent flow causes an additional nongradient contribution to the turbulent flux of particles that is proportional to a product of the mean particle number density and effective pumping velocity. In a density stratified turbulence, the effective pumping velocity of particles is proportional to the gradient of the mean fluid density multiplied by the turbulent diffusion coefficient [10,11]. The pumping effect results in an accumulation of particles in regions of maximum mean fluid density.

In a temperature stratified turbulence, a similar effect referred to as turbulent thermal diffusion results in a turbulent nondiffusive flux of particles in the direction of the turbulent heat flux, so that particles are accumulated in the vicinity of the mean temperature minimum [10,11]. This phenomenon has been studied theoretically [14–18], found in direct numerical simulations [7,19,20], detected in different laboratory experiments [18,21–23], and atmospheric turbulence with temperature inversions [24]. This effect has been shown to be important for concentrating dust in protoplanetary disks [25]. Density stratification which causes the turbulent pumping of particles becomes weaker with increasing compressibility, i.e., with increasing the Mach number [7].

The compressibility of a fluid flow in inhomogeneous turbulence also results in a new pumping effect of particles from regions of low to high turbulent intensity both for small and large Péclet numbers. This effect has been interpreted in Ref. [7] as a compressible turbophoresis of noninertial

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particles and gaseous admixtures, while the classical turbophoresis effect for incompressible inhomogeneous turbulence [26–30] exists only for inertial particles and causes them to be pumped to regions with lower turbulent intensity.

The compressibility of a turbulent fluid flow affects also passive scalar fluctuations. In particular, it results in a slow scale-dependent turbulent diffusion of small-scale passive scalar fluctuations for large Pèclet numbers [9]. In addition, the level of passive scalar fluctuations in the presence of a gradient of the mean passive scalar field in compressible turbulent flow can be fairly strong. On the other hand, passive scalar transport in a density stratified turbulent fluid flow is accompanied by the formation of large-scale structures due to the instability of the mean passive scalar field in an inhomogeneous turbulent velocity field [9].

Another interesting feature for a compressible temperature stratified turbulence is that the turbulent flux of entropy is different from the turbulent convective flux of the fluid internal energy [31,32]. In particular, in a low-Mach-number approximation as well as in the framework of the mean-field approach, the turbulent flux of entropy is given by $\boldsymbol{F}_s = \overline{\rho}\langle s'\boldsymbol{u}\rangle$, where $\overline{\rho}$ is the mean fluid density and s' and $\boldsymbol{u}$ are fluctuations of entropy and velocity, respectively, and the angular brackets $\langle\cdots\rangle$ denote ensemble averaging. On the other hand, the turbulent convective flux of the fluid internal energy is $\boldsymbol{F}_c = \overline{T}\,\overline{\rho}\langle s'\boldsymbol{u}\rangle$, where $\overline{T}$ is the mean fluid temperature. This turbulent convective flux is well known in the astrophysical and geophysical literature, and it cannot be used as a turbulent flux in the equation for the mean entropy. This is the exact result for low-Mach-number temperature stratified turbulence and is independent of the turbulence model used [32].

Temperature fluctuations and anomalous scaling in a low-Mach-number compressible turbulent flow have been studied in Ref. [33]. Due to compressibility and external pressure fluctuations, the anomalous scaling (i.e., the violation of the dimensional analysis predictions for the scaling laws) may occur in the second moment of the temperature field. The cause of the anomalous behavior is a compressibility-induced depletion of the turbulent diffusion of the second moment of the temperature field [33].

In spite of the many studies of turbulent transport of passive scalar, some large-scale (mean-field) features related to the compressibility effects on the turbulent transport of a temperature field are not known. In the present paper, we study the compressibility effects in the turbulent transport of the mean temperature field, i.e., we consider here mean-field effects. This paper is organized as follows. In Sec. II we outline the governing equations. The turbulent heat flux and level of temperature fluctuations are determined for small Péclet numbers in Sec. III and for large Péclet numbers in Sec. IV. In Secs. III and IV we also outline the method of derivations and approximations made for the study of the compressibility effects. In Sec. V we discuss how a homogeneous compressible turbulence can cause a turbulent cooling of the surrounding fluid. Finally, conclusions are drawn in Sec. VI. In Appendix A we outline the multiscale approach used in the present study. Details of the derivation of the turbulent heat flux and level of temperature fluctuations are given in Appendix B for small Péclet numbers and in Appendix C for large Péclet numbers.

II. GOVERNING EQUATIONS

The evolution of the temperature field $T(t,\boldsymbol{r})$ in a compressible fluid velocity field $\mathbf{U}(t,\boldsymbol{r})$ is given by [34]

$$\frac{\partial T}{\partial t} + (\mathbf{U}\cdot\nabla)T + (\gamma-1)T(\nabla\cdot\boldsymbol{U}) = D\Delta T + J_\nu, \quad (1)$$

where D is the molecular thermal conductivity, $\gamma = c_{\rm p}/c_{\rm v}$ is the ratio of specific heats, and J_ν is the heating source caused, e.g., by a viscous dissipation of the kinetic energy.

In a compressible flow, Eq. (1) for the temperature field is different from the equation for the particle number density $n(t,\boldsymbol{r})$ [35,36],

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\boldsymbol{U}) = D_n\,\Delta n, \quad (2)$$

where D_n is the coefficient of the Brownian (molecular) diffusion of particles.

We consider a compressible turbulent flow when the Mach number can be not small. To derive equations for the turbulent heat flux and the level of temperature fluctuations, we apply the mean-field approach. In particular, the fluid temperature, pressure, density, and velocity are decomposed into mean and fluctuating parts, where the fluctuating parts have zero mean values, i.e., the Reynolds averaging is applied here, which easily separates fluctuations from mean fields. For example, the density-weighted averaging quantities [37,38] are usually difficult to extract from laboratory and atmospheric measurements or from astrophysical observations.

In the framework of the mean-field approach, the fluid temperature is $T = \overline{T} + \theta$, the fluid pressure is $P = \overline{P} + p$, and the fluid density is $\rho = \overline{\rho} + \rho'$, where $\overline{T} = \langle T\rangle$ is the mean fluid temperature, $\overline{P} = \langle P\rangle$ is the mean fluid pressure, and $\overline{\rho} = \langle\rho\rangle$ is the mean fluid density, θ are temperature fluctuations, p are pressure fluctuations, and ρ' are density fluctuations. The angular brackets denote an ensemble averaging. Similarly, $\boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u}$, where $\overline{\boldsymbol{U}} = \langle\boldsymbol{U}\rangle$ is the mean fluid velocity, and $\boldsymbol{u}$ are velocity fluctuations. For simplicity, we consider the case $\overline{\boldsymbol{U}} = 0$.

Averaging Eq. (1) over an ensemble of turbulent velocity field, we arrive at the equation for the mean temperature field as

$$\frac{\partial\overline{T}}{\partial t} + \nabla\cdot\langle\theta\,\boldsymbol{u}\rangle = -(\gamma-2)\langle\theta(\nabla\cdot\boldsymbol{u})\rangle + D\,\Delta\overline{T} + \overline{J}_\nu, \quad (3)$$

where $\boldsymbol{F} = \langle\theta\,\boldsymbol{u}\rangle$ is the turbulent heat flux, $\overline{J}_\nu$ is the mean heating source caused by the viscous dissipation of the turbulent kinetic energy, and $I_{\rm S} = -(\gamma-2)\langle\theta(\nabla\cdot\boldsymbol{u})\rangle$ is the mean sink term resulting in a turbulent cooling due to compressibility effects (see Sec. V). Using Eqs. (1) and (3), we obtain the equation for temperature fluctuations, $\theta(\boldsymbol{x},t) = T - \overline{T}$,

$$\frac{\partial\theta}{\partial t} + \mathcal{Q} - D\nabla\theta = -(\boldsymbol{u}\cdot\nabla)\overline{T} - (\gamma-1)\,\overline{T}\,\nabla\cdot\boldsymbol{u}, \quad (4)$$

where

$$\mathcal{Q} = \nabla\cdot(\theta\boldsymbol{u} - \langle\boldsymbol{u}\,\theta\rangle) + (\gamma-2)[\theta\,\nabla\cdot\boldsymbol{u} - \langle\theta\,\nabla\cdot\boldsymbol{u}\rangle]$$

are nonlinear terms and

$$I = -(\boldsymbol{u}\cdot\nabla)\overline{T} - (\gamma-1)\overline{T}\,\nabla\cdot\boldsymbol{u}$$

are the source terms of temperature fluctuations. The ratio of the nonlinear term to the diffusion term is the Péclet number, that is estimated as $\mathrm{Pe} = u_0\, \ell_0/D$, where u_0 is the characteristic turbulent velocity in the integral (energy-containing) scale ℓ_0 of turbulence. We consider a one-way coupling, i.e., we take into account the effect of turbulence on the temperature field, but neglect the feedback effect of the temperature on the turbulence.

To determine the turbulent heat flux and the level of temperature fluctuations, and to take into account the small-scale properties of the turbulence, we use two-point correlation functions. For a fully developed turbulence, scalings for the turbulent correlation time and the turbulent kinetic energy spectrum are related via the Kolmogorov scalings [1,2,39–41]. We consider the cases with small and large Péclet and Reynolds numbers.

In the framework of the mean-field approach, we assume that there is a separation of spatial and temporal scales, i.e., $\ell_0 \ll L_T$ and $\tau_0 \ll t_T$, where L_T and t_T are the characteristic spatial and temporal scales characterizing the variations of the mean temperature field and $\tau_0 = \ell_0/u_0$. The mean fields depend on "slow" variables, while fluctuations depend on "fast" variables. Separation into slow and fast variables is widely used in theoretical physics, and all calculations are reduced to the Taylor expansions of all functions using small parameters ℓ_0/L_T and τ_0/t_T. The findings are further truncated to leading-order terms. Separation to slow and fast variables is performed by means of a standard multiscale approach [42] discussed in detail in Appendix A.

III. TURBULENT HEAT FLUX AND LEVEL OF TEMPERATURE FLUCTUATIONS FOR SMALL PÉCLET NUMBERS

In this section we derive the equations for the turbulent heat flux and the level of temperature fluctuations for small Péclet numbers using the quasilinear approach. For a random flow with small Péclet and Reynolds numbers, there are no universal scalings for the correlation time and the turbulent kinetic energy spectrum. This is the reason why we use noninstantaneous two-point correlation functions in this case. In the framework of the quasilinear approach, we neglect the nonlinear term $\mathcal{Q}$, but keep the molecular diffusion term in Eq. (4). We rewrite this equation in Fourier space and find the solution of this equation, given by Eq. (B1) in Appendix B. Using this solution and applying the multiscale approach (see Ref. [42] and Appendix A), we arrive at the expressions for the turbulent heat flux and the level of temperature fluctuations in Fourier space for small Péclet numbers as

$$\begin{aligned}\langle \theta\, u_j\rangle = &-\frac{\gamma-1}{2}\biggl[\overline{T}\int G_D(\nabla_i - 2Dk^2 G_D\, k_{im}\nabla_m + 2ik_i) f_{ij}\, d\boldsymbol{k}\, d\omega \\ &-(\nabla_i \overline{T})\int G_D\left(\frac{\gamma-3}{\gamma-1}\delta_{im} + 2Dk^2 G_D\, k_{im} + k_m\frac{\partial}{\partial k_i}\right) f_{mj}\biggr] d\boldsymbol{k}\, d\omega, \end{aligned} \tag{5}$$

$$\begin{aligned}\langle \theta^2\rangle = &\frac{\gamma-1}{4}\biggl\{\overline{T}\int G_D[(2Dk^2 G_D\, k_{jn}\nabla_n - \nabla_j)F_j^{(+)} + 2ik_j\, F_j^{(-)}] d\boldsymbol{k}\, d\omega \\ &+(\nabla_n \overline{T})\int G_D\left[\frac{\gamma-3}{\gamma-1}\delta_{jn} + 2Dk^2 G_D\, k_{jn} + k_j\frac{\partial}{\partial k_n}\right] F_j^{(+)}\biggr\} d\boldsymbol{k}\, d\omega. \end{aligned} \tag{6}$$

Details of the derivations of Eqs. (5) and (6) are given in Appendix B. Here, $G_D \equiv G_D(\boldsymbol{k},\omega) = (D\boldsymbol{k}^2 + i\omega)^{-1}$, $f_{ij} \equiv f_{ij}(\boldsymbol{k},\omega) = \langle u_i(\boldsymbol{k},\omega) u_j(-\boldsymbol{k},-\omega)\rangle$, $F_j^{(\pm)} = F_j(\boldsymbol{k},\omega) \pm F_j(-\boldsymbol{k},\omega)$, where $F_j(\boldsymbol{k},\omega) = \langle \theta(\boldsymbol{k},\omega)\, u_j(-\boldsymbol{k},-\omega)\rangle$ is the turbulent heat flux in Fourier space, δ_{ij} is the Kronecker unit tensor, and $k_{ij} = k_i k_j/k^2$. Since we consider a one-way coupling, the correlation function f_{ij} in Eqs. (5) and (6) should be replaced by $f_{ij}^{(0)}$ for the background random flow with zero turbulent heat flux.

We use a statistically stationary, density stratified, inhomogeneous, compressible, and nonhelical background random flow determined by the following correlation function in Fourier space [5,7],

$$\begin{aligned}f_{ij}^{(0)}(\boldsymbol{k},\omega) = &\frac{\Phi(\omega)}{8\pi k^2(1+\sigma_c)}\biggl\{E(k)\left[\delta_{ij} - k_{ij} + \frac{i}{k^2}(k_j\lambda_i - k_i\lambda_j) + \frac{i}{2k^2}(k_i\nabla_j - k_j\nabla_i)\right] \\ &+2\sigma_c\, E_c(k)\left[k_{ij} + \frac{i}{2k^2}(k_i\nabla_j - k_j\nabla_i)\right]\biggr\}\langle \boldsymbol{u}^2\rangle, \end{aligned} \tag{7}$$

where $\boldsymbol{\lambda} = -\boldsymbol{\nabla} \ln \overline{\rho}$ characterizes the fluid density stratification, $\sqrt{\langle \boldsymbol{u}^2\rangle}$ is the characteristic turbulent velocity at the maximum scale ℓ_0 of random motions, and the parameter

$$\sigma_c = \frac{\langle (\boldsymbol{\nabla}\cdot \boldsymbol{u})^2\rangle}{\langle (\boldsymbol{\nabla}\times \boldsymbol{u})^2\rangle} \tag{8}$$

is the degree of compressibility of the turbulent velocity field. We considered a weakly anisotropic background turbulence. In particular, in the derivation of Eq. (7), we assumed that $\ell_0 \ll H_\rho$ and $\ell_0 \ll L_u$, where $L_u = |\boldsymbol{\nabla} \ln \langle \boldsymbol{u}^2\rangle|^{-1}$ is the characteristic scale of the inhomogeneity of turbulence, and $H_\rho = |\boldsymbol{\lambda}|^{-1} = |\boldsymbol{\nabla} \ln \overline{\rho}|^{-1}$ is the mean density stratification scale, which is assumed to be constant. These conditions allow us to take into account the leading effects in Eq. (7), which are linear in stratification, $\propto|\boldsymbol{\lambda}|$, and the inhomogeneity of turbulence, $\propto|\boldsymbol{\nabla} \ln \langle \boldsymbol{u}^2\rangle|$. We neglect in Eq. (7) the high-order effects which are of the order of $\mathrm{O}(\lambda^2\langle \boldsymbol{u}^2\rangle)$, $\mathrm{O}(\nabla^2\langle \boldsymbol{u}^2\rangle)$, $\mathrm{O}(\lambda_i\nabla_i\langle \boldsymbol{u}^2\rangle)$.

Generally, stratification also contributes to div $\boldsymbol{u}$, i.e., it contributes to the parameter σ_c. Since this contribution is small, i.e., it is of the order of $\sim$O$(\lambda^2 \langle \boldsymbol{u}^2 \rangle)$, we neglect this contribution in Eq. (7). This allows us to separate the effects of the arbitrary Mach number, characterized by the parameter σ_c, and density stratification, described by $\boldsymbol{\lambda}$. The degree of compressibility σ_c depends on the Mach number. This dependence is not known for arbitrary Mach numbers and can be determined, e.g., in direct numerical simulations.

In Eq. (7), $E(k)$ and $E_c(k)$ are the spectrum functions for incompressible and compressible parts of a random flow. We assume that the random flows have a power-law spectrum for incompressible $E(k) = (q-1)(k/k_0)^{-q}\, k_0^{-1}$ and compressible $E_c(k) = (q_c-1)(k/k_0)^{-q_c}\, k_0^{-1}$ parts, where the wave number varies in the range $k_0 \leqslant k \leqslant k_\nu$. Here, $k_\nu = 1/\ell_\nu$ is the wave number based on the viscous scale ℓ_ν, and $k_0 = 1/\ell_0 \ll k_\nu$. We assume also that there are no random motions for $k < k_0$. In the model of a compressible background turbulence used in Ref. [7], the exponents $q = q_c$. In the present study, we consider the case when the spectrum exponents of the incompressible and compressible parts of random motions are different, i.e., $q \neq q_c$.

We assume that the frequency function $\Phi(\omega)$ has a Lorentz profile, $\Phi(\omega) = [\pi\, \tau_0(\omega^2 + \tau_0^{-2})]^{-1}$, which corresponds to the correlation function $\langle u_i(t) u_j(t+\tau)\rangle \propto \exp(-\tau/\tau_0)$. Here, the correlation time for small Péclet numbers $\tau_0 \equiv \ell_0/u_0 \gg (Dk^2)^{-1}$ for all turbulent scales. To derive Eq. (7) we use the identities given in Appendix B. Different contributions to Eq. (7) have been discussed in Refs. [5,7,9,43].

Integration in ω and $\boldsymbol{k}$ space in Eq. (5) yields an equation for the turbulent heat flux for small Péclet numbers,

$$\langle \theta\, \boldsymbol{u} \rangle = \overline{T}\, \boldsymbol{V}^{\rm eff} - D_T\, \boldsymbol{\nabla} \overline{T}, \tag{9}$$

where the turbulent diffusivity D_T and the effective pumping velocity $\boldsymbol{V}^{\rm eff}$ are given by

$$D_T = \frac{(q-1)}{3(q+1)} \frac{\tau_0 \langle \boldsymbol{u}^2 \rangle}{1+\sigma_c} {\rm Pe} \left[\gamma - \frac{1}{2}(3\gamma - 5)\sigma_c\, C_\sigma\right], \tag{10}$$

$$\boldsymbol{V}^{\rm eff} = (\gamma - 1) \frac{(q-1)}{3(q+1)} \frac{\tau_0 \langle \boldsymbol{u}^2 \rangle}{(1+\sigma_c)}\, {\rm Pe} \left[\frac{3}{2} C_\sigma\, \sigma_c\, \boldsymbol{\lambda}_u + \boldsymbol{\lambda}_P\right]. \tag{11}$$

Here, $\boldsymbol{\lambda}_u = \boldsymbol{\nabla} \ln \langle \boldsymbol{u}^2 \rangle$, $\boldsymbol{\lambda}_P = \boldsymbol{\nabla} \ln \overline{P}$, and

$$C_\sigma = \frac{(q_c - 1)(q+1)}{(q_c+1)(q-1)}. \tag{12}$$

We take into account that the equation of state for an ideal gas yields $\boldsymbol{\lambda} = -\boldsymbol{\lambda}_P + \boldsymbol{\nabla} \ln \overline{T}$. Since $\tau_0\, {\rm Pe} = \ell_0^2/D$, the turbulent transport coefficients given by Eqs. (10) and (11) are determined only by the microphysical diffusion timescale ℓ_0^2/D for small Péclet numbers. Equation (10) implies that for small Péclet numbers, the compressibility effects in most of the cases decrease the turbulent diffusivity. Indeed, for $\gamma \geqslant 5/3$, the derivative $\partial D_T/\partial \sigma_c$ is always negative, i.e., the compressibility effects decrease the turbulent diffusivity. When $1 < \gamma < 5/3$, the derivative $\partial D_T/\partial \sigma_c$ is negative when $C_\sigma < 2\gamma/(5 - 3\gamma)$. For example, for $q = q_c$, the derivative $\partial D_T/\partial \sigma_c$ is always negative. When $q = 5/3$ (i.e., for the Kolmogorov spectrum) and $q_c = 2$ (i.e., for the Burgers turbulence with shock waves), the derivative $\partial D_T/\partial \sigma_c$ is negative when $10/9 < \gamma < 5/3$. Note that the total diffusivity $D + D_T$ cannot be negative, because for ${\rm Pe} \ll 1$ the molecular diffusivity is much larger than the turbulent one, $D \gg |D_T|$.

The first term ($\propto \sigma_c\, \boldsymbol{\nabla} \langle \boldsymbol{u}^2 \rangle$) in Eq. (11) for the effective pumping velocity $\boldsymbol{V}^{\rm eff}$ of the mean temperature field describes a combined effect of the compressibility of fluid flow and the inhomogeneity of turbulence. This effect increases the mean temperature field in the regions with more intense velocity fluctuations due to turbulent pumping. This effect is similar to a phenomenon of compressible turbophoresis found previously for noninertial particles or gaseous admixtures [7].

The second term ($\propto \boldsymbol{\lambda}_P$) in Eq. (11) describes an additional turbulent pumping effect due to the gradient of the mean fluid pressure. This effect is similar to the turbulent barodiffusion of particles and gaseous admixtures [11]. The physics of these effects is discussed in the next section. Note that the expressions for turbulent diffusion and the effective pumping velocity for the mean temperature field in a compressible turbulence are different from those for the particle number density and magnetic field (see Ref. [7]), because equations for the particle number density or magnetic field are different from those for the fluid temperature (see the discussion at the end of Sec. IV).

Integration in ω and $\boldsymbol{k}$ space in Eq. (6) yields the expression for the level of temperature fluctuations for small Péclet numbers as

$$\langle \theta^2 \rangle = (\gamma - 1)^2 \left(\frac{q_c - 1}{q_c + 1}\right)\left(\frac{\sigma_c}{1+\sigma_c}\right) {\rm Pe}^2\; \overline{T}^2 + \frac{q-1}{3(q+3)} {\rm Pe}^2\, \ell_0^2 \left\{(\boldsymbol{\nabla} \overline{T})^2 + \frac{1}{8}(\gamma - 1)\left[6(\boldsymbol{\lambda} \cdot \boldsymbol{\nabla}) + (\gamma + 3)(\boldsymbol{\lambda}_u \cdot \boldsymbol{\nabla})\right]\overline{T}^2\right\}. \tag{13}$$

The first term on the right-hand side of Eq. (13) determines a dominant contribution of the compressible part of velocity fluctuations to the level of temperature fluctuations. Here, we neglect much smaller contributions $\sim$O$[\ell_0^2/(L_T L_u)]$, O$[\ell_0^2/(L_T H_\rho)]$, O$[\ell_0^2/L_T^2]$, caused by the compressible part of velocity fluctuations, where L_T is the characteristic scale of the mean temperature field variations. For small σ_c, the level of temperature fluctuations is determined by the terms given by the second and third lines of Eq. (13) and caused by the mean temperature gradient and the density stratified and inhomogeneous velocity fluctuations.

IV. TURBULENT HEAT FLUX AND LEVEL OF TEMPERATURE FLUCTUATIONS FOR LARGE PÉCLET NUMBERS

In this section we determine the turbulent heat flux and the level of temperature fluctuations for large Péclet and Reynolds numbers. We consider fully developed turbulence, where the Strouhal number is of the order of unity and the turbulent correlation time is scale dependent, so we apply the Fourier transformation only in $\boldsymbol{k}$ space, where the Strouhal number is the ratio of the correlation time τ_0 to the turn-over time ℓ_0/u_0 of turbulent eddies.

The procedure of the derivations of the expressions for the turbulent heat flux and the level of temperature fluctuations includes (i) a derivation of equations for the second moments in $\boldsymbol{k}$ space using the multiscale approach, (ii) application of the spectral τ approach (see below) which allows us to relate the deviations of the third moments (appearing due to nonlinear terms) from those of the background turbulence with the deviations of the second moments, (iii) a solution of the equations for the second moments in the $\boldsymbol{k}$ space, and (iv) an inverse transformation to the physical space to obtain formulas for the turbulent heat flux and the level of temperature fluctuations.

Starting with Eq. (4) for the temperature fluctuations θ and the Navier-Stokes equation for the velocity $\boldsymbol{u}$ written in Fourier space, we derive the dynamic equations for the turbulent heat flux and level of temperature fluctuations as

$$\frac{\partial F_j}{\partial t} = -\frac{1}{2}(\gamma-1)\bigg[\overline{T}(2ik_i+\nabla_i)f_{ij} - (\nabla_i\overline{T}) \times \left(\frac{\gamma-3}{\gamma-1}\delta_{im} + k_m\frac{\partial}{\partial k_i}\right)f_{mj}\bigg] + \hat{\mathcal{M}}F_j^{(\mathrm{III})}, \tag{14}$$

$$\frac{\partial E_\theta}{\partial t} = \frac{1}{2}(\gamma-1)\bigg[\overline{T}(2ik_jF_j^{(-)} - \nabla_jF_j^{(+)}) + (\nabla_m\overline{T}) \times \left(k_j\frac{\partial}{\partial k_m} + \frac{\gamma-3}{\gamma-1}\delta_{jm}\right)F_j^{(+)}\bigg] + \hat{\mathcal{M}}E_\theta^{(\mathrm{III})}. \tag{15}$$

Details of the derivations of Eqs. (14) and (15) are given in Appendix C. Here, $F_j(\boldsymbol{k}) = \langle\theta(\boldsymbol{k})u_j(-\boldsymbol{k})\rangle$, $E_\theta(\boldsymbol{k}) = \langle\theta(\boldsymbol{k})\theta(-\boldsymbol{k})\rangle$, $f_{ij}(\boldsymbol{k}) = \langle u_i(\boldsymbol{k})u_j(-\boldsymbol{k})\rangle$, and $F_j^{(\pm)} = F_j(\boldsymbol{k}) \pm F_j(-\boldsymbol{k})$, and the third-order moment terms $\hat{\mathcal{M}}F_j^{(\mathrm{III})}$ and $\hat{\mathcal{M}}E_\theta^{(\mathrm{III})}$ written in $\boldsymbol{k}$ space and appearing due to the nonlinear terms are given by Eqs. (C8) and (C9) in Appendix C.

Equations (14) and (15) for the second moment include first-order spatial differential operators $\hat{\mathcal{M}}$ applied to the third-order moments $F^{(\mathrm{III})}$. The problem arises how to close Eqs. (14) and (15), i.e., how to express the third-order terms $\hat{\mathcal{M}}F^{(\mathrm{III})}$ through the lower moments [1,2,39,44]. We use the spectral τ approach which is a universal tool in turbulent transport for strongly nonlinear systems. The spectral τ approximation postulates that the deviations of the third-moment terms $\hat{\mathcal{M}}F^{(\mathrm{III})}(\boldsymbol{k})$ from the contributions to these terms afforded by the background turbulence $\hat{\mathcal{M}}F^{(\mathrm{III},0)}(\boldsymbol{k})$ can be expressed through similar deviations of the second moments $F^{(\mathrm{II})}(\boldsymbol{k}) - F^{(\mathrm{II},0)}(\boldsymbol{k})$ as

$$\hat{\mathcal{M}}F^{(\mathrm{III})}(\boldsymbol{k}) - \hat{\mathcal{M}}F^{(\mathrm{III},0)}(\boldsymbol{k}) = -\frac{F^{(\mathrm{II})}(\boldsymbol{k}) - F^{(\mathrm{II},0)}(\boldsymbol{k})}{\tau_r(k)} \tag{16}$$

(see, e.g., Refs. [44–46]), where $\tau_r(k)$ is the scale-dependent relaxation time which can be identified with the correlation time $\tau(k)$ of the turbulent velocity field for large fluid Reynolds numbers and large Péclet numbers. Here, functions with a superscript (0) correspond to background turbulence with zero turbulent heat flux. Therefore, Eq. (16) is reduced to $\hat{\mathcal{M}}F_i^{(\mathrm{III})}(\boldsymbol{k}) = -F_i(\boldsymbol{k})/\tau(k)$ and $\hat{\mathcal{M}}E_\theta^{(\mathrm{III})}(\boldsymbol{k}) = -E_\theta(\boldsymbol{k})/\tau(k)$. The validation of the τ approximation for different situations has been performed in various numerical simulations [6,7,19,47–53]. We apply the τ approximation only to study the deviations from the background turbulence which are caused by the spatial derivatives of the mean temperature. The background compressible inhomogeneous and density stratified turbulence is assumed to be known (see below).

The τ approximation is a sort of the high-order closure and in general is similar to the eddy damped quasinormal Markovian (EDQNM) approximation. However, some principal difference exists between these two approaches [44,45]. The EDQNM closures do not relax to equilibrium (the background turbulence), and the EDQNM approach does not describe properly the motions in the equilibrium state in contrast to the τ approximation. Within the EDQNM theory, there is no dynamically determined relaxation time, and no slightly perturbed steady state can be approached. In the τ approximation, the relaxation time for small departures from equilibrium is determined by the random motions in the equilibrium state, but not by the departure from the equilibrium. As follows from the analysis in Ref. [44], the τ approximation describes the relaxation to the equilibrium state (the background turbulence) much more accurately than the EDQNM approach.

Next, we assume that the characteristic times of variation of the second moments F_i and E_θ are substantially larger than the correlation time $\tau(k)$ in all turbulence scales. This allows us to get steady-state solutions of Eqs. (14) and (15) as

$$\langle\theta\, u_j\rangle = -\frac{1}{2}(\gamma-1)\int \tau(k)\bigg[\overline{T}(2ik_i+\nabla_i)f_{ij} - (\nabla_i\overline{T}) \times \left(\frac{\gamma-3}{\gamma-1}\delta_{im} + k_m\frac{\partial}{\partial k_i}\right)f_{mj}\bigg]d\boldsymbol{k}, \tag{17}$$

$$\langle\theta^2\rangle = \frac{1}{2}(\gamma-1)\int \tau(k)\bigg[\overline{T}(2ik_jF_j^{(-)} - \nabla_jF_j^{(+)}) + (\nabla_m\overline{T}) \times \left(k_j\frac{\partial}{\partial k_m} + \frac{\gamma-3}{\gamma-1}\delta_{jm}\right)F_j^{(+)}\bigg]d\boldsymbol{k}. \tag{18}$$

In Eqs. (17) and (18) we take into account a one-way coupling, i.e., we neglect the effect of the mean temperature gradients on the turbulent velocity field. This implies that we replace the correlation function f_{ij} in Eqs. (17) and (18) by $f_{ij}^{(0)}$ for the background turbulent flow with zero turbulent heat flux.

We use statistically stationary, density stratified, inhomogeneous, compressible, and nonhelical background turbulence, which is determined by the following correlation function in $\boldsymbol{k}$ space [5,7]:

$$f_{ij}^{(0)}(\boldsymbol{k}) = \frac{1}{8\pi k^2(1+\sigma_c)}\bigg\{E(k)\bigg[\delta_{ij} - k_{ij} + \frac{i}{k^2}(k_j\lambda_i - k_i\lambda_j) + \frac{i}{2k^2}(k_i\nabla_j - k_j\nabla_i)\bigg] + 2\sigma_c\,E_c(k)\bigg[k_{ij} + \frac{i}{2k^2}(k_i\nabla_j - k_j\nabla_i)\bigg]\bigg\}\langle\boldsymbol{u}^2\rangle. \tag{19}$$

We assume here that the background turbulence is of Kolmogorov type with a constant energy flux over the spectrum, i.e., the velocity fluctuation spectrum for the incompressible part of turbulence in the range of wave numbers $k_0 < k < k_\nu$

is $E(k) = -d\bar\tau(k)/dk$, where the function $\bar\tau(k) = (k/k_0)^{1-q}$ with $1 < q < 3$ being the exponent of the turbulent kinetic energy spectrum. The condition $q > 1$ corresponds to a finite turbulent kinetic energy for very large fluid Reynolds numbers, while $q < 3$ corresponds to the finite dissipation of the turbulent kinetic energy at the viscous scale (see, e.g., Refs. [1,2,39–41]). Similarly, the turbulent kinetic energy spectrum for the compressible part of turbulence is $E_c(k) = -d\bar\tau_c(k)/dk$, where the function $\bar\tau_c(k) = (k/k_0)^{1-q_c}$ with $1 < q_c < 3$. For instance, the exponent of the incompressible part of the turbulent kinetic energy spectrum $q = 5/3$ (the Kolmogorov spectrum), while the exponent of the compressible part of the spectrum $q_c = 2$ (for the Burgers turbulence with shock waves). The turbulent correlation time in $\boldsymbol{k}$ space is

$$\tau(k) = \frac{2\tau_0}{1+\sigma_c}[\bar\tau(k) + \sigma_c\,\bar\tau_c(k)]. \tag{20}$$

Note that for fully developed Kolmogorov-like turbulence, $\sigma_c < 1$ [54].

Integration in $\boldsymbol{k}$ space in Eq. (17) yields the turbulent heat flux $\langle\theta\,\boldsymbol{u}\rangle = \overline{T}\,\boldsymbol{V}^{\rm eff} - D_T\,\boldsymbol{\nabla}\overline{T}$, where the turbulent diffusivity D_T and the effective pumping velocity $\boldsymbol{V}^{\rm eff}$ of the mean temperature field for large Péclet numbers are given by

$$D_T = \frac{\tau_0\langle\boldsymbol{u}^2\rangle}{3}\left\{1 + \frac{\gamma-1}{1+\sigma_c}\left[1 - \frac{\sigma_c}{2(1+\sigma_c)}(\tilde C_\sigma\,q + \sigma_c(q_c-1))\right]\right\}, \tag{21}$$

$$\boldsymbol{V}^{\rm eff} = (\gamma-1)\frac{\tau_0\langle\boldsymbol{u}^2\rangle}{3(1+\sigma_c)}\left\{\frac{\sigma_c}{2}\left[1 + \frac{\tilde C_\sigma}{2(1+\sigma_c)}\right]\boldsymbol{\lambda}_u + \left[1 - \frac{\tilde C_\sigma\,\sigma_c}{2(1+\sigma_c)}\right]\boldsymbol{\lambda}_P\right\}, \tag{22}$$

and

$$\tilde C_\sigma = \frac{2(q_c-1)}{q+q_c-2}. \tag{23}$$

Equation (21) implies that for large Péclet numbers, compressibility effects decrease the turbulent diffusivity. Indeed, the derivative $\partial D_T/\partial\sigma_c$ is always negative when $\sigma_c(\tilde C_\sigma q - 2q_c) < \tilde C_\sigma q + 2$. Since $\tilde C_\sigma > 0$ and $\tilde C_\sigma q - 2q_c < 0$ [the latter inequality is reduced to $(q_c-1)^2 + (q-1) > 0$], the derivative $\partial D_T/\partial\sigma_c$ is negative, i.e., compressibility effects do decrease the turbulent diffusivity.

For irrotational flow ($\sigma_c \gg 1$), the turbulent diffusivity and the effective pumping velocity for large Péclet numbers are given by

$$D_T = \frac{1}{3}\tau_0\langle\boldsymbol{u}^2\rangle\left[1 - \frac{1}{2}(\gamma-1)(q_c-1)\right], \tag{24}$$

$$\boldsymbol{V}^{\rm eff} = \left(\frac{\gamma-1}{6}\right)\tau_0\,\boldsymbol{\nabla}\langle\boldsymbol{u}^2\rangle. \tag{25}$$

Equations (22) and (25) determine the effective pumping velocity $\boldsymbol{V}^{\rm eff}$ of the mean temperature field caused by the inhomogeneity of compressible turbulence and the gradient of the fluid pressure. Let us discuss the mechanisms of the turbulent pumping effects. The first term ($\propto\sigma_c\,\boldsymbol{\nabla}\langle\boldsymbol{u}^2\rangle$) in Eq. (22) implies that there is an additional contribution to the turbulent heat flux caused by the combined effect of the inhomogeneity of turbulence and the compressibility of fluid flow. This effect results in an increase of the mean temperature in the region with more intense velocity fluctuations in a compressible turbulence. This effect can be understood using the budget equation for the mean internal energy density $\overline{E} = c_{\rm v}\overline{T}$, where $c_{\rm v}$ is the specific heat at constant volume. In particular, one of the sources in the budget equation for the mean internal energy density is $-\langle p\boldsymbol{\nabla}\cdot\boldsymbol{u}\rangle$ [34], so that $\partial(\overline{\rho}\,\overline{E})/\partial t \sim -\langle p\boldsymbol{\nabla}\cdot\boldsymbol{u}\rangle$, where p are pressure fluctuations. As follows from the Bernoulli law, variations of the sum $\delta(p + \rho\boldsymbol{u}^2/2) \approx 0$, so that $\delta p \approx -\delta(\rho\boldsymbol{u}^2/2)$. This implies that the mean internal energy (and the mean temperature) is larger in the region with more intense compressible velocity fluctuations. The turbulent pumping effect of the mean temperature field caused by the joint effect of compressibility and inhomogeneity of turbulence is similar to a phenomenon of compressible turbophoresis for noninertial particles or gaseous admixtures [7]. In particular, the expression for the effective pumping velocity for particles due to the compressible turbophoresis is proportional to $\boldsymbol{V}^{\rm eff}_{\rm particles} \propto \sigma_c\,\tau_0\,\boldsymbol{\nabla}\langle\boldsymbol{u}^2\rangle$.

The second term ($\propto\boldsymbol{\nabla}\overline{P}$) in Eq. (22) determines an additional contribution to the turbulent heat flux caused by the gradient of the mean fluid pressure. This turbulent pumping increases the mean temperature in the regions with a higher mean fluid pressure. The mechanism of this effect is the following. Since there is an outflow of fluid from the turbulent regions with a higher mean fluid pressure, the fluid density decreases in these regions and temperature increases. This effect is similar to the turbulent barodiffusion [11] of particles or gaseous admixtures.

Note that expressions (10) and (21) for the turbulent diffusion coefficient of the mean temperature field in a compressible turbulence are different from those for the mean particle number density [5,7]. Indeed, Eq. (1) for the temperature field contains an additional term, $(\gamma-2)T\,{\rm div}\,\boldsymbol{U}$, in comparison with Eq. (2) for the particle number density. Even for $\gamma = 2$ when this additional term vanishes and the equations for the temperature field and the particle number density are similar, the expressions for the turbulent diffusion coefficient for the mean temperature field in a compressible turbulence are different from those for the mean particle number density.

The main reason for this difference is as follows. Particles in a fluid flow is a two-phase system, while the turbulent transport of fluid temperature is a one-phase system. Equation (4) for temperature fluctuations $\theta(\boldsymbol{x},t) = T - \overline{T}$ has two source terms $I = -(\boldsymbol{u}\cdot\boldsymbol{\nabla})\overline{T} - (\gamma-1)\overline{T}\,\boldsymbol{\nabla}\cdot\boldsymbol{u}$, where the first term $-(\boldsymbol{u}\cdot\boldsymbol{\nabla})\overline{T}$ contributes to turbulent diffusion D_T, while the second term $-(\gamma-1)\overline{T}\,\boldsymbol{\nabla}\cdot\boldsymbol{u}$ contributes to the effective pumping velocity $\boldsymbol{V}^{\rm eff}$ of the mean temperature, so that the turbulent heat flux in a compressible turbulence is $\langle\theta\,\boldsymbol{u}\rangle = \overline{T}\,\boldsymbol{V}^{\rm eff} - D_T\,\boldsymbol{\nabla}\overline{T}$. The contribution $(\boldsymbol{V}^{\rm eff})_{\boldsymbol{\nabla}\overline{T}} = -D_T^*\,\boldsymbol{\nabla}\overline{T}/\overline{T}$

to the effective pumping velocity $\boldsymbol{V}^{\rm eff}$ of the mean temperature due to the mean temperature gradient $\boldsymbol{\nabla}\overline{T}$ is actually an additional contribution to the turbulent diffusivity D_T. Indeed, we can rewrite this contribution as

$$\overline{T}(\boldsymbol{V}^{\rm eff})_{\boldsymbol{\nabla}\overline{T}} = \overline{T}\left(-D_T^* \frac{\boldsymbol{\nabla}\overline{T}}{\overline{T}}\right) = -D_T^* \boldsymbol{\nabla}\overline{T}, \tag{26}$$

where

$$D_T^* = (\gamma-1)\frac{(q-1)}{3(q+1)}\frac{\tau_0\langle \boldsymbol{u}^2\rangle}{(1+\sigma_c)}{\rm Pe}, \tag{27}$$

for Pe $\ll 1$, and

$$D_T^* = (\gamma-1)\frac{\tau_0\langle \boldsymbol{u}^2\rangle}{3(1+\sigma_c)}\left(1-\frac{\tilde{C}_\sigma\,\sigma_c}{2(1+\sigma_c)}\right), \tag{28}$$

for Pe $\gg 1$. This is the main reason why the expressions for the turbulent diffusion coefficient for the mean temperature field in a compressible turbulence are different from those for the mean particle number density.

Integration in $\boldsymbol{k}$ space in Eq. (18) yields the level of temperature fluctuations for large Péclet numbers

$$\langle\theta^2\rangle = 8\,f_c(\gamma-1)^2\left(\frac{\sigma_c}{1+\sigma_c}\right)^3\overline{T}^2 + \frac{1}{9}\ell_0^2\{8(\boldsymbol{\nabla}\overline{T})^2 + (\gamma-1)[(\gamma+3)(\boldsymbol{\lambda}_u\cdot\boldsymbol{\nabla}) + 2(5-\gamma)(\boldsymbol{\lambda}\cdot\boldsymbol{\nabla})]\overline{T}^2\}, \tag{29}$$

where the function $f_c(q, q_c, \sigma_c)$ depends on the degree of compressibility and the exponents of spectra for the incompressible and compressible parts of velocity fluctuations:

$$f_c = \frac{q_c-1}{3q_c-5} + \frac{2(q_c-1)}{\sigma_c(q+2q_c-5)} + \frac{q_c-1}{\sigma_c^2(2q+q_c-5)}. \tag{30}$$

The first term on the right-hand side of Eq. (29) determines a dominant contribution of the compressible part of the velocity fluctuations to the level of temperature fluctuations. Here, we neglect much smaller contributions $\sim{\rm O}[\ell_0^2/(L_T L_u)]$, ${\rm O}[\ell_0^2/(L_T H_\rho)]$, ${\rm O}[\ell_0^2/L_T^2]$, caused by the compressible part of velocity fluctuations. For small σ_c, the level of temperature fluctuations is determined by the other terms in Eq. (29) which are caused by the mean temperature gradient and the density stratified and inhomogeneous velocity fluctuations.

V. TURBULENT COOLING

In this section we discuss how a homogeneous compressible turbulence can cause a turbulent cooling of the surrounding fluid. Equation (3) for the mean temperature field $\overline{T}$ contains an additional sink term $I_{\rm S} = -(\gamma-2)\langle\theta(\boldsymbol{\nabla}\cdot\boldsymbol{u})\rangle$ which can result in the turbulent cooling of the surrounding fluid for $\gamma < 2$. Indeed, substituting Eq. (9) for the turbulent heat flux into Eq. (3), we obtain the equation for the mean temperature field $\overline{T}$ as

$$\frac{\partial\overline{T}}{\partial t} + \boldsymbol{\nabla}\cdot[\overline{T}\,\boldsymbol{V}^{\rm eff} - (D+D_T)\boldsymbol{\nabla}\overline{T}] = \overline{J}_\nu - (\gamma-2)\langle\theta(\boldsymbol{\nabla}\cdot\boldsymbol{u})\rangle, \tag{31}$$

where the sink term $I_{\rm S}$ in Eq. (31) for small Péclet numbers is given by

$$I_{\rm S} = -(\gamma-1)(2-\gamma)\left(\frac{\sigma_c}{1+\sigma_c}\right){\rm Pe}\,\frac{\overline{T}}{\tau_0}, \tag{32}$$

while for large Péclet numbers it is

$$I_{\rm S} = -6(\gamma-1)(2-\gamma)\frac{\sigma_c}{(1+\sigma_c)^2}\frac{\overline{T}}{\tau_0}\left[{\rm Re}^{1/4} + \frac{\sigma_c}{4}\ln{\rm Re}\right]. \tag{33}$$

In Eq. (33) for simplicity we determine $I_{\rm S}$ when the exponent of the incompressible part of the turbulent kinetic energy spectrum for large Reynolds numbers is $q = 5/3$, while the exponent of the compressible part of the spectrum is $q_c = 2$ [55,56].

Let us consider a simple case with a uniform mean temperature field. The heating source $\overline{J}_\nu$ in Eq. (31) caused by the viscous dissipation of the turbulent kinetic energy is given by

$$\overline{J}_\nu = \frac{\nu}{c_{\rm v}}\left[\langle(\boldsymbol{\nabla}\times\boldsymbol{u})^2\rangle + \frac{4}{3}\langle(\boldsymbol{\nabla}\cdot\boldsymbol{u})^2\rangle\right], \tag{34}$$

where $c_{\rm v}$ is the specific heat at constant volume. Here, we use the equation for the turbulent kinetic energy density $E_K = \langle\rho\,\boldsymbol{u}^2\rangle/2$ for compressible turbulence written as

$$\frac{\partial E_K}{\partial t} + {\rm div}\,\boldsymbol{\Phi}_K = -\varepsilon_K + \Pi_K, \tag{35}$$

where

$$\boldsymbol{\Phi}_K = -\nu\big[\langle\rho\,\boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u})\rangle + \tfrac{4}{3}\langle\rho\,\boldsymbol{u}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\rangle\big] + \langle\boldsymbol{u}(\rho\,\boldsymbol{u}^2/2)\rangle + \langle\boldsymbol{u}\,p\rangle \tag{36}$$

is the flux of the density of turbulent kinetic energy, p are fluid pressure fluctuations,

$$\varepsilon_K = \nu\big[\langle\rho(\boldsymbol{\nabla}\times\boldsymbol{u})^2\rangle + \tfrac{4}{3}\langle\rho(\boldsymbol{\nabla}\cdot\boldsymbol{u})^2\rangle\big] \tag{37}$$

is the dissipation rate of the density of turbulent kinetic energy, and $\Pi_K = \langle\rho\,\boldsymbol{u}\cdot\boldsymbol{f}\rangle + \langle p(\boldsymbol{\nabla}\cdot\boldsymbol{u})\rangle$ is the production rate of the density of turbulent kinetic energy caused by the external force (e.g., by an external large-scale shear). The production term includes also the pressure-dilatation term $\langle p\,(\boldsymbol{\nabla}\cdot\boldsymbol{u})\rangle$ (see, e.g., Refs. [57,58]). In the limit of low Mach numbers, the pressure-dilatation term in Π_K is known to be much smaller than $\langle\rho\,\boldsymbol{u}\cdot\boldsymbol{f}\rangle$, and hence it can be safely neglected.

Using Eq. (19) for the second moment of velocity fluctuations in the background turbulence, we obtain that the viscous heating source $\overline{J}_\nu$ is given by

$$\overline{J}_\nu = \frac{\langle\boldsymbol{u}^2\rangle}{2\tau_0}(1+\sigma_c)^{-1}\left[1 + \frac{8}{3}\sigma_c\,{\rm Re}^{-1/4}\right]. \tag{38}$$

Turbulence can generate acoustic waves, and the rate of the energy radiated by the acoustic waves per unit mass for small Mach numbers is [59,60]

$$E_{\rm w} = \alpha\frac{\langle\boldsymbol{u}^2\rangle}{\tau_0}{\rm Ma}^5, \tag{39}$$

where $\alpha \sim 10$–10^2 is the numerical coefficient, $\mathrm{Ma} = u_{\rm rms}/c_{\rm s}$ is the Mach number, $u_{\rm rms} = \langle \boldsymbol{u}^2 \rangle^{1/2}$, and $c_{\rm s} = (\gamma \overline{P}/\overline{\rho})^{1/2}$ is the sound speed. The second term in Eq. (38) describes the compressibility contribution to the rate of the viscous heating,

$$\overline{J}_{\nu}^{\rm (c)} = \frac{4}{3} \frac{\langle \boldsymbol{u}^2 \rangle}{\tau_0} \frac{\sigma_c}{1+\sigma_c} \mathrm{Re}^{-1/4}. \tag{40}$$

Assuming that the compressibility contribution to the viscous heating of turbulence $\overline{J}_{\nu}^{\rm (c)}$ is compensated by the radiative wave energy density $E_{\rm w}$, we obtain that the degree of compressibility for small Mach numbers is given by

$$\sigma_c = \frac{3\alpha}{4} \mathrm{Ma}^5\, \mathrm{Re}^{1/4}. \tag{41}$$

In equilibrium, the total viscous heating $\overline{J}_{\nu}$ is compensated by the compressible cooling $I_{\rm S}$, so that the increase of the internal thermal energy caused by the viscous heating is given by

$$c_{\rm v}\, \overline{T}_c = \frac{2\langle \boldsymbol{u}^2 \rangle}{9\,\alpha\, \mathrm{Ma}^5\, \mathrm{Re}^{1/2}}. \tag{42}$$

Taking into account that the sound speed $c_{\rm s}$ depends on the mean temperature, we obtain from Eq. (42) that the increase of the internal thermal energy caused by the viscous heating is given by

$$c_{\rm v}\, \overline{T}_c = C_* \langle \boldsymbol{u}^2 \rangle \mathrm{Re}^{1/3}, \tag{43}$$

where $C_* = (9\,\alpha/2)^{2/3}/[\gamma\,(\gamma-1)]^{5/3}$. Equation (43) can be rewritten in terms of the Mach number $\mathrm{Ma} = u_{\rm rms}/c_{\rm s}$ as

$$\mathrm{Ma} = \left[\frac{2\gamma(\gamma-1)}{9\,\alpha}\right]^{1/3} \mathrm{Re}^{-1/6}. \tag{44}$$

For example, taking parameters typical for the atmospheric turbulence, $\ell_0 = 10^2$ cm, $u_{\rm rms} = 2.7 \times 10^2$ cm/s, and $\alpha = 10$, we obtain $\overline{T}_c = 286$ K.

VI. DISCUSSION AND CONCLUSIONS

In the present study we have investigated the compressibility effects on the turbulent transport of the mean temperature field. We use the quasilinear approach to study turbulent transport for small Péclet numbers. When nonlinear effects are much stronger than the molecular diffusion (i.e., for large Péclet numbers), we apply the spectral τ approach. Similarly to the turbulent transport of particles and magnetic fields, the compressibility decreases the turbulent diffusivity of the mean temperature field, but the expression for turbulent diffusivity for the mean temperature field in a compressible turbulence is different from those for the turbulent diffusivity of the mean particle number density and the turbulent magnetic diffusivity of the mean magnetic field.

We have found also turbulent pumping of the mean temperature field due to joint effects of the fluid flow compressibility and inhomogeneity of turbulence. This effect causes an increase of the mean temperature in the regions of more intense velocity fluctuations. A similar compressibility effect, referred to as compressible turbophoresis [7], results in a pumping of noninertial particles or gaseous admixtures from regions of low to high turbulent intensity. Turbulent pumping also can be due to the gradients of the mean fluid pressure resulting in an increase of the mean temperature in the regions with increased mean fluid pressure, similarly to the phenomenon of the turbulent barodiffusion of particles and gaseous admixtures.

Due to compressibility, there is an additional sink term in the equation for the mean fluid temperature, causing a turbulent cooling in homogeneous turbulence. This implies that there can be an equilibrium in a compressible homogeneous turbulence with a uniform mean fluid temperature, where the heating caused by the viscous dissipation of the turbulent kinetic energy can be compensated by the turbulent cooling caused by the fluid compressibility. Such an effect does not exist in the turbulent transport of particles or gaseous admixtures in a compressible fluid flow.

To derive expressions for the turbulent heat flux and the level of temperature fluctuations for large Péclet and Reynolds numbers in a compressible inhomogeneous and density stratified turbulence, we apply the spectral τ approach (see Sec. IV). The τ approach reproduces many well-known phenomena found by other methods in the turbulent transport of particles, temperature, and magnetic fields, in turbulent convection and stably stratified turbulent flows (for a review, see Ref. [5]). In turbulent transport, the τ approach yields correct formulas for turbulent diffusion, turbulent thermal diffusion, and turbulent barodiffusion [9–11,61]. The phenomenon of turbulent thermal diffusion was predicted using stochastic calculus (the path-integral approach). This effect was also reproduced using the quasilinear approach, the spectral τ approach, and the renormalization approach.

The τ approach reproduces the well-known $k^{-7/3}$ spectrum of anisotropic velocity fluctuations in a sheared turbulence (see Ref. [62]). This spectrum was previously found in analytical, numerical, laboratory studies, and was observed in atmospheric turbulence (see, e.g., Ref. [63]). In the turbulent boundary layer problems, the τ approach yields correct expressions for turbulent viscosity, turbulent thermal conductivity, and the turbulent heat flux [2,39]. This approach also describes the counter wind turbulent heat flux and the Deardorff's heat flux in convective boundary layers (see Ref. [62]). These phenomena were studied using different approaches (see, e.g., Refs. [2,39,64]).

In magnetohydrodynamics, the τ approach reproduces many well-known phenomena found by different methods, e.g., the τ approximation yields correct formulas for the α effect, the turbulent diamagnetic and paramagnetic velocities, the turbulent magnetic diffusion, the $\boldsymbol{\Omega}\times\mathbf{J}$ effect, and the κ effect [8,65,66].

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APPENDIX A: MULTISCALE APPROACH

In the framework of the multiscale approach [42], the non-instantaneous two-point second-order correlation functions are written as follows:

$$\langle\theta(\boldsymbol{x},t_1)u_j(\boldsymbol{y},t_2)\rangle = \int \langle\theta(\boldsymbol{k}_1,\omega_1)u_j(\boldsymbol{k}_2,\omega_2)\rangle \exp\left[i(\boldsymbol{k}_1\cdot\boldsymbol{x}+\boldsymbol{k}_2\cdot\boldsymbol{y})+i(\omega_1 t_1+\omega_2 t_2)\right]d\omega_1\, d\omega_2\, d\boldsymbol{k}_1\, d\boldsymbol{k}_2$$

$$= \int F_j(\boldsymbol{k},\omega,t,\boldsymbol{R}) \exp[i\boldsymbol{k}\cdot\boldsymbol{r}+i\omega\,\tilde{\tau}]d\omega\, d\boldsymbol{k}, \tag{A1}$$

$$\langle\theta(\boldsymbol{x},t_1)\theta(\boldsymbol{y},t_2)\rangle = \int \langle\theta(\boldsymbol{k}_1,\omega_1)\theta(\boldsymbol{k}_2,\omega_2)\rangle \exp\left[i(\boldsymbol{k}_1\cdot\boldsymbol{x}+\boldsymbol{k}_2\cdot\boldsymbol{y})+i(\omega_1 t_1+\omega_2 t_2)\right]d\omega_1\, d\omega_2\, d\boldsymbol{k}_1\, d\boldsymbol{k}_2$$

$$= \int E_\theta(\boldsymbol{k},\omega,t,\boldsymbol{R}) \exp[i\boldsymbol{k}\cdot\boldsymbol{r}+i\omega\,\tilde{\tau}]d\omega\, d\boldsymbol{k}, \tag{A2}$$

where

$$F_j(\boldsymbol{k},\omega,\boldsymbol{R},t) = \int \langle\theta(\boldsymbol{k}_1,\omega_1)u_j(\boldsymbol{k}_2,\omega_2)\rangle \exp[i\Omega t+i\boldsymbol{K}\cdot\boldsymbol{R}]d\Omega\, d\boldsymbol{K}, \tag{A3}$$

$$E_\theta(\boldsymbol{k},\omega,\boldsymbol{R},t) = \int \langle\theta(\boldsymbol{k}_1,\omega_1)\theta(\boldsymbol{k}_2,\omega_2)\rangle \exp[i\Omega t+i\boldsymbol{K}\cdot\boldsymbol{R}]d\Omega\, d\boldsymbol{K}. \tag{A4}$$

Here, we introduce large-scale variables $\boldsymbol{R}=(\boldsymbol{x}+\boldsymbol{y})/2$, $\boldsymbol{K}=\boldsymbol{k}_1+\boldsymbol{k}_2$, $t=(t_1+t_2)/2$, $\Omega=\omega_1+\omega_2$, and small-scale variables $\boldsymbol{r}=\boldsymbol{x}-\boldsymbol{y}$, $\boldsymbol{k}=(\boldsymbol{k}_1-\boldsymbol{k}_2)/2$, $\tilde{\tau}=t_1-t_2$, $\omega=(\omega_1-\omega_2)/2$. This implies that $\omega_1=\omega+\Omega/2$, $\omega_2=-\omega+\Omega/2$, $\boldsymbol{k}_1=\boldsymbol{k}+\boldsymbol{K}/2$, and $\boldsymbol{k}_2=-\boldsymbol{k}+\boldsymbol{K}/2$. Mean fields depend on the large-scale variables, while fluctuations depend on the small-scale variables. Similarly to Eqs. (A1)–(A4), the correlation function for velocity fluctuations reads

$$f_{ij}(\boldsymbol{k},\omega,\boldsymbol{R},t) = \int \langle u_i(\boldsymbol{k}_1,\omega_1)u_j(\boldsymbol{k}_2,\omega_2)\rangle \exp[i\Omega t+i\boldsymbol{K}\cdot\boldsymbol{R}]d\Omega\, d\boldsymbol{K}. \tag{A5}$$

After separation into slow and fast variables and calculating the functions $F_j(\boldsymbol{k},\omega,\boldsymbol{R},t)$ and $E_\theta(\boldsymbol{k},\omega,\boldsymbol{R},t)$, Eqs. (A1) and (A2) in the limit of $\boldsymbol{r}\to\boldsymbol{0}$ and $\tilde{\tau}\to 0$ allow us to determine the turbulent flux of the temperature field and the level of temperature fluctuations in physical space:

$$\langle\theta(\boldsymbol{x},t)u_j(\boldsymbol{x},t)\rangle = \int F_j(\boldsymbol{k},\omega,\boldsymbol{R},t)d\omega\, d\boldsymbol{k}, \tag{A6}$$

$$\langle\theta(\boldsymbol{x},t)\theta(\boldsymbol{x},t)\rangle = \int E_\theta(\boldsymbol{k},\omega,\boldsymbol{R},t)d\omega\, d\boldsymbol{k}. \tag{A7}$$

APPENDIX B: DERIVATION OF EQS. (5)–(7)

We rewrite Eq. (4) in Fourier space and find the solution of this equation as

$$\theta(\boldsymbol{k},\omega) = -i\left[(\gamma-1)\int \overline{T}(\boldsymbol{Q})(k_i-Q_i)u_i(\boldsymbol{k}-\boldsymbol{Q},\omega)\, d\boldsymbol{Q} + \int Q_i\,\overline{T}(\boldsymbol{Q})u_i(\boldsymbol{k}-\boldsymbol{Q},\omega)d\boldsymbol{Q}\right]G_D(\boldsymbol{k},\omega), \tag{B1}$$

where $G_D(\boldsymbol{k},\omega)=(Dk^2+i\omega)^{-1}$. Using Eqs. (A6) and (B1), we determine the functions $F_j(\boldsymbol{k},\boldsymbol{R})$ and $E_\theta(\boldsymbol{k},\boldsymbol{R})$ as

$$F_j(\boldsymbol{k},\boldsymbol{R}) = -i\int \left[(\gamma-1)\left(k_i+\frac{K_i}{2}-Q_i\right)+Q_i\right]G_D(\boldsymbol{k}+\boldsymbol{K}/2)\langle u_i(\boldsymbol{k}+\boldsymbol{K}/2-\boldsymbol{Q})u_j(-\boldsymbol{k}+\boldsymbol{K}/2)\rangle$$
$$\times\overline{T}(\boldsymbol{Q})\exp(i\boldsymbol{K}\cdot\boldsymbol{R})d\boldsymbol{K}\, d\boldsymbol{Q}, \tag{B2}$$

$$E_\theta(\boldsymbol{k},\boldsymbol{R}) = -\frac{i}{2}\int \left\{\left[(\gamma-1)\left(k_i+\frac{K_i}{2}-Q_i\right)+Q_i\right]G_D(\boldsymbol{k}+\boldsymbol{K}/2)\langle\theta(-\boldsymbol{k}+\boldsymbol{K}/2)u_i(\boldsymbol{k}+\boldsymbol{K}/2-\boldsymbol{Q})\rangle\right.$$
$$\left.+\left[(\gamma-1)\left(-k_i+\frac{K_i}{2}-Q_i\right)+Q_i\right]G_D(-\boldsymbol{k}+\boldsymbol{K}/2)\langle\theta(\boldsymbol{k}+\boldsymbol{K}/2)\, u_i(-\boldsymbol{k}+\boldsymbol{K}/2-\boldsymbol{Q})\rangle\right\}$$
$$\times\overline{T}(\boldsymbol{Q})\exp(i\boldsymbol{K}\cdot\boldsymbol{R})d\boldsymbol{K}\, d\boldsymbol{Q}, \tag{B3}$$

where the functions F_j, G_D, and u_i depend also on ω, and $\overline{T}$ depend on t as well. To simplify the notations, we do not show these dependencies here. To determine $f_{ij}(\boldsymbol{k},\boldsymbol{K},\boldsymbol{Q})=\langle u_i(\boldsymbol{k}+\boldsymbol{K}/2-\boldsymbol{Q})u_j(-\boldsymbol{k}+\boldsymbol{K}/2)\rangle$, we use the following new variables:

$$\tilde{\boldsymbol{k}} = (\tilde{\boldsymbol{k}}_1-\tilde{\boldsymbol{k}}_2)/2 = \boldsymbol{k}-\boldsymbol{Q}/2, \tag{B4}$$

$$\tilde{\boldsymbol{K}} = \tilde{\boldsymbol{k}}_1+\tilde{\boldsymbol{k}}_2 = \boldsymbol{K}-\boldsymbol{Q}, \tag{B5}$$

where

$$\tilde{\boldsymbol{k}}_1 = \boldsymbol{k} + \boldsymbol{K}/2 - \boldsymbol{Q}, \quad \tilde{\boldsymbol{k}}_2 = -\boldsymbol{k} + \boldsymbol{K}/2. \tag{B6}$$

Since $|\boldsymbol{Q}| \ll |\boldsymbol{k}|$ and $|\boldsymbol{K}| \ll |\boldsymbol{k}|$, we use the Taylor expansion

$$f_{ij}(\boldsymbol{k} - \boldsymbol{Q}/2, \boldsymbol{K} - \boldsymbol{Q}) = f_{ij}(\boldsymbol{k}, \boldsymbol{K} - \boldsymbol{Q}) - \frac{1}{2}\frac{\partial f_{ij}}{\partial k_m} Q_m + O(\boldsymbol{Q}^2), \tag{B7}$$

$$G_D(\boldsymbol{k} + \boldsymbol{K}/2) = G_D(\boldsymbol{k})[1 - D(\boldsymbol{k}\cdot\boldsymbol{K})G_D(\boldsymbol{k})] + O(\boldsymbol{K}^2). \tag{B8}$$

In a similar way we calculate the other terms in Eqs. (B2) and (B3). Using Eqs. (B2)–(B8), we arrive at expressions (5) and (6) for the turbulent heat flux and the level of temperature fluctuations in Fourier space for small Péclet numbers.

To derive Eq. (7), the second rank tensor $f_{ij}^{(0)}$ is constructed as a linear combination of symmetric tensors, δ_{ij} and k_{ij}, with respect to the indices i and j, and nonsymmetric tensors: $k_i\lambda_j$, $k_j\lambda_i$, $k_i\nabla_j\langle\boldsymbol{u}^2\rangle$, and $k_j\nabla_i\langle\boldsymbol{u}^2\rangle$. We consider here only linear effects in $\boldsymbol{\lambda}$ and $\nabla\langle\boldsymbol{u}^2\rangle$. To determine unknown coefficients multiplying by these tensors, we use the following conditions in the derivation of Eq. (7): $\langle\boldsymbol{u}^2\rangle = \int f_{ii}^{(0)}(\boldsymbol{k}, \omega, \boldsymbol{K}) \exp(i\boldsymbol{K}\cdot\boldsymbol{R}) d\boldsymbol{k}\, d\omega\, d\boldsymbol{K}$, $f_{ij}^{(0)}(\boldsymbol{k}, \omega, \boldsymbol{K}) = f_{ji}^{*(0)}(\boldsymbol{k}, \omega, \boldsymbol{K}) = f_{ji}^{(0)}(-\boldsymbol{k}, \omega, \boldsymbol{K})$, and

$$\langle(\mathrm{div}\, \boldsymbol{u})^2\rangle = \int (k_i + K_i/2)(k_j - K_j/2)\, f_{ij}^{(0)}(\boldsymbol{k}, \omega, \boldsymbol{K}) \exp(i\boldsymbol{K}\cdot\boldsymbol{R}) d\boldsymbol{k}\, d\omega\, d\boldsymbol{K}. \tag{B9}$$

The normalization conditions for the functions $\Phi(\omega)$, $E(k)$, and $E_c(k)$ in Eq. (7) are $\int_{-\infty}^{\infty} \Phi(\omega) d\omega = 1$, $\int_{k_0}^{k_d} E(k) dk = 1$, and $\int_{k_0}^{k_d} E_c(k) dk = 1$. For very low Mach numbers, i.e., when the parameter σ_c is very small, the continuity equation can be written in the anelastic approximation, $\mathrm{div}(\overline{\rho}\, \boldsymbol{u}) = 0$, which implies that $(ik_i + iK_i/2 - \lambda_i) f_{ij}^{(0)}(\boldsymbol{k}, \omega, \boldsymbol{K}) = 0$ and $(-ik_j + iK_j/2 - \lambda_j) f_{ij}^{(0)}(\boldsymbol{k}, \omega, \boldsymbol{K}) = 0$.

For the integration over ω in Eqs. (5) and (6), we use the following identities:

$$\int_{-\infty}^{\infty} \frac{d\omega}{(\pm i\omega + Dk^2)(\omega^2 + \tau_0^{-2})} = \frac{\pi\, \tau_0}{\tau_0^{-1} + D\, k^2} \approx \frac{\pi\, \tau_0}{Dk^2},$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{(i\omega + Dk^2)(-i\omega + Dk^2)(\omega^2 + \tau_0^{-2})} = \frac{\pi\, \tau_0}{Dk^2(\tau_0^{-1} + D\, k^2)} \approx \frac{\pi\, \tau_0}{(D\, k^2)^2},$$

which are determined in the limit when the correlation time $\tau_0 \gg (D^{(\theta)} k^2)^{-1}$. For the integration over angles in $\boldsymbol{k}$ space in Eqs. (5) and (6), we use the following identity:

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta\, d\vartheta\, \frac{k_i k_j}{k^2} = \frac{4\pi}{3}\delta_{ij}.$$

For the integration over k in Eqs. (5) and (6), we use the following identities:

$$\int_{k_0}^{k_d} \frac{E(k)}{k^2} dk = \frac{q-1}{q+1}\ell_0^2, \quad \int_{k_0}^{k_d} \frac{E(k)}{k^4} dk = \frac{q-1}{q+3}\ell_0^4.$$

APPENDIX C: DERIVATION OF EQS. (14) and (15)

In this Appendix we derive Eqs. (14) and (15) for large Péclet and Reynolds numbers. Using Eq. (4) for the temperature fluctuations θ and the Navier-Stokes equation for the velocity $\boldsymbol{u}$ written in Fourier space, we derive equations for the following correlation functions:

$$F_j(\boldsymbol{k}, \boldsymbol{R}) = \int \langle\theta(\boldsymbol{k} + \boldsymbol{K}/2) u_j(-\boldsymbol{k} + \boldsymbol{K}/2)\rangle \times \exp[i\boldsymbol{K}\cdot\boldsymbol{R}] d\boldsymbol{K}, \tag{C1}$$

$$E_\theta(\boldsymbol{k}, \boldsymbol{R}) = \int \langle\theta(\boldsymbol{k} + \boldsymbol{K}/2)\theta(-\boldsymbol{k} + \boldsymbol{K}/2)\rangle \times \exp[i\boldsymbol{K}\cdot\boldsymbol{R}] d\boldsymbol{K}. \tag{C2}$$

For brevity of notations we omit the large-scale variable t in the functions $F_j(\boldsymbol{k}, \boldsymbol{R}, t)$, $E_\theta(\boldsymbol{k}, \boldsymbol{R}, t)$, and the mean temperature $\overline{T}(\boldsymbol{R}, t)$.

To derive evolution equations in the Fourier space for the turbulent heat flux $F_j(\boldsymbol{k}, \boldsymbol{R})$ and the level of temperature fluctuations $E_\theta(\boldsymbol{k}, \boldsymbol{R})$, we rewrite Eq. (4) for the temperature fluctuations in $\boldsymbol{k}$ space as

$$\frac{\partial\theta(\boldsymbol{k})}{\partial t} = -i\left[(\gamma - 1)\int \overline{T}(\boldsymbol{Q})(k_i - Q_i)\, u_i(\boldsymbol{k} - \boldsymbol{Q}) d\boldsymbol{Q} + \int Q_i \overline{T}(\boldsymbol{Q}) u_i(\boldsymbol{k} - \boldsymbol{Q}) d\boldsymbol{Q}\right] - \mathcal{Q}(\boldsymbol{k}), \tag{C3}$$

where $\mathcal{Q}(\boldsymbol{k})$ are the nonlinear terms written in $\boldsymbol{k}$ space. For brevity of notations we omit below the variable t in the functions $\overline{T}(\boldsymbol{Q}, t)$, $\theta(\boldsymbol{k}, t)$, $\theta^{(\mathrm{N})}(\boldsymbol{k}, t)$, and $u_i(\boldsymbol{k}, t)$.

Using Eq. (C3) for the temperature fluctuations θ written in Fourier space, we derive equations for the instantaneous two-point correlation functions $F_j(\boldsymbol{k}, \boldsymbol{R})$ and $E_\theta(\boldsymbol{k}, \boldsymbol{R})$ defined by Eqs. (C1) and (C2). To this end we use the identities

$$\frac{\partial}{\partial t}\langle\theta(\boldsymbol{k}_1, t) u_j(\boldsymbol{k}_2, t)\rangle = \left\langle\frac{\partial\theta(\boldsymbol{k}_1, t)}{\partial t} u_j(\boldsymbol{k}_2, t)\right\rangle + \left\langle\theta(\boldsymbol{k}_1, t)\frac{\partial u_j(\boldsymbol{k}_2, t)}{\partial t}\right\rangle, \tag{C4}$$

$$\frac{\partial}{\partial t}\langle\theta(\boldsymbol{k}_1, t)\theta(\boldsymbol{k}_2, t)\rangle = \left\langle\frac{\partial\theta(\boldsymbol{k}_1, t)}{\partial t}\theta(\boldsymbol{k}_2, t)\right\rangle + \left\langle\theta(\boldsymbol{k}_1, t)\frac{\partial\theta(\boldsymbol{k}_2, t)}{\partial t}\right\rangle. \tag{C5}$$

Equations (C3)–(C5) yield the dynamic equations as

$$\frac{\partial F_j(\boldsymbol{k}, \boldsymbol{R})}{\partial t} = J_j(\boldsymbol{k}, \boldsymbol{R}) + \hat{\mathcal{M}} F_j^{(\mathrm{III})}(\boldsymbol{k}, \boldsymbol{R}), \tag{C6}$$

$$\frac{\partial E_\theta(\boldsymbol{k}, \boldsymbol{R})}{\partial t} = S(\boldsymbol{k}, \boldsymbol{R}) + \hat{\mathcal{M}} E_\theta^{(\mathrm{III})}(\boldsymbol{k}, \boldsymbol{R}), \tag{C7}$$

where

$$\hat{\mathcal{M}}F_j^{(\mathrm{III})}(\boldsymbol{k},\boldsymbol{R}) = \int \left[\left\langle\theta(\boldsymbol{k}_1)\frac{\partial u_j(\boldsymbol{k}_2)}{\partial t}\right\rangle - \langle\mathcal{Q}(\boldsymbol{k}_1)u_j(\boldsymbol{k}_2)\rangle\right]\exp[i\boldsymbol{K}\cdot\boldsymbol{R}]d\boldsymbol{K}, \quad (\mathrm{C8})$$

$$\hat{\mathcal{M}}E_\theta^{(\mathrm{III})}(\boldsymbol{k},\boldsymbol{R}) = -\int [\langle\theta(\boldsymbol{k}_1)\mathcal{Q}(\boldsymbol{k}_2)\rangle - \langle\mathcal{Q}(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2)\rangle]\exp[i\boldsymbol{K}\cdot\boldsymbol{R}]d\boldsymbol{K} \quad (\mathrm{C9})$$

are the third-order moment terms in $\boldsymbol{k}$ space appearing due to the nonlinear terms, and

$$J_j(\boldsymbol{k},\boldsymbol{R}) = -i\int [(\gamma-1)(k_i+K_i/2-Q_i)+Q_i] \times \langle u_i(\boldsymbol{k}+\boldsymbol{K}/2-\boldsymbol{Q})u_j(-\boldsymbol{k}+\boldsymbol{K}/2)\rangle\overline{T}(\boldsymbol{Q}) \times \exp(i\boldsymbol{K}\cdot\boldsymbol{R})d\boldsymbol{K}\,d\boldsymbol{Q}, \quad (\mathrm{C10})$$

$$S(\boldsymbol{k},\boldsymbol{R}) = -i\int \{[(\gamma-1)(k_j+K_j/2-Q_j)+Q_j] \times \langle\theta(-\boldsymbol{k}+\boldsymbol{K}/2)u_j(\boldsymbol{k}+\boldsymbol{K}/2-\boldsymbol{Q})\rangle + [(\gamma-1)(-k_j+K_j/2-Q_j)+Q_j] \times\langle\theta(\boldsymbol{k}+\boldsymbol{K}/2)u_j(-\boldsymbol{k}+\boldsymbol{K}/2-\boldsymbol{Q})\rangle\}\overline{T}(\boldsymbol{Q}) \times \exp(i\boldsymbol{K}\cdot\boldsymbol{R})d\boldsymbol{K}\,d\boldsymbol{Q}. \quad (\mathrm{C11})$$

To derive Eq. (14), we perform calculations in Eq. (C10) which are similar to those in Eqs. (B4)–(B7). To determine $\langle\theta(\tilde{\boldsymbol{k}}_1)u_j(\tilde{\boldsymbol{k}}_2)\rangle$ in Eq. (C11), we use new variables:

$$\tilde{\boldsymbol{k}} = (\tilde{\boldsymbol{k}}_1 - \tilde{\boldsymbol{k}}_2)/2 = -\boldsymbol{k}+\boldsymbol{Q}/2, \quad (\mathrm{C12})$$

$$\tilde{\boldsymbol{K}} = \tilde{\boldsymbol{k}}_1 + \tilde{\boldsymbol{k}}_2 = \boldsymbol{K}-\boldsymbol{Q}, \quad (\mathrm{C13})$$

where

$$\tilde{\boldsymbol{k}}_1 = -\boldsymbol{k}+\boldsymbol{K}/2, \quad \tilde{\boldsymbol{k}}_2 = \boldsymbol{k}+\boldsymbol{K}/2-\boldsymbol{Q}. \quad (\mathrm{C14})$$

Since $|\boldsymbol{Q}| \ll |\boldsymbol{k}|$ and $|\boldsymbol{K}| \ll |\boldsymbol{k}|$, we use the Taylor expansion

$$\langle\theta(\tilde{\boldsymbol{k}}_1)u_j(\tilde{\boldsymbol{k}}_2)\rangle = F_j(\tilde{\boldsymbol{k}},\tilde{\boldsymbol{K}}) = F_j(-\boldsymbol{k},\tilde{\boldsymbol{K}}) + \frac{Q_m}{2}\frac{\partial F_j}{\partial\tilde{k}_m} + O(\boldsymbol{Q}^2) = \left(1-\frac{Q_m}{2}\frac{\partial}{\partial k_m}\right)F_j(-\boldsymbol{k},\tilde{\boldsymbol{K}}) + O(\boldsymbol{Q}^2). \quad (\mathrm{C15})$$

Similarly,

$$\langle\theta(\tilde{\boldsymbol{k}}_3)u_j(\tilde{\boldsymbol{k}}_4)\rangle = \left(1+\frac{Q_m}{2}\frac{\partial}{\partial k_m}\right)F_j(\boldsymbol{k},\tilde{\boldsymbol{K}}) + O(\boldsymbol{Q}^2), \quad (\mathrm{C16})$$

where

$$\tilde{\boldsymbol{k}}_3 = \boldsymbol{k}+\boldsymbol{K}/2, \quad \tilde{\boldsymbol{k}}_4 = -\boldsymbol{k}+\boldsymbol{K}/2-\boldsymbol{Q}. \quad (\mathrm{C17})$$

Substituting Eqs. (C15) and (C16) into Eq. (C11), neglecting the terms $O(\boldsymbol{Q}^2;\boldsymbol{K}^2)$, and returning to the physical space in the large-scale variables, we obtain Eqs. (14) and (15).

To determine the turbulent heat flux and the level of temperature fluctuations, we use the following identities for integration over k in Eqs. (17) and (18):

$$\int_{k_0}^{k_\nu} \tau(k)[E(k)+\sigma_c E_c(k)]dk = \tau_0(1+\sigma_c),$$

$$\int_{k_0}^{k_\nu} \tau(k)E(k)dk = \tau_0\left[1-\frac{\tilde{C}_\sigma\,\sigma_c}{2(1+\sigma_c)}\right],$$

$$\int_{k_0}^{k_\nu} \tau(k)E_c(k)dk = \tau_0\left[1+\frac{\tilde{C}_\sigma}{2(1+\sigma_c)}\right],$$

$$\int_{k_0}^{k_\nu} \tau(k)k^2 E_c(k)dk = \frac{6\tau_0}{\ell_0^2}(1+\sigma_c)^{-1}\left[\mathrm{Re}^{1/4}+\frac{\sigma_c}{4}\ln\mathrm{Re}\right],$$

$$\int_{k_0}^{k_\nu} \frac{d\tau(k)}{dk}E_c(k)k\,dk = -\frac{\tau_0(q_c-1)\sigma_c}{1+\sigma_c} \times\left[1+\frac{2(q-1)}{\sigma_c(q+q_c-2)}\right],$$

$$\int_{k_0}^{k_\nu} \tau^2(k)k^2E_c(k)\,dk = 4f_c\left(\frac{\tau_0}{\ell_0}\right)^2\left(\frac{\sigma_c}{1+\sigma_c}\right)^2,$$

$$\int_{k_0}^{k_\nu} \tau^2(k)[E(k)+\sigma_c E_c(k)]dk = \frac{4}{3}\tau_0^2(1+\sigma_c),$$

$$\int_{k_0}^{k_\nu} \tau^2(k)E_c(k)dk = \frac{4}{3}\tau_0^2\, f_*\left(\frac{\sigma_c}{1+\sigma_c}\right)^2,$$

$$\int_{k_0}^{k_\nu} \tau^2(k)E(k)dk = \frac{4}{3}\tau_0^2(1+\sigma_c)\left[1-f_*\left(\frac{\sigma_c}{1+\sigma_c}\right)^3\right],$$

where

$$f_* = 1+\frac{6(q_c-1)}{\sigma_c(q+2q_c-3)}+\frac{3(q_c-1)}{\sigma_c^2(2q+q_c-3)}.$$

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