Generation of a large-scale vorticity in a fast-rotating density-stratified turbulence or turbulent convection

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We find an instability resulting in generation of large-scale vorticity in a fast-rotating small-scale turbulence or turbulent convection with inhomogeneous fluid density along the rotational axis in anelastic approximation. The large-scale instability causes excitation of two modes: (i) the mode with dominant vertical vorticity and with the mean velocity being independent of the vertical coordinate; (ii) the mode with dominant horizontal vorticity and with the mean momentum being independent of the vertical coordinate. The mode with the dominant vertical vorticity can be excited in a fast-rotating density-stratified hydrodynamic turbulence or turbulent convection. For this mode, the mean entropy is depleted inside the cyclonic vortices, while it is enhanced inside the anticyclonic vortices. The mode with the dominant horizontal vorticity can be excited only in a fast-rotating density-stratified turbulent convection. The developed theory may be relevant for explanation of an origin of large spots observed as immense storms in great planets, e.g., the Great Red Spot in Jupiter and large spots in Saturn. It may be also useful for explanation of an origin of high-latitude spots in rapidly rotating late-type stars.

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I. INTRODUCTION

Generations of large-scale vorticity in turbulent flows have been investigated theoretically, experimentally, and numerically in a number of studies due to various applications in geophysical, astrophysical and industrial flows (see, e.g., Refs. [1–3]). Using an analogy between the induction equation for magnetic field and the vorticity equation (see Refs. [4,5]), it has been proposed in Refs. [6–8] that the large-scale vorticity can be generated due to a large-scale instability by the kinetic α effect in a helical turbulence with a net kinetic helicity. The kinetic helicity and the kinetic α effect can be produced in rotating density stratified or inhomogeneous turbulence.

Another possibility for a generation of large-scale vorticity is related to anisotropic kinetic α effect referred as the AKA effect [9–11], which is caused by a non-Galilean invariant forcing. For example, boundaries can break the Galilean invariance which results in an anisotropic kinetic α effect [12], resulting in a large-scale instability. In astrophysics, a turbulence driven by non-Galilean invariant forcing can exist in galaxies (e.g., supernova-driven turbulence [13,14] and the turbulent wakes driven by galaxies moving through the galaxy cluster [15]).

In a nonconducting fluid, a nonhelical turbulence with an imposed large-scale velocity shear can cause a large-scale instability resulting in generation of the large-scale vorticity due to a combined effect of the large-scale shear motions and Reynolds stress-induced production of perturbations of mean vorticity [16,17]. This effect referred as "vorticity dynamo" has been also confirmed in direct numerical simulations (DNS) [18,19]. This mechanism of the generation of the large-scale vorticity is also associated with the Prandtl's first and second kinds of secondary flows [20,21]. In particular, the skew-induced streamwise mean vorticity generation arises at the lateral boundaries of three-dimensional thin shear layers and corresponds to the Prandtl's first kind of secondary flows. In turbulent flows, the streamwise mean vorticity can be generated by the Reynolds stress, and this mechanism is associated with formation of the Prandtl's second kind of turbulent flows [21].

A large-scale vorticity also can be produced by a combined effect of a rotating incompressible turbulence and inhomogeneous kinetic helicity [22–25] or due to a combined action of a density-stratified rotating homogeneous turbulence and uniform kinetic helicity [25]. These effects result in the formation of a large-scale shear, and in turn its interaction with the small-scale turbulence causes an excitation of the large-scale instability (the vorticity dynamo) due to a combined effect of the large-scale shear and Reynolds stress-induced generation of the mean vorticity [25].

Recent DNS have shown that large-scale vortices in rapidly rotating turbulent convection can be formed in compressible [26–28] or Boussinesq fluids [29–33]. The produced large-scale motions include cyclonic vortices and anti-cyclonic vortices. It was found that in the cyclonic vortices the temperature is depleted [27,28].

In the present study we develop a theory of the generation of the large-scale vorticity in a fast-rotating turbulent convection with inhomogeneous fluid density in anelastic

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approximation. A particular case when gravity is along rotational axis has been considered. We have found a large-scale instability which results in an excitation of two modes. For the mode with dominant vertical vorticity, the mean velocity is independent of the vertical coordinate. This mode can be excited in both, a fast-rotating density-stratified hydrodynamic turbulence and fast-rotating density-stratified turbulent convection. We have demonstrated that for this mode, the mean entropy is depleted inside the cyclonic vortices in agreement with [27,28]. For the second mode, the horizontal component of the mean vorticity is dominant, and the mean momentum is independent of the vertical coordinate.

This study may be relevant to formation mechanisms of large spots observed in the form of immense storms in great planets (e.g., the Great Red Spot in Jupiter and large spots in Saturn, see, e.g., Refs. [34–36]), and it may be useful for explanation of an origin of high-latitude spots seen in Doppler imaging in rapidly rotating late-type stars [27,28].

This paper is organized as follows. In Sec. II we consider the effect of fast rotation on the Reynolds stress and the effective force. Here we outline the method of derivations and approximations made for study of this effect. Using mean-field equations and the derived rotational contributions to the Reynolds stress, we study in Sec. III the large-scale instability causing the generation of the large-scale vorticity in a fast-rotating turbulent convection with inhomogeneous fluid density along the rotational axis. Finally, conclusions are drawn in Sec. V. In Appendix A we present details of the derivation of equation for the rotational contributions to the Reynolds stress. In Appendix B we give an explicit form for the mean-field equations describing the large-scale instability which results in generation of the mean vorticity for different modes. In Appendix C we discuss the role of the centrifugal force in the production of large-scale vorticity. The centrifugal force causes the inhomogeneous density distribution in the plane perpendicular to the angular velocity $\boldsymbol{\Omega}$. We have shown in Appendix C that a combined effect of a fast rotation and horizontal inhomogeneity of the fluid density (caused by the centrifugal force) results in the production of the large-scale vertical vorticity in an anisotropic isothermal turbulence.

II. EFFECT OF FAST ROTATION ON THE REYNOLDS STRESS AND THE EFFECTIVE FORCE

To derive mean-field equations which describe generation of the large-scale vorticity, we consider a small-scale low-Mach-number fast-rotating density-stratified turbulent convection in anelastic approximation with equation of state for the ideal gas. To investigate effect of fast rotation on the Reynolds stress in a turbulent convection with inhomogeneous fluid density, we use a mean-field approach, whereby the velocity, pressure, and entropy are decomposed in the mean and fluctuating parts. An ensemble averaging of the momentum and entropy equations yields the equations for mean velocity, $\overline{\boldsymbol{U}}(t,\boldsymbol{x})$, and mean entropy, $\overline{S}(t,\boldsymbol{x})$, in the reference frame rotating with the constant angular velocity $\boldsymbol{\Omega}$:

$$\frac{\partial \overline{U}_i}{\partial t} + (\overline{\boldsymbol{U}}\cdot\nabla)\overline{U}_i = -\nabla_i\left(\frac{\overline{P}}{\rho_0}\right) - g_i\overline{S} + 2(\overline{\boldsymbol{U}}\times\boldsymbol{\Omega})_i - \frac{1}{\rho_0}\nabla_j\big(\rho_0\langle u'_i u'_j\rangle\big), \tag{1}$$

$$\frac{\partial \overline{S}}{\partial t} + (\overline{\boldsymbol{U}}\cdot\nabla)\overline{S} = -(\overline{\boldsymbol{U}}\cdot\nabla)S_0 - \frac{1}{\rho_0}\nabla\cdot(\rho_0\langle\boldsymbol{u}'\,s'\rangle), \tag{2}$$

where $\overline{S} = \overline{T}/T_0 - (1-\gamma^{-1})\overline{P}/P_0$, $\overline{T}$ and $\overline{P}$ are the mean entropy, the mean temperature and the mean pressure, respectively, γ is the ratio of specific heats, $\boldsymbol{u}'$ and s' are fluctuations of the fluid velocity and entropy, $\langle u'_i u'_j\rangle$ is the Reynolds stress describing turbulent viscosity and rotational effects to turbulent convection, $\langle\boldsymbol{u}'\,s'\rangle$ is the turbulent flux of entropy, T_0, P_0, S_0, and ρ_0 are the fluid temperature, pressure, entropy, and density, respectively, in the basic reference state and $\nabla S_0 = (\gamma P_0)^{-1}\nabla P_0 - \rho_0^{-1}\nabla\rho_0$. The variables with the subscript "0" correspond to the hydrostatic nearly isentropic basic reference state defined by $\nabla P_0 = \rho_0\boldsymbol{g}$ and $\boldsymbol{g}\cdot\nabla S_0 \approx 0$, where $\boldsymbol{g}$ is the acceleration due to the gravity. In Eqs. (1) and (2) we neglect small molecular viscosity and heat conductivity terms.

To derive equations for the rotational contributions to the Reynolds stress, we follow the method that is developed in Refs. [25,37] and outlined below (see, for details, Appendix A). We use equations for fluctuations of velocity $\boldsymbol{u}'$ and entropy $s' = \theta/T_0 - (1-\gamma^{-1})p'/P_0$:

$$\frac{\partial \boldsymbol{u}'}{\partial t} = -(\overline{\boldsymbol{U}}\cdot\nabla)\boldsymbol{u}' - (\boldsymbol{u}'\cdot\nabla)\overline{\boldsymbol{U}} - \nabla\left(\frac{p'}{\rho_0}\right) - \boldsymbol{g}\,s' + 2\boldsymbol{u}'\times\boldsymbol{\Omega} + \boldsymbol{U}^N, \tag{3}$$

$$\frac{\partial s'}{\partial t} = -(\boldsymbol{u}'\cdot\nabla)\overline{S} - (\overline{\boldsymbol{U}}\cdot\nabla)s' + S^N, \tag{4}$$

where p' and θ are fluctuations of fluid pressure and temperature, respectively, $\boldsymbol{U}^N = \langle(\boldsymbol{u}'\cdot\nabla)\boldsymbol{u}'\rangle - (\boldsymbol{u}'\cdot\nabla)\boldsymbol{u}'$ and $S^N = \langle(\boldsymbol{u}'\cdot\nabla)s\rangle - (\boldsymbol{u}'\cdot\nabla)s$ are the nonlinear terms, and the angular brackets imply ensemble averaging. In Eqs. (3) and (4) we neglect small molecular viscosity and heat conductivity terms. Equation (3) is written in the reference frame rotating with the constant angular velocity $\boldsymbol{\Omega}$. The turbulent convection is considered as a small deviation from a well-mixed adiabatic reference state. The equations for fluctuations of velocity and entropy are obtained by subtracting Eqs. (1) and (2) for the mean fields from the corresponding equations for the total velocity $\overline{\boldsymbol{U}}+\boldsymbol{u}'$ and entropy $\overline{S}+s'$ fields. The fluid velocity for a low Mach number flows with strong inhomogeneity of the fluid density ρ_0 along the gravity field is assumed to be satisfied to the continuity equation written in the anelastic approximation, $\operatorname{div}(\rho_0\overline{\boldsymbol{U}}) = 0$ and $\operatorname{div}(\rho_0\boldsymbol{u}') = 0$.

To study the effects of fast rotation on the Reynolds stress in density-stratified turbulent convection, we perform the derivations which include the following steps:

(i) using new variables for fluctuations of velocity $\boldsymbol{v} = \sqrt{\rho_0}\,\boldsymbol{u}'$ and entropy $s = \sqrt{\rho_0}\,s'$;

(ii) derivation of the equations for the second-order moments of the velocity fluctuations $\langle v_i\,v_j\rangle$, the entropy fluctuations $\langle s^2\rangle$ and the turbulent flux of entropy $\langle v_i\,s\rangle$ in the $\boldsymbol{k}$ space;

(iii) application of the multiscale approach [38] that allows us to separate turbulent scales from large scales;

(iv) adopting the spectral τ approximation [39–41] (see below);

(v) solution of the derived second-order moment equations in the $\boldsymbol{k}$ space;

(vi) returning to the physical space to obtain expression for the Reynolds stress as the function of the rotation rate Ω.

The derived equations for the second-order moments of the velocity fluctuations $\langle v_i\, v_j\rangle$, the entropy fluctuations $\langle s^2\rangle$ and the turbulent flux of entropy $\langle v_i\, s\rangle$ [see Eqs. (A1)–(A3) in Appendix A], include the first-order spatial differential operators $\hat{\mathcal{N}}$ applied to the third-order moments $M^{(III)}$. A problem arises how to close the system of the second-moment equations, i.e., how to express the set of the third-moment terms $\hat{\mathcal{N}} M^{(III)}(\boldsymbol{k})$ through the lower moments (see, e.g., Refs. [40,42,43]). Various approximate methods have been proposed to solve this problem. In the present study we use the spectral τ approximation (see, e.g., Refs. [39–41]), which postulates that the deviations of the third-moment terms, $\hat{\mathcal{N}} M^{(III)}(\boldsymbol{k})$, from the contributions to these terms afforded by the background fast-rotating turbulent convection, $\hat{\mathcal{N}} M^{(III,0)}(\boldsymbol{k})$, are expressed through the similar deviations of the second-order moments, $M^{(II)}(\boldsymbol{k}) - M^{(II,0)}(\boldsymbol{k})$ in the relaxation form:

$$\hat{\mathcal{N}} M^{(III)}(\boldsymbol{k}) - \hat{\mathcal{N}} M^{(III,0)}(\boldsymbol{k}) = -\frac{M^{(II)}(\boldsymbol{k}) - M^{(II,0)}(\boldsymbol{k})}{\tau_r(k)}, \quad (5)$$

see for details, Eqs. (A17)–(A19) in Appendix A. Here the correlation functions with the superscript (0) correspond to the background fast-rotating turbulent convection with zero spatial derivatives of the mean velocity, $\nabla_i \overline{U}_j = 0$. The time $\tau_r(k)$ is the characteristic relaxation time of the statistical moments, which can be identified with the correlation time $\tau(k)$ of the turbulent velocity field for large Reynolds numbers. Validations of the τ approximation for different situations have been performed in various direct numerical simulations [44–52] (see also discussion in Sec. IV).

The τ approximation is a sort of the high-order closure and in general is similar to eddy-damped quasinormal Markovian (EDQNM) approximation. However, some principle difference exists between these two approaches [40,41]. The EDQNM closures do not relax to equilibrium (the background turbulence), and the EDQNM approach does not describe properly the motions in the equilibrium state in contrast to the τ approximation. Within the EDQNM theory, there is no dynamically determined relaxation time, and no slightly perturbed steady state can be approached. In the τ approximation, the relaxation time for small departures from equilibrium is determined by the random motions in the equilibrium state, but not by the departure from the equilibrium. As follows from the analysis in [40], the τ approximation describes the relaxation to the equilibrium state (the background turbulence) much more accurately than the EDQNM approach.

We apply the τ approximation only to study the deviations from the background turbulent convection which are caused by the spatial derivatives of the mean velocity. The background fast-rotating turbulent convection is assumed to be known (see below). The τ approximation is only valid for large Reynolds numbers, where the relaxation time can be clearly identified with the turbulence correlation time.

We use the model of the background homogeneous turbulent convection with inhomogeneous fluid density distribution along the gravity field which takes into account an anisotropy of turbulent convection caused by the fast rotation [see Eqs. (A25) and (A26) in Appendix A]. We assume that the background turbulent convection is of Kolmogorov type with constant flux of energy over the spectrum, i.e., the kinetic energy spectrum function for the range of wave numbers $k_0 < k < k_\nu$ is $E(k) = -d\bar{\tau}(k)/dk$, the function $\bar{\tau}(k) = (k/k_0)^{1-q}$ with $1 < q < 3$ being the exponent of the kinetic energy spectrum ($q = 5/3$ for a Kolmogorov spectrum). Here $k_\nu = 1/\ell_\nu$ is the wave number based on the viscous scale ℓ_ν, and $k_0 = 1/\ell_0 \ll k_\nu$, where ℓ_0 is the integral (energy containing) scale of turbulent motions. The turbulent correlation time in $\boldsymbol{k}$ space is $\tau(k) = 2\tau_{_\Omega}\, \bar{\tau}(k)$, where the effect of rotation on the turbulent correlation time, $\tau_{_\Omega}$, is described just by an heuristic argument. In particular, we assume that

$$\tau_{_\Omega} = \frac{\tau_0}{\left[1 + C_\tau\, \Omega^2\, \tau_0^2\right]^{1/2}}. \quad (6)$$

Here the dimensionless constant $C_\tau \sim 1$ and $\tau_0 = \ell_0/u_0$ with the characteristic turbulent velocity u_0 in the integral scale of turbulence ℓ_0. In particular, the squared inverse timescale $\tau_{_\Omega}^{-2}$ is considered as a linear combination of the two simple squared inverse timescales: τ_0^{-2} and Ω^2:

$$\tau_{_\Omega}^{-2} = \tau_0^{-2} + C_\tau \Omega^2. \quad (7)$$

For fast rotation, $\Omega\, \tau_0 \gg 1$, the parameter $\Omega\, \tau_{_\Omega}$ tends to be limiting value $C_\tau^{-1/2}$.

The above described procedure yields the rotational contribution to the Reynolds stress, and the effective force, $\mathcal{F}_i^\Omega = \rho_0\, \langle v_i\, v_j\rangle^\Omega e_j/H_\rho$ for a fast-rotating density-stratified turbulent convection or for a fast-rotating density-stratified anisotropic homogeneous turbulence, where $H_\rho = (|\boldsymbol{\nabla} \rho_0|/\rho_0)^{-1}$ is the density stratification hight, $\langle v_i\, v_j\rangle^\Omega$ are the rotational contributions to the Reynolds stress given by Eqs. (A33)–(A34) in Appendix A and $\boldsymbol{e}$ is the vertical unit vector along the z axis (in the direction opposite to the gravity acceleration). The components of the effective force are given by

$$\mathcal{F}_x^\Omega = -2(A_F - A_u)\, \rho_0\, \nu_{_T}\, \Omega\tau_0\, \frac{\ell_0^2}{H_\rho^3}\, \nabla_z \overline{U}_y, \quad (8)$$

$$\mathcal{F}_y^\Omega = -2\, \rho_0\, \nu_{_T}\, \Omega\tau_0\, \frac{\ell_0^2}{H_\rho^3}\, [(A_F + A_u)\nabla_x \overline{U}_z - (A_F - A_u)\overline{W}_y], \quad (9)$$

$$\mathcal{F}_z^\Omega = -(5A_F + 4A_u)\, \rho_0\, \nu_{_T}\, \Omega\tau_0\, \frac{\ell_0^2}{H_\rho^3}\, \nabla_x \overline{U}_y, \quad (10)$$

where $\overline{\boldsymbol{W}} = \boldsymbol{\nabla} \times \overline{\boldsymbol{U}}$ is the mean vorticity, $\nu_{_T} = \tau_0 u_0^2/6$ is the turbulent viscosity,

$$A_F = \frac{9(q-1)}{2(2q-1)}\, \frac{\varepsilon_{_F}\, \tau_0\, F_* g}{\rho_0 u_0^2}, \quad (11)$$

$$A_u = \frac{3(q-1)}{3q-1}\, \frac{\varepsilon_u}{1+\varepsilon_u}, \quad (12)$$

$F_* = \rho_0\, \langle u'_z\, s'\rangle$, the parameter ε_u is the degree of anisotropy of turbulent velocity field in the background turbulence, and the parameter $\varepsilon_{_F}$ is the degree of thermal anisotropy of the background turbulence [see Eqs. (A25) and (A26) in Appendix A]. The details of the derivation of Eqs. (8)–(10) are given in Appendix A. These equations are derived using the following conditions: $\Omega\tau_0 \gg 1$ and the turbulent integral scale ℓ_0 is much smaller than the density stratification scale H_ρ and the characteristic horizontal scale L_x of variations of the mean velocity $\overline{\boldsymbol{U}}$ (i.e., $\ell_0 \ll H_\rho; L_x$). We also assumed that the density stratification scale H_ρ is much smaller than the characteristic vertical scale L_z of variations of the mean velocity $\overline{\boldsymbol{U}}$.

To introduce anisotropy of turbulent velocity field in the background turbulence caused by a fast rotation, we consider an anisotropic turbulence as a combination of a three-dimensional isotropic turbulence and two-dimensional turbulence in the plane perpendicular to the rotational axis. The degree of anisotropy ε_u is defined as the ratio of turbulent kinetic energies of two-dimensional to three-dimensional motions. The degree of thermal anisotropy $\varepsilon_{_F}$ determines the contribution of the two-dimensional turbulence to the heat flux.

The anisotropy parameters ε_u and $\varepsilon_{_F}$ appeared in the model of the background turbulent rotating convection depend on the Coriolis number $\mathrm{Co} = 2\Omega\tau_0$. For a slow rotation (small Coriolis numbers or large Rossby numbers), the parameters $\varepsilon_u \to 0$ and $\varepsilon_{_F} \to 0$. For a fast rotation (very large Coriolis numbers or very small Rossby numbers), the parameters $\varepsilon_u \gg 1$ and $\varepsilon_{_F} \sim 1$. In this case the background turbulent convection is a highly anisotropic nearly two-dimensional turbulence, and the main rotational contributions to the Reynolds stress are from the two-dimensional part of turbulence. Formally, in the present study where we investigate a fast-rotating turbulent convection, these parameters are not specified, but they should satisfy the following conditions $\varepsilon_u \gg 1$ and $\varepsilon_{_F} \sim 1$.

In the derivation of the expressions for the Reynolds stress and the effective force, we take into account the terms which are linear in the angular velocity and drop the terms that are quadratic in the angular velocity. The reason is that the terms that are proportional to the angular velocity causes generation of large-scale vorticity, while the terms that are quadratic in the angular velocity yield small contributions to the anisotropic part of the turbulent viscosity. The latter effect is neglected in the present study. However, we have taken into account the dominant contributions to the Reynolds stress and the effective force which are caused by the effect of fast rotation on turbulent convection.

III. MEAN-FIELD DYNAMICS AND LARGE-SCALE INSTABILITY

In this section we study large-scale instability resulting in generation of the large-scale vorticity. Using the derived Eqs. (8)–(10) for the effective force, the Navier-Stokes Eq. (1) for the mean velocity $\overline{\boldsymbol{U}}$, and the equation for the mean vorticity $\overline{\boldsymbol{W}} = \nabla\times\overline{\boldsymbol{U}}$, we investigate the large-scale instability. For simplicity, we consider the case with the angular velocity along z axis (opposite to the gravity field). The linearized equations for $\overline{U}_y$ and $\overline{W}_y$ are given by

$$\frac{\partial \overline{U}_y}{\partial t} = -2\,\overline{U}_x\Omega + \frac{\mathcal{F}_y^\Omega}{\rho_0} + \frac{\nu_{_T}}{\rho_0}\nabla\cdot(\rho_0\nabla\overline{U}_y), \tag{13}$$

$$\frac{\partial \overline{W}_y}{\partial t} = 2\Omega\,\nabla_z\overline{U}_y + \left(\nabla\times\frac{\boldsymbol{\mathcal{F}}^\Omega}{\rho_0}\right)_y + \frac{\nu_{_T}}{\rho_0}\nabla\cdot(\rho_0\nabla\overline{W}_y) - g\nabla_z\overline{S}. \tag{14}$$

We introduce new variables $\overline{V}(t,x,z)$ and $\overline{\Phi}(t,x,z)$:

$$\rho_0\overline{\boldsymbol{U}} = [\overline{V}(t,x,z)\rho_0^{1/2}]\boldsymbol{e}_y + \nabla\times[\overline{\Phi}(t,x,z)\rho_0^{1/2}]\boldsymbol{e}_y, \tag{15}$$

which corresponds to axi-symmetric problem. In the new variables Eqs. (13) and (14) are given by Eqs. (B2) and (B3) (see Appendix B).

First, we consider a mode with the mean velocity that is independent of z, i.e., we seek for a solution of Eqs. (B2) and (B3) in the following form: $\overline{V}, \overline{\Phi} \propto \exp(-\lambda z/2)\exp(\gamma_{\rm inst}t + iK_xX)$. Substituting this solution into Eqs. (B2) and (B3), we obtain the growth rate of the large-scale instability resulting in the generation of this mode:

$$\gamma_{\rm inst} = \Omega\,\frac{\ell_0^2}{H_\rho^2}\left[\frac{3(q-1)}{2(2q-1)}\left(\frac{5\varepsilon_{_F}\tau_0 F_*\, g}{\rho_0 u_0^2} + \frac{4(2q-1)}{3(3q-1)}\,\frac{\varepsilon_u}{1+\varepsilon_u}\right)\right]^{1/2} - \nu_{_T}K_x^2. \tag{16}$$

This mode is with a dominant vertical mean vorticity, $\overline{W}_z/\overline{W}_y \sim (H_\rho L_x)/\ell_0^2 \gg 1$, where $L_x = 2\pi/K_x$. It follows from Eq. (16) that the large-scale instability for this mode can be excited even for a hydrodynamic anisotropic turbulence (i.e., when there is no turbulent convection, $F_* = 0$). The mechanism of the large-scale instability resulting in the generation of the dominant vertical mean vorticity, $\overline{W}_z = \nabla_x\overline{U}_y$, is as follows. The Coriolis force for a fast rotation strongly modifies turbulence and the Reynolds stress, so that the second term in Eq. (14) does not vanish, $[\nabla\times(\boldsymbol{\mathcal{F}}^\Omega/\rho_0)]_y \neq 0$. This term depends on $\overline{U}_y$ [see Eqs. (10) and (14)]. The horizontal component of the mean vorticity $\overline{W}_y$ is produced by this key term, $[\nabla\times(\boldsymbol{\mathcal{F}}^\Omega/\rho_0)]_y$, which is caused by the effective force, i.e., $\partial\overline{W}_y/\partial t \sim [\nabla\times(\boldsymbol{\mathcal{F}}^\Omega/\rho_0)]_y$; see Eq. (14). However, the velocity component $\overline{U}_y$ is produced by the Coriolis force, $\partial\overline{U}_y/\partial t \sim -2\,\overline{U}_x\Omega$ [see Eq. (13)], which closes the generation loop. Here we took into account that the Coriolis force is much larger than the effective force, i.e., the ratio $|2\,\overline{U}_x\Omega|/|\mathcal{F}_y^\Omega/\rho_0| \sim L_xH_\rho^3/\ell_0^4 \gg 1$.

Usually for a fast rotation, inertial waves characterised by the dispersion relation, $\omega = 2(\boldsymbol{\Omega}\cdot\boldsymbol{K})/K$, are dominant and they decrease the growth rate of instabilities for different modes. However, since for the considered mode the vertical derivative $\nabla_z\overline{U}_y = 0$, the contribution of this effect (caused by the inertial waves) to the growth rate of the large-scale instability for this mode vanishes.

Let us study the evolution of the mean entropy $\overline{S}$ in this mode. The linearized Eq. (2) for $\overline{S}$ reads:

$$\frac{\partial\overline{S}}{\partial t} = -\overline{U}_z\nabla_z S_0 + \rho_0^{-1}\nabla\cdot(\rho_0\kappa_{_T}\nabla\overline{S}), \tag{17}$$

where $\kappa_{_T}$ is the coefficient of turbulent diffusion. This implies that

$$\overline{S} = \frac{\overline{U}_z|\nabla_z S_0|}{\gamma_{\rm inst}+\kappa_{_T}K_x^2} = -\frac{\overline{W}_z H_\rho}{2\Omega}\left(\frac{\gamma_{\rm inst}+\nu_{_T}K_x^2}{\gamma_{\rm inst}+\kappa_{_T}K_x^2}\right)|\nabla_z S_0|, \tag{18}$$

where we use the solutions for the vertical mean velocity $\overline{U}_z = K_x \Phi_* \cos(K_x X + \varphi) \exp(\gamma_{\rm inst} t)$, and the vertical mean vorticity $\overline{W}_z = K_x V_* \cos(K_x X + \varphi) \exp(\gamma_{\rm inst} t)$. Here the ratio of amplitudes V_*/Φ_* for this mode is

$$\frac{V_*}{\Phi_*} = -\frac{2\Omega}{H_\rho(\gamma_{\rm inst} + \kappa_{_T} K_x^2)}. \qquad (19)$$

In Eq. (19) we neglect the small terms $\sim \mathrm{O}(\ell_0^2/H_\rho^2)$. Thus, the solution for the mean entropy is $\overline{S} = -S_* \cos(K_x X + \varphi) \exp(\gamma_{\rm inst} t)$. Equation (18) implies that inside the cyclonic vortices where the perturbations of the vertical mean vorticity are positive ($\overline{W}_z > 0$), the perturbations of the mean entropy are negative ($\overline{S} < 0$). Therefore, inside the cyclonic vortices the mean entropy is reduced. However, inside the anticyclonic vortices where the perturbations of the vertical mean vorticity are negative ($\overline{W}_z < 0$), the perturbations of the mean entropy are positive ($\overline{S} > 0$). Therefore, inside the anticyclonic vortices the mean entropy is increased.

There is also another mode for which the negative contribution caused by the inertial waves to the growth rate of the instability for this mode vanishes. Indeed, for this mode a solution of Eqs. (B2) and (B3) has the following form: $\overline{V}$, $\overline{\Phi} \propto \exp(\lambda z/2) \exp(\gamma_{\rm inst} t + i K_x X)$. This is a mode with the mean momentum, $\rho_0 \overline{\boldsymbol{U}}$, that is independent of z. Substituting this solution into Eqs. (B2) and (B3), we obtain the growth rate of the large-scale instability resulting in the generation of this mode:

$$\gamma_{\rm inst} = \Omega \, \frac{\ell_0^2}{H_\rho^2} \left[\frac{6(q-1)\, \varepsilon_{_F}\, \tau_0\, F_*\, g}{(2q-1)\, \rho_0 u_0^2}\right]^{1/2} - \nu_{_T} K_x^2. \qquad (20)$$

This mode is with a dominant horizontal mean vorticity, i.e., $\overline{W}_z/\overline{W}_y \sim \ell_0^2/(H_\rho L_x) \ll 1$. It follows from Eq. (20) that the large-scale instability for this mode can be excited only in turbulent convection (when $F_* \neq 0$). For this mode the component of the mean velocity $\overline{U}_x = 0$, and the component $\overline{U}_y$ is produced by the effective force $\mathcal{F}_y^\Omega/\rho_0$ [see Eq. (13)]. However, the dominant horizontal mean vorticity $\overline{W}_y$ is produced by the term $2\Omega\, \nabla_z \overline{U}_y$ caused by the Coriolis force [see Eq. (14)], which closes the generation loop.

Let us check if the obtained results are consistent with the Taylor-Proudman theorem. For a fast-rotating laminar flow, the Taylor-Proudman theorem implies that the leading-order balance in the equation for the vorticity for large Coriolis number (small Rossby numbers) is $(\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})\boldsymbol{U} = 0$. This implies that the velocity is independent of the vertical coordinate z, where $\boldsymbol{\Omega} = \Omega \boldsymbol{e}_z$. For the mode with the dominant vertical mean vorticity, the mean velocity is independent of z. This implies that this mode is consistent with the Taylor-Proudman theorem. However, for the mode with the dominant horizontal mean vorticity, the mean momentum is independent of z, while the mean velocity depends on z, so that this mode is not consistent with the Taylor-Proudman theorem.

IV. DISCUSSION AND CONCLUSIONS

In the present study we have considered a fast-rotating turbulence or turbulent convection with inhomogeneous fluid density along the rotational axis in anelastic approximation. A large-scale instability exciting at large Coriolis number has been found, which causes generation of large-scale vorticity for two key modes with dominant vertical or horizontal components. The effective force caused by the rotational contribution to the Reynolds stress in small-scale turbulent convection in combination with the Coriolis force in the mean-field momentum equation are the main effects resulting in the generation of the large-scale vorticity due to the excitation of the large-scale instability. The mode with the vertical vorticity can be generated in both, a fast-rotating density-stratified hydrodynamic turbulence and turbulent convection, while the mode with the dominant horizontal vorticity can be excited only in a fast-rotating density-stratified turbulent convection. When the density stratification hight $H_\rho \to \infty$ (i.e., when the fluid density is uniform), the large-scale instability found in the present study cannot be excited [see Eqs. (16) and (20) for the growth rates of the instability]. This implies that this theory cannot describe formation of large-scale vortices observed in the Boussinesq turbulent convection with div $\boldsymbol{u} = 0$ (see Refs. [29–33]).

Our theory is developed for a low-Mach number fast-rotating turbulent convection with inhomogeneous fluid density, which corresponds to the set-ups of DNS described in Refs. [27,28]. However, in DNS on the large-scale vorticity growth, it is very difficult to observe the kinematic stage of the evolution of the large-scale vorticity with an exponential growth. Usually in DNS it is only seen the nonlinear evolution of the large-scale vorticity. This implies that it is very difficult to make quantitative comparisons between the kinematic mean-field theory for the large-scale vorticity growth and DNS. We have only performed a qualitative comparison with the DNS described in Refs. [27,28]. In particular, we confirm the existence of the threshold in the Coriolis number for the generation of the large-scale vorticity. The critical Coriolis number should be much larger than 1. The derived mean-field equations describe formations of both, cyclonic and anticyclonic large-scale vortices in the kinematic (linear) stage of the instability. As in the DNS, we also find the similar behavior of the mean entropy or temperature inside cyclonic and anticyclonic vortices. For example, we have shown that for the mode with the dominant vertical mean vorticity, the mean entropy is decreased inside the cyclonic vortices and increased inside the anticyclonic vortices in agreement with [27,28].

To derive equations for the rotational contribution to the Reynolds stress and the effective force in fast-rotating density-stratified turbulent convection, we apply the spectral τ approximation (see Sec. II). The τ approximation is an universal tool in turbulent transport that allows to obtain closed results and compare them with the results of laboratory experiments, observations and numerical simulations. The τ approximation reproduces many well-known phenomena found by other methods in turbulent transport of particles, temperature and magnetic fields, in turbulent convection and stably stratified turbulent flows (see below).

In turbulent transport, the τ approximation yields correct formulas for turbulent diffusion, turbulent thermal diffusion and turbulent barodiffusion [53,54]. The phenomenon of turbulent thermal diffusion (a nondiffusive streaming of particles in the direction of the mean heat flux), has been predicted using the stochastic calculus (the path integral approach), the quasilinear approach and the τ approximation. This phenomenon has been already detected in laboratory experiments in oscillating grids turbulence [55] and in a multifan produced turbulence [56] in both, stably and unstably stratified fluid flows. The phenomenon of turbulent thermal diffusion has been also detected in direct numerical simulations [48,49,52]. The numerical and experimental results are in a good agreement with the theoretical studies performed by means of different approaches (see Refs. [53,57]).

The τ approximation reproduces the well-known $k^{-7/3}$-spectrum of anisotropic velocity fluctuations in a sheared turbulence (see Ref. [58]). This spectrum was previously found in analytical, numerical, laboratory studies and was observed in the atmospheric turbulence (see, e.g., Ref. [59]). In the turbulent boundary layer problems, the τ approximation yields correct expressions for turbulent viscosity, turbulent thermal conductivity and the classical heat flux. This approach also describes the counter wind heat flux and the Deardorff's heat flux in convective boundary layers (see Ref. [58]). These phenomena have been previously studied using different approaches (see, e.g., Refs. [42,43,60]).

The theory of turbulent convection [58] based on the τ approximation explains the recently discovered hysteresis phenomenon in laboratory turbulent convection [61]. The results obtained using the τ approximation allow also to explain the most pronounced features of typical semiorganized coherent structures observed in the atmospheric convective boundary layers ("cloud cells" and "cloud streets") [62]. The theory [58] based on the τ approximation predicts realistic values of the following parameters: the aspect ratios of structures, the ratios of the minimum size of the semiorganized structures to the maximum scale of turbulent motions and the characteristic lifetime of the semiorganized structures. The theory [58] also predicts excitation of convective-shear waves propagating perpendicular to the convective rolls ("cloud streets"). These waves have been observed in the atmospheric convective boundary layers with cloud streets [62]. A theory [63–67] for stably stratified atmospheric turbulent flows based on both, the budget equations for the key second moments, turbulent kinetic and potential energies and vertical turbulent fluxes of momentum and buoyancy, and the τ approximation is in a good agrement with data from atmospheric and laboratory experiments, direct numerical simulations and large-eddy simulations (see detailed comparison in Refs. [63,66]).

The detailed verification of the τ approximation in the direct numerical simulations of turbulent transport of passive scalar has been performed in Ref. [44]. In particular, the results on turbulent transport of passive scalar obtained using direct numerical simulations of homogeneous isotropic turbulence have been compared with that obtained using a closure model based on the τ approximation. The numerical and analytical results are in a good agreement.

In magnetohydrodynamics, the τ approximation reproduces many well-known phenomena found by different methods, e.g., the τ approximation yields correct formulas for the α-effect [68–71], the turbulent diamagnetic and paramagnetic velocities [71–73], the turbulent magnetic diffusion [68,71,73,74], the $\mathbf{\Omega}\times\mathbf{J}$ effect and the κ-effect [68,71].

The developed theory in the present study may be important for interpretation of origin of large spots in the great planets (e.g., the Great Red Spot in Jupiter [34] and large spots in Saturn [35]). The giant planets Jupiter and Saturn have outer convection zones of rapidly rotating convection [36]. The spots on giant planets are not of magnetic origin and may be related to the large-scale instability excited in the convective turbulence. The developed theory may be also useful for explanation of an origin of high-latitude spots in rapidly rotating late-type stars [27,28].

We have also discuss a role of the centrifugal force in production of large-scale vorticity by a fast-rotating homogeneous anisotropic turbulence in a special case when the gravity force is small (see Appendix C). In this case the centrifugal force should be taken into account, which causes an inhomogeneous fluid density distribution in the plane perpendicular to the angular velocity. As a result, the large-scale vertical vorticity is produced by a combined effect of a fast rotation and horizontal inhomogeneity of the fluid density.

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APPENDIX A: DERIVATION OF EQUATION FOR THE ROTATIONAL CONTRIBUTIONS TO THE REYNOLDS STRESS

In this Appendix we derive equation for the rotational contributions to the Reynolds stress. We follow the approach developed in Refs. [25,37]. Fluctuations of velocity $\boldsymbol{u}'$ and entropy s' are given by Eqs. (3) and (4). We rewrite these equations in the $\boldsymbol{k}$ space using new variables for fluctuations of velocity $\boldsymbol{v} = \sqrt{\rho_0}\,\boldsymbol{u}'$ and entropy $s = \sqrt{\rho_0}\, s'$, and derive equations for the following correlation functions: $f_{ij}(\boldsymbol{k},\boldsymbol{K}) = \langle v_i(t,\boldsymbol{k}_1)v_j(t,\boldsymbol{k}_2)\rangle$, $F_i(\boldsymbol{k},\boldsymbol{K}) = \langle s(t,\boldsymbol{k}_1)v_i(t,\boldsymbol{k}_2)\rangle$ and $\Theta_i(\boldsymbol{k},\boldsymbol{K}) = \langle s(t,\boldsymbol{k}_1)s(t,\boldsymbol{k}_2)\rangle$. Here we apply multiscale approach [38], where $\boldsymbol{k}_1 = \boldsymbol{k} + \boldsymbol{K}/2$, $\boldsymbol{k}_2 = -\boldsymbol{k} + \boldsymbol{K}/2$, the wave vector $\boldsymbol{K}$ and the vector $\boldsymbol{R} = (\boldsymbol{x} + \boldsymbol{y})/2$ correspond to the large scales, while $\boldsymbol{k}$ and $\boldsymbol{r} = \boldsymbol{y} - \boldsymbol{x}$ correspond to the small ones. Hereafter, we omitted argument t in the correlation functions. The equations for these correlation functions are given by

$$\frac{\partial f_{ij}(\boldsymbol{k},\boldsymbol{K})}{\partial t} = (I^U_{ijmn} + L^\Omega_{ijmn}) f_{mn} + M^F_{ij} + \hat{\mathcal{N}}\tilde{f}_{ij}, \tag{A1}$$

$$\frac{\partial F_i(\boldsymbol{k},\boldsymbol{K})}{\partial t} = (J^U_{im} + D^\Omega_{im}) F_m + g e_m P_{im}(\boldsymbol{k}_1)\Theta + \hat{\mathcal{N}}\tilde{F}_i, \tag{A2}$$

$$\frac{\partial \Theta(\boldsymbol{k},\boldsymbol{K})}{\partial t} = -\mathrm{div}\,[\overline{\boldsymbol{U}}\,\Theta] + \hat{\mathcal{N}}\Theta, \tag{A3}$$

where $D^\Omega_{ij}(\boldsymbol{k}) = 2\varepsilon_{ijm}\Omega_n k_{mn}$, $L^\Omega_{ijmn} = D^\Omega_{im}(\boldsymbol{k}_1)\,\delta_{jn} + D^\Omega_{jn}(\boldsymbol{k}_2)\,\delta_{im}$, δ_{ij} is the Kronecker unit tensor, $k_{ij} = k_i k_j/k^2$, ε_{ijk} is the Levi-Civita fully antisymmetric tensor, $\boldsymbol{e}$ is the unit vector directed opposite to the acceleration due to the gravity,

$$I^U_{ijmn} = J^U_{im}(\boldsymbol{k}_1)\,\delta_{jn} + J^U_{jn}(\boldsymbol{k}_2)\,\delta_{im} = \left[2k_{iq}\delta_{mp}\delta_{jn} + 2k_{jq}\delta_{im}\delta_{pn} - \delta_{im}\delta_{jq}\delta_{np} - \delta_{iq}\delta_{jn}\delta_{mp} + \delta_{im}\delta_{jn}k_q\frac{\partial}{\partial k_p}\right]\nabla_p \overline{U}_q - \delta_{im}\delta_{jn}\,[\mathrm{div}\,\overline{\boldsymbol{U}} + \overline{\boldsymbol{U}}\boldsymbol{\cdot}\boldsymbol{\nabla}], \tag{A4}$$

and

$$M^F_{ij} = ge_m[P_{im}(\boldsymbol{k}_1)F_j(\boldsymbol{k},\boldsymbol{K}) + P_{jm}(\boldsymbol{k}_2)F_i(-\boldsymbol{k},\boldsymbol{K})], \tag{A5}$$

$$J^U_{ij}(\boldsymbol{k}) = 2k_{in}\nabla_j U_n - \nabla_j U_i - \delta_{ij}[(1/2)\,\mathrm{div}\,\boldsymbol{U} + i(\boldsymbol{U}\boldsymbol{\cdot}\boldsymbol{k})], \tag{A6}$$

$P_{ij}(\boldsymbol{k}) = \delta_{ij} - k_{ij}$ and $F_i(-\boldsymbol{k},\boldsymbol{K}) = \langle s(\boldsymbol{k}_2)v_i(\boldsymbol{k}_1)\rangle$. Note that the correlation functions f_{ij}, F_i and Θ are proportional to the fluid density $\rho_0(\boldsymbol{R})$. Here the third-order moments appearing due to the nonlinear terms, $\hat{\mathcal{N}}\tilde{f}_{ij}$, $\hat{\mathcal{N}}\tilde{F}_i$, and $\hat{\mathcal{N}}\Theta$, are given by

$$\hat{\mathcal{N}}\tilde{f}_{ij} = \big\langle P_{im}(\boldsymbol{k}_1)v^N_m(\boldsymbol{k}_1)v_j(\boldsymbol{k}_2)\big\rangle + \big\langle v_i(\boldsymbol{k}_1)P_{jm}(\boldsymbol{k}_2)v^N_m(\boldsymbol{k}_2)\big\rangle, \tag{A7}$$

$$\hat{\mathcal{N}}\tilde{F}_i = \big\langle s^N(\boldsymbol{k}_1)u_j(\boldsymbol{k}_2)\big\rangle + \big\langle s(\boldsymbol{k}_1)P_{im}(\boldsymbol{k}_2)v^N_m(\boldsymbol{k}_2)\big\rangle, \tag{A8}$$

$$\hat{\mathcal{N}}\Theta = \langle s^N(\boldsymbol{k}_1)s(\boldsymbol{k}_2)\rangle + \langle s(\boldsymbol{k}_1)s^N(\boldsymbol{k}_2)\rangle, \tag{A9}$$

where $\boldsymbol{v}^N(\boldsymbol{k})$ and $s^N(\boldsymbol{k})$ are the nonlinear terms related to $\boldsymbol{U}^N$ and S^N and rewritten in new variables.

In tensors D^Ω_{ij} and L^Ω_{ijmn} we extract the parts which depend on the density stratification effects, characterised by the vector $\boldsymbol{\lambda} = -(\boldsymbol{\nabla}\rho_0)/\rho_0$, i.e.,

$$D^\Omega_{ij} = \tilde{D}_{ij} + D^\lambda_{ij} + D^{\lambda^2}_{ij} + O(\lambda^3), \tag{A10}$$

$$L^\Omega_{ijmn} = \tilde{L}_{ijmn} + L^\lambda_{ijmn} + L^{\lambda^2}_{ijmn} + O(\lambda^3), \tag{A11}$$

where $\tilde{D}_{ij} = 2\varepsilon_{ijp}\Omega_q k_{pq}$, $D^\lambda_{ij} = 2\varepsilon_{ijp}\Omega_q k^\lambda_{pq}$, $D^{\lambda^2}_{ij} = 2\varepsilon_{ijp}\Omega_q k^{\lambda^2}_{pq}$,

$$\tilde{L}_{ijmn} = 2\,\Omega_q\,(\varepsilon_{imp}\,\delta_{jn} + \varepsilon_{jnp}\,\delta_{im})\,k_{pq}, \tag{A12}$$

$$L^\lambda_{ijmn} = -2\,\Omega_q\left[(\varepsilon_{imp}\,\delta_{jn} - \varepsilon_{jnp}\,\delta_{im})\,k^\lambda_{pq} + \frac{i}{k^2}(\varepsilon_{ilq}\,\delta_{jn}\,\lambda_m - \varepsilon_{jlq}\,\delta_{im}\,\lambda_n)\,k_l\right], \tag{A13}$$

$$L^{\lambda^2}_{ijmn} = 2\,\Omega_q\,(\varepsilon_{imp}\,\delta_{jn} + \varepsilon_{jnp}\,\delta_{im})\,k^{\lambda^2}_{pq}, \tag{A14}$$

$$k^\lambda_{ij} = \frac{i}{2k^2}\,[k_i\lambda_j + k_j\lambda_i - 2k_{ij}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{\lambda})], \tag{A15}$$

$$k^{\lambda^2}_{ij} = \frac{1}{4k^2}\,[\lambda_i\lambda_j - k_{ij}\lambda^2 + 4k_{ijpq}\lambda_p\lambda_q]. \tag{A16}$$

Next, we apply the spectral τ approximation [see Eq. (5)], i.e.,

$$\hat{\mathcal{N}}f_{ij}(\boldsymbol{k}) - \hat{\mathcal{N}}f^{(0)}_{ij}(\boldsymbol{k}) = -\frac{f_{ij}(\boldsymbol{k}) - f^{(0)}_{ij}(\boldsymbol{k})}{\tau(k)}, \tag{A17}$$

$$\hat{\mathcal{N}}F_i(\boldsymbol{k}) - \hat{\mathcal{N}}F^{(0)}_i(\boldsymbol{k}) = -\frac{F_i(\boldsymbol{k}) - F^{(0)}_i(\boldsymbol{k})}{\tau(k)}, \tag{A18}$$

$$\hat{\mathcal{N}}\Theta(\boldsymbol{k}) - \hat{\mathcal{N}}\Theta^{(0)}(\boldsymbol{k}) = -\frac{\Theta(\boldsymbol{k}) - \Theta^{(0)}(\boldsymbol{k})}{\tau(k)}, \tag{A19}$$

where $\hat{\mathcal{N}}f_{ij} = \hat{\mathcal{N}}\tilde{f}_{ij} + M^F_{ij}(F^{\Omega=0})$ and $\hat{\mathcal{N}}F_i = \hat{\mathcal{N}}\tilde{F}_i + ge_nP_{in}(k)\Theta^{\Omega=0}$. The quantities $F^{\Omega=0}$ and $\Theta^{\Omega=0}$ are for a nonrotating turbulent convection with nonzero spatial derivatives of the mean velocity. The superscript (0) corresponds to the rotating background turbulent convection with $\nabla_i\overline{U}_j = 0$.

Equations (A1)–(A3) in a steady state read

$$f_{ij}(\boldsymbol{k}) = L^{-1}_{ijmn}\big[f^{(0)}_{mn} + \tau\,\tilde{M}^F_{mn} + \tau\,\big(I^U_{mnpq} + L^\lambda_{mnpq} + L^{\lambda^2}_{mnpq}\big)\,f_{pq}\big], \tag{A20}$$

$$F_i(\boldsymbol{k}) = D^{-1}_{im}\big[F^{(0)}_m(\boldsymbol{k}) + \tau\,\big(J^U_{mn} + D^\lambda_{mn} + D^{\lambda^2}_{mn}\big)F_n\big], \tag{A21}$$

where

$$\tilde{M}^F_{ij} = ge_m\big\{\big[P_{im}(\boldsymbol{k}) + k^\lambda_{im} + k^{\lambda^2}_{im}\big]\tilde{F}_j(\boldsymbol{k}) + \big[P_{jm}(\boldsymbol{k}) - k^\lambda_{jm} + k^{\lambda^2}_{jm}\big]\tilde{F}_i(-\boldsymbol{k})\big\}, \tag{A22}$$

$\tilde{F}_i = F_i - F^{\Omega=0}_i$ and we neglected small terms $\sim O(\lambda^3)$, see [37]. In Eqs. (A20) and (A21), the operator D^{-1}_{ij} is the inverse of $\delta_{ij} - \tau\tilde{D}_{ij}$ and the operator $L^{-1}_{ijmn}(\boldsymbol{\Omega})$ is the inverse of $\delta_{im}\delta_{jn} - \tau\,\tilde{L}_{ijmn}$, where

$$D^{-1}_{ij} = \chi(\psi)\,(\delta_{ij} + \psi\,\varepsilon_{ijm}\,\hat{k}_m + \psi^2\,k_{ij}), \tag{A23}$$

and

$$L^{-1}_{ijmn}(\boldsymbol{\Omega}) = \tfrac{1}{2}[B_1\,\delta_{im}\delta_{jn} + B_2\,k_{ijmn} + B_3\,(\varepsilon_{imp}\delta_{jn} + \varepsilon_{jnp}\delta_{im})\hat{k}_p + B_4\,(\delta_{im}k_{jn} + \delta_{jn}k_{im}) + B_5\,\varepsilon_{ipm}\varepsilon_{jqn}k_{pq} + B_6\,(\varepsilon_{imp}k_{jpn} + \varepsilon_{jnp}k_{ipm})], \tag{A24}$$

$\hat{k}_i = k_i/k$, $\chi(\psi) = 1/(1+\psi^2)$, $\psi = 2\tau(k)\,(\boldsymbol{k}\cdot\boldsymbol{\Omega})/k$, $B_1 = 1 + \chi(2\psi)$, $B_2 = B_1 + 2 - 4\chi(\psi)$, $B_3 = 2\psi\,\chi(2\psi)$, $B_4 = 2\chi(\psi) - B_1$, $B_5 = 2 - B_1$, and $B_6 = 2\psi\,[\chi(\psi) - \chi(2\psi)]$; see Ref. [75].

We use the following model of the background homogeneous stratified turbulence or turbulent convection which takes into account an increase of the anisotropy of turbulence with increase of the rate of rotation:

$$f^{(0)}_{ij} \equiv \langle v_i(\boldsymbol{k}_1)\,v_j(\boldsymbol{k}_2)\rangle = \frac{E(k)\,[1 + 2k\,\varepsilon_u\,\delta(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{\Omega}})]}{8\pi\,k^2\,(k^2+\tilde{\lambda}^2)\,(1+\varepsilon_u)}\,[\delta_{ij}\,(k^2+\tilde{\lambda}^2) - k_i\,k_j - \tilde{\lambda}_i\,\tilde{\lambda}_j + i\,(\tilde{\lambda}_i\,k_j - \tilde{\lambda}_j\,k_i)]\langle\boldsymbol{v}^2\rangle, \tag{A25}$$

$$F_i^{(0)} \equiv \langle v_i(\boldsymbol{k}_1)\, s(\boldsymbol{k}_2)\rangle = \frac{3\, E(k)\, [1 + k\, \varepsilon_{_F}\, \delta(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{\Omega}})]}{8\pi\, k^2\, (k^2 + \tilde{\lambda}^2)} \Big[k^2\, e_j\, P_{ij}(\boldsymbol{k}) + i\tilde{\lambda}\, k_j\, P_{ij}(\boldsymbol{e})\Big] F_* \tag{A26}$$

(see Ref. [37]), and $\Theta^{(0)} \equiv \langle s(\boldsymbol{k}_1)\, s(\boldsymbol{k}_2)\rangle = \Theta_*\, E(k)/4\pi k^2$, where $F_* = \rho_0\, \langle u'_z\, s'\rangle$, $\Theta_* = \rho_0\, \langle (s')^2\rangle$, δ_{ij} is the Kronecker tensor, $P_{ij}(\boldsymbol{e}) = \delta_{ij} - e_i e_j$, $\delta(x)$ is the Dirac delta function, $\hat{\boldsymbol{k}} = \boldsymbol{k}/k$, and $\hat{\boldsymbol{\Omega}} = \boldsymbol{\Omega}/\Omega$. Here we have taken into account that in the anelastic approximation the velocity fluctuations $\boldsymbol{v} = \sqrt{\rho_0}\, \boldsymbol{u}'$ satisfy the equation $\nabla \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \tilde{\boldsymbol{\lambda}}$, where $\tilde{\lambda} \equiv \boldsymbol{\lambda}/2 = -(\nabla \rho_0)/2\rho_0$. To derive Eqs. (A25) and (A26) we use the following conditions: (i) the anelastic approximation in the Fourier space implies that $(ik_i^{(1)} - \tilde{\lambda}_i) f_{ij}^{(0)}(\boldsymbol{k}, \boldsymbol{K}) = 0$, $(ik_j^{(2)} - \tilde{\lambda}_j) f_{ij}^{(0)}(\boldsymbol{k}, \boldsymbol{K}) = 0$ and $(ik_i^{(1)} - \tilde{\lambda}_i) F_i^{(0)}(\boldsymbol{k}, \boldsymbol{K}) = 0$, where $\boldsymbol{k}_1 \equiv \boldsymbol{k}^{(1)} = \boldsymbol{k} + \boldsymbol{K}/2$ and $\boldsymbol{k}_2 \equiv \boldsymbol{k}^{(2)} = -\boldsymbol{k} + \boldsymbol{K}/2$; (ii) $\int f_{ii}^{(0)}(\boldsymbol{k}, \boldsymbol{K}) \exp{[i\boldsymbol{K} \cdot \boldsymbol{R}]}\, d\boldsymbol{k}\, d\boldsymbol{K} = \rho_0\, \langle \boldsymbol{u}^2\rangle^{(0)}$; (iii) $f_{ij}^{(0)}(\boldsymbol{k}, \boldsymbol{K}) = f_{ji}^{*(0)}(\boldsymbol{k}, \boldsymbol{K}) = f_{ji}^{(0)}(-\boldsymbol{k}, \boldsymbol{K})$.

Solution of Eq. (A21) for fast rotation by iterations in small parameter $\ell_0\lambda$ reads

$$\hat{F}^{(1,U)} = \tau^2 \big(\hat{J}^U \hat{D}^\lambda + \hat{D}^\lambda \hat{J}^U\big) \hat{F}^{(0)}, \tag{A27}$$

and

$$\hat{F}^{(2,U)} = \tau^2 (\hat{J}^U \hat{D}^\lambda + \hat{D}^\lambda \hat{J}^U) \hat{F}^{(0,\lambda)} + \tau^2 \big(\hat{J}^U \hat{D}^{\lambda^2} + \hat{D}^{\lambda^2} \hat{J}^U\big) \hat{F}^{(0)} + \tau^3 (\hat{J}^U \hat{D}^\lambda \hat{D}^\lambda + \hat{D}^\lambda \hat{J}^U \hat{D}^\lambda + \hat{D}^\lambda \hat{D}^\lambda \hat{J}^U) \hat{F}^{(0)}. \tag{A28}$$

Here the contribution $\hat{F}^{(1,U)}$ is linear in the ratio ℓ_0/H_ρ (i.e., it is linear in the parameter $\ell_0\lambda$), while the contributions $\hat{F}^{(2,U)}$ is quadratic in ℓ_0/H_ρ, where $H_\rho = \lambda^{-1}$, $\hat{J}^U \equiv J_{ij}^U(\boldsymbol{k})$, $\hat{D}^\lambda \equiv D_{ij}^\lambda$, $\hat{D}^{\lambda^2} \equiv D_{ij}^{\lambda^2}$, the vector $\hat{F}^{(0)}$ is the part of $F_i^{(0)}$ that is a zero order in λ [i.e., it is proportional to $k^2\, e_j\, P_{ij}(\boldsymbol{k})$], while the operator $\hat{F}^{(0,\lambda)}$ is the part of $F_i^{(0)}$ that is linear in λ [i.e., it is proportional to $i\tilde{\lambda}\, k_j\, P_{ij}(\boldsymbol{e})$]. Solution of Eq. (A20) for fast rotation by iterations in small parameter $\ell_0\lambda$ up to the second-order in this parameter is given by

$$\hat{f}^{(1,F)} = \tau g (\hat{\boldsymbol{e}} \hat{P} \hat{F}^{(1,U)} + \hat{I}^U \tau^2 \hat{\boldsymbol{e}} \hat{P} \hat{D}^\lambda \hat{F}^{(0)}), \tag{A29}$$

$$\hat{f}^{(1,u)} = \tau (\hat{I}^U \tau \hat{L}^\lambda + \hat{L}^\lambda \tau \hat{I}^U) \hat{f}^{(0)}, \tag{A30}$$

and

$$\begin{aligned} \hat{f}^{(2,F)} &= \tau g \hat{\boldsymbol{e}} (\hat{P} \hat{F}^{(2,U)} + \hat{k}^\lambda \hat{F}^{(1,U)}) + \tau \hat{L}^\lambda (\hat{f}^{(1,F)} + \tau \hat{I}^U \tau \hat{D}^\lambda \hat{F}^{(0)}) + \tau g \hat{I}^U \tau^2 \hat{\boldsymbol{e}} \hat{P} \big[\big(\hat{D}^{\lambda^2} + \hat{k}^\lambda \hat{D}^\lambda\big) \hat{F}^{(0)} + \hat{D}^\lambda \hat{F}^{(0,\lambda)}\big] \\ &= \tau g \hat{I}^U \tau^2 \hat{\boldsymbol{e}} \hat{P} \big[\big(\hat{k}^\lambda \hat{D}^\lambda + \hat{D}^{\lambda^2}\big) \hat{F}^{(0)} + \hat{D}^\lambda \hat{F}^{(0,\lambda)}\big] + \tau^3 g \big[\hat{L}^\lambda \hat{\boldsymbol{e}} (\hat{k}^\lambda \hat{J}^U \hat{F}^{(0)} + \hat{P} \hat{J}^U \hat{F}^{(0,)}) + \hat{L}^{\lambda^2} \hat{\boldsymbol{e}} \hat{P} \hat{J}^U \hat{F}^{(0)}\big] \\ &\quad + \tau^3 g \hat{\boldsymbol{e}} \big\{\hat{P} \big[(\hat{J}^U \hat{D}^\lambda + \hat{D}^\lambda \hat{J}^U) \hat{F}^{(0,\lambda)} + \big(\hat{J}^U \hat{D}^{\lambda^2} + \hat{D}^{\lambda^2} \hat{J}^U\big) \hat{F}^{(0)}\big] + \hat{k}^\lambda (\hat{J}^U \hat{D}^\lambda + \hat{D}^\lambda \hat{J}^U) \hat{F}^{(0)}\big\}, \end{aligned} \tag{A31}$$

$$\hat{f}^{(2,u)} = \tau (\hat{I}^U \tau \hat{L}^\lambda + \hat{L}^\lambda \tau \hat{I}^U) \hat{f}^{(0,\lambda)} + \tau \big(\hat{I}^U \tau \hat{L}^{\lambda^2} + \hat{L}^{\lambda^2} \tau \hat{I}^U\big) \hat{f}^{(0)} + \tau^2 \hat{L}^\lambda (\hat{I}^U \tau \hat{L}^\lambda + \hat{L}^\lambda \tau \hat{I}^U) \hat{f}^{(0)}. \tag{A32}$$

Here the contributions $\hat{f}^{(1,F)}$ and $\hat{f}^{(1,u)}$ are linear in the ratio ℓ_0/H_ρ, while the contributions $\hat{f}^{(2,F)}$ and $\hat{f}^{(2,u)}$ are quadratic in ℓ_0/H_ρ, and $\hat{\boldsymbol{e}} \equiv e_i$, $\hat{I}^U \equiv I_{ijmn}^U$, $\hat{P} \equiv P_{ij}(\boldsymbol{k})$, $\hat{k}^\lambda \equiv k_{ij}^\lambda$, $\hat{L}^\lambda \equiv L_{ijmn}^\lambda$, $\hat{L}^{\lambda^2} \equiv L_{ijmn}^{\lambda^2}$, and the tensor $\hat{f}^{(0)}$ is the part of $f_{ij}^{(0)}$ that is a zero order in λ [i.e., it is proportional to $k^2\, P_{ij}(\boldsymbol{k})$], while the tensor $\hat{f}^{(0,\lambda)}$ is the part of $f_{ij}^{(0)}$ that is linear in λ [i.e., it is proportional to $i\, (\tilde{\lambda}_i\, k_j - \tilde{\lambda}_j\, k_i)$].

After integration in $\boldsymbol{k}$ space in Eqs. (A29)–(A32) we obtain the rotational contributions to the Reynolds stresses, $f_{ij} = f_{ij}^{(F,\Omega)} + f_{ij}^{(u,\Omega)}$, for the fast-rotating stratified anisotropic homogeneous turbulence or density-stratified turbulent convection for the fast rotation, where $f_{ij}^{(F,\Omega)}$ and $f_{ij}^{(u,\Omega)}$:

$$\begin{aligned} f_{ij}^{(F,\Omega)} &= -A_F\, \rho_0\, \nu_{_T}\, \Omega\tau_0\, \frac{\ell_0^2}{H_\rho^2}\, \{e_i e_j \overline{W}_z + 2(\overline{W}_i e_j + \overline{W}_j e_i) + 6[(\boldsymbol{e}\times\nabla)_i e_j + (\boldsymbol{e}\times\nabla)_j e_i]\, \overline{U}_z + (\boldsymbol{e}\times\nabla)_i \overline{U}_j^\perp \\ &\quad + (\boldsymbol{e}\times\nabla)_j \overline{U}_i^\perp + 2[\nabla_i^\perp (\boldsymbol{e}\times\overline{\boldsymbol{U}})_j + \nabla_j^\perp (\boldsymbol{e}\times\overline{\boldsymbol{U}})_i] - 4\nabla_z [(\boldsymbol{e}\times\overline{\boldsymbol{U}})_i e_j + (\boldsymbol{e}\times\overline{\boldsymbol{U}})_j e_i]\}, \end{aligned} \tag{A33}$$

$$\begin{aligned} f_{ij}^{(u,\Omega)} &= -\frac{A_u}{2}\, \rho_0\, \nu_{_T}\, \Omega\tau_0\, \frac{\ell_0^2}{H_\rho^2}\, \{4(\overline{W}_i e_j + \overline{W}_j e_i) + 4[(\boldsymbol{e}\times\nabla)_i e_j + (\boldsymbol{e}\times\nabla)_j e_i]\, \overline{U}_z \\ &\quad + 3(q+1)[(\boldsymbol{e}\times\nabla)_i \overline{U}_j^\perp + (\boldsymbol{e}\times\nabla)_j \overline{U}_i^\perp] + (3q+7)[\nabla_i^\perp (\boldsymbol{e}\times\overline{\boldsymbol{U}})_j + \nabla_j^\perp (\boldsymbol{e}\times\overline{\boldsymbol{U}})_i]\}. \end{aligned} \tag{A34}$$

Note that the contributions $\hat{f}^{(1,F)}$ and $\hat{f}^{(1,u)}$ (which are linear in ℓ_0/H_ρ) to $f_{ij}^{(F,\Omega)}$ and $f_{ij}^{(u,\Omega)}$ vanish. This implies that only the quadratic contributions, $\hat{f}^{(2,F)}$ and $\hat{f}^{(2,u)}$, in ℓ_0/H_ρ are the leading-order contributions to $f_{ij}^{(F,\Omega)}$ and $f_{ij}^{(u,\Omega)}$.

To integrate over the angles in $\boldsymbol{k}$-space, we use the following integrals:

$$\int k_{ij}^{\perp}\, d\varphi = \pi \delta_{ij}^{(2)}, \quad \int k_{ijmn}^{\perp}\, d\varphi = \frac{\pi}{4} \Delta_{ijmn}^{(2)}, \tag{A35}$$

$$\int k_{ijmnpq}^{\perp}\, d\varphi = \frac{\pi}{24} \Delta_{ijmnpq}^{(2)}, \tag{A36}$$

where $\delta_{ij}^{(2)} \equiv P_{ij}(\Omega) = \delta_{ij} - \Omega_i \Omega_j/\Omega^2$, $\Delta_{ijmn}^{(2)} = \delta_{ij}^{(2)}\delta_{mn}^{(2)} + \delta_{im}^{(2)}\delta_{jn}^{(2)} + \delta_{in}^{(2)}\delta_{jm}^{(2)}$, and

$$\Delta_{ijmnpq}^{(2)} = \Delta_{mnpq}^{(2)}\delta_{ij}^{(2)} + \Delta_{jmnp}^{(2)}\delta_{iq}^{(2)} + \Delta_{imnp}^{(2)}\delta_{jq}^{(2)} + \Delta_{jmnq}^{(2)}\delta_{ip}^{(2)} + \Delta_{imnq}^{(2)}\delta_{jp}^{(2)} + \Delta_{ijmn}^{(2)}\delta_{pq}^{(2)} - \Delta_{ijpq}^{(2)}\delta_{mn}^{(2)}. \tag{A37}$$

Here $\boldsymbol{k}^{\perp} = \boldsymbol{k} - \boldsymbol{k} \cdot \hat{\boldsymbol{\Omega}}$ is the wave vector in the plane perpendicular to the angular velocity $\boldsymbol{\Omega}$ with the polar angle φ in this plane and the corresponding unit vector $\hat{\boldsymbol{k}}^{\perp} = \boldsymbol{k}^{\perp}/k^{\perp}$. Thus, the following symmetric tensors are defined as $k_{ij}^{\perp} = \hat{k}_i^{\perp}\hat{k}_j^{\perp}$, $k_{ijmn}^{\perp} = k_{ij}^{\perp}\, k_{mn}^{\perp}$, and $k_{ijmnpq}^{\perp} = k_{ij}^{\perp}\, k_{mn}^{\perp}\, k_{pq}^{\perp}$.

APPENDIX B: EQUATIONS DESCRIBING THE LARGE-SCALE INSTABILITY

To solve system of Eqs. (13) and (14), we introduce new variables $\overline{V}(t,x,z)$ and $\overline{\Phi}(t,x,z)$:

$$\rho_0 \overline{\boldsymbol{U}} = [\overline{V}(t,x,z)\rho_0^{1/2}]\boldsymbol{e}_y + \nabla \times [\overline{\Phi}(t,x,z)\rho_0^{1/2}]\boldsymbol{e}_y, \tag{B1}$$

which corresponds to axisymmetric problem. In the new variables Eqs. (13) and (14) read

$$\left[\frac{\partial}{\partial t} - \nu_{_T}\left(\Delta - \frac{1}{4H_\rho^2}\right)\right]\overline{V} = 2\Omega\left\{\nabla_z - \frac{1}{2H_\rho} - \nu_{_T}\, \tau_0\, \frac{\ell_0^2}{H_\rho^3}\left[2A_F\nabla_x^2 + (A_F - A_u)\left(\nabla_z^2 - \frac{1}{4H_\rho^2}\right)\right]\right\}\overline{\Phi}, \tag{B2}$$

$$\left(\Delta - \frac{1}{4H_\rho^2}\right)\left[\frac{\partial}{\partial t} - \nu_{_T}\left(\Delta - \frac{1}{4H_\rho^2}\right)\right]\overline{\Phi} = -\Omega\left\{2\left(\nabla_z + \frac{1}{2H_\rho}\right) + \nu_{_T}\, \tau_0\, \frac{\ell_0^2}{H_\rho^3}\left[(5A_F + 4A_u)\nabla_x^2 - 2(A_F - A_u)\left(\nabla_z + \frac{1}{2H_\rho}\right)^2\right]\right\}\overline{V}. \tag{B3}$$

These equations allow us to study the large-scale instability which results in generation of the mean vorticity for different modes (see Sec. III).

APPENDIX C: THE ROLE OF THE CENTRIFUGAL FORCE IN PRODUCTION OF LARGE-SCALE VORTICITY FOR VANISHING GRAVITY

In this section we study production of large-scale vorticity by fast-rotating homogeneous anisotropic turbulence for vanishing gravity. An ensemble averaging of the momentum equation yields the equation for the mean velocity field, $\overline{\boldsymbol{U}}(t,\boldsymbol{x})$, in the reference frame rotating with the constant angular velocity $\boldsymbol{\Omega}$:

$$\frac{\partial \overline{U}_i}{\partial t} + (\overline{\boldsymbol{U}} \cdot \nabla)\overline{U}_i = -\frac{\nabla_i \overline{P}}{\overline{\rho}} + \Omega^2 r_i + 2(\overline{\boldsymbol{U}} \times \boldsymbol{\Omega})_i - \frac{1}{\overline{\rho}}\nabla_j \overline{\rho\, u_i'\, u_j'}, \tag{C1}$$

Here $\overline{P}$ is the mean fluid pressure, $\boldsymbol{u}'$ are fluctuations of fluid velocity, $\overline{\rho}$ is the mean fluid density that satisfies the continuity equation written in the anelastic approximation, $\mathrm{div}\,(\overline{\rho}\,\overline{\boldsymbol{U}}) = 0$, and the vector $\boldsymbol{r}$ is perpendicular to $\boldsymbol{\Omega}$. The basic equilibrium is determined by $\overline{\boldsymbol{U}}_0 = 0$ and $(\nabla \overline{P}_0)/\overline{\rho}_0 = \Omega^2 \boldsymbol{r}$ for fast rotation, where the equilibrium fluid pressure $\overline{P}_0$ and density $\overline{\rho}_0$ are related by the isothermal equation of state $\overline{P}_0 = c_s^2 \overline{\rho}_0$ with a constant sound speed c_s. We use the cylindrical coordinates (r, φ, z), where the angular velocity $\boldsymbol{\Omega}$ is directed along the z axis. The equilibrium profile of the fluid density is given by

$$\overline{\rho}_0(r) = \rho_* \exp\left(\frac{r^2}{L_\Omega^2}\right), \tag{C2}$$

where $L_\Omega = \sqrt{2} c_s/\Omega$. The second term, $\Omega^2 r_i$, in the right hand side of Eq. (C1) for the mean fluid velocity is the centrifugal force, which causes the inhomogeneous density distribution (C2) in the plane perpendicular to the angular velocity $\boldsymbol{\Omega}$. In the previous sections, we consider a fast-rotating turbulent convection, where in the momentum equation we have taken into the Coriolis force, but neglected the centrifugal force. The centrifugal force should be taken into account only when $\Omega \geqslant (g/R)^{1/2}$, where R is the radius (or a typical horizontal scale of the motions).

To obtain the rotational contribution to the Reynolds stress, we use the same approach which has been applied in previous sections, but for isothermal turbulence (i.e., in the absence of the heat flux $\boldsymbol{F}$) and with the inhomogeneous fluid density in radial direction (perpendicular to $\boldsymbol{\Omega}$). The equation for the Reynolds stress in the $\boldsymbol{k}$ space coincides with Eq. (A20) in Appendix A with the vanishing term $\tau\, \tilde{M}_{mn}^F$. Integrating in $\boldsymbol{k}$ space in this equation, we obtain the contribution to the Reynolds stress caused by a fast rotation:

$$f_{ij}^{\Omega} = \left[(\boldsymbol{\Omega}\times\boldsymbol{\lambda}^{(\Omega)})_i \lambda_j^{(\Omega)} + (\boldsymbol{\Omega}\times\boldsymbol{\lambda}^{(\Omega)})_j \lambda_i^{(\Omega)}\right] \frac{\overline{\rho}_0\, u_0\, \ell_0^3\, \varepsilon_u}{5(1+\varepsilon_u)}. \tag{C3}$$

This equation has been derived in Ref. [37] (see the first two terms in the right hand side of Eq. (B13) in Appendix B of

Ref. [37], where now the unit vector $\boldsymbol{e}$ is perpendicular to $\boldsymbol{\Omega}$). Here $\boldsymbol{\lambda}^{(\Omega)} = -(\boldsymbol{\nabla}\overline{\rho}_0)/\overline{\rho}_0 = -\boldsymbol{r}/L_\Omega^2$ [see Eq. (C2)] and the radius-vector $\boldsymbol{r}$ is perpendicular to $\boldsymbol{\Omega}$.

In the cylindrical coordinates (r, φ, z), the φ-component of the mean velocity is determined by the following equation:

$$\overline{\rho}_0 \, \frac{\partial \overline{U}_\varphi}{\partial t} = -\frac{1}{r^2} \frac{\partial}{\partial r}\left[r^2 \left(f_{r\varphi}^{\Omega} + f_{r\varphi}^{\nu}\right)\right] + 2\, \overline{\rho}_0 \, (\overline{\boldsymbol{U}} \times \boldsymbol{\Omega})_\varphi, \quad \text{(C4)}$$

where the contribution to the Reynolds stress caused by uniform rotation is given by

$$f_{r\varphi}^{\Omega} = \overline{\rho}_0 \nu_{_T} \, \frac{3\Omega\, \varepsilon_u}{5(1+\varepsilon_u)} \left(\frac{\ell_0^2}{L_\Omega^4}\right) r^2, \quad \text{(C5)}$$

while the contribution to the Reynolds stress caused by turbulent viscosity is

$$f_{r\varphi}^{\nu} = \overline{\rho}_0 \nu_{_T} \, r \frac{\partial}{\partial r}\left(\frac{\overline{U}_\varphi}{r}\right). \quad \text{(C6)}$$

For simplicity we have considered the case when the radial dependence of the mean velocity is the strongest one, i.e., $\overline{U}_\varphi = \overline{U}_\varphi(t, r)$. This implies that the last term in the right hand side of Eq. (C4) vanishes. We also neglect here a small kinematic viscosity in comparison with the turbulent viscosity.

The steady-state solution of Eq. (C4) reads

$$\overline{U}_\varphi^{(\mathrm{steady})}(r) = \frac{3\Omega\, \varepsilon_u}{10(1+\varepsilon_u)} \left(\frac{\ell_0^2}{L_\Omega^4}\right) r^3, \quad \text{(C7)}$$

which yields the vertical mean vorticity as

$$\overline{W}_z^{(\mathrm{steady})}(r) \equiv \frac{1}{r} \frac{\partial}{\partial r}\left(r\, \overline{U}_\varphi^{(\mathrm{steady})}\right) = \frac{6\Omega\, \varepsilon_u}{5(1+\varepsilon_u)} \left(\frac{\ell_0^2}{L_\Omega^4}\right) r^2. \quad \text{(C8)}$$

Therefore, the balance between the contributions $f_{r\varphi}^{\Omega}$ to the Reynolds stress caused by a fast rotation and that caused by the turbulent viscosity, $f_{r\varphi}^{\nu}$, determines the produced time-independent large-scale vorticity, $\overline{W}_z^{(\mathrm{steady})}(r)$; see Eq. (C8).

In the absence of the contribution $f_{r\varphi}^{\Omega}$ to the Reynolds stress caused by a fast uniform rotation, Eq. (C4) for $\overline{\Omega}_\varphi(t, r) \equiv \overline{U}_\varphi / r$ reads

$$\frac{\partial \overline{\Omega}_\varphi}{\partial t} = \nu_{_T} \left[\frac{\partial^2 \overline{\Omega}_\varphi}{\partial r^2} + \frac{3}{r}\left(1 + \frac{2\, r^2}{3 L_\Omega^2}\right) \frac{\partial \overline{\Omega}_\varphi}{\partial r}\right]. \quad \text{(C9)}$$

This equation has a decaying solution for $\overline{\Omega}_\varphi$ caused by the turbulent viscosity:

$$\overline{\Omega}_\varphi(t, r) = 2C_*\, \Omega\, \exp(-\gamma_{\mathrm{dec}} t)\, \Phi\left(\frac{\gamma_{\mathrm{dec}} L_\Omega^2}{2\nu_{_T}}, 2, -\frac{r^2}{2L_\Omega^2}\right), \quad \text{(C10)}$$

where $\Phi(a, b, z)$ is the degenerate hypergeometric function, γ_{dec} is the damping rate due to the turbulent viscosity and C_* is a free constant. For $r \ll L_\Omega$, this solution reads

$$\overline{\Omega}_\varphi(t, r) = 2C_*\, \Omega\, \exp(-\gamma_{\mathrm{dec}} t)\left(1 - \frac{\gamma_{\mathrm{dec}} r^2}{4\nu_{_T}}\right). \quad \text{(C11)}$$

In Eq. (C11) we have to exclude a uniform rotation, so that the vertical mean vorticity corresponding to the decaying solution is given by

$$\overline{W}_z^{(\mathrm{decay})}(t, r) \equiv \frac{1}{r} \frac{\partial}{\partial r}\left(r^2\, \overline{\Omega}_\varphi\right) = -C_*\, \Omega\, \frac{\gamma_{\mathrm{dec}} r^2}{4\nu_{_T}} \exp(-\gamma_{\mathrm{dec}} t). \quad \text{(C12)}$$

The total mean vertical vorticity, $\overline{W}_z^{(\mathrm{tot})}$, is determined by the sum of homogeneous and inhomogeneous solutions of Eq. (C4), i.e., $\overline{W}_z^{(\mathrm{tot})}$ is given by the sum of the stationary and decaying solutions, $\overline{W}_z^{(\mathrm{tot})} \equiv \overline{W}_z^{(\mathrm{steady})} + \overline{W}_z^{(\mathrm{decay})}$. The free constant C_* is determined by the initial condition: $\overline{W}_z^{(\mathrm{tot})}(t=0) = 0$, so that

$$C_* = \frac{6\varepsilon_u}{5(1+\varepsilon_u)} \left(\frac{\ell_0^2\, \nu_{_T}}{L_\Omega^4 \gamma}\right). \quad \text{(C13)}$$

Therefore, the total mean vertical vorticity for $r \ll L_\Omega$ reads

$$\overline{W}_z^{(\mathrm{tot})} = \frac{6\, \Omega\, \varepsilon_u}{5(1+\varepsilon_u)} \left(\frac{\ell_0^2\, r^2}{L_\Omega^4}\right) [1 - \exp(-\gamma_{\mathrm{dec}} t)]. \quad \text{(C14)}$$

At small times, $\gamma_{\mathrm{dec}} t \ll 1$, we obtain a linear in time growing solution for the total mean vertical vorticity:

$$\overline{W}_z^{(\mathrm{tot})} = \frac{6\, \Omega\, \varepsilon_u}{5(1+\varepsilon_u)} \left(\frac{\ell_0^2\, r^2}{L_\Omega}\right) \gamma_{\mathrm{dec}} t. \quad \text{(C15)}$$

Therefore, a combined effect of a fast rotation and horizontal inhomogeneity of the fluid density (caused by the centrifugal force) results in the production of the large-scale vertical vorticity in an anisotropic turbulence. A balance between the effective force caused by the rotational contributions to the Reynolds stress and that due to the turbulent viscosity determines the vertical component of the large-scale vorticity given by Eq. (C8).

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