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Nonlinear waves and pattern formation in multiphase flows in porous media

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Abstract

The paper analyzes pattern formation in initially homogeneous one-dimensional two-phase flows in porous medium. It is shown that generally these flows are unstable. The mechanism of the instabilities is associated with inertial effects. Such instabilities are of explosive type and are probably important in various engineering applications and natural phenomena. In small-amplitude finite approximation the evolution of patterns is governed by the Korteweg-de Vries-Burgers equation. Pattern formation occurs when the coefficient multiplying the Burgers term becomes negative. During nonlinear evolution a soliton with a tail is formed. The amplitude of the soliton increases while the tail decreases. These results can be regarded as a generalization of results by Harris and Crighton (1994) to the case of two-phase flows in porous medium. The obtained solution in form of soliton with a tail can be interpreted as initial phase of formation of the phase composition inhomogeneities in porous media. In the case of fluidized beds this pattern can be regarded as initial phase of bubble formation in a fluidized bed of granular material. The characteristic size of bubbles and time of its formation are estimated.

1. Introduction

Patterns in multiphase flows are generally formed due to different instabilities. The well known example is Saffman-Taylor instability [1] of interface between different fluids during two-phase filtration in porous media. This instability leads to formation of viscous fingers. Another example is an instability of flows with initially homogeneous phase composition, i.e. without distinct phase interface. This instability was studied in linear approximation for fluidized bed by [2–5]. A mechanism of this instability is associated with inertial terms in the Navier-Stokes equation.

However, the inertial effects are commonly not considered in filtration problems but in many cases their effect becomes crucial. Indeed, inertial effects in multiphase filtration are generally small. Actually this condition implies the main equation of filtration flows - Darcy's law. However the analysis of long time behavior of the system requires to take these effects into account. The instability of phase composition in multiphase flows in porous media is studied in the present paper. Thus, e.g., during long time operation of any device with multiphase filtration flows this instability is generally excited and may consequently lead to an accident. The analysis presented below probably can be of relevance in this respect.

Simultaneously with inertial terms we also consid-

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ered viscous type (Stokes) terms. It has been shown recently that these viscous effects can cause the cubic dispersion of both 'magma' waves in the Earth [6,7] and nonlinear waves in fluidized beds [8] and in porous media [9]. When nonlinear effects are taken into account this results in appearance of soliton type solutions, which can be described in the small-amplitude finite approximation by the Korteweg-de Vries equation. For the case of two-phase filtration it was found in [9] that the perturbations of the phase composition result in formation of structures similar to nonstationary nondissipative shock waves [10]. The nondissipative shock wave emerges as a nonstationary multi-soliton solution of the Korteweg-de Vries equation that describes evolution of small perturbations of phase composition.

When inertial effects in two-phase flow are taken into account the small-amplitude finite approximation yields generally the Korteweg-de Vries-Burgers equation for fluidized beds [4,5] and in suspensions of particles in fluids [11,36]. The instability and pattern formation occur when the coefficient multiplying the Burgers term becomes negative [4,5].

The present paper can be considered as a certain generalization of results [4,5] to the case of two-phase flows in porous medium. We have shown that the developing instability of two-phase flows in porous medium is of explosive type and of the nonlinear stage of its evolution a soliton with a tail is formed. The amplitude of the soliton increases while the tail decreases. This solution can be interpreted as initial phase of formation of the phase composition inhomogeneities in porous media. In the case of fluidized beds this pattern can be regarded as initial phase of bubble formation in a fluidized bed of granular material (see, e.g., [4,5]). The characteristic size of bubbles and time of its formation are estimated. For simplicity we consider the one-dimensional case since it allows us to demonstrate the principal effects. This paper presents in detail the results reported previously in [13].

The instability discussed here is similar to the phenomenon of spinodal decomposition whereby an initially homogeneous binary system decomposes into its constituents. The origin of the phenomenon of spinodal decomposition is associated with metastability of the initial homogeneous state. Remarkably this phase transition is similar to the macroscopic effect of phase decomposition of the homogeneous two-phase flow (see, e.g., [14]).

2. The governing equations

Let us analyze the case of one-dimensional filtration of two liquids through porous medium. The continuity equations for both phases and their equations of motion in non-dimensional form are given by:

$$\frac{\partial \Phi_i}{\partial t} + \frac{\partial}{\partial x}(\Phi_i v_i) = 0\ , \tag{1}$$

$$\varepsilon\rho_i\Big(\frac{\partial v_i}{\partial t} + v_i\frac{\partial v_i}{\partial x}\Big) = -T\frac{\partial p_i}{\partial x} + G\rho_i - \alpha_i\Phi_i v_i + \frac{1}{\Phi_i}\frac{\partial}{\partial x}\Big(\beta_i\Phi_i\frac{\partial v_i}{\partial x}\Big) + S(\Phi_{i,j})\Phi_j(v_j - v_i)\ , \tag{2}$$

where ρ_i, p_i, v_i are the non-dimensional density, pressure and velocity of the i-th phase ($i = 1,2$) measured in units of ρ_*, p_*, v_*, respectively; Φ_i is the volume fraction of the i-th phase. For the two-phase media $\Phi_1 + \Phi_2 = 1$. The coordinate x and time t are measured in units of L_* and L_*/v_*, respectively. The basic dimensionless parameters of the problem are

$$\varepsilon = \frac{\rho_* v_* K_*}{L_*\mu_*},\quad F = \frac{v_*^2}{gL_*},\quad T = \varepsilon\frac{p_*}{\rho_* v_*^2},$$
$$G = \frac{\varepsilon}{F},\quad \alpha_i = \frac{\mu_i}{K},\quad \beta_i = \frac{\eta_i K_*}{L_*^2}\ . \tag{3}$$

Here F is the Froude number, μ_i and η_i are the dynamic viscosities measured in units of μ_*. Dynamic viscosities μ_i are determined by the filtration of each of the liquids through the porous medium (the Darcy's term), while η_i corresponds to the Brinkman's (Stokes) term (see, e.g., [15]). The function $S(\Phi) = \sigma(\Phi)\Phi(1-\Phi)$ determines the mutual friction of the two filtrating liquids, K is the permeability of the porous medium measured in the units K_*. A form of the function $\sigma(\Phi)$ depends on the model of the medium. For example, for a porous medium with Carman-Kozeny law (see, e.g., [16]) $\sigma(\Phi) = \kappa\Phi^{-3}$, where the coefficient κ depends upon the geometry of the porous medium. The numerical values of the parameter κ for various types of porous media can be found in [16]. The parameters ε (multiplying the inertial terms) and β_i (multiplying the Brinkman term) are small since $\sqrt{K_*} \ll L_*$, but for slow processes these terms can be essential as shown below. The value of $\sqrt{K_*}$ is determined by the microscale of pores. The appearance of the Brinkman and inertial terms in the governing equations results in a qualitative change of the evolution of finite perturbations in multiphase flows in porous media (see below).

The system of Eqs. (1),(2) is widely used for analysis of multiphase flows of different nature. Examples of these flows are pneumatic transport, fluidized beds, multiphase hydrodynamics, problems of viscous consolidation, flows of magma through viscous deformable rock, two-fluids magnetohydrodynamics (with the Ampere force), and other media (see, e.g., [17–21]). The governing equations of two-phase flows through porous media can be derived from mass and momentum conservation equations for a three-phase continuum (see, e.g., [17]) where one of the phases is a rigid porous matrix and the densities ρ_i of the liquid phases are constant. A unique nontrivial solution of these equations for the velocity $\mathbf{v}_m$ of the rigid porous matrix is given by $\mathbf{v}_m = const$. Transition to a frame moving with the velocity $\mathbf{v}_m$ yields the system of Eqs. (1),(2). Here the velocities of the liquid phases are measured relative to the velocity $\mathbf{v}_m$.

The unknowns in Eqs. (1),(2) are v_1, v_2, p_1, p_2 and Φ_2. Here $\Phi_1 = 1 - \Phi_2$. In such systems the pressures of phases are either equal or the pressure difference is considered as a known function of v_i and Φ_2 (see below). Thus we arrive at a system of four equations for four unknown functions.

In the new variables $\Phi(x,t)$ and $\Psi(x,t)$ Eqs. (1),(2) can be reduced to

$$\frac{\partial \Phi}{\partial t} + W\frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial x} = 0 , \tag{4}$$

$$W\{\Phi(\alpha_1+\alpha_2) - \alpha_1\} + \Psi\varphi + G(\rho_1 - \rho_2) = \varepsilon \hat{F}(\Psi, \Phi) , \tag{5}$$

where $W = (1-\Phi)v_1 + \Phi v_2$ is the total volume flow rate which is independent of the coordinate x, the relative velocity of two phases is $V = v_2 - v_1$, $\Psi(x,t) = V\Phi(1-\Phi)$, $\Phi \equiv \Phi_2$, $\varphi = \alpha_1 + \alpha_2 + \sigma(\Phi)$. The function $\hat{F}(\Psi,\Phi)$ is given by

$$\hat{F}(\Psi,\Phi) = -\rho_a\left(\frac{\partial \Psi}{\partial t} + W\frac{\partial \Psi}{\partial x}\right) - 2\rho_b\Psi\frac{\partial \Psi}{\partial x} + \rho_c\Psi^2\frac{\partial \Phi}{\partial x} + \varepsilon^{-1}\left(\beta_a\frac{\partial^2 \Psi}{\partial x^2} + \beta_b\frac{\partial^2 \Phi}{\partial x^2}\right) - \frac{\partial Q(\Phi,\Psi^2)}{\partial x} , \tag{6}$$

where we take into account a possible difference between the pressures of phases

$$Q(\Phi,\Psi^2) = \frac{p_*}{\rho_* v_*^2}(p_1(x) - p_2(x)) , \tag{7}$$

$$\rho_a = \frac{\rho_1}{1-\Phi} + \frac{\rho_2}{\Phi}, \quad \rho_b = \frac{\rho_1}{(1-\Phi)^2} - \frac{\rho_2}{\Phi^2},$$
$$\rho_c = \frac{\rho_1}{(1-\Phi)^3} + \frac{\rho_2}{\Phi^3} ,$$
$$\beta_a = \frac{\beta_1}{1-\Phi} + \frac{\beta_2}{\Phi}, \quad \beta_b = \frac{\beta_1}{(1-\Phi)^2} - \frac{\beta_2}{\Phi^2},$$
$$\beta_c = \frac{\beta_1}{(1-\Phi)^3} + \frac{\beta_2}{\Phi^3}, . \tag{8}$$

Since we assume that $\beta \le \varepsilon$, i.e. β is sufficiently small, the function $\hat{F}(\Psi,\Phi)$ defined above is of order 1. In the opposite case, i.e. $\beta > \varepsilon$, the parameter β can be considered as a small parameter instead of ε and the final results do not change. The explicit form of the function $Q(\Phi,\Psi^2)$ depends on the specific model of interaction between phases. For the analysis of stability of two-phase flows we have to know only derivatives of the function Q which is supposed to be known. It is important to note that in case of single phase filtration Eqs. (4),(5) are not valid. All the effects discussed below occur only in the case of multiphase filtration.

In equilibrium a homogeneous and steady flow of a two-phase medium filtrated through the matrix is determined by

$$W\{\alpha_1 - \Phi_0(\alpha_1+\alpha_2)\} = \Psi_0\varphi_0 + G(\rho_1 - \rho_2) , \tag{9}$$

where the subscript 0 corresponds to equilibrium values. The condition (9) relates the total volume flow rate W, the equilibrium volume fraction Φ_0, and the relative velocity of two phases V_0 for given parameters of the porous medium and two liquid phases. This equilibrium flow can be unstable, i.e., small perturbations can grow.

For small ε and β_i we can study this instability (see Section 3) and evolution of finite amplitude perturbations (see Section 4). For this purpose we represent the volume fraction Φ and the function Ψ as a sum of two terms:

$$\Phi = \Phi_0 + f, \quad \Psi = \Psi_0 + \psi \, . \tag{10}$$

A nonlinear equation for the finite perturbation of volume fraction f in a frame moving with velocity V_* is given by (for details see Appendix)

$$\frac{\partial f}{\partial t} + bf\frac{\partial f}{\partial x} + c\frac{\partial^3 f}{\partial x^3} = -\varepsilon\Gamma_0\frac{\partial^2 f}{\partial x^2} - \varepsilon a\frac{\partial^2 f}{\partial x \partial t} \, , \tag{11}$$

where

$$a = \rho_a\frac{V_d}{\varphi_0}, \quad b = \frac{1}{\varphi_0}(2V_d\sigma_0' - \Psi_0\sigma_0''),$$

$$c = \frac{1}{\varphi_0}(\beta_b - V_d\beta_a), \tag{12}$$

$$\Gamma_0 = \frac{1}{\varphi_0}[\rho_a V_d^2\chi(\rho, \Psi_0, \Phi_0) + 2\Psi_0 V_d Q_\Psi - Q_\Phi] \, , \tag{13}$$

and

$$V_d = W - V_*,$$

$$V_* = \frac{1}{\varphi_0}[W\sigma_0 - V_0\sigma_0'\Phi_0(1-\Phi_0)] \, ,$$

$$\varphi_0 = \varphi(\Phi = \Phi_0), \quad \sigma_0' = \frac{d\sigma_0}{d\Phi} \, ,$$

$$\chi(\rho, \Psi_0, \Phi_0) = 1 + 2\frac{\rho_b}{\rho_a}\left(\frac{\Psi_0}{V_d}\right) + \frac{\rho_c}{\rho_a}\left(\frac{\Psi_0}{V_d}\right)^2 \, ,$$

$$Q_\Phi = \frac{\partial Q}{\partial \Phi}, \quad Q_\Psi = \frac{\partial Q}{\partial \Psi^2} \, .$$

In the next section we study the instability of two-phase flow by means of Eq. (11).

3. Instability of two-phase flow

We seek a solution of Eq. (11) in the form

$$\Phi = \Phi_0 + f\exp\{i(kx - \omega t)\},$$

$$\Psi = \Psi_0 + \psi\exp\{i(kx - \omega t)\} \, , \tag{14}$$

where k is the wave number, and ω is the frequency. Substituting (14) into the linearized version of equation (11) yields the growth rate of the instability

$$\gamma = \varepsilon\frac{k^2}{\varphi_0}[\rho_a V_d^2\chi(\rho, \Psi_0, \Phi_0) + 2\Psi_0 V_d Q_\Psi - Q_\Phi] \, , \tag{15}$$

and the frequency of the growing volume fraction waves

$$\omega_R = kV_* + \frac{k^3}{\varphi_0}(\beta_a V_d - \beta_b) \, , \tag{16}$$

where $\omega = \omega_R + i\gamma$. Eqs. (15) and (16) can be regarded as the generalization of results [2,4,5] to the case of two-phase flows in porous medium (see below). The first term ($\propto \chi$) in Eq. (15) is always positive. This means that for $Q = 0$ (i. e. when the pressures of phases are equal) the equilibrium homogeneous flow of the two-phase medium through the matrix is unstable. In a general case $Q \neq 0$ and there exists a threshold for excitation of the instability. The excitation of the instability (if $\gamma > 0$) results in the formation of inhomogeneities in the phase composition of the mixture.

Eq. (16) for $\beta_i = 0$ describes the kinematic waves of phase composition of the mixture and of the relative velocity with linear dispersion law (see, e.g., [22]). It is the well known result of the classical theory of filtration in which the inertial term $\sim \varepsilon$ and the dynamic viscosity $\sim \beta_i$ are neglected.

Now we consider several characteristic cases which are important in view of their applications.

1. $\Psi_0 = 0$, i.e. zero relative velocity between the phases. This is interesting for the analysis of flow patterns during pneumatic transport of granular or powdered materials (see, e.g., [18]) since the wall friction in pneumatic transport systems can be modeled by local Darcy-type friction term. Waves with frequency

$$\omega_R = k\frac{W\sigma_0}{\varphi_0} + \frac{k^3}{\varphi_0}(\beta_a W\varphi_c - \beta_b) \tag{17}$$

are excited and the growth rate is

$$\gamma = \varepsilon\frac{k^2}{\varphi_0}[\rho_a(W\varphi_c)^2 - Q_\Phi] \, . \tag{18}$$

Here $\varphi_c = (\alpha_1 + \alpha_2)/\varphi_0$. The zero slip equilibrium flow in this case can exist even without gravitation [see Eq. (9)]. The above instability is similar to the Saffman-Taylor instability of two-phase flow in porous media [1] which arises due to the different viscosities of each of the liquids filtrated through the porous medium. It results in the appearance of a relative velocity between the two phases (i.e., excitation of the instability) and formation of fingers. However the above discussed instability is different from the Saffman-Taylor instability since pattern formation in Saffman-Taylor instability is associated with the growth of interface perturbations. In our case the heterogeneities are formed in the volume with initially homogeneous composition and uniform velocity. This result means that zero-slip two-phase flow is unstable (at least if $Q_\Phi \leq 0$). The presence of the gravitational field promotes the growth of the instability.

2. If the volume flow rate $W = 0$, the growth rate of the instability and the frequency of the excited waves are given by Eqs. (15),(16) with $V_d = \Psi_0\sigma_0'/\varphi_0$. This case is important, for example, for filtration of viscous melt (magma) through very viscous deformable rock and for the problem of viscous consolidation (see, e.g., [7,19,23,24]). When $W = 0$ the directions of the flows of two phases are opposite and the heavier liquid moves in the direction of the gravitational field. The relative velocity of the phases increases and the instability is excited. In this case also the difference between the viscosities of the liquids promotes the instability.

3. In the absence of solid matrix ($\alpha_1 = 0, \quad \alpha_2 = 0$) the oscillations of the phase composition and the relative velocity are amplified in gravitation field with the growth rate determined by Eq. (15) with $V_d = \Psi_0\sigma_0'/\sigma_0, \quad \varphi_0 = \sigma_0$. The frequency of the oscillations is given by Eq. (16) with corresponding values of V_d and φ_0 . The equilibrium exists only in the gravitational field: the equilibrium condition is $G(\rho_2 - \rho_1) = \sigma_0\Psi_0$. This equilibrium is unstable since the work performed by the gravitational force is positive for the period of oscillations. This situation is typical for fluidized beds and was investigated in detail in [2,4,5]. Notably the growth rate of instability derived in [2,4,5] coincides with our more general expression (15) with $\alpha_1 = \alpha_2 = 0$ and $Q_\Psi = 0$.

In all the above cases the mutual friction of the two filtrating liquids results in the decrease of the growth rate of the instability. Note that the instability of the two-phase flow in porous media in the general case arises both due to the different viscosities of the liquids and the gravitational field. This case is important for chemical catalytic bed reactors (hot spot formation) and the two-phase filtration in natural porous formations with applications in petroleum industry and irrigation (see, e.g., [16]). In the next section we consider the evolution of finite perturbations.

4. Nonlinear waves: solitons and 'oscillating' shock waves

In new variables Eq. (11) for the case $\varepsilon \ll \beta_i^{1/3} \ll 1$ reduces to the well known Korteweg-de Vries-Burgers equation:

$$\frac{\partial \hat{f}}{\partial t} + \hat{f}\frac{\partial \hat{f}}{\partial X} + \frac{\partial^3 \hat{f}}{\partial X^3} = \mu\frac{\partial^2 \hat{f}}{\partial X^2}\,, \tag{19}$$

where

$$\hat{f} = bc^{-1/3}f, \quad X = c^{-1/3}x, \quad \mu = -\varepsilon\Gamma_0 c^{-2/3}.$$

This nonlinear partial differential equation is encountered in various fundamental and applied fields of science (see, e.g., [10,25–33]). It is interesting to note that the Burgers term in Eq. (19) arises due to inertial effects, i.e. in two-phase flows inertial terms are responsible for dissipation or excitation of disturbances. On the other hand, the Stokes friction terms result in effective dispersion while the interphase friction terms describe nonlinear terms in the Korteweg-de Vries-Burgers equation. Eq. (19) can be considered as a generalization of results [11,36,4,5] to the case of two-phase flows in porous medium ($\alpha_1 \neq \alpha_1 \neq 0$).

For small μ a solution of Eq. (19) in form of 'oscillating' shock wave can be described as a solution of the Korteweg-de Vries equation perturbed by $\mu\partial^2\hat{f}/\partial X^2$ (see, [31–33,5]). The shape of the head (with largest amplitude) soliton in the oscillating shock wave can be described by the following equation:

$$\hat{f}_s = 12k^2(t)\cosh^{-2}\{k(t)[X - \xi(t)]\}\,, \tag{20}$$

with slowly varying amplitude

$$k(t) = k_0\left(1 + \frac{t}{\tau}\right)^{-\frac{1}{2}}, \quad \tau = \tfrac{15}{16}(\mu k_0^2)^{-1}, \tag{21}$$

where $k_0 = k(t = 0)$ (see [34,35,32]). The function $\xi(t)$ satisfies the following differential equation:

$$\frac{d\xi}{dt} = 4k^2(t) + \tfrac{8}{5}\mu k(t) \tag{22}$$

(see [31–33]).

For small negative values of the parameter μ (in the case of an unstable two-phase flow) the solution of Eq. (19) can be obtained via a simple modification of the solution derived in [34,35,32]. This solution can be represented as the sum of a soliton with a slowly varying amplitude and a long negative 'tail', $\hat{f} = \hat{f}_s + \hat{f}_t$. The shape of this soliton is described by Eq. (20) with

$$k(t) = k_0\left(1 - \frac{t}{\tau_0}\right)^{-\frac{1}{2}}, \quad \tau_0 = \tfrac{15}{16}(|\mu|k_0^2)^{-1}. \tag{23}$$

The above solution exists at times $t < \tau_0$. At $t \to \tau_0$ the Korteweg-de Vries-Burgers equation (19) is not valid for description of an unstable two-phase flow. The correct description of the two-phase flow can be achieved by abandoning the small amplitude approximation. The shape of the 'tail' far from the soliton ($-z \gg 1$) and at times $t < \tau_0$ can be described by the following expression:

$$\hat{f}_t \simeq -\tfrac{24}{5}|\mu|k(t)\left[1 + z^2\exp(2z)\right]. \tag{24}$$

where $z = k(t)[X - \xi(t)]$. Remarkably the solution obtained satisfies the following conservation law:

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \hat{f}(t, X)\, dX = 0. \tag{25}$$

Eq. (23) was derived first by [5] in the theory of gas-fluidized beds. We generalize this result to the case of two-phase flows in porous media.

In the case $\varepsilon^{1/2} \ll 1$ and $\beta_i \ll \varepsilon^{3/2}$, i.e., inertial effects prevail, Eq. (11) reduces to the Burgers equation (see, e.g., [10,26]). Thus these nonlinear equations can predict complex behavior and pattern formation in two-phase flows in porous media.

5. Discussion

It has been demonstrated here that multiphase filtration flows are generally unstable. The nature of this instability is associated with inertial effects. The instability can be considered as a 'volumetric' analog of the well known Saffman-Taylor instability and is of explosive type. In the small-amplitude finite approximation the evolution of this instability is governed by the Korteweg-de Vries-Burgers equation. The analysis allows one to estimate the characteristic size and velocity of the heterogeneities that are formed. Certainly a complete investigation of the problem requires at least a two-dimensional analysis and the abandonment of the small-amplitude finite approximation. Recently the consistent theory of inhomogeneities formation in fluidized beds without small-amplitude finite approximation was developed in [5].

Now we discuss an application of the results described in Sections 3 and 4 to the problem of bubble formation in fluidized beds. Fluidization is a process in which a bed of solid particles is subjected to a vertical, upward flow of fluid (see, e.g., [20]). If a fluid is passed upward through a bed of fine particles at a low flow rate, the fluid merely percolates through the void spaces between stationary particles. At a higher velocity, a point is reached where all the particles are just suspended by the upward-flowing gas or liquid. At this point the frictional force between particle and fluid just counterbalances the weight of the particles and fluidized bed is formed (see, e.g., [36]). In view of applications it is important to create an homogeneous and stable fluidized bed. However, experiments (see, e.g., [20]) show that inhomogeneous structures, so called bubbles are formed in fluidized beds. The bubbles emerge as cavities with lower concentration of particles of granular material and are filled with gas. The bubbles are frequently accompanied by a long tail (stem) with decreasing gas volume fraction: clusters of solid particles are observed in the flow behind the rising bubble. The obtained solution in form of a soliton with a tail with slowly changed parameters can be interpreted as initial phase of bubble formation in granular media (see [5]).

Now let us estimate a characteristic size of the bubble. The characteristic size can be estimated as a characteristic dispersion length of the soliton. We rewrite Eq. (19) in dimensional form for the function $f_* = bv_* f$:

$$\frac{\partial f_*}{\partial t} + f_* \frac{\partial f_*}{\partial x} + \lambda \frac{\partial^3 f_*}{\partial x^3} = -\delta_* \frac{\partial^2 f_*}{\partial x^2} , \tag{26}$$

where $\lambda = cL_* v_*$, $\delta_* = \varepsilon \Gamma_0 L_* v_*$, and c and Γ_0 are given by Eqs. (12) and (13), respectively. The dispersion scale of the soliton is given by $\Delta = 2(\lambda / L_* v_*)^{1/2}$ (see, e.g., [10,26]). In explicit form the dispersion scale of the soliton is given by

$$\Delta = \frac{2}{1-\Phi_0} \left\{ \frac{K_* \eta_1 v_*}{\mu_* \sigma_0 v_2} \left[1 - \frac{\sigma_0' v_2}{\sigma_0 v_*} \Phi_0 (1-\Phi_0)^2 \right] \right\}^{1/2} , \tag{27}$$

where we take into account that for the particles $v_1 = 0$. The characteristic velocity is $v_* = g\rho_2 \rho_* K_* / \mu_*$. Using the equation of equilibrium (9) we obtain the characteristic size of the bubble

$$\Delta = D_*^{5/2} \left\{ \frac{\pi \rho_s \nu_s n_s [1 + \zeta(\Phi_0)]}{6\mu_* (1-\Phi_0)} \right\}^{1/2} , \tag{28}$$

where

$$\zeta(y) = (1-y)[2y-1 + y(1-y)P'/P(y)]P(y),$$
$$\eta_1 = \pi \rho_s \nu_s n_s D_*^3 / 6,$$
$$P(y) = (9/2) K_* / S(y) D_*^2 .$$

Here ν_s is an effective kinematic viscosity of solid phase, n_s is the number density of solid particles, D_* is a diameter of solid particles, ρ_s and ρ_f are densities of solid particles and gas , respectively, the function $P(\Phi)$ is determined by the dependence of the friction force between solid particles and gas on the volume fraction of the solid phase. For instance, this function determined experimentally is given by [37]

$$P(\Phi) = \exp\left[(0.3652\Phi - 4.093)^{1/3} \left(\frac{1-\Phi}{\Phi} \right)^{1/2} \right] .$$

Using typical parameters for fluidized bed ($\rho_s/\rho_f \sim 10^3; D_* \sim 5 \times 10^{-2}$ cm; $\Phi_0 = 0.5; v_* \sim 10^2$ cm/s (see, e.g., [20,36]) and assuming that $\nu_s \propto v_*/(\pi D_*^2 n_s)$ we obtain $\Delta \simeq 10$ cm. It is in agreement with experiments [20] and numerical simulations [38]. Note that the characteristic size of a bubble scales as $D_*^{3/2}$ where D_* is a particle diameter.

Characteristic time of the bubbles formation is of the order of the characteristic time of the instability $\tau \sim L_*^2/\delta_* \sim (L_*/u_*)\varepsilon^{-1}$. The values of $\varepsilon \sim 10^{-4}$–10^{-2} and the characteristic time of the bubbles formation is of the order of $10^2 - 10^4$ of the residence time of the gas in the fluidized bed.

Note that the condition (25) has the clear physical meaning of mass conservation when the soliton with tail solution describes bubble like patterns in fluidized beds of a granular material. Indeed, experiments (see, e.g., [20]) show that bubbles formed in fluidized beds are frequently accompanied by a long tail (stem) with decreasing gas volume fraction: clusters of solid particles are observed in the flow behind the rising bubble. The evolution of the bubble like flow pattern described by the soliton with a tail is accompanied by the slow growth of its amplitude and velocity. Although the latter property is the natural behavior of the soliton type solutions, it of interest to note that it can be validated by experiments.

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Appendix. Nonlinear equation

We represent the volume fraction Φ and the function Ψ as a sum of two terms:

$$\Phi = \Phi_0 + f, \quad \Psi = \Psi_0 + \psi . \tag{A.1}$$

Substitution (A.1) into Eqs. (4),(5) yields

$$\frac{\partial f}{\partial t} + W\frac{\partial f}{\partial x} + \frac{\partial \psi}{\partial x} = 0\,, \tag{A.2}$$

$$\begin{aligned} &W\{(\alpha_1+\alpha_2)(\Phi_0+f)-\alpha_1\} \\ &\quad + (\Psi_0+\psi)\varphi(\Phi_0+f) + G(\rho_1-\rho_2) \\ &\quad = \varepsilon\hat{F}(\Psi_0+\psi,\Phi_0+f)\,, \end{aligned} \tag{A.3}$$

We search for a solution of Eq. (A.3) for ψ by iterations. The small parameter here is ε. Note that the zero order in ε approximation of Eq. (A.3) yields the equilibrium solution (9). The first iteration yields

$$\psi_1 = -V_d f_1\,. \tag{A.4}$$

Here $\psi_1, f_1 \sim \varepsilon$. The second iteration yields

$$\begin{aligned} \psi_2 &= -V_d f_2 + \frac{1}{2\varphi_0}(2V_d\sigma_0' - \Psi_0\sigma_0'')f_1^2 \\ &\quad + \frac{\varepsilon}{\varphi_0}\hat{F}(\Psi_0+\psi_1,\Phi_0+f_1)\,, \end{aligned} \tag{A.5}$$

where $\psi_2, f_2 \sim \varepsilon^2$, $\hat{F}(\Psi_0+\psi_1,\Phi_0+f_1) \sim O(f_1) \sim \varepsilon$ and

$$\begin{aligned} &V_d = W - V_*, \\ &\quad V_* = \frac{1}{\varphi_0}[W\sigma_0 - V_0\sigma_0'\Phi_0(1-\Phi_0)]\,, \end{aligned}$$

$$\varphi_0 = \varphi(\Phi=\Phi_0), \quad \sigma_0' = \frac{d\sigma_0}{d\Phi},$$
$$f = f_1 + f_2, \quad \psi = \psi_1 + \psi_2\,.$$

Combination of (A.2)–(A.5) yields a nonlinear equation for the finite perturbation of volume fraction f in a frame moving with velocity V_*:

$$\frac{\partial f}{\partial t} + bf\frac{\partial f}{\partial x} + c\frac{\partial^3 f}{\partial x^3} = -\varepsilon\Gamma_0\frac{\partial^2 f}{\partial x^2} - \varepsilon a\frac{\partial^2 f}{\partial x\partial t}\,, \tag{A.6}$$

where

$$a = \rho_a\frac{V_d}{\varphi_0}, \quad b = \frac{1}{\varphi_0}(2V_d\sigma_0' - \Psi_0\sigma_0''),$$
$$c = \frac{1}{\varphi_0}(\beta_b - V_d\beta_a),$$
$$\Gamma_0 = \frac{1}{\varphi_0}[\rho_a V_d^2\chi(\rho,\Psi_0,\Phi_0) + 2\Psi_0 V_d Q_\Psi - Q_\Phi]\,,$$

$$\chi(\rho,\Psi_0,\Phi_0) = 1 + 2\frac{\rho_b}{\rho_a}\left(\frac{\Psi_0}{V_d}\right) + \frac{\rho_c}{\rho_a}\left(\frac{\Psi_0}{V_d}\right)^2,$$

$$Q_\Phi = \frac{\partial Q}{\partial \Phi}, \quad Q_\Psi = \frac{\partial Q}{\partial \Psi^2}.$$

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