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ABSTRACT

The energy- and flux-budget (EFB) theory developed previously for atmospheric stably stratified turbulence is extended to the surface layer in atmospheric convective turbulence. This theory is based on budget equations for turbulent energies and fluxes in the Boussinesq approximation. In the lower part of the surface layer in the atmospheric convective boundary layer, the rate of turbulence production of the turbulent kinetic energy (TKE) caused by the surface shear is much larger than that caused by the buoyancy, which results in three-dimensional turbulence of very complex nature. In the upper part of the surface layer, the rate of turbulence production of TKE due to the shear is much smaller than that caused by the buoyancy, which causes unusual strongly anisotropic buoyancy-driven turbulence. Considering the applications of the obtained results to the atmospheric convective boundary-layer turbulence, the theoretical relationships potentially useful in modeling applications have been derived. The developed EFB theory allows us to obtain a smooth transition between a stably stratified turbulence to a convective turbulence. The EFB theory for the surface layer in a convective turbulence provides an analytical expression for the entire surface layer including the transition range between the lower and upper parts of the surface layer, and it allows us to determine the vertical profiles for all turbulent characteristics, including TKE, the intensity of turbulent potential temperature fluctuations, the vertical turbulent fluxes of momentum and buoyancy (proportional to potential temperature), the integral turbulence scale, the turbulence anisotropy, the turbulent Prandtl number, and the flux Richardson number.

I. INTRODUCTION

Despite turbulent transport having been studied theoretically, in laboratory and field experiments and numerical simulations during a century,[1–8] some crucial questions remain. This is particularly true in applications such as geophysics and astrophysics, where the governing parameter values are too large to be modeled either experimentally or numerically. Classical Kolmogorov's theory has been formulated for a neutrally stratified homogeneous and isotropic turbulence.[9–12] This turbulence is different from convective and stably stratified turbulence.

Various aspects of the atmospheric turbulent convection have been studied theoretically,[13–30] numerically,[25,29,31,32] and in the field experiments,[21,25,33–36] see also books and reviews,[1,2,19,37–42] and references therein. The atmospheric turbulent convective boundary layer (CBL) consists of three basic parts:

- Surface layer strongly unstably stratified and dominated by small-scale turbulence of very complex nature including usual 3D turbulence, generated by mean-flow surface shear and structural shears (the lower part of the surface layer), and unusual strongly anisotropic buoyancy-driven turbulence (the upper part of the surface layer);
- CBL core dominated by the structural energy, momentum, and mass transport, with only minor contribution from usual 3D turbulence generated by local structural shears on the background of almost zero vertical gradient of potential temperature (or buoyancy);
- turbulent entrainment layer at the CBL upper boundary, characterized by essentially stable stratification with negative (downward) turbulent flux of potential temperature (or buoyancy).

The classical theory of surface layer in convective turbulence is based on seminal papers by Prandtl,[13] Obukhov,[14,15] and Zilitinkevich.[17–19] In particular, according to this theory based on the dimensional analysis, anisotropic buoyancy-driven turbulence in the upper part of the surface layer is determined by the following relations for the vertical profiles of velocity fluctuations, $\sqrt{\langle \boldsymbol{u}^2\rangle}/u_* \propto (z/|L|)^{1/3}$, potential temperature fluctuations, $\sqrt{\langle \theta^2\rangle}/\theta_* \propto (z/|L|)^{-1/3}$, ratio of the horizontal to vertical components of the turbulent flux of potential temperature, $F_x/F_z \propto (z/|L|)^{-2/3}$, vertical gradients of the mean potential temperature, $\nabla_z \bar{\Theta} \propto (\theta_*/|L|)(z/|L|)^{-4/3}$, and mean wind velocity $\nabla_z \bar{U} \propto (u_*/|L|)(z/|L|)^{-4/3}$. Here, u_* is the friction velocity, $\theta_* = F_z/u_*$ is the characteristic level of potential temperature fluctuations, $|L| = u_*^3/(\beta F_z)$ is the Obukhov length scale, and β is the buoyancy parameter (see Sec. II for definitions).

The goal of this paper is to develop the energy- and flux-budget (EFB) turbulence closure theory for the surface layer in convective turbulence using budget equations for turbulent energies and fluxes. The EFB theory has been previously developed for stably stratified dry atmospheric flows[43–48] and for passive scalar transport in stratified turbulence.[49] The EFB theory is based on the budget equations for the densities of turbulent kinetic and potential energies, and turbulent fluxes of momentum and heat.

In agreement with wide experimental evidence,[50–58] the EFB theory for the stably stratified turbulence[43–48] demonstrates that strong turbulence is maintained by large-scale shear for any stratification, and the "critical Richardson number," treated many years as a threshold between the turbulent and laminar regimes, actually separates two turbulent regimes: the strong turbulence typical of atmospheric boundary layers and the weak three-dimensional turbulence typical of the free atmosphere, and characterized by strong decrease in heat transfer in comparison with momentum transfer.

The physical mechanism of self-existence of a stably stratified turbulence is as follows.[47,49] The increase in the vertical gradient of the mean potential temperature (i.e., the increase in the buoyancy) causes a conversion of turbulent kinetic energy into turbulent potential energy. On the other hand, the negative down-gradient vertical turbulent heat flux is decreased by the counteracting positive non-gradient heat flux that is increased with the increase in the turbulent potential energy. The latter is the mechanism of the self-control feedback resulting in a decrease in the buoyancy. Due to this feedback, the stably stratified turbulence is maintained up to strongly supercritical stratifications. The EFB theory has been verified against scarce data from the atmospheric experiments, direct numerical simulations (DNS), large-eddy simulations (LES), and laboratory experiments relevant to the steady-state turbulence regime.

The EFB theory is a sort of the turbulence closure. Previously, various closure models have been adopted in turbulence and turbulent transport.[1,2,6–8,59–62] Some of the turbulent closure models for stably stratified atmospheric turbulence also do not imply a critical Richardson number,[58,63–72] see also Ref. 73.

In the present paper, we have extended the energy- and flux-budget (EFB) theory developed previously for an atmospheric stably stratified turbulence to the surface layer in an atmospheric convective turbulence. This theory allows us to obtain a smooth transition between a stably stratified turbulence to a convective turbulence. The EFB theory for the surface layer in a convective turbulence (that is based on the budget equations for the turbulent energies and turbulent fluxes of momentum and heat) provides an analytical expression for the entire surface layer including the transition range between the lower and upper parts of the surface layer.

This paper is organized as follows: In Sec. II, we formulate governing equations for the energy- and flux-budget turbulence-closure theory for convective and stably stratified turbulence. In this section, we also discuss assumptions used in the EFB theory. In Sec. III, we develop the EFB theory for surface layers in stratified turbulence considering the steady-state and homogeneous regime of turbulence. In Sec. IV, we apply the EFB theory to surface layers in turbulent convection. Finally, conclusions are drawn in Sec. V. In Appendix A, we derive equation for the turbulent Prandtl number, and in Appendix B, we derive equations for the vertical and horizontal shares of TKE (anisotropy parameters). In Appendix C, we discuss the EFB theory for the atmospheric stably stratified boundary-layer turbulence mainly developed in Refs. 47 and 49.

II. ENERGY- AND FLUX-BUDGET EQUATIONS AND BASIC ASSUMPTIONS

We consider plain-parallel, unstably, and stably stratified dry-air flow and employ the budget equations underlying turbulence-closure theory in the Boussinesq approximation. We assume that vertical component of the mean-wind velocity is negligibly small compared to horizontal component, and horizontal gradients of all properties of the mean flow (the mean velocity and the mean potential temperature) are negligibly small compared to vertical gradients.

In this section, we outline the energy- and flux-budget (EFB) closure theory based on the budget equations for the density of turbulent kinetic energy, the intensity of potential temperature fluctuations, and turbulent fluxes of momentum and heat. In our analysis, we use budget equations for the one-point second moments to develop a mean-field theory. We do not study small-scale structure of turbulence like intermittency described by high-order moments for turbulent quantities. We are interested by large-scale long-term dynamics and consider effects in the spatial scales, which are much larger than the integral scale of turbulence, and in timescales, which are much longer than the turbulent timescales.

We start with the basic equations of the EFB theory. The budget equation for the density of turbulent kinetic energy (TKE), $E_{\rm K} = \langle \boldsymbol{u}^2\rangle/2$, reads

$$\frac{DE_{\rm K}}{Dt} + \nabla_z \, \Phi_{\rm K} = -\tau_{iz} \, \nabla_z \bar{U}_i + \beta \, F_z - \varepsilon_{\rm K}, \tag{1}$$

where the first term, $-\tau_{iz} \, \nabla_z \bar{U}_i$, in the right-hand side of Eq. (1) is the rate of production of TKE by the vertical gradient of horizontal mean velocity $\bar{\boldsymbol{U}}(z)$, $D/Dt = \partial/\partial t + \bar{\boldsymbol{U}} \cdot \nabla$ is the convective derivative, $\tau_{iz} = \langle u_i \, u_z\rangle$ with $i = x, y$ are the off-diagonal components of the Reynolds stress describing the vertical turbulent flux of momentum, and the angular brackets imply ensemble averaging. The second term $\beta \, F_z$ in Eq. (1) describes buoyancy, $\beta = g/T_*$ is the buoyancy parameter, $\boldsymbol{g}$ is the gravity acceleration, $F_z = \langle u_z \, \theta\rangle$ is the vertical component of the turbulent flux of potential temperature, $\Theta = T(P_*/P)^{1-\gamma^{-1}}$ is the potential temperature, T is the fluid temperature with the reference value T_*, P is the fluid pressure with the reference value P_*, and $\gamma = c_{\rm p}/c_{\rm v}$ is the specific heat ratio.

The potential temperature $\Theta = \bar{\Theta} + \theta$ is characterized by the mean potential temperature $\bar{\Theta}(z)$ and fluctuations θ; the fluid velocity $\bar{\boldsymbol{U}} + \boldsymbol{u}$ is characterized by the mean fluid velocity, which generally includes the mean-wind velocity $\bar{\boldsymbol{U}}^{(\mathrm{w})}(z) = (\bar{U}_x, \bar{U}_y, 0)$; and the local three-dimensional mean velocity $\bar{\boldsymbol{U}}^{(\mathrm{s})}$ related to the large-scale semi-organized coherent structures in a convective turbulence and small-scale fluctuations $\boldsymbol{u} = (u_x, u_y, u_z)$.

The last term, $\varepsilon_{\rm K} = \nu \langle (\nabla_j u_i)^2 \rangle$, in the right-hand side of Eq. (1) is the dissipation rate of the density of the turbulent kinetic energy, where ν is the kinematic viscosity of fluid. The term $\Phi_{\rm K} = \rho_0^{-1} \langle u_z\, p\rangle + (\langle u_z\, \mathbf{u}^2\rangle - \nu \nabla_z \langle \mathbf{u}^2 \rangle)/2$ determines the flux of $E_{\rm K}$, where the fluid pressure $P = \bar{P} + p$ is characterized by the mean pressure $\bar{P}$ and fluctuations p, and ρ_0 is the fluid density.

The budget equation for the intensity of potential temperature fluctuations $E_\theta = \langle \theta^2 \rangle /2$ is

$$\frac{DE_\theta}{Dt} + \nabla_z \Phi_\theta = -F_z \nabla_z \bar{\Theta} - \varepsilon_\theta, \tag{2}$$

where $\Phi_\theta = (\langle u_z\, \theta^2\rangle - \chi \nabla_z \langle \theta^2 \rangle)/2$ describes the flux of E_θ and $\varepsilon_\theta = \chi \langle (\nabla \theta)^2 \rangle$ is the dissipation rate of the intensity of potential temperature fluctuations E_θ, and χ is the molecular temperature diffusivity.

The budget equation for the turbulent flux $F_i = \langle u_i\, \theta \rangle$ of potential temperature is given by

$$\frac{\partial F_i}{\partial t} + \nabla_z \boldsymbol{\Phi}_i^{(\mathrm{F})} = -\tau_{iz} \nabla_z \bar{\Theta} + 2\beta\, E_\theta\, \delta_{i3} - \frac{1}{\rho_0} \langle \theta\, \nabla_i p \rangle - F_z \nabla_z \bar{U}_i - \varepsilon_i^{(\mathrm{F})}, \tag{3}$$

where δ_{ij} is the Kronecker unit tensor, $\boldsymbol{\Phi}_i^{(\mathrm{F})} = \langle u_i\, u_z\, \theta \rangle - \nu \langle \theta\, (\nabla_z u_i) \rangle - \chi \langle u_i\, (\nabla_z \theta) \rangle$ determines the flux of F_i, and $\varepsilon_i^{(\mathrm{F})} = (\nu + \chi) \langle (\nabla_j u_i)\, (\nabla_j \theta) \rangle$ is the dissipation rate of the turbulent heat flux. The first term, $-\tau_{iz} \nabla_z \bar{\Theta}$, in the right-hand side of Eq. (3) contributes to the traditional turbulent flux of potential temperature, which describes the classical gradient mechanism of the turbulent heat transfer. The second and third terms in the right-hand side of Eq. (3) describe a non-gradient contribution to the turbulent flux of potential temperature. The budget equation for the vertical turbulent flux $F_z = \langle u_z\, \theta \rangle$ of potential temperature is given by

$$\frac{\partial F_z}{\partial t} + \nabla_z \boldsymbol{\Phi}_z^{(\mathrm{F})} = -2E_z \nabla_z \bar{\Theta} + 2\beta\, E_\theta - \frac{1}{\rho_0} \langle \theta\, \nabla_z p \rangle - \varepsilon_z^{(\mathrm{F})}, \tag{4}$$

where $E_z = \langle u_z^2 \rangle /2$ is the density of the vertical turbulent kinetic energy.

The budget equation for the off-diagonal components of the Reynolds stress $\tau_{iz} = \langle u_i\, u_z \rangle$ with $i = x, y$ reads

$$\frac{D\tau_{iz}}{Dt} + \nabla_z \Phi_i^{(\tau)} = -2\, E_z \nabla_z \bar{U}_i + \beta\, F_i + Q_{iz} - \varepsilon_{iz}^{(\tau)}, \tag{5}$$

where $\Phi_i^{(\tau)} = \langle u_i\, u_z^2 \rangle + \rho_0^{-1} \langle p\, u_i \rangle - \nu\, [\langle u_i\, (\nabla_z u_z) \rangle + \langle u_z\, (\nabla_z u_i) \rangle]$ describes the flux of τ_{iz}, the tensor $Q_{ij} = \rho_0^{-1} (\langle p \nabla_i u_j \rangle + \langle p \nabla_j u_i \rangle)$, and $\varepsilon_{iz}^{(\tau)} = 2\nu \langle (\nabla_j u_i)\, (\nabla_j u_z) \rangle$ is the molecular-viscosity dissipation rate.

The budget equations for the horizontal and vertical turbulent kinetic energies $E_\alpha = \langle u_\alpha^2 \rangle /2$ can be written as follows:

$$\frac{DE_\alpha}{Dt} + \nabla_z \Phi_\alpha = -\tau_{\alpha z} \nabla_z \bar{U}_\alpha + \delta_{\alpha 3}\, \beta\, F_z + \frac{1}{2} Q_{\alpha\alpha} - \varepsilon_\alpha, \tag{6}$$

where $\alpha = x, y, z$, the term $\varepsilon_\alpha = \nu \langle (\nabla_j u_\alpha)^2 \rangle$ is the dissipation rate of E_α, and Φ_α determines the flux of E_α. Here, $\Phi_z = \rho_0^{-1} \langle u_z\, p \rangle + (\langle u_z^3 \rangle - \nu \nabla_z \langle u_z^2 \rangle)/2$ and $\Phi_{x,y} = (\langle u_z\, u_{x,y}^2 \rangle - \nu \nabla_z \langle u_{x,y}^2 \rangle)/2$. The terms $Q_{\alpha\alpha} = 2\rho_0^{-1} (\langle p \nabla_\alpha u_\alpha \rangle$ are the diagonal terms of the tensor Q_{ij}. In Eq. (6), we do not apply the summation convention for the double Greek indices. Different aspects related to budget equations (1)–(6) have been discussed in a number of publications.[38,43–49,65,73,74]

The energy- and flux-budget turbulence closure theory assumes the following. The characteristic times of variations of the densities of the turbulent kinetic energies $E_{\rm K}$ and E_α, the intensity of potential temperature fluctuations E_θ, the turbulent flux F_i of potential temperature, and the turbulent flux τ_{iz} of momentum (i.e., the off-diagonal components of the Reynolds stress) are much larger than the turbulent timescale. This allows us to obtain steady-state solutions of the budget equations (1)–(6).

Dissipation rates of the turbulent kinetic energies $E_{\rm K}$ and E_α, and the intensity of potential temperature fluctuations E_θ and F_i are expressed using the Kolmogorov hypothesis, that is, $\varepsilon_{\rm K} = E_{\rm K}/t_{\rm T}$, $\varepsilon_\theta = E_\theta/(C_{\rm p}\, t_{\rm T})$, and $\varepsilon_i^{(\mathrm{F})} = F_i/(C_{\rm F}\, t_{\rm T})$, where $t_{\rm T} = \ell_z/E_z^{1/2}$ is the turbulent dissipation timescale, ℓ_z is the vertical integral scale, and $C_{\rm p}$ and $C_{\rm F}$ are dimensionless empirical constants.[1,2,8,9,11] Note also that the dissipation rate of the TKE components E_α (where $\alpha = x, y, z$) is $\varepsilon_\alpha = E_{\rm K}/3t_{\rm T}$. This is because the main contribution to the rate of dissipation of the TKE components is from the Kolmogorov viscous scale where turbulence is nearly isotropic, so that $\varepsilon_x = \varepsilon_y = \varepsilon_z = E_{\rm K}/3t_{\rm T}$.

The term $\varepsilon_i^{(\tau)} = \varepsilon_{iz}^{(\tau)} - \beta\, F_i - Q_{iz}$ in Eq. (5) is the effective dissipation rate of the off-diagonal components of the Reynolds stress τ_{iz},[43,47,49] where $\varepsilon_{iz}^{(\tau)}$ is the molecular-viscosity dissipation rate of τ_{iz} that is small because the smallest eddies associated with viscous dissipation are nearly isotropic.[75] In the framework of EFB theory, the role of the dissipation rate of τ_{iz} is assumed to be played by the combination of terms $-\beta\, F_i - Q_{iz}$, and it is assumed that $\varepsilon_i^{(\tau)} = \tau_{iz}/(C_\tau\, t_{\rm T})$, where C_τ is the effective-dissipation timescale empirical constant for stably stratified turbulence,[43,47,49] while for a convective turbulence C_τ is a function of the flux Richardson number (see Sec. V).

The effective dissipation rate assumption has been justified by large-eddy simulations (see Fig. 1 in Ref. 47), where LES data in Refs. 76 and 77 have been used for the two types of atmospheric boundary layer: "nocturnal stable" (with essentially negative buoyancy flux at the surface and neutral stratification in the free flow) and "conventionally neutral" (with a negligible buoyancy flux at the surface and essentially stably stratified turbulence in the free flow). The effective dissipation rate assumption was based on our prior analysis of the Reynolds stress equation in the $\boldsymbol{k}$ space using the spectral τ approach.[22,23] Remarkably, the effective dissipation assumption directly yields the familiar down-gradient formulation of the vertical turbulent flux of momentum [see Eq. (7) below] that is well-known result, which is valid for any turbulence with a non-uniform mean velocity field.

Note that the diagonal and off-diagonal components of the Reynolds stress have different physical meaning. The diagonal components of the Reynolds stress describe turbulent kinetic energy components. They have the Kolmogorov spectrum $\propto k^{-5/3}$ that is related to the direct energy cascade. The latter is the main reason for turbulent viscosity and turbulent diffusivity. The off-diagonal components of the Reynolds stress are related to the tangling mechanism of the generation of anisotropic velocity fluctuations. They have different spectrum $\propto k^{-7/3}$ (see Refs. 78–81). The off-diagonal components of the Reynolds stress are determined by spatial derivatives of the mean velocity field. The diagonal components of the Reynolds stress are much larger than the off-diagonal components.

We assume that the term $\rho_0^{-1}\,\langle\theta\,\nabla_z p\rangle$ in Eq. (4) for the vertical turbulent flux of potential temperature is parameterized so that $\beta\,\langle\theta^2\rangle - \rho_0^{-1}\,\langle\theta\,\nabla_z p\rangle = 2C_\theta\,\beta\,E_\theta$, with the positive dimensionless empirical constant C_θ, which is less than 1. This assumption has been justified by large-eddy simulations (see Fig. 2 in Ref. 47), where LES data by Refs. 76 and 77 have been used for the two types of atmospheric boundary layer: nocturnal stable and conventionally neutral. In addition, this assumption has been justified analytically (see Appendix A in Ref. 43).

III. THE EFB THEORY FOR SURFACE LAYERS IN STRATIFIED TURBULENCE

In this section, we develop the EFB theory for surface layers in convective and stably stratified turbulence. We use the down-gradient formulation of the vertical turbulent flux of momentum, which follows from Eq. (5); that is, the turbulent fluxes of the momentum are

$$\tau_{iz} = -K_{\rm M}\,\nabla_z \bar{U}_i, \quad i = x, y, \tag{7}$$

$$K_{\rm M} = 2C_\tau\, t_{\rm T}\, E_z = 2C_\tau\, \ell_z\, E_z^{1/2}, \tag{8}$$

where $K_{\rm M}$ is the turbulent (eddy) viscosity, $t_{\rm T} = \ell_z / E_z^{1/2}$ is the turbulent dissipation timescale, ℓ_z is the vertical integral scale, and E_z is the vertical turbulent kinetic energy. The production rate, $\Pi_{\rm K} = -\tau_{iz}\,\nabla_z \bar{U}_i$, of the turbulent kinetic energy by the vertical gradient of horizontal mean velocity [see Eq. (1)] can be rewritten by means of Eq. (7) as $\Pi_{\rm K} = -(\tau_{xz}\,\nabla_z \bar{U}_x + \tau_{yz}\,\nabla_z \bar{U}_y) = K_{\rm M}\,S^2$, where $S = [(\nabla_z \bar{U}_x)^2 + (\nabla_z \bar{U}_y)^2]^{1/2}$ is the mean velocity shear caused by the horizontal mean wind velocity.

The steady-state version of the budget equations for the density of turbulent kinetic energy $E_{\rm K} = \langle \boldsymbol{u}^2 \rangle / 2$ reads

$$\nabla_z\, \Phi_{\rm K} = K_{\rm M} S^2 + \beta\, F_z - \frac{E_{\rm K}}{t_{\rm T}}, \tag{9}$$

where the dissipation rate $\varepsilon_{\rm K}$ of the turbulent kinetic energy is expressed using the Kolmogorov hypothesis, $\varepsilon_{\rm K} = E_{\rm K}/t_{\rm T}$. We stress that all results obtained in the present study are mainly valid for temperature-stratified turbulence (convective turbulence or stably stratified turbulence), where fluctuations of the vertical velocity u_z depend on the buoyancy, $\beta\, F_z$. Since for temperature-stratified turbulence, $\rho_0^{-1}\langle u_z\, p\rangle$ and $\langle u_z\, \mathbf{u}^2\rangle$ do depend on the buoyancy, the third-order moments $\Phi_{\rm K}$ should depend on buoyancy. We assume that the vertical gradient $\nabla_z\, \Phi_{\rm K}$ of the flux of $E_{\rm K}$ is determined by the buoyancy, that is, $\nabla_z \Phi_{\rm K} = -C_\Phi\, \beta\, F_z$, where C_Φ is the dimensionless empirical constant. The justification of this assumption for a convective turbulence has been performed in Ref. 30, where experimental data obtained from meteorological observations at the Eureka station (located in the Canadian territory of Nunavut) in conditions of the long-lived convective boundary layer typical of the Arctic summer have been used for the validation of the assumption $\nabla_z \Phi_{\rm K} = -C_\Phi\, \beta\, F_z$ (see the right panel in Fig. 1 in Ref. 30). Turbulent fluxes were calculated directly from the measured velocity and temperature fluctuations. In these meteorological observations, warming of the convective layer from the surface is balanced by pumping of colder air into the layer via the general-circulation mechanisms. Note also that no principal contradictions have been found between the available data from observations at mid- or low latitudes and the data from Eureka.[36]

Using the expression

$$\tau = \left(\tau_{xz}^2 + \tau_{yz}^2\right)^{1/2} = K_{\rm M}\, S, \tag{10}$$

and taking into account that for any boundary layer turbulence $\tau = u_*^2$, Eq. (9) is reduced by simple algebraic calculations to a nonlinear equation for the vertical profile of the normalized TKE, $\tilde{E}_{\rm K}(\tilde{Z}) = E_{\rm K}(\tilde{Z})/E_{\rm K0}$ as

$$\tilde{E}_{\rm K}^2 + \tilde{Z}\, \tilde{E}_{\rm K}^{1/2} - 1 = 0, \tag{11}$$

where the normalized height $\tilde{Z} = \ell_z/(C_*\, L)$, $E_{\rm K0} = u_*^2/(2C_\tau\, A_z)^{1/2}$, $C_*^{-1} = (1 + C_\Phi)\,(2C_\tau)^{3/4} A_z^{1/4}$, $A_z = E_z/E_{\rm K}$ is the vertical share of TKE (vertical anisotropy parameter), u_* is the local (z-dependent) friction velocity, and L is the local Obukhov length defined as

$$L = -\frac{\tau^{3/2}}{\beta\, F_z}, \tag{12}$$

and F_z is the local vertical turbulent flux of potential temperature. For stably stratified turbulence, the vertical turbulent flux F_z of potential temperature is negative, and the local Obukhov length L is positive. For stably stratified turbulence ($\tilde{Z} > 0$), Eq. (11) has two asymptotic solutions:

(i) for a lower part ($\tilde{Z} \ll 1$) of the surface layer, Eq. (11) yields

$$\tilde{E}_{\rm K} = 1 - \frac{\tilde{Z}}{2} + \frac{\tilde{Z}^2}{8}, \tag{13}$$

(ii) for an upper part ($\tilde{Z} \gg 1$) of the surface layer, it is

$$\tilde{E}_{\rm K} = \tilde{Z}^{-2}\left(1 - 2\tilde{Z}^{-4}\right). \tag{14}$$

In the framework of the EFB theory of surface layers, we use the same definition (12) for L in convective turbulence as well, where the vertical turbulent flux F_z of potential temperature is positive, and L is negative. Equation (11) for the surface layer in convective turbulence ($\tilde{Z} < 0$) reads

$$\tilde{E}_{\rm K}^2 - |\tilde{Z}|\, \tilde{E}_{\rm K}^{1/2} - 1 = 0, \tag{15}$$

and it has two asymptotic solutions:

(i) for a lower part ($|\tilde{Z}| \ll 1$) of the surface convective layer, Eq. (15) yields

$$E_{\rm K} = E_{\rm K0}\left(1 + \frac{1}{2}\,|\tilde{Z}|\right), \tag{16}$$

(ii) for an upper part ($|\tilde{Z}| \gg 1$) of the surface convective layer, the balance of the first and the second terms in Eq. (11) yields $\tilde{E}_{\rm K} = \tilde{Z}^{2/3}$, that is,

$$E_{\rm K} = E_{\rm K0}\, \tilde{Z}^{2/3}. \tag{17}$$

Note that as follows from the definition of $\tilde{Z} = \ell_z/(C_* L)$, the ratio z/L for convective turbulence is

$$\frac{z}{L} = \frac{\tilde{Z}}{\kappa_0\,(1 + C_\Phi)}, \tag{18}$$

where $\ell_z = C_\ell\, z$ with $C_\ell = \kappa_0\,(2C_\tau)^{-3/4} A_z^{-1/4}$ and $\kappa_0 = 0.4$ is the von Karman constant. Note that generally, Eq. (18) can be valid also for arbitrary z, but in this case, C_ℓ should be a function of height (see Sec. V).

In Fig. 1, we show a numerical solution of Eq. (11). In particular, in Fig. 1 we plot the normalized turbulent kinetic energy $\tilde{E}_{\rm K} = E_{\rm K}/E_{\rm K0}$ vs $\tilde{Z}$ for convective ($\tilde{Z} < 0$) and stably stratified ($\tilde{Z} > 0$) turbulence. This numerical solution is in a good agreement with the above asymptotic solutions for convective and stably stratified turbulence.

Now we define the flux Richardson number as

$$\mathrm{Ri_f} = -\frac{\beta\, F_z}{\tau\, S} = -\frac{\beta\, F_z}{K_{\rm M} S^2} \tag{19}$$

[see Eq. (10)], so that for stably stratified turbulence, $\mathrm{Ri_f}$ is positive and varies from 0 to the limiting value $R_\infty = 0.2$ at very large gradient Richardson number $\mathrm{Ri} \gg 1$. Here, the gradient Richardson number, Ri, is defined as

$$\mathrm{Ri} = \frac{N^2}{S^2}, \tag{20}$$

where $N^2 = \beta\, \nabla_z \bar{\Theta}$ and N is the Brunt–Väisälä frequency. In the framework of the EFB theory of surface layers, we use the same definition (19) for the flux Richardson number in turbulent convection, so that $\mathrm{Ri_f}$ is negative in turbulent convection, and its absolute value is not limited and can be large.

Equations (12) and (19) allow us to relate the turbulent viscosity $K_{\rm M}$ with the flux Richardson number $\mathrm{Ri_f}$ as[49]

$$K_{\rm M} = \mathrm{Ri_f}\, \tau^{1/2}\, L, \tag{21}$$

where τ is given by Eq. (10). Using Eqs. (8) and (21), we rewrite the flux Richardson number as

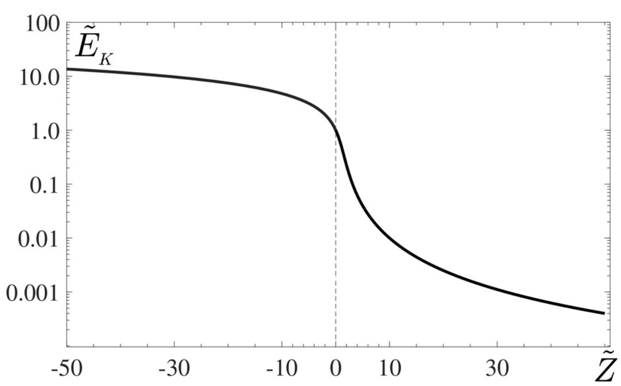

FIG. 1. The normalized turbulent kinetic energy $\tilde{E}_{\rm K} = E_{\rm K}/E_{\rm K0}$ vs $\tilde{Z}$ for convective ($\tilde{Z} < 0$) and stably stratified turbulence ($\tilde{Z} > 0$).

$$\mathrm{Ri_f} = (1 + C_\Phi)^{-1}\, \tilde{Z}\, \tilde{E}_{\rm K}^{1/2}. \tag{22}$$

Equations (10) and (21) allow us to relate the large-scale shear S with the flux Richardson number as

$$S = \frac{\tau^{1/2}}{L\, \mathrm{Ri_f}}. \tag{23}$$

Using Eqs. (8), we rewrite Eq. (9) as the dimensionless ratio

$$\left(\frac{E_{\rm K}}{\tau}\right)^2 = \frac{1 - \mathrm{Ri_f}\,(1 + C_\Phi)}{2C_\tau\, A_z}. \tag{24}$$

In addition, by means of Eqs. (8), (21), and (24), we obtain the normalized vertical integral scale ℓ_z as the function of the flux Richardson number

$$\frac{\ell_z}{L} = \frac{(2C_\tau)^{-3/4}\, A_z^{-1/4}\, \mathrm{Ri_f}}{[1 - \mathrm{Ri_f}\,(1 + C_\Phi)]^{1/4}}, \tag{25}$$

where $1 - \mathrm{Ri_f}\,(1 + C_\Phi) > 0$. This condition implies that $C_\Phi < R_\infty^{-1} - 1$. For stably stratified turbulence, $R_\infty = 0.2$, so that $C_\Phi < 4$. Thus, the normalized height $\tilde{Z} = \ell_z/(C_* L)$ as the function of the flux Richardson number reads

$$\tilde{Z} = \frac{\mathrm{Ri_f}\,(1 + C_\Phi)}{[1 - \mathrm{Ri_f}\,(1 + C_\Phi)]^{1/4}}. \tag{26}$$

Note also that using Eq. (26), we can rewrite Eq. (25) as

$$\frac{\ell_z}{L} = \frac{C_\ell\, \tilde{Z}}{\kappa_0\,(1 + C_\Phi)}. \tag{27}$$

Since convective turbulence is essentially different from stably stratified turbulence, the behavior of the flux Richardson number $\mathrm{Ri_f} \propto \tilde{Z}\, \tilde{E}_{\rm K}^{1/2}$ is also different for these two kinds of turbulence [see Eq. (22)]. In particular, in convection both, the buoyancy and large-scale shear produce convective turbulence, so that the flux Richardson number can be enough large. On contrary, in stably stratified turbulence, the large-scale shear produces turbulence, while the buoyancy decreases TKE, so that the flux Richardson number is limited by some value, $\mathrm{R}_\infty = 0.2$. However, in the presence of internal gravity waves the maximum value of the flux Richardson number can be larger in several times in comparison with the case without waves.[45,48]

Equations (11) and (22) yield the normalized turbulent kinetic energy $\tilde{E}_{\rm K} = E_{\rm K}/E_{\rm K0}$ as a function of the flux Richardson number as

$$\tilde{E}_{\rm K} = [1 - (1 + C_\Phi)\, \mathrm{Ri_f}]^{1/2}. \tag{28}$$

This implies that the normalized turbulent kinetic energy $\tilde{E}_{\rm K}$ in stably stratified turbulence decreases up to the minimum value

$$\tilde{E}_{\rm K}^{\rm min} = [1 - (1 + C_\Phi)\, \mathrm{R}_\infty]^{1/2}. \tag{29}$$

As follows from Eq. (22), the function $\tilde{Z}\, \tilde{E}_{\rm K}^{1/2} \le (1 + C_\Phi)\, \mathrm{R}_\infty$, so that according to Eq. (26), the maximum value of the height $\tilde{Z}^{\rm max}$ in stably stratified turbulence is

$$\tilde{Z}^{\rm max} = \frac{\mathrm{R}_\infty\,(1 + C_\Phi)}{[1 - \mathrm{R}_\infty\,(1 + C_\Phi)]^{1/4}}. \tag{30}$$

Since in convective turbulence, the flux Richardson number is negative, the normalized turbulent kinetic energy, $\tilde{E}_{\rm K} = [1 + (1 + C_\Phi)\, |\mathrm{Ri_f}|]^{1/2}$, increases with the flux Richardson number.

Next, we derive the expression for the vertical turbulent heat flux F_z using the steady-state versions of Eqs. (2) and (4),

$$F_z = -K_{\rm H}\,\nabla_z \bar{\Theta}, \tag{31}$$

where the eddy diffusivity is given by

$$K_{\rm H} = 2C_{\rm F}\, t_{\rm T}\, E_z \left[1 - \frac{C_\theta\, C_{\rm p}\, {\rm Ri_f}}{A_z[1 - {\rm Ri_f}\,(1 + C_\Phi)]}\right]. \tag{32}$$

The details of derivation of Eq. (32) are given in Appendix A. The turbulent Prandtl number, ${\rm Pr_T} = K_{\rm M}/K_{\rm H}$, follows from Eqs. (8) and (32)

$${\rm Pr_T} = \frac{C_\tau}{C_{\rm F}} \left[1 - \frac{C_\theta\, C_{\rm p}\, {\rm Ri_f}}{A_z[1 - {\rm Ri_f}\,(1 + C_\Phi)]}\right]^{-1}, \tag{33}$$

where ${\rm Pr}_{\rm T}^{(0)} = C_\tau/C_{\rm F}$ is the turbulent Prandtl number for a non-stratified turbulence when the mean potential temperature gradient vanishes. The gradient Richardson number Ri and the flux Richardson number ${\rm Ri_f}$ are related as ${\rm Ri} = {\rm Ri_f}\,{\rm Pr_T}$.

Using Eq. (A1) in Appendix A, and Eqs. (8) and (32), we determine the level of temperature fluctuations characterized by the dimensional ratio E_θ/θ_*^2 as

$$\frac{E_\theta}{\theta_*^2} = C_{\rm p}\,(2C_\tau\, A_z)^{-1/2}\, {\rm Pr_T}\, \tilde{E}_{\rm K}^{-1}, \tag{34}$$

where $\theta_* = |F_z|/u_* = u_*^2/\beta\,|L|$. Equation (23), expression for the friction velocity, $u_*^2 = K_{\rm M}\, S$, and Eq. (31) yield the vertical gradient of the mean potential temperature as

$$\nabla_z \bar{\Theta} = \frac{\theta_*\,{\rm Pr_T}}{|L|\,{\rm Ri_f}}. \tag{35}$$

The steady-state version of Eq. (3) for homogeneous turbulence yields the horizontal turbulent flux $F_i = F_{x,y}$ of potential temperature:

$$F_i = -C_{\rm F}\, t_{\rm T}\,(1 + {\rm Pr_T})\, F_z\, \nabla_z \bar{U}_i, \quad i = x, y. \tag{36}$$

Since in convective turbulence the vertical turbulent flux F_z is positive, the horizontal turbulent flux $F_i = F_{x,y}$ of potential temperature is directed opposite to the wind velocity $\bar{U}_i$; that is, Eq. (36) describes the counter-wind horizontal turbulent flux in convective turbulence. On contrary, in a stably stratified turbulence the vertical turbulent flux F_z is negative, so that Eq. (36) determines the co-wind horizontal turbulent flux.

The physics of the counter-wind turbulent flux in a convective turbulence is as follows:[49] In horizontally homogeneous, convective turbulence with a large-scale shear velocity (e.g., directed along the x axis), the mean shear velocity $\bar{U}_x$ increases with increasing height, while the mean potential temperature $\bar{\Theta}$ decreases with height. Uprising fluid particles produce positive fluctuations of potential temperature ($\theta > 0$) since $\partial\theta/\partial t \propto -(\boldsymbol{u}\cdot\nabla)\bar{\Theta}$, and negative fluctuations of horizontal velocity ($u_x < 0$) since $\partial u_x/\partial t \propto -(\boldsymbol{u}\cdot\nabla)\bar{U}_x$. It results in negative horizontal temperature flux, $u_x\,\theta < 0$. Similarly, sinking fluid particles cause negative fluctuations of potential temperature ($\theta < 0$), and positive fluctuations of horizontal velocity ($u_x > 0$), that implies negative horizontal temperature flux $u_x\,\theta < 0$. Therefore, the net horizontal turbulent flux is negative ($\langle u_x\,\theta\rangle < 0$) even for a zero horizontal mean temperature gradient. This is the counter-wind turbulent flux of potential temperature that results in modification of the potential-temperature flux by the non-uniform velocity field.

Let us find dependence of the horizontal turbulent flux F_i of potential temperature on the flux Richardson number. To this end, we use the identity,

$$(S\, t_{\rm T})^2 = \frac{1}{2\, C_\tau\, A_z\,[1 - {\rm Ri_f}\,(1 + C_\Phi)]}, \tag{37}$$

that is derived by means of Eqs. (8), (10), and (24). Therefore, the ratio of the horizontal and vertical turbulent fluxes of potential temperature, F_x/F_z, is given by

$$\frac{F_x}{F_z} = -C_{\rm F}\,(1 + {\rm Pr_T})\,(2\, C_\tau\, A_z)^{-1/2}[1 - {\rm Ri_f}\,(1 + C_\Phi)]^{-1/2}. \tag{38}$$

Most of the results obtained in this section depend on the vertical anisotropy parameter, $A_z \equiv E_z/E_{\rm K}$. The mean shear velocity $\bar{U}_x(z)$ produces the energy of longitudinal velocity fluctuations E_x, which in turns feeds the transverse E_y and the vertical E_z components of turbulent kinetic energy. The inter-component energy exchange term $Q_{\alpha\alpha}$ in the right-hand side of Eq. (6) is traditionally parameterized through the "return-to-isotropy" hypothesis.[82] On the other hand, the temperature-stratified turbulence is usually anisotropic, and the inter-component energy exchange term $Q_{\alpha\alpha}$ should depend on the flux Richardson number ${\rm Ri_f}$. Analysis performed in Appendix B allows us to determine dependence of the vertical anisotropy parameter $A_z({\rm Ri_f})$ on the flux Richardson number ${\rm Ri_f}$ in a stably stratified turbulence

$$A_z({\rm Ri_f}) = A_z^{(0)} - {\rm Ri_f}\left[\frac{1 - A_z^{(0)}}{(1 + C_\Phi)^{-1} - {\rm Ri_f}} - \frac{2A_z^{(0)}}{{\rm R}_\infty}\right], \tag{39}$$

while in a convective turbulence (where $|{\rm Ri_f}| \ll |{\rm R}_\infty|$ and ${\rm Ri_f} < 0$), the vertical anisotropy parameter is given by

$$A_z({\rm Ri_f}) = A_z^{(0)} + \frac{(1 - A_z^{(0)})\,|{\rm Ri_f}|}{(1 + C_\Phi)^{-1} + |{\rm Ri_f}|}. \tag{40}$$

When turbulence is isotropic in the horizontal plane, the horizontal shares of TKE are $A_x = A_y = 1 - A_z$, where $A_x = E_x/E_{\rm K}$ and $A_y = E_y/E_{\rm K}$. For anisotropic turbulence in the horizontal plane, the horizontal anisotropy parameters are given by

$$A_x({\rm Ri_f}) = A_z^{(0)}\left[1 - C_1 - \frac{{\rm Ri_f}}{{\rm R}_\infty}\,(1 - C_2)\right] + \frac{1 - 3A_z^{(0)}}{1 - {\rm Ri_f}\,(1 + C_\Phi)}, \tag{41}$$

and $A_y = 1 - A_x - A_z$, where the vertical anisotropy parameter A_z is given by Eq. (39). The free constants C_1 and C_2 are determined by the values $A_x^{(0)}$ at ${\rm Ri_f} = 0$ and $A_x^{(\infty)}$ at ${\rm Ri_f} \to {\rm R}_\infty$. The details of derivations of Eq. (41) are given in Appendix B.

Let us consider stably stratified turbulence. Neglecting the term $\nabla_z\,\Phi_{\rm K}$ in Eq. (9), we rewrite this equation as ${\rm Ri_f}^{-1} = 1 - \varepsilon_{\rm K}/\beta\,F_z$, where we use the definition (19) for the flux Richardson number. By means of this equation and the expressions for the squared Brunt–Väisälä frequency, $N^2 = \beta\,\nabla_z\bar{\Theta}$, and the turbulent heat flux, $F_z = -K_{\rm H}\,\nabla_z\bar{\Theta}$, we obtain equation for the turbulent heat conductivity $K_{\rm H}$ as

$$K_{\rm H} = \left({\rm Ri_f}^{-1} - 1\right)^{-1} \frac{\varepsilon_{\rm K}}{N^2}. \tag{42}$$

In very strong stable stratification, the gradient Richardson number admits a limit ${\rm Ri} \to \infty$ and the flux Richardson number ${\rm Ri_f} \to 0.2$.

This implies that the turbulent heat conductivity for a very strong stable stratification $K_{\rm H} \approx 0.25\, \varepsilon_{\rm K}/N^2$. This is a well-known Cox–Osborn equation[83,84] that plays an important role in physical oceanography.

IV. SURFACE LAYERS IN CONVECTIVE TURBULENCE

In this section, we apply results obtained in Sec. III to surface layer of a convective turbulence. In this case, the nonlinear equation for the vertical profile of the normalized TKE, $\tilde E_{\rm K}(\tilde Z) = E_{\rm K}(\tilde Z)/E_{\rm K0}$, is given by Eq. (15). Asymptotic solutions of Eq. (15) for the normalized TKE, $\tilde E_{\rm K}(\tilde Z)$, are given by Eq. (16) for a lower part ($|\tilde Z| \ll 1$) of the surface convective layer, and by Eq. (17) for an upper part ($|\tilde Z| \gg 1$) of the surface convective layer. Below we present asymptotic formulas for various turbulent characteristics based on Eqs. (21), (22)–(26), (33)–(35), (38), and (40) for the lower and upper parts of the surface layer in convective turbulence. In particular, the turbulence characteristics for a lower part ($|\tilde Z| \ll 1$) of the surface convective layer are given by

- the flux Richardson number,

$$\mathrm{Ri_f} = -(1+C_\Phi)^{-1}\, |\tilde Z|, \tag{43}$$

- the large-scale shear,

$$S = \frac{u_*}{\kappa_0\, z}, \tag{44}$$

- the turbulent viscosity,

$$K_{\rm M} = \kappa_0\, u_*\, z, \tag{45}$$

- the vertical anisotropy parameter,

$$A_z(\mathrm{Ri_f}) = A_z^{(0)} + (1 - A_z^{(0)})\, |\tilde Z|, \tag{46}$$

- the turbulent Prandtl number,

$$\mathrm{Pr_T} = \mathrm{Pr_T^{(0)}} \left[1 - \frac{C_\theta\, C_{\rm p}}{A_z^{(0)}\,(1+C_\Phi)}\, |\tilde Z|\right], \tag{47}$$

- the level of temperature fluctuations,

$$\frac{E_\theta}{\theta_*^2} = C_{\rm p} \left(2C_\tau\, A_z^{(0)}\right)^{-1/2} \mathrm{Pr_T^{(0)}}\, \left(1 - C_{\rm E}\, |\tilde Z|\right), \tag{48}$$

- the vertical gradient of the mean potential temperature,

$$\nabla_z \bar\Theta = -\frac{\theta_*\, \mathrm{Pr_T^{(0)}}}{\kappa_0\, z}, \tag{49}$$

- the ratio of the horizontal and vertical turbulent fluxes of potential temperature,

$$\frac{F_i}{F_z} = -C_{\rm F} \left(1 + \mathrm{Pr_T^{(0)}}\right) \left(2\, C_\tau\, A_z^{(0)}\right)^{-1/2}, \tag{50}$$

- the horizontal components of TKE,

$$E_x = E_y = \frac{1}{2}\, E_{\rm K0} \left(1 - A_z^{(0)}\right) \left(1 - \frac{1}{2}\, |\tilde Z|\right), \tag{51}$$

where

$$C_{\rm E} = \frac{1}{6}\left[1 + 2\left(A_z^{(0)}\right)^{-1}\right] + \frac{C_\theta\, C_{\rm p}}{A_z^{(0)}\,(1+C_\Phi)}. \tag{52}$$

In Eq. (44), we take into account that for the surface layer in convective turbulence, the vertical integral turbulent scale, $\ell_z = C_\ell\, z$, and in Eq. (50), we consider the case when the mean velocity $\bar U_i$ is directed along the x axis.

For an upper part ($|\tilde Z| \gg 1$) of the surface convective layer, the turbulence characteristics are given by

- the flux Richardson number,

$$\mathrm{Ri_f} = -(1+C_\Phi)^{-1}\, \tilde Z^{4/3}, \tag{53}$$

- the large-scale shear,

$$S = \frac{u_*}{|L|}\,(1+C_\Phi)\, \tilde Z^{-4/3}, \tag{54}$$

- the turbulent viscosity,

$$K_{\rm M} = (1+C_\Phi)^{-1}\, u_*\, |L|\, \tilde Z^{4/3}, \tag{55}$$

- the normalized vertical integral scale ℓ_z,

$$\frac{\ell_z}{|L|} = (2C_\tau)^{-3/4}\, \tilde Z^{4/3}, \tag{56}$$

- the normalized TKE,

$$\frac{E_{\rm K}}{u_*^2} = (2C_\tau)^{-1/2}\, \tilde Z^{2/3}, \tag{57}$$

- the vertical anisotropy parameter,

$$A_z(\mathrm{Ri_f}) = 1 - (1 - A_z^{(0)})\, \tilde Z^{-4/3}, \tag{58}$$

- the turbulent Prandtl number,

$$\mathrm{Pr_T} = \mathrm{Pr_T^{(\infty)}} \left[1 - \left(1 - \frac{\mathrm{Pr_T^{(\infty)}}}{\mathrm{Pr_T^{(0)}}}\right) \tilde Z^{-4/3}\right], \tag{59}$$

- the level of temperature fluctuations E_θ/θ_*^2,

$$\frac{E_\theta}{\theta_*^2} = C_{\rm p}\ (2C_\tau)^{-1/2}\, \mathrm{Pr_T^{(\infty)}}\, \tilde Z^{-2/3}, \tag{60}$$

- the vertical gradient of the mean potential temperature,

$$\nabla_z \bar\Theta = -\frac{\theta_*}{|L|}\, \mathrm{Pr_T^{(\infty)}}\, \tilde Z^{-4/3}, \tag{61}$$

- the ratio of the horizontal and vertical turbulent fluxes of potential temperature,

$$\frac{F_x}{F_z} = -C_{\rm F}\,(1 + \mathrm{Pr_T})\,(2\, C_\tau)^{-1/2}\, \tilde Z^{-2/3}, \tag{62}$$

- the horizontal components of TKE,

$$E_x = E_y = \frac{1}{2}\, E_{\rm K0} \left(1 - A_z^{(0)}\right) \tilde Z^{-2/3}, \tag{63}$$

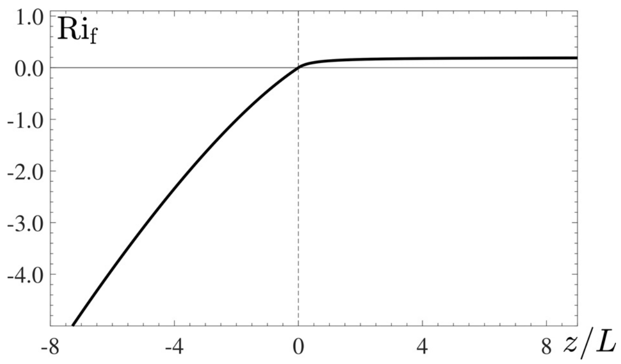

FIG. 2. The flux Richardson number $\mathrm{Ri_f}$ vs z/L for convective $(z/L < 0)$ and stably stratified $(z/L > 0)$ turbulence.

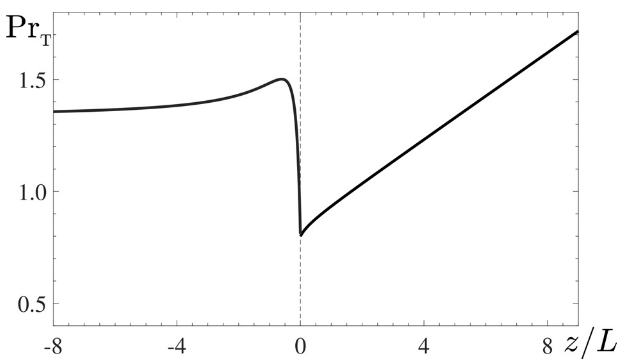

FIG. 3. The turbulent Prandtl number Pr_T vs z/L for convective $(z/L < 0)$ and stably stratified $(z/L > 0)$ turbulence.

where

$$\mathrm{Pr}_\mathrm{T}^{(\infty)} = \mathrm{Pr}_\mathrm{T}^{(0)} \left[1 + \frac{C_\theta\, C_\mathrm{p}}{1 + C_\Phi}\right]^{-1}, \tag{64}$$

and in Eq. (62), we consider the case when the mean velocity $\bar{U}_i$ is directed along the x axis.

Substituting Eq. (12) and the relation $\ell_z = C_\ell\, z$ into Eq. (57), we arrive at the famous expression for the convective turbulent energy,

$$E_\mathrm{K} = C_\mathrm{c}\, (\beta\, F_z\, z)^{2/3}, \tag{65}$$

obtained using dimension analysis in Ref. 13. Scalings for convective turbulence similar to Eqs. (53)–(55), (60), and (61) (where $\ell_z = C_\ell\, z$) were obtained in Refs. 13–15 using dimensional analysis (see also for a review books by Refs. 1, 2, and 19). The scalings similar to Eqs. (62) and (63) were derived using dimensional analysis in Refs. 17 and 18. Most of the above scalings are in agreement with the data of the atmospheric observations discussed in Ref. 34.

For illustration, in Figs. 2–10 we show vertical profiles of various turbulent characteristics and mean velocity and potential temperature for convective $(z/L < 0)$ and stably stratified $(z/L > 0)$ turbulence. These dependencies are based on Eqs. (11), (15), (18), (21)–(26), (33)–(35), (38)–(40), and (C7), see Appendix C. Three basic dimensional numbers, $\mathrm{Ri_f}$, Pr_T, and Ri, plotted in Figs. 2–4, are related by the expression $\mathrm{Ri} = \mathrm{Ri_f}\, \mathrm{Pr}_T$.

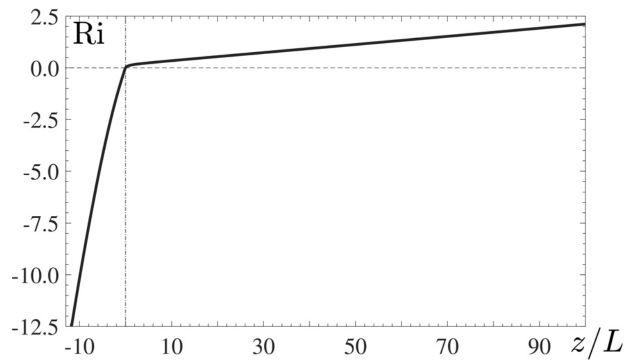

FIG. 4. The gradient Richardson number Ri vs z/L for convective $(z/L < 0)$ and stably stratified $(z/L > 0)$ turbulence.

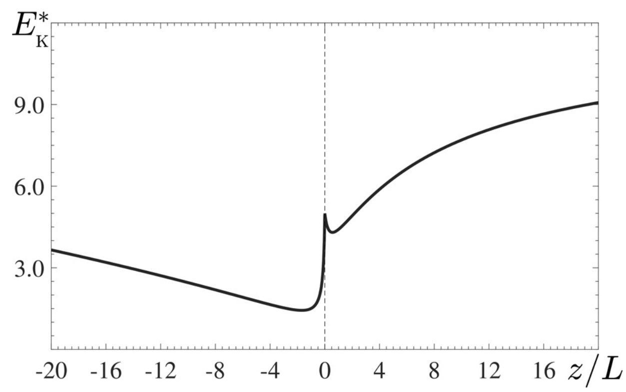

FIG. 5. The normalized turbulent kinetic energy $E_\mathrm{K}^* = E_\mathrm{K}/u_*^2$ vs z/L for convective $(z/L < 0)$ and stably stratified $(z/L > 0)$ turbulence.

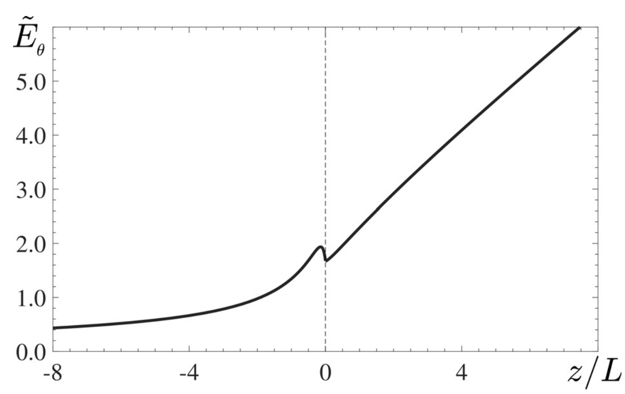

FIG. 6. The normalized intensity of potential temperature fluctuations $\tilde{E}_\theta = E_\theta/\theta_*^2$ vs z/L for convective $(z/L < 0)$ and stably stratified $(z/L > 0)$ turbulence.

Absolute values of the gradient Richardson number Ri (see Fig. 4) and the flux Richardson numbers $\mathrm{Ri_f}$ (see Fig. 2) in convective turbulence are much larger than in stably stratified turbulence. The reason is that the large-scale shear inside large-scale circulations in a

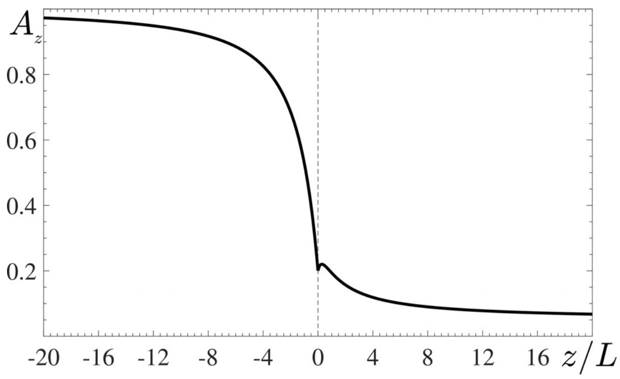

FIG. 7. The vertical anisotropy parameter A_z vs z/L for convective ($z/L < 0$) and stably stratified ($z/L > 0$) turbulence.

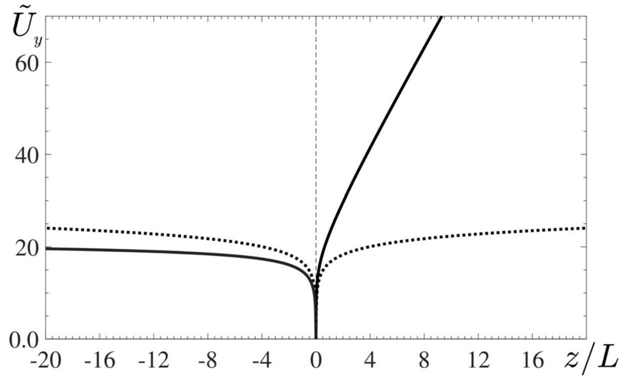

FIG. 9. The normalized mean velocity $\tilde{U}_y = \bar{U}_y/u_*$ (solid line) vs z/L for convective ($z/L < 0$) and stably stratified ($z/L > 0$) turbulence, where the normalized height of the roughness elements is $z_*/L = 1.64 \times 10^{-3}$. The dotted line corresponds to $\tilde{U}_y = \kappa^{-1} \ln(z/z_*)$.

convective turbulence is much smaller than in stably stratified turbulence. TKE in convective turbulence is much larger than in stably stratified turbulence (see Fig. 1), because in convection, both the buoyancy and large-scale shear produce turbulence. On contrary, in stably stratified turbulence, the large-scale shear produces TKE, while the buoyancy decreases TKE and produces the temperature fluctuations.

On the other hand, the normalized intensity of potential temperature fluctuations $\tilde{E}_\theta = E_\theta/\theta_*^2$ (see Fig. 6) in convective turbulence is much weaker than in stably stratified turbulence. The latter is caused by a weak gradient of the mean potential temperature in convective turbulence in comparison with that of stably stratified turbulence (see Fig. 10). The vertical anisotropy parameter A_z in stably stratified turbulence is changed stronger than in the surface layers of convective turbulence (see Fig. 7). Indeed, turbulence tends to be two-dimensional one for very large gradient Richardson number in stably stratified turbulence; that is, A_z becomes very small. On contrary, in convection the buoyancy is dominated in the energy production in the upper part of the surface layer, resulting in a strong increase in the vertical TKE, that is, the vertical anisotropy parameter $A_z \to 1$. In Fig. 8, we show the normalized vertical integral scale ℓ_z/L vs z/L for convective and stably stratified turbulence. In stably stratified turbulence, the vertical integral scale reaches the Obukhov length scale at high gradient Richardson numbers. On contrary, in convective turbulence the ratio $\ell_z/|L|$ is strongly increases with height.

Let us discuss the choice of the dimensionless empirical constants. We will start with stably stratified turbulence.[47,49] There are two well-known universal constants: the limiting value of the flux Richardson number $R_\infty = 0.2$ for an extremely strongly stratified turbulence (i.e., for $\mathrm{Ri} \to \infty$)[50,52,55] and the turbulent Prandtl number $\mathrm{Pr}_T^{(0)} = 0.8$ for a nonstratified turbulence (i.e., for $\mathrm{Ri} \to 0$).[85–87] The constant C_p describes the deviation of the dissipation timescale of $E_\theta = \langle\theta^2\rangle/2$ from the dissipation timescale of TKE. The constant C_θ is given by $C_\theta = A_z^{(\infty)}[(1 - \mathrm{R}_\infty(1 + C_\phi)]/(C_\mathrm{p}\mathrm{R}_\infty)$ [see Eq. (33)]. We use here the following values of the non-dimensional empirical constants: $C_\mathrm{p} = 0.417$, $C_\theta = 0.744$, $C_\phi = 0.899$, $\kappa_0 = 0.4$, $\mathrm{Pr}_\mathrm{T}^{(0)} = 0.8$, $A_z^{(0)} = 0.2$, $A_z^{(\infty)} = 0.1$. The parameter $C_\tau = 0.1$ for stably stratified turbulence and $C_\mathrm{F} = C_\tau/\mathrm{Pr}_T^{(0)} = 0.125$.[43,47,49]

To determine these parameters for convective turbulence, we use well-known expressions (obtained by dimensional analysis[1,2,13–15,17–19] and verified against data of various field experiments[33]) for the upper part of the surface convective layer

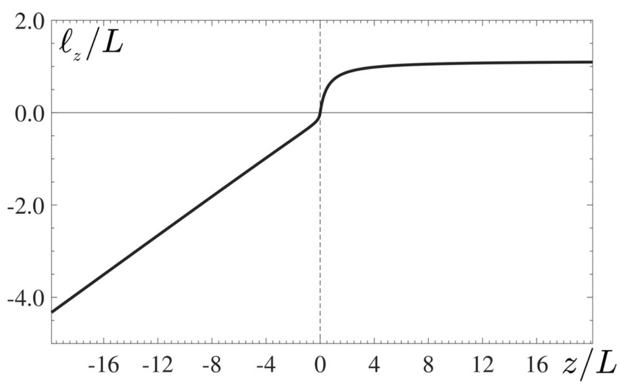

FIG. 8. The normalized vertical integral scale ℓ_z/L vs z/L for convective ($z/L < 0$) and stably stratified ($z/L > 0$) turbulence.

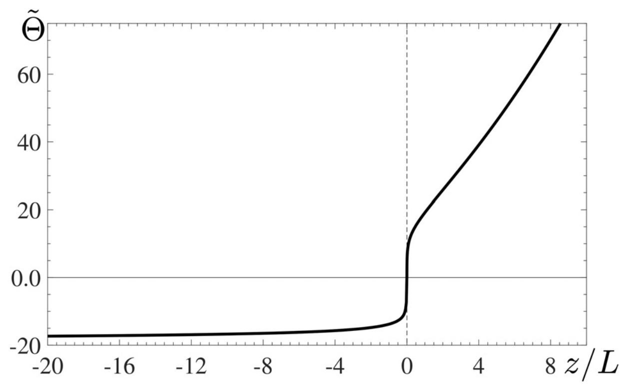

FIG. 10. The normalized mean temperature difference $\tilde{\Theta} = (\bar{T} - \bar{T}_b)/\theta_*$ vs z/L for convective ($z/L < 0$) and stably stratified ($z/L > 0$) turbulence, where $\bar{T}_b$ is the mean temperature at the lower boundary. Here, the normalized height of the roughness elements is $z_*/L = 1.64 \times 10^{-3}$.

$$\frac{\sqrt{\langle \boldsymbol{u}^2 \rangle}}{u_*} = B_3 \left(\frac{z}{|L|} \right)^{1/3}, \tag{66}$$

$$\frac{\sqrt{\langle \theta^2 \rangle}}{\theta_*} = B_4 \left(\frac{z}{|L|} \right)^{-1/3}, \tag{67}$$

$$\frac{F_x}{F_z} = -B_1 \left(\frac{z}{|L|} \right)^{-2/3}, \tag{68}$$

$$\nabla_z \bar{\Theta} = -B_2 \frac{\theta_*}{|L|} \left(\frac{z}{|L|} \right)^{-4/3} = -\frac{\theta_*}{z} \Phi_T(z/|L|), \tag{69}$$

where

$$\Phi_T(z/|L|) = B_2 \left(\frac{z}{|L|} \right)^{-1/3}. \tag{70}$$

According to the data of the field experiments presented in Ref. 33, the constants are $B_1 = 1.3$, $B_2 = 0.9$, $B_3 = 1.35$, and $B_4 = 1.55$.

Comparison of Eqs. (66)–(70) with Eqs. (57)–(62) yields

$$B_1 = (2C_\tau)^{-1/2} C_{\rm F} \left(1 + {\rm Pr}_{\rm T}^{(\infty)} \right) [\kappa_0 (1 + C_\Phi)]^{-2/3}, \tag{71}$$

$$B_2 = {\rm Pr}_{\rm T}^{(\infty)} [\kappa_0 (1 + C_\Phi)]^{-4/3}, \tag{72}$$

$$B_3 = (2/C_\tau)^{1/4} [\kappa_0 (1 + C_\Phi)]^{1/3}, \tag{73}$$

$$B_4 = (2/C_\tau)^{1/4} \left(C_{\rm p} {\rm Pr}_{\rm T}^{(\infty)} \right)^{1/2} [\kappa_0 (1 + C_\Phi)]^{-1/3}, \tag{74}$$

where we take into account that $C_\ell / C_* = \kappa_0 (1 + C_\Phi)$. Equations (71)–(74) yield

$$C_{\rm p} = \frac{B_4^2}{B_3^2 B_2}, \tag{75}$$

$$C_\tau = 2\kappa_0^{4/3} Y^4 / B_2, \tag{76}$$

$$C_\Phi = Y^3 - 1, \tag{77}$$

$$C_{\rm F} = \frac{2B_1}{B_3^2 B_2} \left[1 + \left({\rm Pr}_{\rm T}^{(\infty)} \right)^{-1} \right]^{-1}, \tag{78}$$

where Y is determined by an equation

$$Y^4 + C_{\rm p} C_\theta Y - \frac{{\rm Pr}_{\rm T}^{(0)}}{B_2 \kappa_0^{4/3}} = 0. \tag{79}$$

Equation (79) follows from Eqs. (59) and (72).

Numerical solution of Eq. (79) allows us to determine $Y = 1.152$, so that Eqs. (75)–(78) yield $C_{\rm F}^{(\infty)} = 0.505$, $C_\tau^{(\infty)} = 1.153$, $C_{\rm p}^{(\infty)} = 1.465$, ${\rm Pr}_{\rm T}^{(\infty)} = 1.335$, and $C_\Phi^{(\infty)} = 0.529$. Also, we choose the constant coefficient $C_\theta = 0.744$ for a convective turbulence to be the same as in a stably stratified turbulence.

Since the values of $C_{\rm F}^{(\infty)}$, $C_\tau^{(\infty)}$, and $C_{\rm p}^{(\infty)}$ are different from the values of these parameters at neutral and stable stratifications, we choose a smooth function $f({\rm Ri_f}) = -\alpha\, {\rm Ri_f} / (1 - \alpha\, {\rm Ri_f})$ with $\alpha = 3$, which allows us to obtain a smooth transition between the values of these parameters from neutral stratification to the end of the surface convective layer. For example, $C_{\rm F}({\rm Ri_f}) - C_{\rm F}^{(0)} = f({\rm Ri_f}) \, (C_{\rm F}^{(\infty)} - C_{\rm F}^{(0)})$, $C_\tau({\rm Ri_f}) - C_\tau^{(0)} = f({\rm Ri_f}) \, (C_\tau^{(\infty)} - C_\tau^{(0)})$, and $C_{\rm p}({\rm Ri_f}) - C_{\rm p}^{(0)} = f({\rm Ri_f}) \, (C_{\rm p}^{(\infty)} - C_{\rm p}^{(0)})$, where $C_{\rm F}^{(0)}$, $C_\tau^{(0)}$, and $C_{\rm p}^{(0)}$ are the values of these parameters at neutral stratification.

Finally, we can suggest some contributions of the developed EFB theory to the RANS and LES modeling. As to RANS modeling, we suggest to use Eq. (21) for the turbulent viscosity $K_{\rm M}(z/L) = u_* L\, {\rm Ri_f}(z/L)$ and for the eddy diffusivity $K_{\rm H}(z/L) = K_{\rm M}(z/L)/{\rm Pr_T}(z/L)$ in RANS modeling, where the vertical profiles of the flux Richardson number ${\rm Ri_f}(z/L)$ (see Fig. 2) and the turbulent Prandtl number ${\rm Pr}_T(z/L)$ (see Fig. 3) are given by Eqs. (22) and (33), respectively. Next step is to obtain numerical solutions for the mean velocity and mean potential temperature for the surface layers using RANS modeling with these vertical profiles of turbulent viscosity $K_{\rm M}(z/L)$ and the eddy diffusivity $K_{\rm H}(z/L)$, and to compare the obtained numerical solutions in a steady state with those shown in Figs. 9 and 10.

In a similar way, in LES modeling, we suggest to use Eq. (21) for the turbulent viscosity $K_{\rm M}(z/L) = u_* L\, {\rm Ri_f^*}(z/L)$ and for the eddy diffusivity $K_{\rm H}(z/L) = K_{\rm M}(z/L)/{\rm Pr_T^*}(z/L)$, where the vertical profile of the flux Richardson number ${\rm Ri_f^*}(z/L)$ is obtained from a solution of an equation

$$\frac{z}{L} = \frac{{\rm Ri_f^*}(z/L)}{\kappa_0 \left[1 - {\rm Ri_f^*}(z/L) \, (1 + C_\Phi) \right]^{1/4}}, \tag{80}$$

which follows from Eq. (26), where $z \ge z_*$ and z_* is the filtering scale used in the LES. The vertical profile of the turbulent Prandtl number ${\rm Pr}_T^*(z/L)$ is obtained from Eq. (33), where ${\rm Ri_f}$ is replaced by ${\rm Ri_f^*}(z/L)$ and $A_z(z/L)$ (see Fig. 7) is obtained from Eq. (40) for a convective turbulence. Next, to get numerical solutions for the mean velocity and mean potential temperature for the surface layers using LES with the vertical profiles of turbulent viscosity $K_{\rm M}^*(z/L)$ and the eddy diffusivity $K_{\rm H}^*(z/L)$, and to compare the obtained numerical solutions in a steady state with those shown in Figs. 9 and 10.

V. DISCUSSION AND CONCLUSIONS

We extended the energy- and flux-budget theory to the atmospheric convective surface layers. This theory applies the budget equations for turbulent energies and turbulent fluxes of momentum and heat. The EFB theory yields analytical expressions for the entire surface layer including the transition region between the lower and upper parts of the surface layer. In the framework of this theory, we determine the vertical profiles for all turbulent characteristics and for the mean velocity and mean potential temperature. In particular, we find the vertical profiles of turbulent kinetic energy, the intensity of turbulent potential temperature fluctuations, the vertical turbulent fluxes of momentum and buoyancy (proportional to potential temperature), the integral turbulence scale, the turbulent anisotropy, the turbulent Prandtl number, and the flux Richardson number.

Since the large-scale shear in convective turbulence is much smaller than in stably stratified turbulence, the absolute values of the gradient Richardson number in convective turbulence are much larger than in stably stratified turbulence. This is natural result, since turbulent kinetic energy (produced by both the buoyancy and large-scale shear) in convective turbulence is much stronger than in stably stratified turbulence. On the other hand, the large-scale shear produces turbulent kinetic energy in stably stratified turbulence, and the buoyancy decreases TKE and produces the temperature fluctuations. In convective turbulence, the gradient of the mean potential temperature is usually small in comparison with stably stratified turbulence. Therefore, potential temperature fluctuations in convective turbulence are much smaller than in stably stratified turbulence. The vertical integral scale in stably stratified turbulence can only reach the Obukhov length scale at large gradient Richardson numbers. On the other hand, the vertical integral scale in convective turbulence can be much larger than the absolute value of the Obukhov length scale.

Finally, let us discuss how the energy- and flux-budget theory can be extended to the core of the atmospheric convective boundary layer. To this end, the following basic features can be taking into account. The core of the atmospheric convective boundary layer (the CBL-core) involves three principally different types of motion:

- regular plain-parallel mean flow that can be homogeneous in the horizontal plain, but heterogeneous in the vertical direction;
- vertically and horizontally heterogeneous long-lived CBL-scale self-organized convective structures (large-scale circulations or large-scale convective cells);
- small-scale turbulent fluctuations.

Turbulence in the CBL-core is generated largely by local shears of the self-organized convective structures. Kinetic energy of such turbulence should be low compared to kinetic energy of large-scale structural motions (see, e.g., Ref. 88). About 80% of vertical heat transport is due to structural motions and only 20% due to turbulence.

VI. DEDICATION

This paper was dedicated to Prof. Sergej Zilitinkevich (1936–2021) who initiated this work and discussed some of the obtained results.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Igor Rogachevskii: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal). **Nathan Kleeorin:** Conceptualization (equal); Formal analysis (equal); Investigation (equal); Writing – original draft (equal). **Sergej Zilitinkevich:** Conceptualization (equal); Investigation (equal); Methodology (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

NOMENCLATURE

$A_{x,y} = E_{x,y}/E_{\rm K}$	Horizontal anisotropy parameters
$A_z = E_z/E_{\rm K}$	Vertical anisotropy parameter
$E_{\rm K} = \langle \boldsymbol{u}^2 \rangle/2$	Density of turbulent kinetic energy (TKE)
$\tilde{E}_{\rm K} = E_{\rm K}/E_{\rm K0}$	Normalized density of turbulent kinetic energy
$E_{\rm K0} = u_*^2/(2C_\tau A_z)^{1/2}$	Density of turbulent kinetic energy at the surface
$E_z = \langle u_z^2 \rangle/2$	Density of the vertical turbulent kinetic energy
$E_\alpha = \langle u_\alpha^2 \rangle/2$	Horizontal and vertical turbulent kinetic energies ($\alpha = x, y, z$)
$E_\theta = \langle \theta^2 \rangle/2$	Intensity of potential temperature fluctuations
$F_i = \langle u_i\, \theta \rangle$	Turbulent flux of potential temperature
$F_{x,y}$	Horizontal turbulent flux of potential temperature
$F_z = \langle u_z\, \theta \rangle$	Vertical turbulent flux of potential temperature
$\boldsymbol{g}$	Gravity acceleration
$K_{\rm M}$	Turbulent (eddy) viscosity
$K_{\rm H}$	Turbulent heat conductivity
$L = -\tau^{3/2}/(\beta\, F_z)$	Local Obukhov length
ℓ_z	Vertical integral scale
$N = (\beta\, \lvert\nabla_z \bar{\Theta}\rvert)^{1/2}$	Brunt–Väisälä frequency
P	Fluid pressure with the reference value P_*
$\bar{P}$	Mean fluid pressure
${\rm Pr}_{\rm T} = K_{\rm M}/K_{\rm H}$	Turbulent Prandtl number
${\rm Pr}_{\rm T}^{(0)} = C_\tau/C_{\rm F}$	Turbulent Prandtl number for a non-stratified turbulence when the mean potential temperature gradient vanishes
p	Fluctuations of the fluid pressure
$Q_{ij} = \rho_0^{-1}(\langle p \nabla_i u_j \rangle + \langle p \nabla_j u_i \rangle)$	Inter-component energy exchange term
$Q_{\alpha\alpha} = 2\rho_0^{-1}(\langle p \nabla_\alpha u_\alpha \rangle$	Diagonal terms of the tensor Q_{ij}
${\rm Ri} = N^2/S^2$	Gradient Richardson number
${\rm Ri}_{\rm f} = -\beta\, F_z/(K_{\rm M} S^2)$	Flux Richardson number
$R_\infty = {\rm Ri}_{\rm f}({\rm Ri} \to \infty) = 0.2$	Flux Richardson number at very large gradient Richardson number
$S = [(\nabla_z \bar{U}_x)^2 + (\nabla_z \bar{U}_y)^2]^{1/2}$	Mean velocity shear caused by the horizontal mean wind velocity
T	Fluid temperature with the reference value T_*
$t_{\rm T} = \ell_z/E_z^{1/2}$	Turbulent dissipation timescale
$\bar{\boldsymbol{U}} = \bar{\boldsymbol{U}}^{(\rm w)}(z) + \bar{\boldsymbol{U}}^{(\rm s)}$	Mean fluid velocity
$\bar{\boldsymbol{U}}^{(\rm s)}$	Three-dimensional mean velocity related to the large-scale semi-organized coherent structures in a convective turbulence
$\bar{\boldsymbol{U}}^{(\rm w)}(z) = (\bar{U}_x, \bar{U}_y, 0)$	Mean-wind velocity
$\boldsymbol{u} = (u_x, u_y, u_z)$	Fluctuations of the fluid velocity
u_*	Local (z-dependent) friction velocity
$\tilde{Z} = \ell_z/(C_*\, L)$	Normalized height
$\beta = g/T_*$	Buoyancy parameter
$\gamma = c_{\rm p}/c_{\rm v}$	Specific heat ratio
δ_{ij}	Kronecker unit tensor

$\varepsilon_i^{(\mathrm{F})} = (\nu + \chi)\,\langle(\nabla_j u_i)\,(\nabla_j \theta)\rangle$	Dissipation rate of the turbulent heat flux
$\varepsilon_{\mathrm{K}} = \nu\,\langle(\nabla_j u_i)^2\rangle$	Dissipation rate of E_{K}
$\varepsilon_\alpha = \nu\,\langle(\nabla_j u_\alpha)^2\rangle$	Dissipation rate of horizontal and vertical turbulent kinetic energy components E_α with $\alpha = x, y, z$
$\varepsilon_\theta = \chi\,\langle(\nabla\theta)^2\rangle$	Dissipation rate of E_θ
$\varepsilon_i^{(\tau)} = \varepsilon_{iz}^{(\tau)} - \beta\,F_i - Q_{iz}$	Effective dissipation rate of the off-diagonal components of the Reynolds stress τ_{iz}
$\varepsilon_{iz}^{(\tau)} = 2\nu\,\langle(\nabla_j u_i)\,(\nabla_j u_z)\rangle$	Molecular-viscosity dissipation rate of the off-diagonal components of the Reynolds stress τ_{iz}
$\Theta = T(P_*/P)^{1-\gamma^{-1}}$	Potential temperature
$\bar{\Theta}(z)$	Mean potential temperature
θ	Fluctuations of the potential temperature
$\theta_* = \|F_z\|/u_* = u_*^2/(\beta\,\|L\|)$	Level of potential temperature fluctuations
$\kappa_0 = 0.4$	von Karman constant
ν	Kinematic viscosity of fluid
$\Pi_{\mathrm{K}} = -\tau_{iz}\,\nabla_z \bar{U}_i = -(\tau_{xz}\,\nabla_z \bar{U}_x + \tau_{yz}\,\nabla_z \bar{U}_y) = K_{\mathrm{M}}\,S^2$	Production rate of the turbulent kinetic energy
ρ_0	Fluid density
$\varsigma = z/L$	Dimensionless height
$\tau_{iz} = \langle u_i\,u_z\rangle$ with $i = x, y$	Off-diagonal components of the Reynolds stress
$\tau = (\tau_{xz}^2 + \tau_{yz}^2)^{1/2} = K_{\mathrm{M}}\,S \equiv u_*^2$	Vertical turbulent flux of momentum
$\mathbf{\Phi}_i^{(\mathrm{F})} = \langle u_i\,u_z\,\theta\rangle - \nu\,\langle\theta\,(\nabla_z u_i)\rangle - \chi\,\langle u_i\,(\nabla_z\theta)\rangle$	Flux of F_i
$\Phi_i^{(\tau)} = \langle u_i\,u_z^2\rangle + \rho_0^{-1}\,\langle p\,u_i\rangle - \nu\,\nabla_z \tau_{iz}$	Flux of τ_{iz}
$\Phi_{\mathrm{K}} = \rho_0^{-1}\langle u_z\,p\rangle + (\langle u_z\,\mathbf{u}^2\rangle - \nu\,\nabla_z\langle\mathbf{u}^2\rangle)/2$	Flux of E_{K}
$\Phi_{x,y} = (\langle u_z\,u_{x,y}^2\rangle - \nu\,\nabla_z\langle u_{x,y}^2\rangle)/2$	Flux of the horizontal turbulent kinetic energy components $E_{x,y}$
$\Phi_z = \rho_0^{-1}\langle u_z\,p\rangle + (\langle u_z^3\rangle - \nu\,\nabla_z\langle u_z^2\rangle)/2$	Flux of the vertical turbulent kinetic energy components E_z
Φ_α	Flux of E_α with $\alpha = x, y, z$
$\Phi_\theta = (\langle u_z\,\theta^2\rangle - \chi\,\nabla_z\langle\theta^2\rangle)/2$	Flux of E_θ
χ	Molecular temperature diffusivity

APPENDIX A: DERIVATION OF EQ. (32) FOR THE EDDY DIFFUSIVITY

In this Appendix, we derive the expression for the turbulent Prandtl number, $\mathrm{Pr_T} = K_{\mathrm{M}}/K_{\mathrm{H}}$. To this end, we use the steady-state versions of Eqs. (2) and (4),

$$F_z\,\nabla_z\bar{\Theta} + \frac{E_\theta}{C_{\mathrm{p}}\,t_{\mathrm{T}}} = 0, \tag{A1}$$

$$2E_z\,\nabla_z\bar{\Theta} - 2C_\theta\,\beta\,E_\theta + \frac{F_z}{C_{\mathrm{F}}\,t_{\mathrm{T}}} = 0. \tag{A2}$$

Equations (A1) and (A2) and the expression for the vertical turbulent heat flux, $F_z = -K_{\mathrm{H}}\,\nabla_z\bar{\Theta}$, yield the turbulent heat conductivity K_{H} as

$$K_{\mathrm{H}} = 2C_{\mathrm{F}}\,t_{\mathrm{T}}\,E_z\left[1 + \frac{C_\theta\,C_{\mathrm{p}}\,t_{\mathrm{T}}\,\beta F_z}{E_z}\right]. \tag{A3}$$

By means of Eq. (9) for TKE,

$$E_{\mathrm{K}} = K_{\mathrm{M}}\,S^2\,t_{\mathrm{T}}\,[1 - \mathrm{Ri_f}\,(1 + C_\Phi)], \tag{A4}$$

and by means of Eq. (19) for $\mathrm{Ri_f}$, we derive the identity for the dimensionless ratio as

$$\frac{\beta\,F_z\,t_{\mathrm{T}}}{E_z} = -\frac{\mathrm{Ri_f}}{A_z[1 - \mathrm{Ri_f}\,(1 + C_\Phi)]}. \tag{A5}$$

Thus, Eqs. (A3) and (A5) yield Eq. (32) for the eddy diffusivity.

APPENDIX B: DERIVATION OF EQS. (39)–(41) FOR THE VERTICAL AND HORIZONTAL ANISOTROPY PARAMETERS

In this appendix, we derive equations for the vertical anisotropy parameter, $A_z \equiv E_z/E_{\mathrm{K}}$. The mean shear S generates the energy of longitudinal velocity fluctuations E_x. Due to inter-component energy exchange term $Q_{\alpha\alpha}$, the transverse E_y and the vertical E_z components of turbulent kinetic energy are produced. The inter-component energy exchange term $Q_{\alpha\alpha}$ is usually parameterized using the return-to-isotropy hypothesis.[82] However, the temperature-stratified turbulence is anisotropic, and the inter-component energy exchange term $Q_{\alpha\alpha}$ should depend on the flux Richardson number $\mathrm{Ri_f}$.

Here, we adopt the following model for the inter-component energy exchange term $Q_{\alpha\alpha}$, which generalizes the return-to-isotropy hypothesis to the case of the convective and stably stratified turbulence. We use the normalized flux Richardson number $\mathrm{Ri_f}/R_\infty$ that is varying from 0 for a non-stratified turbulence to 1 for a strongly stratified turbulence, where the limiting value of the flux Richardson number, $\mathrm{R}_\infty \equiv \mathrm{Ri_f}|_{\mathrm{Ri}\to\infty}$, is defined for very strong stratifications when the gradient Richardson number $\mathrm{Ri} \to \infty$. The model for the inter-component energy exchange term $Q_{\alpha\alpha}$ is described by

$$Q_{xx} = -\frac{2(1 + C_{\mathrm{r}})}{t_{\mathrm{T}}}\left(E_x - \frac{1}{3}\,E_{\mathrm{int}}\right), \tag{B1}$$

$$Q_{yy} = -\frac{2(1 + C_{\mathrm{r}})}{t_{\mathrm{T}}}\left(E_y - \frac{1}{3}\,E_{\mathrm{int}}\right), \tag{B2}$$

$$Q_{zz} = -\frac{2(1 + C_{\mathrm{r}})}{t_{\mathrm{T}}}\left(E_z - E_{\mathrm{K}} + \frac{2}{3}\,E_{\mathrm{int}}\right), \tag{B3}$$

where

$$E_{\mathrm{int}} = E_{\mathrm{K}}\left[1 - \frac{\mathrm{Ri_f}}{R_\infty}\left(\frac{C_{\mathrm{r}}}{1 + C_{\mathrm{r}}}\right)\right], \tag{B4}$$

and C_{r} is the dimensionless empirical constant. When $\mathrm{Ri_f} = 0$, Eqs. (B1)–(B4) describe the return-to-isotropy hypothesis.[82] To derive equation for the vertical anisotropy parameter in a stratified turbulence, we use the steady-state version of Eq. (6) for vertical TKE E_z as

$$\nabla_z\,\Phi_z = \beta\,F_z + \frac{1}{2}Q_{zz} - \frac{E_{\mathrm{K}}}{3t_{\mathrm{T}}}. \tag{B5}$$

We assume that the vertical gradient $\nabla_z \Phi_z$ of the flux of E_z is determined by the buoyancy, that is, $\nabla_z \Phi_z = -C_z\, \beta\, F_z$, where C_z is the dimensionless empirical constant. The justification of this assumption for a convective turbulence has been performed in Ref. 30, where experimental data obtained from meteorological observations at the Eureka station have been used for the validation of this assumption (see the left panel in Fig. 1 in Ref. 30). Thus, by means of Eqs. (A5) and (B3)–(B5), we determine the vertical anisotropy parameter $A_z \equiv E_z/E_{\rm K}$ as a function of the flux Richardson number

$$A_z({\rm Ri_f}) = A_z^{(0)} - {\rm Ri_f} \left[\frac{(1-3A_z^{(0)})\,(1+C_z)}{1-{\rm Ri_f}\,(1+C_\Phi)} - \frac{2A_z^{(0)}}{{\rm R}_\infty}\right]. \tag{B6}$$

According to Eq. (B6), the vertical anisotropy parameter A_z for a non-stratified turbulence is $(A_z)_{{\rm Ri}\to 0} \equiv A_z^{(0)} = C_{\rm r}/3(1+C_{\rm r})$. Usually in surface layers in convective turbulence, $|{\rm Ri_f}| \ll |{\rm R}_\infty|$. This implies that the vertical anisotropy parameter in a convective turbulence is given by

$$A_z({\rm Ri_f}) = A_z^{(0)} + (1-3A_z^{(0)})\, \frac{|{\rm Ri_f}|\,(1+C_z)}{1+|{\rm Ri_f}|\,(1+C_\Phi)}. \tag{B7}$$

In convective turbulence for large $|{\rm Ri_f}| \gg 1$, the vertical anisotropy parameter $A_z \to 1$.[30] This condition yields

$$\frac{1+C_z}{1+C_\Phi} = \frac{1-A_z^{(0)}}{1-3A_z^{(0)}}. \tag{B8}$$

Substituting Eq. (B8) into Eq. (B7), we obtain the vertical anisotropy parameter in a stably stratified turbulence given by Eq. (39). In a convective turbulence where $|{\rm Ri_f}| \ll |{\rm R}_\infty|$ and ${\rm Ri_f} < 0$, the vertical anisotropy parameter is given by Eq. (40).

Note that Eqs. (B1)–(B4) describes a simple generalization of the return-to-isotropy hypothesis.[82] These equations affect only Eq. (39) for the dependence of the vertical anisotropy parameter on the flux Richardson number, $A_z({\rm Ri_f})$. This function is the most unknown in observations. The return-to-isotropy hypothesis[82] implies that the inter-component energy exchange $E_{\rm int} = E_{\rm K}$ [see Eq. (B4)]. However, the return-to-isotropy hypothesis yields the results for stably stratified turbulence, which are in a disagreement with observations for very large gradient Richardson number. On the other hand, the main results obtained in the present study are weakly dependent on the model for the inter-component energy exchange term $Q_{\alpha\alpha}$. In particular, in our previous studies,[43,47,49] we used different models for $Q_{\alpha\alpha}$ and obtained similar results.

When turbulence is isotropic in the horizontal plane, the horizontal shares of TKE are $A_x = A_y = 1 - A_z$. This yields the horizontal components of TKE as

$$E_x = E_y = \frac{1}{2}\, E_{\rm K}\,(1-A_z), \tag{B9}$$

where $A_x = E_x/E_{\rm K}$ and $A_y = E_y/E_{\rm K}$. When turbulence is anisotropic in the horizontal plane, the model for the inter-component energy exchange term $Q_{\alpha\alpha}$ is given by

$$Q_{xx} = -\frac{2(1+C_{\rm r})}{t_{\rm T}} \left[E_x - \frac{E_{\rm int}}{3} + A_z^{(0)} \left(C_1 + C_2 \frac{{\rm Ri_f}}{R_\infty}\right)\right], \tag{B10}$$

$$Q_{yy} = -\frac{2(1+C_{\rm r})}{t_{\rm T}} \left[E_y - \frac{E_{\rm int}}{3} - A_z^{(0)} \left(C_1 + C_2 \frac{{\rm Ri_f}}{R_\infty}\right)\right]. \tag{B11}$$

Using Eq. (6) for E_x, we obtain that the horizontal anisotropy parameter, $A_x = E_x/E_{\rm K}$, for stably stratified turbulence is given by Eq. (41) and $A_y = 1 - A_x - A_z$, where the vertical anisotropy parameter A_z is given by Eq. (39).

APPENDIX C: THE ATMOSPHERIC STABLY STRATIFIED BOUNDARY-LAYER TURBULENCE

In view of the applications of the obtained results to the atmospheric stably stratified boundary-layer turbulence, we outline below the useful in modeling theoretical relationships.[47,49] It is known that the wind shear in stably stratified turbulence has two asymptotic results: (i) $S = \tau^{1/2}/(\kappa_0\, z)$ at $\varsigma \ll 1$, which describes the log-profile for the mean velocity, and (ii) $S = \tau^{1/2}/({\rm R}_\infty\, L)$ when $\varsigma \gg 1$. The latter result follows from Eq. (23), where $\varsigma = \int_0^z dz'/L(z')$ is the dimensionless height based on the local Obukhov length scale $L(z)$, and $\kappa_0 = 0.4$ is the von Karman constant. For surface layer in stably stratified turbulence (defined as the lower layer which is 10% of the turbulent boundary layer), the Obukhov length scale L is independent of z and the dimensionless height $\varsigma = z/L$. Interpolating these two asymptotic results, we obtain that the wind shear $S(\varsigma)$ can be written as

$$S = \frac{\tau^{1/2}}{L} \left(R_\infty^{-1} + \frac{1}{\kappa_0\, \varsigma}\right). \tag{C1}$$

The latter allows us to get the vertical profile of the turbulent viscosity $K_{\rm M}(\varsigma) = \tau/S$ as

$$K_{\rm M} = \tau^{1/2}\, L\, \frac{\kappa_0\, \varsigma}{1 + R_\infty^{-1}\, \kappa_0\, \varsigma}. \tag{C2}$$

Using Eqs. (21) and (C2), we arrive at the expression for the vertical profile of the flux Richardson number ${\rm Ri_f}(\varsigma)$ as

$${\rm Ri_f} = \frac{\kappa_0\, \varsigma}{1 + R_\infty^{-1}\, \kappa_0\, \varsigma}. \tag{C3}$$

Equation (C3) yields the expression for ς as

$$\varsigma = \frac{{\rm Ri_f}}{\kappa_0\,(1 - {\rm Ri_f}/R_\infty)}. \tag{C4}$$

In this case, the vertical anisotropy parameter $A_z(\varsigma) \equiv E_z/E_{\rm K}$ reads

$$A_z = A_z^{(0)} + \frac{1 - A_z^{(0)}}{1 - (1+C_\Phi)^{-1}\left[(\kappa_0\,\varsigma)^{-1} + {\rm R}_\infty^{-1}\right]} + \frac{2A_z^{(0)}}{1 + {\rm R}_\infty\,(\kappa_0\,\varsigma)^{-1}}, \tag{C5}$$

and the vertical profile of the turbulent Prandtl number ${\rm Pr}_T(\varsigma)$ is given by

$${\rm Pr_T} = {\rm Pr_T^{(0)}} \left[1 - \frac{C_\theta\, C_{\rm p}}{A_z\left[{\rm R}_\infty^{-1} + (\kappa_0\,\varsigma)^{-1} - (1+C_\Phi)\right]}\right]^{-1}. \tag{C6}$$

Note that the gradient Richardson number Ri and the flux Richardson number $\mathrm{Ri_f}$ are related as $\mathrm{Ri}(\varsigma) = \mathrm{Ri_f}(\varsigma)\,\mathrm{Pr_T}(\varsigma)$.

Equations (C2)–(C6) are in agreement with Monin–Obukhov–Nieuwstadt similarity theories.[16,89] In the Monin–Obukhov similarity theory,[16] the turbulent fluxes of momentum τ, heat F_z, the Obukhov length scale L, and other scalars are approximated by their surface values, while the similarity theory by Nieuwstadt[89] is extended to the entire stably stratified boundary layer employing local z-dependent values of the turbulent fluxes $\tau(z)$ and $F_z(z)$, and the length $L(z)$ instead of their surface values.

Using Eqs. (22) and (C4), we can relate ς and $\tilde{Z}$ for stably stratified turbulence as

$$\varsigma = \frac{\tilde{Z}\,\tilde{E}_{\mathrm{K}}^{1/2}}{\kappa_0\,(1 + C_\Phi - \tilde{Z}\,\tilde{E}_{\mathrm{K}}^{1/2}/R_\infty)}. \tag{C7}$$

For the surface layer ($\tilde{Z} \ll 1$) of the stably stratified turbulence, the dimensionless height is $\varsigma = z/L$, and the normalized TKE is $\tilde{E}_{\mathrm{K}} \approx 1$ [see Eq. (16)]. Therefore, Eq. (C7) in this case is reduced to

$$\frac{z}{L} = \frac{\tilde{Z}}{\kappa_0\,(1 + C_\Phi)}. \tag{C8}$$

This equation coincides with Eq. (18) derived for the low part ($|\tilde{Z}| \ll 1$) of the surface layer in convective turbulance. The EFB theory for stably stratified turbulence described in Secs. II and III, and Appendix C, has been verified against scarce data[90–95] from the atmospheric and laboratory experiments, DNS and LES.

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