

Alpha effect and dynamo in density-stratified turbulence with large-scale shear: applications to protoplanetary discs and astrophysical clouds

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ABSTRACT

A joint effect of the density-stratified turbulence (or inhomogeneous turbulence) and a large-scale shear for arbitrary Mach numbers results in the α effect and mean-field dynamo action. These effects also produce the effective pumping velocity of a large-scale magnetic field. Compressibility of the turbulent velocity field (i.e. finite Mach number effect) does not affect the contributions to the α tensor caused by the joint effect of inhomogeneity of turbulence and a large-scale shear, but it influences the effective pumping velocity of the mean magnetic field. The isotropic part of the α tensor is independent of the exponent of the turbulent kinetic energy spectrum, while its anisotropic part depends on this exponent. This anisotropic part of the α tensor depends on the latitudinal profile of the large-scale shear velocity (differential rotation), which may be important for dynamo operation in the upper parts of the solar and stellar convection zones. There is also an additional contribution to the effective pumping velocity of the mean magnetic field that is proportional to the product of the fluid density gradient and the divergence of the mean fluid velocity caused, e.g. by collapsing (or expanding) astrophysical clouds. Applications of these effects to protoplanetary discs, protogalactic and protostellar clouds are discussed.

Key words: dynamo – MHD – turbulence – ISM: clouds.

1 INTRODUCTION

Large-scale magnetic fields in astrophysical turbulence can be generated by combined effect of kinetic helicity and non-uniform (differential) rotation or large-scale shear flow (see e.g. H. K. Moffatt 1978; E. N. Parker 1979; F. Krause & K.-H. Rädler 1980; Y. B. Zeldovich, A. A. Ruzmaikin & D. D. Sokoloff 1983; A. Ruzmaikin, A. M. Shukurov & D. D. Sokoloff 1988; G. Rüdiger, R. Hollerbach & L. L. Kitchatinov 2013; H. K. Moffatt & E. Dormy 2019; I. Rogachevskii 2021; A. Shukurov & K. Subramanian 2021). The kinetic helicity in turbulence can be produced in rotating inhomogeneous or density stratified turbulence. The alternative to rotation is the large-scale shear which is in combination with the density-stratified turbulence (or inhomogeneous turbulence) can produce the kinetic helicity and the α effect.

Examples of astrophysical systems where large-scale shear motions place an important role are protoplanetary discs (PPD; see e.g. T. Elperin, N. Kleeorin & I. Rogachevskii 1998; L. S. Hodgson & A. Brandenburg 1998; L. Pan et al. 2011; A. Hubbard 2016; P. F. Hopkins 2016a, b; N. Kleeorin & I. Rogachevskii 2025), colliding protogalactic clouds (PGC) and merging protostellar clouds (PSC; see e.g. A. D. Chernin 1991, 1993; H. Wiechen, G. Birk & H. Lesch 1998; G. Birk, H. Wiechen & H. Lesch 2002; I. Rogachevskii

et al. 2006), as well as solar and stellar convective zones (see e.g. E. N. Parker 1979; F. Krause & K.-H. Rädler 1980). In such systems, large-scale shear motions coexist with small-scale turbulence. Interaction between large-scale shearing motions and density stratified or inhomogeneous turbulence causes a non-zero α effect and generation of large-scale magnetic field. In addition to the large-scale shear motions, there can be collapsing or expanding astrophysical clouds or discs. Typical examples of such astrophysical systems are gravitational collapse of young stars, expanding Universe, and supernova explosions resulting to production of turbulence in galaxies and formation of expanding astrophysical clouds.

The α effect and effective pumping velocity in *inhomogeneous and incompressible turbulence with a large-scale shear* have been determined by I. Rogachevskii & N. Kleeorin (2003) using the spectral τ approach for large fluid and magnetic Reynolds numbers. These effects have been also studied applying the quasi-linear approach (or the second-order correlation approximation) by K.-H. Rädler & R. Stepanov (2006). This approach is valid for small fluid and magnetic Reynolds numbers or for high conductivity limit and small Strouhal numbers.

In addition, a mean-field theory for a pumping effect of the mean magnetic field in *helical homogeneous turbulence with large-scale shear* has been also developed by I. Rogachevskii et al. (2011), applying various analytical methods. In particular, they

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have used the quasi-linear approach, the path-integral technique, and the spectral τ approach, and have found that the effective pumping velocity is proportional to the product of α effect and large-scale vorticity associated with the large-scale shear. Direct numerical simulations of helical turbulence with large-scale shear in different ranges of hydrodynamic and magnetic Reynolds numbers have found the effective pumping velocity of the mean magnetic field by a kinematic test-field method in agreement with the theoretical predictions by I. Rogachevskii et al. (2011).

However, the α effect and effective pumping velocity have not yet been derived for a density-stratified non-helical background turbulence with large-scale shear and for arbitrary Mach numbers. In the present study, we investigate these effects applying the spectral τ approach. We find that the isotropic part of the α tensor is independent of the exponent of the turbulent kinetic energy spectrum. The joint effect of density-stratified turbulence and large-scale shear also produce the effective pumping velocity of a large-scale magnetic field.

There are also additional contributions to the effective pumping velocity $\mathbf{V}^{\text{eff}} \propto \lambda \text{div} \bar{\mathbf{U}}$ in density stratified turbulence, or $\mathbf{V}^{\text{eff}} \propto \Lambda \text{div} \bar{\mathbf{U}}$ in inhomogeneous turbulence, which can arise in collapsing (or expanding) astrophysical turbulent clouds. Here, $\lambda = -\nabla \ln \bar{\rho}$ describes the fluid density stratification, $\Lambda = \nabla \ln \langle \mathbf{u}^2 \rangle^{(0)}$ describes inhomogeneous background turbulence, where the angular brackets imply an ensemble averaging, $\bar{\rho}$ is the mean fluid density and $\bar{\mathbf{U}}$ is the mean velocity. Note that magnetic field amplification during a turbulent collapse have been recently studied by A. Brandenburg & E. Ntormousi (2025) and Irshad et al. (2025) (see also references therein). The α tensor is independent of $\text{div} \bar{\mathbf{U}}$, i.e. it is independent of the effects of collapsing or expanding of clouds. We apply the obtained results related to the α tensor and effective pumping velocity of the large-scale magnetic field to protoplanetary discs, colliding PGC and merging PSC.

This paper is organized as follows. In Section 2 we discuss the assumptions and the procedure for the derivation of the turbulent electromotive force (EMF). In Section 3 we determine the α effect and the effective pumping velocity of the large-scale magnetic field in a density-stratified turbulence with a large-scale shear flow. For comparison, in Section 4 we find the α effect and the effective pumping velocity in an inhomogeneous turbulence with a large-scale shear flow. In Section 5, we consider compressible turbulence with large-scale shear and calculate the α effect and the effective pumping velocity for arbitrary Mach numbers. In Section 6, we discuss applications of the obtained results to PPD and astrophysical clouds. Finally, in Section 7, we draw the concluding remarks. In Appendix A, we give identities used for derivation of the α effect and the effective pumping velocity of the mean magnetic field.

2 GOVERNING EQUATIONS AND METHOD OF DERIVATIONS

To determine the α effect and effective pumping velocity of the mean magnetic field in a density-stratified, inhomogeneous and compressible turbulence with non-uniform large-scale flow, we follow the approach described by I. Rogachevskii & N. Kleeorin (2004) and N. Kleeorin & I. Rogachevskii (2022) [see also book by I. Rogachevskii (2021)]. We consider turbulence with large fluid and magnetic Reynolds numbers, so that the Strouhal number (the ratio of the characteristic turbulent time τ_0 to the turn-over

time ℓ_0/u_0) is of the order of unity. Here, ℓ_0 and u_0 are the integral turbulence scale and characteristic turbulent velocity at the integral scale.

We apply the mean-field approach, i.e. we assume that there is a separation of spatial and temporal scales, $\ell_0 \ll L_B$ and $\tau_0 \ll t_B$, where L_B and t_B are the spatial and temporal scales characterizing the variations of the mean magnetic field. We also use the multiscale approach (P. H. Roberts & A. M. Soward 1975). In the framework of the mean-field approach, we separate magnetic field $\mathbf{B} = \bar{\mathbf{B}} + \mathbf{b}$ and velocity field $\mathbf{U} = \bar{\mathbf{U}} + \mathbf{u}$ into mean field and fluctuations, where $\bar{\mathbf{B}} = \langle \mathbf{B} \rangle$ is the mean magnetic field, $\bar{\mathbf{U}} = \langle \mathbf{U} \rangle$ is the mean fluid velocity, $\mathbf{b}$ and $\mathbf{u}$ are magnetic and velocity fluctuations, respectively. In similar fashion, we separate fluid density and pressure. Here, we use the Reynolds averaging, which implies that $\langle \mathbf{u} \rangle = 0$, $\langle \mathbf{b} \rangle = 0$, etc.

We determine contributions to the turbulent electromotive force $\langle \mathbf{u} \times \mathbf{b} \rangle$ caused by the non-uniform large-scale flow $\bar{\mathbf{U}}$. To this end, we use the momentum equation for velocity fluctuations $\mathbf{u}$ and the induction equation for magnetic fluctuations $\mathbf{b}$ as

$$\frac{\partial \mathbf{u}}{\partial t} = -(\bar{\mathbf{U}} \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \bar{\mathbf{U}} - \frac{\nabla p_{\text{tot}}}{\bar{\rho}} + \mathbf{F}_v + \mathbf{F} + \frac{1}{\mu_0 \bar{\rho}} [(\mathbf{b} \cdot \nabla) \bar{\mathbf{B}} + (\bar{\mathbf{B}} \cdot \nabla) \mathbf{b}] + \mathbf{u}^{(N)}, \quad (1)$$

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times [\bar{\mathbf{U}} \times \mathbf{b} + \mathbf{u} \times \bar{\mathbf{B}} - \eta \nabla \times \mathbf{b}] + \mathbf{b}^{(N)}, \quad (2)$$

where η is the magnetic diffusion due to electrical conductivity of fluid, $\bar{\rho}$ is the mean fluid density, μ_0 is the magnetic permeability of the fluid, $\mathbf{F}$ is a random external stirring force, $p_{\text{tot}} = p + \mu_0^{-1} (\bar{\mathbf{B}} \cdot \mathbf{b})$ are fluctuations of the total pressure, p are fluctuations of the fluid pressure, $\mathbf{u}^{(N)}$ and $\mathbf{b}^{(N)}$ are the non-linear terms. The velocity $\mathbf{u}$ satisfies to the continuity equation. Generally, all mean quantities depend on coordinate and time. Equation (1) is written for the case when fluctuations of the fluid density are much smaller in comparison with the mean fluid density. For simplicity, the mean fluid velocity in this study describes only two effects: (i) the imposed large-scale shear and (ii) the effects of collapsing or expanding of clouds which can be described by large-scale motions with a non-zero $\text{div} \bar{\mathbf{U}}$ (see Section 5).

Using equations (1)–(2), we derive equations for the cross-helicity tensor $g_{ij}(\mathbf{k}) = \langle u_i(t, \mathbf{k}) b_j(t, -\mathbf{k}) \rangle$ and the tensor $f_{ij}(\mathbf{k}) = \langle u_i(t, \mathbf{k}) u_j(t, -\mathbf{k}) \rangle$ for the velocity fluctuations in a Fourier space. Since our goal is to derive only expressions for the α effect and the effective pumping velocity of the mean magnetic field, we neglect terms proportional to spatial derivatives of the mean magnetic field in these equations. We consider the kinematic dynamo problem and do not discuss the non-linear effects, so we do not need evolutionary equation for the tensor for magnetic fluctuations $b_{ij}(\mathbf{k}) = \langle b_i(t, \mathbf{k}) b_j(t, -\mathbf{k}) \rangle$. Equations for the second-order moments include the first-order spatial differential operators applied to the third-order moments $\hat{\mathcal{M}} g_{ij}^{(III)}(\mathbf{k})$ and $\hat{\mathcal{M}} f_{ij}^{(III)}(\mathbf{k})$ appearing due to the non-linear terms. Therefore, a problem arises how to close the system, i.e. how to express the third-order moments through the second-order moments, g_{ij} and f_{ij} denoted as $F^{(II)}$. We use the spectral τ approach (S. A. Orszag 1970; A. Pouquet, U. Frisch & J. Léorat 1976; I. Rogachevskii 2021), which postulates that the deviations of the third-order moments, denoted as $\hat{\mathcal{M}} F^{(III)}(\mathbf{k})$, from the contributions to these terms afforded by a background turbulence, $\hat{\mathcal{M}} F^{(III,0)}(\mathbf{k})$, can be expressed through the similar

deviations of the second moments, $F^{(II)}(\mathbf{k}) - F^{(II,0)}(\mathbf{k})$ as

$$\hat{\mathcal{M}}F^{(III)}(\mathbf{k}) - \hat{\mathcal{M}}F^{(III,0)}(\mathbf{k}) = -\frac{1}{\tau_r(k)} \left[F^{(II)}(\mathbf{k}) - F^{(II,0)}(\mathbf{k}) \right], \quad (3)$$

where $\tau_r(k)$ is the scale-dependent relaxation time, which can be identified with the turbulent time $\tau(k)$ of velocity fluctuations for large fluid and magnetic Reynolds numbers (see e.g. I. Rogachevskii 2021). The functions with the superscript (0) correspond to the background turbulence with a zero mean magnetic field and a zero large-scale shear. The background turbulence is assumed to be stationary in statistical sense.

The turbulent time in the $\mathbf{k}$ space is defined as in the Kolmogorov-like turbulence: $\tau(k) = 2\tau_0 \bar{\tau}(k)$ (see e.g. I. Rogachevskii 2021), where $\tau_0 = \ell_0/u_0$ is the characteristic turbulent time, the function $\bar{\tau}(k) = (k/k_0)^{1-q}$, and the turbulent kinetic energy spectrum in the inertial range of wave numbers $k_0 < k < k_v$ is $E(k) = -d\bar{\tau}(k)/dk = (q-1)k_0^{-1}(k/k_0)^{-q}$. Here the wave number $k_0 = 1/\ell_0$, and $u_0 = \left[\langle \mathbf{u}^2 \rangle^{(0)} \right]^{1/2} \equiv u_{\text{rms}}$, the wave number $k_v = \ell_v^{-1}$, the length ℓ_v is the Kolmogorov (viscous) scale. The exponent $q = 5/3$ corresponds to the Kolmogorov spectrum. Generally, the exponent q can be varied within the interval $1 < q < 3$. The condition $q > 1$ corresponds to finite kinetic energy for very large fluid Reynolds numbers, while $q < 3$ corresponds to finite dissipation of the turbulent kinetic energy at the viscous scale.

Validation of the τ approaches applied in the $\mathbf{k}$ space (so-called the spectral τ approximation) or in the physical space (so-called the minimal τ approximation) for different situations has been performed in various numerical simulations (A. Brandenburg & K. Subramanian 2005a, b; I. Rogachevskii et al. 2011; A. Brandenburg, K.-H. Rädler & K. Kemel 2012; I. Rogachevskii et al. 2012; A. Brandenburg et al. 2013; A. Brandenburg, I. Rogachevskii & N. Kleeorin 2016; I. Rogachevskii et al. 2017; I. Rogachevskii, N. Kleeorin & A. Brandenburg 2018; J. Schober et al. 2018).

When the mean magnetic field is zero, $g_{ij}^{(0)}(\mathbf{k}) = 0$ and the turbulent electromotive force vanishes. Consequently, equation (3) reduces to $\hat{\mathcal{M}}g_{ij}^{(III)}(\mathbf{k}) = -g_{ij}(\mathbf{k})/\tau(k)$ and $\hat{\mathcal{M}}f_{ij}^{(III)}(\mathbf{k}) - \hat{\mathcal{M}}f_{ij}^{(III,0)}(\mathbf{k}) = -[f_{ij}(\mathbf{k}) - f_{ij}^{(0)}(\mathbf{k})]/\tau(k)$. We assume that the characteristic time of variation of the second-order moments $g_{ij}(\mathbf{k})$ and $f_{ij}(\mathbf{k})$ are substantially larger than the correlation time $\tau(k)$ for all turbulence scales.

Therefore, the contributions $g_{ij}^{(S)}(\mathbf{k})$ to the cross-helicity tensor $\langle u_i(\mathbf{k}) b_j(-\mathbf{k}) \rangle$ caused by turbulence with a non-zero large-scale shear are given by

$$\begin{aligned} g_{ij}^{(S)}(\mathbf{k}) = \tau(k) & \left\{ -i(\mathbf{k} \cdot \bar{\mathbf{B}}) \left[\tau(k) J_{ijmn}(\bar{\mathbf{U}}) \left(f_{mn}^{(0)}(\mathbf{k}) \right. \right. \right. \\ & \left. \left. - b_{mn}^{(0)}(\mathbf{k}) \right) + f_{ij}^{(S)}(\mathbf{k}) - b_{ij}^{(S)}(\mathbf{k}) \right] + \frac{1}{2}(\bar{\mathbf{B}} \cdot \nabla) f_{ij}^{(S)}(\mathbf{k}) \right. \\ & \left. + \bar{B}_j \left[ik_n - \lambda_n - \frac{1}{2} \nabla_n \right] f_{in}^{(S)} \right\}, \end{aligned} \quad (4)$$

where the tensor $f_{ij}^{(0)}(\mathbf{k}, \mathbf{R}) = \langle u_i(\mathbf{k}) u_j(-\mathbf{k}) \rangle^{(0)}$ describes velocity fluctuations in the background turbulence. Here, we also take into account small-scale dynamo that generates magnetic fluctuations in the background turbulence with a zero mean magnetic field, and characterized by the tensor $b_{ij}^{(0)}(\mathbf{k}) = \langle b_i(\mathbf{k}) b_j(-\mathbf{k}) \rangle^{(0)}$. The contributions $f_{ij}^{(S)}(\mathbf{k})$ and $b_{ij}^{(S)}(\mathbf{k})$ of the large-scale shear on velocity and magnetic fluctuations are given by

$$f_{ij}^{(S)}(\mathbf{k}) = \tau(k) I_{ijmn}(\bar{\mathbf{U}}) f_{mn}^{(0)}(\mathbf{k}), \quad (5)$$

$$b_{ij}^{(S)}(\mathbf{k}) = \tau(k) E_{ijmn}(\bar{\mathbf{U}}) b_{mn}^{(0)}(\mathbf{k}), \quad (6)$$

where the tensors $I_{ijmn}(\bar{\mathbf{U}})$, $J_{ijmn}(\bar{\mathbf{U}})$, and $E_{ijmn}(\bar{\mathbf{U}})$ are given by equations (A1)–(A3) in Appendix A.

In the next sections, using equations (4)–(6), we determine the α effect and effective pumping velocity of the mean magnetic field produced by a large-scale shear imposed on various kinds of small-scale background turbulence:

- (i) a density-stratified turbulence (Section 3),
- (ii) inhomogeneous turbulence (Section 4), and
- (iii) compressible density-stratified and inhomogeneous turbulence (Section 5).

3 DENSITY-STRATIFIED TURBULENCE WITH LARGE-SCALE SHEAR

In this section, we consider a density-stratified turbulence with large-scale shear and low-Mach-number flows. The velocity $\mathbf{u}$ satisfies to the continuity equation applied in the anelastic approximation:

$$\nabla \cdot (\bar{\rho} \mathbf{u}) = 0, \quad (7)$$

so that $\text{div } \mathbf{u} = \lambda \cdot \mathbf{u}$, where $\lambda = -(\nabla \bar{\rho})/\bar{\rho}$. We remind that we consider the case when fluctuations ρ' of the fluid density are much smaller in comparison with the mean fluid density ($|\rho'| \ll \bar{\rho}$) and $|\rho' \bar{\mathbf{U}}| \ll \bar{\rho} |\mathbf{u}'|$.

We use the following model for the second moment, $f_{ij}^{(0)}(\mathbf{k}, \mathbf{R}) = \langle u_i(\mathbf{k}) u_j(-\mathbf{k}) \rangle^{(0)}$ of velocity fluctuations in a density-stratified homogeneous background turbulence in a Fourier space:

$$f_{ij}^{(0)} = \frac{\langle \mathbf{u}^2 \rangle^{(0)} E(k)}{8\pi k^2} \left[\delta_{ij} - k_{ij} + \frac{i}{k^2} (\lambda_i k_j - \lambda_j k_i) \right] \quad (8)$$

(see e.g. I. Rogachevskii 2021), where δ_{ij} is the Kronecker unit tensor and $k_{ij} = k_i k_j / k^2$.

We also take into account the small-scale dynamo in the background turbulence. To this end, we use the following model for the second moment, $b_{ij}^{(0)}(\mathbf{k}) = \langle b_i(\mathbf{k}) b_j(-\mathbf{k}) \rangle^{(0)}$ of magnetic fluctuations:

$$b_{ij}^{(0)} = \frac{\langle \mathbf{b}^2 \rangle^{(0)} E_M(k)}{8\pi k^2} (\delta_{ij} - k_{ij}), \quad (9)$$

where $E_M(k)$ is the energy spectrum function of magnetic fluctuations. For simplicity, we assume that $E_M(k) = E(k)$ for $k \geq \ell_M^{-1}$ and $E_M(k) = 0$ for $\ell_0^{-1} < k < \ell_M^{-1}$, where ℓ_M is the characteristic scale of the localization of the maximum of magnetic energy caused by the small-scale dynamo. Note that equation (9) for magnetic fluctuations of the background turbulence does not contain terms proportional to λ since $\text{div } \mathbf{b} = 0$, while equation (8) for velocity fluctuations includes terms proportional to λ due to the continuity equation (7): $\text{div } (\bar{\rho} \mathbf{u}) = 0$.

In the present study we only derive expressions for the α effect and the effective pumping velocity, so we neglect terms proportional to spatial derivatives of the mean magnetic field in the turbulent electromotive force. We take into account that the terms in $g_{ij}^{(S)}(\mathbf{k})$ with symmetric tensors with respect to the indexes 'i' and 'j' do not contribute to the turbulent electromotive force because $\mathcal{E}_m = \varepsilon_{mij} \int g_{ij}^{(S)}(\mathbf{k}) d\mathbf{k}$.

The contributions to the turbulent electromotive force caused by density-stratified turbulence with large-scale shear can be written as $\mathcal{E}_i^{(\lambda)} = a_{ij}^{(\lambda)} \bar{B}_j$, where $a_{ij}^{(\lambda)}$ is given by equation (A19) in

Appendix A. Now, we determine the tensor $\alpha_{ij}^{(\lambda)} = (a_{ij}^{(\lambda)} + a_{ji}^{(\lambda)})/2$ and the effective pumping velocity $V_n^{\text{eff}}(\lambda) = -\varepsilon_{ijn} a_{ij}^{(\lambda)}/2$ of the mean magnetic field in a density-stratified turbulence with a large-scale shear:

$$\begin{aligned} \alpha_{ij}^{(\lambda)} &= \frac{\ell_0^2}{45} \left[13 (\lambda \cdot \overline{\mathbf{W}}) \delta_{ij} - 2 (\lambda_i \overline{W}_j + \lambda_j \overline{W}_i) \right. \\ &\quad \left. + (4q - 7) \lambda_m (\varepsilon_{imn} (\partial \overline{U})_{nj} + \varepsilon_{jmn} (\partial \overline{U})_{ni}) \right], \\ V_i^{\text{eff}}(\lambda) &= \frac{\ell_0^2}{45} \left[5 (\lambda \times \overline{\mathbf{W}})_i + 2(8q + 11) \lambda_m (\partial \overline{U})_{mi} \right], \end{aligned} \quad (10)$$

where $\overline{\mathbf{W}} = \nabla \times \overline{\mathbf{U}}$ is the mean vorticity, $(\partial \overline{U})_{ij} = (\nabla_i \overline{U}_j + \nabla_j \overline{U}_i)/2$, and ε_{ijk} is the fully antisymmetric Levi-Civita tensor. Here, the gradient of the mean velocity $\nabla_i \overline{U}_j$ is decomposed into symmetric $(\partial \overline{U})_{ij}$ and antisymmetric $\varepsilon_{ijp} \overline{W}_p/2$ parts, i.e. $\nabla_i \overline{U}_j = (\partial \overline{U})_{ij} + \varepsilon_{ijp} \overline{W}_p/2$. In this study, we consider only a weak large-scale shear ($\overline{W} \tau_0 \ll 1$), and neglect the second-order derivatives of the mean velocity $\overline{\mathbf{U}}$. Equations (10)–(11) are given in the absence of the small-scale dynamo. The tensor $\alpha_{ij}^{(\lambda, b)} = (a_{ij}^{(M, \lambda)} + a_{ji}^{(M, \lambda)})/2$ and the effective pumping velocity $V_n^{\text{eff}}(\lambda, b) = -\varepsilon_{ijn} a_{ij}^{(M, \lambda)}/2$ caused by the small-scale dynamo are given by

$$\begin{aligned} \alpha_{ij}^{(\lambda, b)} &= -\frac{\ell_0^2}{45} \left(\frac{\ell_M}{\ell_0} \right)^{3(q-1)} \left[\frac{\langle \mathbf{b}^2 \rangle^{(0)}}{\mu_0 \bar{\rho} \langle \mathbf{u}^2 \rangle^{(0)}} \right] \left[(\lambda \cdot \overline{\mathbf{W}}) \delta_{ij} \right. \\ &\quad \left. + \lambda_i \overline{W}_j + \lambda_j \overline{W}_i - \lambda_m (\varepsilon_{imn} (\partial \overline{U})_{nj} + \varepsilon_{jmn} (\partial \overline{U})_{ni}) \right], \end{aligned} \quad (12)$$

$$V_i^{\text{eff}}(\lambda, b) = \frac{\ell_0^2}{45} \left(\frac{\ell_M}{\ell_0} \right)^{3(q-1)} \left[\frac{\langle \mathbf{b}^2 \rangle^{(0)}}{\mu_0 \bar{\rho} \langle \mathbf{u}^2 \rangle^{(0)}} \right] \lambda_m (\partial \overline{U})_{mi}, \quad (13)$$

where $a_{ij}^{(M, \lambda)}$ is given by equation (A21) in Appendix A. As follows from equations (12)–(13), magnetic fluctuations caused by the small-scale dynamo decrease the α effect. However, the contributions due to the small-scale dynamo are smaller than those caused by the velocity fluctuations since $\ell_M \ll \ell_0$ (A. Brandenburg, I. Rogachevskii & J. Schober 2023) and $\langle \mathbf{b}^2 \rangle^{(0)}/\mu_0 \leq \bar{\rho} \langle \mathbf{u}^2 \rangle^{(0)}$. Indeed, as follows from direct numerical simulations by A. Brandenburg et al. (2023), the ratio of the wave numbers $k_M/k_v \propto \text{Pm}^{0.6}$ for $\text{Pm} \geq 2$ and $k_M/k_v \propto \text{Pm}$ for $\text{Pm} < 1$ (see fig. 2 in A. Brandenburg et al. 2023). Here, Pm is the magnetic Prandtl number, $\ell_M = k_M^{-1}$ and the wavenumber k_v is based on the Kolmogorov scale. However, $\ell_M \ll \ell_0$ (see table 1 in A. Brandenburg et al. 2023).

The kinetic α effect based on the isotropic part ($\propto \delta_{ij}$) of the alpha tensor is given by

$$\alpha^{(\lambda)} = \frac{13}{45} \ell_0^2 (\lambda \cdot \overline{\mathbf{W}}). \quad (14)$$

Therefore, the α effect and the effective pumping velocity are caused by a combined effect of the density-stratified turbulence and a large-scale shear.

We consider the background turbulence being a non-helical, but the resulting turbulence becomes helical due to the joint effects of the large-scale shear and density stratification. Indeed, let us determine the kinetic helicity $H_u^{(\lambda)} = \langle \mathbf{u} \cdot (\nabla \times \mathbf{u}) \rangle$ in a density-stratified turbulence with a large-scale shear: $H_u^{(\lambda)} = i \varepsilon_{ijs} \int k_s f_{ij}^{(S)}(\mathbf{k}) d\mathbf{k}$, where $f_{ij}^{(S)}(\mathbf{k})$ is given by equation (5). After integration in $\mathbf{k}$ space, we obtain that the kinetic helicity in a

density-stratified turbulence with non-uniform large-scale flow is given by

$$H_u^{(\lambda)} = -\frac{5}{2} \eta_T (\lambda \cdot \overline{\mathbf{W}}), \quad (15)$$

where $\eta_T = (\tau_0/3) \langle \mathbf{u}^2 \rangle$ is the turbulent diffusion coefficient. Equation for η_T has been first obtained by the quasi-linear approach (the second-order correlation approximation, SOCA) in high conductivity limit (see e.g. F. Krause & K.-H. Rädler 1980), and it has been also reproduced using the τ approaches and the path-integral approach (see e.g. I. Rogachevskii 2021). Note that the effect of large-scale shear on the turbulent diffusion coefficient is neglected for a weak large-scale shear ($\overline{W} \tau_0 \ll 1$). To derive equations (10)–(15), we used equations (8)–(9). Applying the classical expression for the kinetic α effect, we obtain

$$\alpha^{(\text{cl}, \lambda)} \equiv -\frac{\tau_0}{3} H_u^{(\lambda)} = \frac{5}{18} \ell_0^2 (\lambda \cdot \overline{\mathbf{W}}), \quad (16)$$

which is in a qualitative agreement with equation (14), but the coefficients in equations (14) and (16) do not coincide. This is not surprising, because the classical expression $\alpha = -(\tau_0/3) H_u$ for the kinetic α effect is only valid for a homogeneous and isotropic helical background turbulence.

The joint action of the α effect and the large-scale shear results in the generation of the large-scale magnetic field due to the α - shear mean-field dynamo in the small-scale turbulence with large fluid and magnetic Reynolds numbers (see e.g. H. K. Moffatt 1978; F. Krause & K.-H. Rädler 1980; Y. B. Zeldovich et al. 1983; I. Rogachevskii 2021). Indeed, let us consider the following equilibrium: $\alpha = \text{const}$ and $\overline{\mathbf{U}} = (0, Sx, 0)$, and take into account that turbulent diffusion coefficient $\eta_T \gg \eta$. We seek for a solution of the induction equation for perturbations of the large-scale magnetic field as

$$\overline{\mathbf{B}}(t, x, z) = \overline{B}_y(t, x, z) \mathbf{e}_y + \nabla \times [\overline{A}(t, x, z) \mathbf{e}_y], \quad (17)$$

where $\mathbf{e}_y$ is the unit vector directed along y axis, and the functions $\overline{B}_y(t, x, z)$ and $\overline{A}(t, x, z)$ are determined by the following equations:

$$\frac{\partial \overline{A}(t, x, z)}{\partial t} = \alpha \overline{B}_y + \eta_T \Delta \overline{A}, \quad (18)$$

$$\frac{\partial \overline{B}_y(t, x, z)}{\partial t} = -\alpha \Delta \overline{A} - S \nabla_z \overline{A} + \eta_T \Delta \overline{B}_y, \quad (19)$$

where $\alpha \equiv \alpha^{(\lambda)}$ is given by equation (14). The second term, $-S \nabla_z \overline{A}$, in the right-hand side of equation (19) is originated from the term $(\overline{\mathbf{B}} \cdot \nabla) \overline{U}_y$. We consider the case when $|\alpha \Delta \overline{A}| \ll |S \nabla_z \overline{A}|$, which is valid when $\ell_0^2 \ll L_B H_\rho$, where L_B is the characteristic scale of the mean magnetic field variations and $H_\rho = |\lambda|^{-1}$ is the mean density variation scale, which is assumed to be constant. The growth rate of the dynamo instability and the frequency of the dynamo waves are given by

$$\gamma = \left(\frac{|\alpha S K_z|}{2} \right)^{1/2} - \eta_T K^2, \quad (20)$$

$$\omega = \left(\frac{|\alpha S K_z|}{2} \right)^{1/2}. \quad (21)$$

The maximum growth rate of the dynamo instability and the maximum frequency of the dynamo waves are given by

$$\gamma^{\text{max}} \approx \frac{3}{8} \left(\frac{\alpha^2 S^2}{2\eta_T} \right)^{1/3}, \quad (22)$$

$$\omega^{\text{max}} = \frac{1}{2} \left(\frac{\alpha^2 S^2}{2\eta_T} \right)^{1/3}. \quad (23)$$

The maximum growth rate of the dynamo instability and the maximum frequency of the dynamo waves are attained at $K_x^{\max} = 0$ and

$$K_z^{\max} = \frac{1}{2} \left(\frac{|\alpha| S}{4\eta_T^2} \right)^{1/3}. \quad (24)$$

The dynamo instability $\gamma > 0$ implies that

$$\frac{\ell_0}{L_B} \left(\frac{H_\rho}{L_B} \right)^{1/2} < S\tau_0 \ll 1. \quad (25)$$

4 INHOMOGENEOUS TURBULENCE WITH LARGE-SCALE SHEAR

For comparison, in this section we consider an inhomogeneous and incompressible turbulence with a large-scale shear. In this case, the model for the second moment $f_{ij}^{(0)}$ of velocity fluctuations in an inhomogeneous and incompressible background turbulence in a Fourier space is given by

$$f_{ij}^{(0)} = \frac{\langle \mathbf{u}^2 \rangle^{(0)} E(k)}{8\pi k^2} \left[\delta_{ij} - k_{ij} - \frac{i}{2k^2} (\Lambda_i k_j - \Lambda_j k_i) \right] \quad (26)$$

(see e.g. I. Rogachevskii 2021), where $\Lambda = \nabla \ln \langle \mathbf{u}^2 \rangle^{(0)}$. The contributions to the turbulent electromotive force caused by an inhomogeneous and incompressible turbulence with a large-scale shear are $\mathcal{E}_i^{(\Lambda)} = a_{ij}^{(\Lambda)} \bar{B}_j$, where $a_{ij}^{(\Lambda)}$ is given by equation (A20) in Appendix A.

We also take into account the small-scale dynamo with inhomogeneous magnetic fluctuations in the background turbulence. To this end, we use the following model for the second moment $b_{ij}^{(0)}(\mathbf{k})$ of magnetic fluctuations:

$$b_{ij}^{(0)} = \frac{\langle \mathbf{b}^2 \rangle^{(0)} E_M(k)}{8\pi k^2} \left[\delta_{ij} - k_{ij} - \frac{i}{2k^2} (\Lambda_i^{(M)} k_j - \Lambda_j^{(M)} k_i) \right], \quad (27)$$

where $\Lambda^{(M)} = \nabla \ln \langle \mathbf{b}^2 \rangle^{(0)}$. For simplicity, we assume that $E_M(k) = E(k)$ for $k \geq \ell_M^{-1}$ and $E_M(k) = 0$ for $\ell_0^{-1} < k < \ell_M^{-1}$.

Using this equation, we determine the tensor $\alpha_{ij}^{(\Lambda)} = (a_{ij}^{(\Lambda)} + a_{ji}^{(\Lambda)})/2$ and the effective pumping velocity $V_n^{\text{eff}}(\Lambda) = -\varepsilon_{ijn} a_{ij}^{(\Lambda)}/2$ of the mean magnetic field in an inhomogeneous and incompressible turbulence with a large-scale shear:

$$\alpha_{ij}^{(\Lambda)} = -\frac{\ell_0^2}{9} \left[(\Lambda \cdot \bar{\mathbf{W}}) \delta_{ij} - \frac{1}{2} (\Lambda_i \bar{W}_j + \Lambda_j \bar{W}_i) + \frac{1}{5} (4q - 2) \Lambda_m (\varepsilon_{imn} (\partial \bar{U})_{nj} + \varepsilon_{jmn} (\partial \bar{U})_{ni}) \right], \quad (28)$$

$$V_i^{\text{eff}}(\Lambda) = -\frac{\ell_0^2}{18} \left[(\Lambda \times \bar{\mathbf{W}})_i - 4\Lambda_m (\partial \bar{U})_{mi} \right]. \quad (29)$$

Equations (28)–(29) are given in the absence of the small-scale dynamo. The tensor $\alpha_{ij}^{(\Lambda_M)} = (a_{ij}^{(\Lambda_M)} + a_{ji}^{(\Lambda_M)})/2$ and the effective pumping velocity $V_n^{\text{eff}}(\Lambda_M) = -\varepsilon_{ijn} a_{ij}^{(\Lambda_M)}/2$ caused by the small-scale dynamo are given by

$$\alpha_{ij}^{(\Lambda_M)} = -\frac{4(q-1)}{45} \ell_0^2 \left(\frac{\ell_M}{\ell_0} \right)^{3(q-1)} \left[\frac{\langle \mathbf{b}^2 \rangle^{(0)}}{\mu_0 \bar{\rho}} \langle \mathbf{u}^2 \rangle^{(0)} \right] \times \Lambda_m^{(M)} (\varepsilon_{imn} (\partial \bar{U})_{nj} + \varepsilon_{jmn} (\partial \bar{U})_{ni}), \quad (30)$$

$$V_i^{\text{eff}}(\Lambda_M) = \frac{\ell_0^2}{45} \left(\frac{\ell_M}{\ell_0} \right)^{3(q-1)} \left[\frac{\langle \mathbf{b}^2 \rangle^{(0)}}{\mu_0 \bar{\rho}} \langle \mathbf{u}^2 \rangle^{(0)} \right] \times \left[5 (\Lambda^{(M)} \times \bar{\mathbf{W}})_i + 2(2q-7) \Lambda_m^{(M)} (\partial \bar{U})_{mi} \right], \quad (31)$$

where $a_{ij}^{(\Lambda_M)}$ is given by equation (A22) in Appendix A.

The kinetic α effect based on the isotropic part ($\propto \delta_{ij}$) of the alpha tensor in an inhomogeneous turbulence with a large-scale shear is given by

$$\alpha^{(\Lambda)} = -\frac{\ell_0^2}{9} (\Lambda \cdot \bar{\mathbf{W}}). \quad (32)$$

Therefore, the α effect and the effective pumping velocity are caused by a joint effect of the inhomogeneous turbulence and a non-uniform large-scale flow. Equations (28)–(29) are in agreement with those derived by I. Rogachevskii & N. Kleeorin (2003) using the spectral τ approach and by K.-H. Rädler & R. Stepanov (2006) applied the quasi-linear approach for high conductivity limit but small Strouhal numbers.

Note that equations (28)–(29) for the tensor $\alpha_{ij}^{(\Lambda)}$ and the effective pumping velocity $\mathbf{V}^{\text{eff}}(\Lambda)$ are different from equations (10)–(11) derived for a density-stratified turbulence with a large-scale shear. The reason is caused by a difference between the effects of the density stratification and the inhomogeneity of turbulence on the tensor α_{ij} and the effective pumping velocity $\mathbf{V}^{\text{eff}}$. Indeed, the density stratification affects

- (i) the background turbulence [see equations (8) and (35)];
- (ii) the tensors $I_{ijmn}(\bar{\mathbf{U}})$ and $J_{ijmn}(\bar{\mathbf{U}})$, which describe the effect of the large-scale shear on turbulence [see equations (A1) and (A2)];
- (iii) and the term $-\bar{\mathbf{B}} \text{div} \mathbf{u}$ in the induction equation (2) which causes the appearance of the term $-\lambda_n \bar{B}_j f_{in}^{(S)}$ in equation (4) for the cross-helicity tensor $g_{ij}^{(S)}(\mathbf{k})$.

On the other hand, the parameter Λ_i characterizing the inhomogeneous turbulence affects only the background turbulence [see equations (26) and (35)]. This causes the difference between the effects of the density stratification and the inhomogeneity of turbulence on the tensor α_{ij} and the effective pumping velocity V_i^{eff} [compare equations (10)–(11) with (28)–(29)].

We consider the background non-helical turbulence, but the resulting turbulence becomes helical due to the joint effects of the large-scale shear and inhomogeneity of turbulence. Indeed, let us determine the kinetic helicity $H_u^{(\Lambda)} = \langle \mathbf{u} \cdot (\nabla \times \mathbf{u}) \rangle$ in an inhomogeneous turbulence with a large-scale shear. Using equations (5) and (26), and integrating in $\mathbf{k}$ space in expression $H_u^{(\Lambda)} = i \varepsilon_{ijs} \int k_s f_{ij}^{(S)}(\mathbf{k}) d\mathbf{k}$, we obtain that the kinetic helicity in an inhomogeneous turbulence with a large-scale shear is given by

$$H_u^{(\Lambda)} = \eta_T (\Lambda \cdot \bar{\mathbf{W}}), \quad (33)$$

where η_T is the turbulent diffusion coefficient. Applying a classical expression for the kinetic α effect, $\alpha^{(\text{cl}, \Lambda)} \equiv -(\tau_0/3) H_u^{(\Lambda)}$, we obtain that

$$\alpha^{(\text{cl}, \Lambda)} = -\frac{\ell_0^2}{9} (\Lambda \cdot \bar{\mathbf{W}}), \quad (34)$$

which coincides with equation (32). May be it is due to the fact that inhomogeneity of turbulence is a more simple effect that only is determined by the background turbulence.

5 COMPRESSIBLE TURBULENCE WITH LARGE-SCALE SHEAR

In this section, we consider five simple independent effects:

- (i) The stratification of a small-scale background turbulence described by the parameter λ ($\text{div} \mathbf{u} = \lambda \cdot \mathbf{u}$).

(ii) The finite Mach number effects for a compressible small-scale background turbulence described by the parameter σ_c .

(iii) The imposed large-scale shear described by a non-zero mean vorticity $\bar{\mathbf{W}} = \text{rot } \bar{\mathbf{U}}$.

(iv) The imposed mean fluid motion with a non-zero $\text{div } \bar{\mathbf{U}}$, which allows to describe collapsing (or expanding) astrophysical clouds.

(v) The inhomogeneity of a small-scale background turbulence described by the parameter Λ .

For simplicity, we assume that these five given parameters are independent. We investigate how these independent parameters affect the kinetic α tensor and the effective pumping velocity. Generally, the mean velocity field $\bar{\mathbf{U}}$ is determined by the mean Navier–Stokes equation and the mean continuity equation.

The tensor $f_{ij}^{(0)}(\mathbf{k})$ for a density-stratified, inhomogeneous and compressible non-helical background turbulence for arbitrary Mach numbers in the $\mathbf{k}$ space is given by

$$f_{ij}^{(0)} = \frac{\langle \mathbf{u}^2 \rangle^{(0)} E(k)}{8\pi k^2} \left[(\delta_{ij} - k_{ij} + 2\sigma_c k_{ij})(1 + \sigma_c)^{-1} + \frac{i}{k^2} \left(\lambda_i k_j - \lambda_j k_i + \frac{1}{2} (k_i \Lambda_j - k_j \Lambda_i) \right) \right] \quad (35)$$

(see e.g. I. Rogachevskii 2021), where the parameter

$$\sigma_c = \frac{\langle (\nabla \cdot \mathbf{u})^2 \rangle}{\langle (\nabla \times \mathbf{u})^2 \rangle} \quad (36)$$

describes the degree of compressibility of a turbulent velocity field. The background turbulence model given by equation (35) is derived from the symmetry arguments under the condition $\ell_0 \ll H_\rho$ and $\ell_0 \ll L_u$. Here $L_u = |\Lambda|^{-1} = \left| \nabla \ln \langle \mathbf{u}^2 \rangle^{(0)} \right|^{-1}$ is the characteristic scale of the inhomogeneity of turbulence, and $H_\rho = |\lambda|^{-1}$ is the mean density variation scale, which is assumed to be constant. This implies that we use the perturbation approach, i.e. equation (35) takes into account leading-order effects, which are linear in stratification ($\propto \ell_0/H_\rho$) and inhomogeneity of turbulence ($\propto \ell_0/L_u$), and the higher-order effects $\sim O(\ell_0^2/H_\rho^2, \ell_0^2/L_u^2)$ are neglected.

Generally, stratification also contributes to the parameter σ_c . However, this contribution is small [$\sim O(\ell_0^2/H_\rho^2)$], and neglected in equation (35). This implies that the effects of the arbitrary Mach number, characterized by the parameter σ_c , and density stratification, described by λ are separated. The degree of compressibility σ_c depends on the Mach number, but an analytical dependence σ_c on the Mach number is not known for arbitrary Mach numbers and it can be determined in numerical simulations. For small Mach numbers $\text{Ma} \ll 1$, the parameter $\sigma_c \sim \text{Ma}^3 \text{Re}^{1/4}$ (I. Rogachevskii & N. Kleeorin 2021a, b), where Re is the Reynolds number based on the integral scale and turbulent velocity.

Since we consider only linear effects in λ and Λ , the tensor $f_{ij}^{(0)}$ is constructed as a linear combination of symmetric tensors, δ_{ij} and k_{ij} , with respect to the indexes i and j , and non-symmetric tensors, $k_i \lambda_j$, $k_j \lambda_i$, and $k_i \Lambda_j$, $k_j \Lambda_i$. To determine unknown coefficients multiplying by these tensors, we use the following conditions in the derivation of equation (35):

$$\langle \mathbf{u}^2 \rangle^{(0)} = \int f_{ii}^{(0)}(\mathbf{k}, \mathbf{R}) d\mathbf{k}, \quad (37)$$

$$\langle (\text{div } \mathbf{u})^2 \rangle = \int k_i k_j f_{ij}^{(0)}(\mathbf{k}, \mathbf{R}) d\mathbf{k}, \quad (38)$$

$$\langle (\text{rot } \mathbf{u})^2 \rangle = \int k^2 f_{ii}^{(0)}(\mathbf{k}, \mathbf{R}) d\mathbf{k} - \langle (\text{div } \mathbf{u})^2 \rangle, \quad (39)$$

where $\mathbf{R}$ corresponds to large scales.

We assume here that the background turbulence is of Kolmogorov type with constant energy flux over the spectrum, i.e. the turbulent kinetic energy spectrum in the range of wave numbers $k_0 < k < k_v$ is $E(k) = -d\bar{\tau}(k)/dk$, where the function $\bar{\tau}(k) = (k/k_0)^{1-q}$ with $1 < q < 3$ (see e.g. I. Rogachevskii 2021). The exponent $q = 5/3$ corresponds to the Kolmogorov spectrum, while the exponent $q = 2$ corresponds to the spectrum of the Burgers turbulence. The turbulent time in the $\mathbf{k}$ space is $\tau(k) = 2\tau_0 \bar{\tau}(k)$.

Let us first consider a density-stratified and inhomogeneous non-helical background turbulence with very small degree of compressibility $\sigma_c \ll 1$. In this case, the total kinetic helicity $H_u^{(\text{tot})} = H_u^{(\lambda)} + H_u^{(\Lambda)}$ is given by

$$H_u^{(\text{tot})} = 2\eta_r (\bar{\mathbf{W}} \cdot \nabla) \ln(\bar{\rho}^{5/4} u_{\text{rms}}), \quad (40)$$

where u_{rms} is defined as $u_{\text{rms}} = \sqrt{\langle \mathbf{u}^2 \rangle}$, and the angular brackets (...) denote the ensemble averaging. The total kinetic α tensor is given by $\alpha_{ij} = \alpha_{ij}^{(\lambda)} + \alpha_{ij}^{(\Lambda)}$ and the effective pumping velocity is $V_i^{\text{eff}} = V_i^{\text{eff}}(\lambda) + V_i^{\text{eff}}(\Lambda)$. In this case, the kinetic α effect based on the isotropic part ($\propto \delta_{ij}$) of the kinetic α tensor is given by

$$\alpha^{(\text{tot})} = -\frac{2}{9} \ell_0^2 (\bar{\mathbf{W}} \cdot \nabla) \ln(\bar{\rho}^{13/10} u_{\text{rms}}), \quad (41)$$

where we used equations (14) and (32). On the other hand, the classical expression for the kinetic α effect, $\alpha^{(\text{cl}, \text{tot})} = -(\tau_0/3) H_u^{(\text{tot})}$, is given by

$$\alpha^{(\text{cl}, \text{tot})} = -\frac{2}{9} \ell_0^2 (\bar{\mathbf{W}} \cdot \nabla) \ln(\bar{\rho}^{5/4} u_{\text{rms}}), \quad (42)$$

where we used equations (16) and (34). Equations (41) and (42) are not coincided. This is not surprising, because the classical expression $\alpha = -(\tau_0/3) H_u$ for the kinetic α effect is only valid for a homogeneous and isotropic helical turbulence.

Let us compare equation (41) for turbulence with a large-scale shear with that for a slowly rotating turbulence ($\bar{\Omega}\tau_0 \ll 1$), where the following scaling for the α effect have been obtained in different studies:

$$\alpha^{(\Omega)} \propto -\ell_0^2 (\bar{\Omega} \cdot \nabla) \ln(\bar{\rho}^{\mu_*} u_{\text{rms}}), \quad (43)$$

with $\mu_* = 1$ (M. Steenbeck, F. Krause & K.-H. Rädler 1966; F. Krause & K.-H. Rädler 1980) by means of the quasi-linear approach, $\mu_* = 3/2$ (G. Rüdiger & L. L. Kichatinov 1993) applying the modified quasi-linear approach, and $\mu_* = 1/2$ (A. Brandenburg et al. 2013) using the spectral τ approach. Here, $\bar{\Omega}$ is the mean angular velocity describing a uniform rotation.

Note that the cases of uniform rotation and large-scale shear with a density-stratified or inhomogeneous turbulence are physically two different cases. However, this comparison shows that in both cases (uniform rotation and large-scale shear) with density-stratified or inhomogeneous turbulence, there is a production of the kinetic helicity and the α effect, and the form of the alpha effect are similar in both cases with the replacement the angular velocity $\bar{\Omega}$ by the large-scale vorticity $\bar{\mathbf{W}}$.

The physics of this effect is the following. Both, the angular velocity $\bar{\Omega}$ and the large-scale vorticity $\bar{\mathbf{W}}$ produce the left-handed and right-handed rotating turbulent eddies. A non-zero kinetic helicity implies that a number of the left-handed eddies at a given instant does not exactly equal the number of right-handed eddies. This breaking a symmetry between the numbers

of the left-handed and right-handed turbulent eddies is caused by density-stratified or inhomogeneous turbulence. A mechanism of the α effect is as follows. Deformation of the original magnetic field is caused by both, the left-handed and the right-handed rotating eddies. Due to the breaking a symmetry, the total effect of deformation of the original magnetic field line is not zero, which causes the generation of a large-scale magnetic field.

Now we consider a density-stratified, homogeneous and compressible turbulence. In this case, the kinetic α tensor $\alpha_{ij}^{(\lambda, \sigma_c)} = (a_{ij}^{(\text{tot})} + a_{ji}^{(\text{tot})})/2$ and the effective pumping velocity $V_n^{\text{eff}}(\lambda, \sigma_c) = -\varepsilon_{ijn} a_{ij}^{(\text{tot})}/2$ of the mean magnetic field are given by

$$\alpha_{ij}^{(\lambda, \sigma_c)} = -\frac{\ell_0^2}{45} \left\{ \left(2 + \frac{15\sigma_c}{2(1+\sigma_c)} \right) (\lambda_i \bar{W}_j + \lambda_j \bar{W}_i) - 13(\lambda \cdot \bar{W}) \delta_{ij} + \left[\left(4q - 7 - \frac{6\sigma_c}{1+\sigma_c} \right) (\varepsilon_{imn} (\partial \bar{U})_{nj} + \varepsilon_{jmn} (\partial \bar{U})_{ni}) \lambda_m \right] \right\}, \quad (44)$$

$$V_i^{\text{eff}}(\lambda, \sigma_c) = \frac{\ell_0^2}{45} \left[5 \left(1 - \frac{\sigma_c}{2(1+\sigma_c)} \right) (\lambda \times \bar{W})_i + 2 \left(8q + 11 - (12q + 13) \frac{\sigma_c}{1+\sigma_c} \right) \lambda_m (\partial \bar{U})_{mi} + 4 \left(11 - 7q + (2q - 3) \frac{\sigma_c}{1+\sigma_c} \right) \lambda_i \text{div} \bar{U} \right], \quad (45)$$

where $a_{ij}^{(\text{tot})}$ is given by equation (A37) in Appendix A. Equations (44)–(45) are obtained for homogeneous turbulence ($\Lambda = 0$). For an inhomogeneous, non-stratified and compressible turbulence (with a non-zero parameter σ_c , i.e. in turbulence with a finite Mach numbers), the kinetic α tensor is independent of the Mach number [i.e. is independent of the parameter σ_c and is given by equation (28)]. On the other hand, the effective pumping velocity $V_n^{\text{eff}}(\Lambda, \sigma_c)$ of the mean magnetic field depends on the Mach number (i.e. it depends on the parameter σ_c):

$$V_i^{\text{eff}}(\Lambda, \sigma_c) = \frac{2\ell_0^2}{45} \left[\left(5 - 2(2q + 1) \frac{\sigma_c}{1+\sigma_c} \right) \Lambda_m (\partial \bar{U})_{mi} - \frac{5}{4} (\Lambda \times \bar{W})_i - \left(4q + 5 + (2q + 1) \frac{\sigma_c}{1+\sigma_c} \right) \Lambda_i \text{div} \bar{U} \right]. \quad (46)$$

Equation (46) is obtained for non-stratified turbulence ($\lambda = 0$).

Note that the parameter σ_c does not affect the terms $\propto \lambda_i$ and $\propto \Lambda_i$ in equation (35) that takes into account only the leading-order effects. However, the parameter σ_c affects the contributions caused by the density stratifications to the α tensor and effective pumping velocity V^{eff} , because the density stratifications influence the large-scale shear contributions [see comment after equation (32)].

In this study, we also take into account a possibility for collapsing (or expanding) astrophysical clouds, which can be described by a non-zero $\text{div} \bar{U}$. This implies that we consider a large-scale dynamo with a large-scale shear (a non-zero large-scale vorticity $\bar{W}$) and collapsing (or expanding) large-scale motions with a non-zero $\text{div} \bar{U}$. This effect causes new contributions to the effective pumping velocity of the mean magnetic field $V^{\text{eff}} \propto \lambda \text{div} \bar{U}$ in density-stratified turbulence, or $V^{\text{eff}} \propto \Lambda \text{div} \bar{U}$ in inhomogeneous turbulence, which can arise in collapsing (or expanding) astrophysical turbulent clouds. However, the α_{ij} tensor is independent of $\text{div} \bar{U}$, i.e. it is independent of the effects of collapsing or expanding of clouds. The isotropic part of the α tensor ($\propto \delta_{ij}$)

is independent of the exponent q of the turbulence energy spectrum.

For illustration various contributions to the α tensor and effective pumping velocity V^{eff} , we consider a small-scale turbulence with large-scale linear velocity $\bar{U} = (a_U x/3, Sx + a_U y/3, a_U z/3)$ in the Cartesian coordinates (x, y, z) , where the large-scale vorticity is $\bar{W} = (0, 0, S)$ and $\text{div} \bar{U} = a_U$. The stress tensor $(\partial \bar{U})_{ij} = (S/2)(e_i^x e_j^y + e_j^x e_i^y) + a_U \delta_{ij}/3$, where e^x , e^y and e^z are the unit vectors. The vector λ that describes the stratification of the mean fluid density, is $\lambda = \lambda(0, 0, 1)$, and the vector Λ that determines the inhomogeneity of turbulence, is $\Lambda = \Lambda(0, 0, 1)$. The tensor $C_{ij}^{(\lambda)} = \lambda_m [\varepsilon_{imn} (\partial \bar{U})_{nj} + \varepsilon_{jmn} (\partial \bar{U})_{ni}]$ entering in equation (44), has the following diagonal components: $C_{xx}^{(\lambda)} = -2\lambda(\partial \bar{U})_{xy} = -\lambda S$, $C_{yy}^{(\lambda)} = 2\lambda(\partial \bar{U})_{xy} = \lambda S$ and $C_{zz}^{(\lambda)} = 0$. This yields the following diagonal components of the α tensor:

$$\alpha_{xx}^{(\lambda, \Lambda)} = -\frac{2(4q+7)}{45} \ell_0^2 S \nabla_z \ln [\bar{\rho}^{\mu_1} u_{\text{rms}}], \quad (47)$$

$$\alpha_{yy}^{(\lambda, \Lambda)} = -\frac{2(4q+3)}{45} \ell_0^2 S \nabla_z \ln [\bar{\rho}^{\mu_2} u_{\text{rms}}], \quad (48)$$

$$\alpha_{zz}^{(\lambda, \Lambda)} = -\frac{2}{9} \ell_0^2 S \nabla_z \ln [\bar{\rho}^{\mu_3} u_{\text{rms}}], \quad (49)$$

where

$$\mu_1 = -\frac{1}{4q+7} \left(2q + 3 - \frac{3\sigma_c}{1+\sigma_c} \right), \quad (50)$$

$$\mu_2 = \frac{1}{4q+3} \left(10 - 2q + \frac{3\sigma_c}{1+\sigma_c} \right). \quad (51)$$

and $\mu_3 = 13/10$. The effective pumping velocity V^{eff} is given by

$$V_x^{\text{eff}} = -\frac{2}{9} \left[1 - \frac{2}{5} (2q+1) \frac{\sigma_c}{1+\sigma_c} \right] \ell_0^2 S \nabla_z \ln [\bar{\rho}^{\mu_x} u_{\text{rms}}], \quad (52)$$

$$V_y^{\text{eff}} = \frac{2}{9} \left[1 - \frac{2}{5} (2q+1) \frac{\sigma_c}{1+\sigma_c} \right] \ell_0^2 S \nabla_z \ln [\bar{\rho}^{\mu_y} u_{\text{rms}}], \quad (53)$$

where

$$\mu_y = -2\mu_x = -\left[8q + 11 - (12q + 13) \frac{\sigma_c}{1+\sigma_c} \right] \times \left[5 - 2(2q+1) \frac{\sigma_c}{1+\sigma_c} \right]^{-1}. \quad (54)$$

The contribution to the effective pumping velocity of the mean magnetic field caused by collapsing (or expanding) clouds described by the divergence of the mean fluid velocity is given by

$$V_z^{\text{eff}} = -\frac{8}{21} \ell_0^2 \text{div} \bar{U} \left[\frac{6q}{5} + 1 + \left(q + \frac{1}{2} \right) \frac{\sigma_c}{1+\sigma_c} \right] \times \nabla_z \ln [\bar{\rho}^{\mu_z} u_{\text{rms}}], \quad (55)$$

where

$$\mu_z = \frac{1}{20} \left[77 - 34q - 31 \frac{\sigma_c}{1+\sigma_c} \right] \times \left[\frac{6q}{5} + 1 + \left(q + \frac{1}{2} \right) \frac{\sigma_c}{1+\sigma_c} \right]^{-1}. \quad (56)$$

6 APPLICATIONS TO PROTOPLANETARY DISCS AND ASTROPHYSICAL CLOUDS

In this section, we consider applications of the obtained results related to the α tensor and effective pumping velocity V^{eff} to PPD and astrophysical clouds. For simplicity, we

consider here the background turbulence without small-scale dynamo.

6.1 PROTOPLANETARY DISCS

In this section, we determine the α tensor and effective pumping velocity $\mathbf{V}^{\text{eff}}$ in protoplanetary discs. We use the cylindrical coordinates (r, φ, z) with corresponding units vectors $\mathbf{e}^r$, $\mathbf{e}^\varphi$, and $\mathbf{e}^z$ along these axes. We consider a small-scale turbulence with the large-scale non-uniform axisymmetric velocity $\bar{\mathbf{U}} = (0, r\delta\Omega(r), 0)$, where $\delta\Omega$ describes differential rotation. In this case, the large-scale vorticity is $\bar{\mathbf{W}} = \mathbf{e}^z r^{-1} (\partial/\partial r)(r^2\delta\Omega)$. Thus, $(\partial\bar{\mathbf{U}})_{r\varphi} = (r/2)(\partial/\partial r)\delta\Omega$. The vector λ that describes the non-uniform mean fluid density, is $\lambda = (\lambda_r, 0, \lambda_z)$, and the vector Λ that determines the inhomogeneity of turbulence, is $\Lambda = (\Lambda_r, 0, \Lambda_z)$. Thus, we obtain that the $\alpha_{\varphi\varphi}$ component of the α tensor is given by

$$\alpha_{\varphi\varphi} = -\frac{4}{9} \left[1 + \frac{D_r}{10}(4q+3) \right] \ell_0^2 \delta\Omega(r) \nabla_z \ln(\bar{\rho}^{\mu_\alpha} u_{\text{rms}}), \quad (57)$$

where

$$\mu_\alpha = \frac{13}{10} \left[1 + \frac{D_r}{13} \left(10 - 2q + \frac{3\sigma_c}{1+\sigma_c} \right) \right] \times \left(1 + \frac{D_r}{10}(4q+3) \right)^{-1}, \quad (58)$$

and the parameter characterizing the differential rotation defined as

$$D_r = \frac{\partial \ln \delta\Omega}{\partial \ln r}. \quad (59)$$

The φ component of the effective pumping velocity is

$$V_\varphi^{\text{eff}} \approx \frac{2\ell_0^2}{9} \delta\Omega(r) \left[1 + \frac{D_r}{10} \left(15 - 4(2q+1) \frac{\sigma_c}{1+\sigma_c} \right) \right] \times \nabla_r \ln(\bar{\rho}^{\mu_\nu} u_{\text{rms}}), \quad (60)$$

where

$$\mu_\nu = \left[1 - \frac{\sigma_c}{10(1+\sigma_c)} - \frac{D_r}{5} \left(3 + 2q - \left(6q - \frac{25}{4} \right) \frac{\sigma_c}{1+\sigma_c} \right) \right] \left[1 + \frac{D_r}{10} \left(15 - 4(2q+1) \frac{\sigma_c}{1+\sigma_c} \right) \right]^{-1}, \quad (61)$$

To derive equations (57) and (60), we use equations (44) and (45).

The analysed effects are important for generation of large-scale magnetic fields in PPD. The typical parameters of the protosolar nebula (see e.g. T. Elperin et al. 1998; L. S. Hodgson & A. Brandenburg 1998; L. Pan et al. 2011; A. Hubbard 2016; P. F. Hopkins 2016a, b; N. Kleeorin & I. Rogachevskii 2025) are as follows: the angular velocity $\Omega \sim 2 \times 10^{-7} r_{\text{AU}}^{-3/2} \text{ s}^{-1}$ (where r_{AU} is the radial coordinate measured in the astronomical units $L_{\text{AU}} = 1.5 \times 10^{13} \text{ cm}$); the shear $\delta\Omega \sim 5 \times 10^{-8} r_{\text{AU}}^{-5/2} \text{ s}^{-1}$; the sound speed $c_s = 6.4 \times 10^4 r_{\text{AU}}^{-3/14} \text{ cm s}^{-1}$; the integral scale of turbulence $\ell_0 = \sqrt{\alpha_{\text{PPD}}} c_s / \Omega = 3 \times 10^{10} r_{\text{AU}}^{9/7} \text{ cm}$; the turbulent velocity $u_0 = \alpha_{\text{PPD}} c_s \approx (65 - 650) r_{\text{AU}}^{9/14} \text{ cm s}^{-1}$; the turbulent time $\tau_0 = \ell_0 / u_0 = (\sqrt{\alpha_{\text{PPD}}} \Omega)^{-1}$, the kinematic viscosity $\nu = c_s \lambda_{\text{mfp}} / 2 = 1.6 \times 10^5 r_{\text{AU}}^{18/7} \text{ cm}^2 \text{ s}^{-1}$, so the Reynolds number $\text{Re} = \ell_0 u_0 / \nu$ varies in the range $\text{Re} = (10^6 - 10^8) r_{\text{AU}}^{-3/2}$. Here, $\lambda_{\text{mfp}} = 5 r_{\text{AU}}^{39/14} \text{ cm}$ is the mean-free path of the gas molecules, and parameter α_{PPD} varies from 10^{-3} to 10^{-2} . The mean fluid density is $\bar{\rho} = 2 \times 10^{-9} r_{\text{AU}}^{-11/4} \text{ g cm}^{-3}$, and the density stratification scale is $H_g = c_s / \Omega = 3 \times 10^{11} r_{\text{AU}}^{9/7} \text{ cm}$. This implies that the $\alpha_{\varphi\varphi}$ component of

the α tensor is estimated as $|\alpha_{\varphi\varphi}| \approx 10 r_{\text{AU}}^{-17/14} \text{ cm s}^{-1}$, and the φ component of the effective pumping velocity is $|V_\varphi^{\text{eff}}| \approx 15 r_{\text{AU}}^{-17/14} \text{ cm s}^{-1}$.

6.2 Colliding protogalactic clouds and merging protostellar clouds

Next, we consider astrophysical clouds, and use the spherical coordinates (r, θ, φ) with corresponding units vectors $\mathbf{e}^r$, $\mathbf{e}^\theta$, and $\mathbf{e}^\varphi$ along these axes. This may have relevance to colliding PGC and merging PSC. Interaction of the merging clouds causes large-scale shear motions which are superimposed on small-scale turbulence. We consider a small-scale turbulence with the large-scale non-uniform axisymmetric velocity $\bar{\mathbf{U}} = (0, 0, r \sin \theta \delta\Omega(r, \theta))$, where $\delta\Omega$ determines differential rotation. Thus, the large-scale vorticity is $\bar{\mathbf{W}} = \mathbf{e}^r (\sin \theta)^{-1} (\partial/\partial \theta)(\sin^2 \theta \delta\Omega) + \mathbf{e}^\theta \sin \theta r^{-1} (\partial/\partial r)(r^2 \delta\Omega)$. Therefore, $(\partial\bar{\mathbf{U}})_{\theta\varphi} = (\sin \theta/2)(\partial/\partial \theta)\delta\Omega$. The vector λ that describes the non-uniform mean fluid density, is $\lambda = \lambda(1, 0, 0)$, and the vector Λ that determines the inhomogeneity of turbulence, is $\Lambda = \Lambda(1, 0, 0)$. Thus, the $\alpha_{\varphi\varphi}$ component of the α tensor is given by

$$\alpha_{\varphi\varphi} = -\frac{4}{9} \left[1 + \frac{D_\theta}{10}(7-4q) \right] \ell_0^2 \delta\Omega(r) \cos \theta \times \nabla_r \ln(\bar{\rho}^{\mu_\alpha} u_{\text{rms}}), \quad (62)$$

where

$$\mu_\alpha = \frac{13}{10} \left[1 + \frac{D_\theta}{26} \left(5 + 4q - \frac{6\sigma_c}{1+\sigma_c} \right) \right] \times \left[1 + \frac{D_\theta}{10}(7-4q) \right]^{-1}, \quad (63)$$

and the parameter D_θ characterizing the latitudinal differential rotation, is defined as

$$D_\theta = \tan \theta \frac{\partial}{\partial \theta} \ln \delta\Omega. \quad (64)$$

The φ component of the effective pumping velocity is

$$V_\varphi^{\text{eff}} \approx \frac{2\ell_0^2}{9} \delta\Omega(r) \sin \theta \left[1 - \frac{D_r}{2} \left(1 - \frac{4}{5}(2q+1) \frac{\sigma_c}{1+\sigma_c} \right) \right] \times \nabla_r \ln(\bar{\rho}^{\mu_\nu} u_{\text{rms}}), \quad (65)$$

where

$$\mu_\nu = \left\{ 1 - \frac{\sigma_c}{2(1+\sigma_c)} - \frac{2D_r}{5} \left[\left(3q + \frac{31}{8} \right) \frac{\sigma_c}{1+\sigma_c} - 4 - 2q \right] \right\} \left[1 - \frac{D_r}{2} \left(1 - \frac{4}{5}(2q+1) \frac{\sigma_c}{1+\sigma_c} \right) \right]^{-1}. \quad (66)$$

To derive equations (62) and (65), we use equations (44) and (45). The parameter D_r characterizing the radial differential rotation, is defined as

$$D_r = \frac{\partial \ln \delta\Omega}{\partial \ln r}. \quad (67)$$

The joint action of the α effect and the large-scale shear causes the dynamo resulting in the generation of the large-scale magnetic field. For illustration, we consider the axisymmetric mean-field $\alpha^2 \Omega$ dynamo, so that the large-scale magnetic field can be written as $\bar{\mathbf{B}} = \bar{B}_\varphi \mathbf{e}_\varphi + \nabla \times (\bar{\mathbf{A}} \mathbf{e}_\varphi)$. For simplicity, we study the mean-field dynamo in a thin shell, neglecting the curvature of the

shell and replace it by a flat slab. We consider a kinematic dynamo problem, assuming for simplicity that the kinetic α effect is a constant. The mean-field dynamo equations in a dimensionless form are given by:

$$\frac{\partial \bar{B}_\varphi}{\partial t} = \left[R_\alpha R_\omega \sin \theta \frac{\partial}{\partial \theta} - R_\alpha^2 \left(\frac{\partial^2}{\partial \theta^2} - \kappa^2 \right) \right] \bar{A} + \left(\frac{\partial^2}{\partial \theta^2} - \kappa^2 \right) \bar{B}_\varphi, \quad (68)$$

$$\frac{\partial \bar{A}}{\partial t} = \alpha \bar{B}_\varphi + \left(\frac{\partial^2}{\partial \theta^2} - \kappa^2 \right) \bar{A}. \quad (69)$$

where $\alpha \equiv \alpha_{\varphi\varphi}$, for simplicity we average the dynamo equations over r and use the no- r model. In particular, the terms describing turbulent diffusion of the mean magnetic field in the radial direction in equations (68) and (69) in the framework of the no- r model are given as $-\kappa^2 \bar{B}_\varphi$ and $-\kappa^2 \bar{A}$ (Y. Kleeorin et al. 2016), where the parameter κ is determined by the following equation: $\int_{r_c}^1 (\partial^2 \bar{B}_\varphi / \partial r^2) dr = -(\kappa^2/3) \bar{B}_\varphi$. Here the radius r varies from r_c to 1 inside the convective shell. Equations (68)–(69) are written in dimensionless variables: the coordinate r is measured in the units of the radius R_* , the time t is measured in the units of turbulent magnetic diffusion time R_*^2/η_T , and the toroidal component $\bar{B}_\varphi(t, \theta)$ of the mean magnetic field is measured in the units of $\bar{B}_{eq} = u_0 \sqrt{\mu_0 \bar{\rho}_*}$. The magnetic potential $\bar{A}(t, \theta)$ of the poloidal field is measured in the units of $R_\alpha R_* \bar{B}_{eq}$, where

$$R_\alpha = \frac{\alpha_* R_*}{\eta_T} = \frac{\ell_0^2 \delta \Omega R_*}{H_\rho \eta_T}, \quad (70)$$

the fluid density $\bar{\rho}$ is measured in the units $\bar{\rho}_*$, the differential rotation $\delta \Omega$ is measured in units of the maximal value of the angular velocity Ω , the α effect is measured in units of the maximum value of the kinetic α effect α_* , the integral scale of the turbulent motions ℓ_0 and the characteristic turbulent velocity u_0 at the scale ℓ_0 are measured in units of their maximum values in the turbulent region, and the turbulent magnetic diffusion coefficient is $\eta_T = \ell_0 u_0/3$. The dynamo number is defined as $D = R_\alpha R_\omega$, where $R_\omega = \delta \Omega R_*^2/\eta_T$. Equations (68) and (69) describe the dynamo waves propagating from the central latitudes towards the equator when the dynamo number is negative. We seek a solution for equations (68)–(69) as a real part of the following functions: $\bar{A} = A_0 \exp(\gamma t - i K \theta)$ and $\bar{B}_\varphi = B_0 \exp(\gamma t - i K \theta)$, where $\gamma = \gamma + i \omega$. The growth rate of the dynamo instability and the frequency of the dynamo waves are given by (N. Kleeorin et al. 2023):

$$\gamma = \frac{R_\alpha R_\alpha^{\text{cr}}}{\sqrt{2}} \left[\left[1 + \left(\frac{\zeta R_\omega}{R_\alpha R_\alpha^{\text{cr}}} \right)^2 \right]^{1/2} + 1 \right]^{1/2} - (R_\alpha^{\text{cr}})^2, \quad (71)$$

$$\omega = -\text{sgn}(R_\omega) \frac{R_\alpha R_\alpha^{\text{cr}}}{\sqrt{2}} \left[\left[1 + \left(\frac{\zeta R_\omega}{R_\alpha R_\alpha^{\text{cr}}} \right)^2 \right]^{1/2} - 1 \right]^{1/2}, \quad (72)$$

where $\zeta^2 = 1 - (\kappa R_\alpha^{\text{cr}})^2$. Here we took into account that $(x + iy)^{1/2} = \pm(X + iY)$, where $X = 2^{-1/2} [(x^2 + y^2)^{1/2} + x]^{1/2}$ and $Y = \text{sgn}(y) 2^{-1/2} [(x^2 + y^2)^{1/2} - x]^{1/2}$. The threshold R_α^{cr} for the mean-field dynamo instability, defined by the conditions $\gamma = 0$ and $R_\omega = 0$, is given by $R_\alpha^{\text{cr}} = (K^2 + \kappa^2)^{1/2}$. The energy ratio of poloidal $\bar{B}_{\text{pol}} = R_\alpha R_\alpha^{\text{cr}} \bar{A}$ and toroidal $\bar{B}_\varphi$ mean magnetic field com-

ponents are given by

$$\frac{\bar{B}_{\text{pol}}^2}{\bar{B}_\varphi^2} = \left[1 + \left(\frac{\zeta R_\omega}{R_\alpha R_\alpha^{\text{cr}}} \right)^2 \right]^{-1/2}, \quad (73)$$

and the phase shift δ between the toroidal field $\bar{B}_\varphi$ and the magnetic vector potential $\bar{A}$ is

$$\sin(2\delta) = -\zeta R_\omega \left[(R_\alpha R_\alpha^{\text{cr}})^2 + \zeta^2 R_\omega^2 \right]^{-1/2}. \quad (74)$$

Now we apply the developed theory to the various astrophysical turbulent clouds. Let us first discuss a scenario of formation the large-scale shear motions in colliding PGC (see e.g. A. D. Chernin 1991, 1993; H. Wiechen et al. 1998; G. Birk et al. 2002; I. Rogachevskii et al. 2006). Jean's process of gravitational instability and fragmentation can cause a very clumsy state of cosmic matter at the epoch of galaxy formation. A complex system of rapidly moving gaseous fragments embedded into rare gas might appear in some regions of protogalactic matter. Supersonic contact collisions of these PGC might play a role of an important elementary process in a complex non-linear dynamics of protogalactic medium. The supersonic contact non-central collisions of these PGC could lead to their coalescence, formation of large-scale shear motions and transformation of their initial orbital momentum into the spin momentum of the merged condensations bound by its condensations (A. D. Chernin 1993).

Two-dimensional hydrodynamical models for inelastic non-central cloud-cloud collisions in the protogalactic medium have been developed by A. D. Chernin (1993). An evolutionary picture of the collision is as follows. At the first stage of the process the standard dynamical structure, i.e. two shock fronts and tangential discontinuity between them arise in the collision zone. Compression and heating of gas which crosses the shock fronts occurs. The heating entails intensive radiation emission and considerable energy loss by the system which promotes gravitational binding of the cloud material.

At the second stage of the process a dense core forms at the central part of the clump. In the vicinity of the core two kinds of jets form: 'flyaway' jets of the material (which does not undergo the direct contact collision) and internal jets sliding along the curved surface of the tangential discontinuity. The flyaway jets are subsequently torn off, having overcome the gravitational attraction of the clump whereas the internal jets remain bound in the clump. When the shock fronts reach the outer boundaries of the clump, the third stage of the process starts. Shocks are replaced by the rarefaction waves and overall differential rotation and large-scale shear motions arise. This structure can be considered as a model of the protogalactic condensation (A. D. Chernin 1993).

The formed large-scale sheared motions are superimposed on small-scale turbulence. There are two important characteristics of the protogalactic cloud—cloud collisions: the mass bound in the resulting clump and the spin momentum acquired by it. These characteristics depend on the relative velocity and impact parameter of the collision (A. D. Chernin 1993).

The parameters of PGC are as follows (see e.g. A. D. Chernin 1991, 1993; H. Wiechen et al. 1998; G. Birk et al. 2002; I. Rogachevskii et al. 2006): the mass is $M \leq 10^{10} M_\odot$, the radius is $R \sim 10^{23}$ cm, the internal temperature is $\bar{T} \sim 10^4$ K, the mean radial velocity of the cloud is $\bar{U} \sim 10^6$ – 10^7 cm s $^{-1}$, where $M_\odot$ is

Table 1. Table 1 The parameters of clouds.

	PGC	PSC
Mass	$M \leq 10^{10} M_{\odot}$	$M \leq M_{\odot}$
R (cm)	10^{23}	10^{17}
$\bar{U}$ (cm s $^{-1}$)	10^6 – 10^7	10^5 – 10^6
$\bar{\rho}$ (g cm $^{-3}$)	10^{-26}	$(1-5) \times 10^{-19}$
$\Delta\bar{U}$ (cm s $^{-1}$)	10^6 – 10^7	10^5
ΔR (cm)	2×10^{23}	10^{16} – 10^{17}
S (s $^{-1}$)	$(0.5-5) \times 10^{-16}$	10^{-12} – 10^{-11}
u_0 (cm s $^{-1}$)	10^6 – 10^7	10^4
ℓ_0 (cm)	10^{22}	10^{15} – 10^{16}
τ_0 (years)	$(0.3-3) \times 10^8$	$(0.3-3) \times 10^4$
η_T (cm 2 s $^{-1}$)	$(0.3-3) \times 10^{28}$	$(0.3-3) \times 10^{19}$
t_η (years)	$(0.3-3) \times 10^9$	10^6 – 10^7
α (cm s $^{-1}$)	10^4 – 10^5	10^2 – 10^4
V_φ^{eff} (cm s $^{-1}$)	10^3 – 10^4	10^2 – 10^4

the solar mass. Some other parameters for the PGC are given in Table 1.

We use the following notations: ΔR is the characteristic scale of the mean velocity inhomogeneity, $\Delta\bar{U}$ is the typical velocity change across ΔR , $S = \Delta\bar{U}/\Delta R$ is the mean velocity shear, u_0 is the characteristic turbulent velocity, ℓ_0 is the integral scale of turbulent motions, $\tau_0 = \ell_0/u_0$ is the characteristic turbulent time, η_T is the turbulent magnetic diffusivity, and $t_\eta = (\Delta R)^2/\eta_T$ is the turbulent diffusion time. Using the parameters given in Table 1, we estimate the α effect as $\alpha \equiv |\alpha_{\varphi\varphi}| \sim 10^4$ – 10^5 cm s $^{-1}$ and the effective pumping velocity of the mean magnetic field is estimated as $|V_\varphi^{\text{eff}}| \sim 10^3$ – 10^4 cm s $^{-1}$. The α effect in combination with the large-scale shear motions can cause generation of large-scale magnetic field.

An important feature of the dynamics of the interstellar matter is fairly rapid motions of relatively dense matter fragments (PSC) embedded in to rare gas. The origin of PSC might be a result of fragmentation of the core of large molecular clouds. Supersonic and inelastic collisions of the PSC can cause merging of the clouds and formation of a condensation. A non-central collision of the PSC can cause conversion of initial orbital momentum of the clouds in to spin momentum and formation of differential rotation and shear motions (A. D. Chernin 1991).

The internal part of the condensation would have only slow rotation because the initial matter motions could be almost stopped in the zone of direct cloud contact. On the other hand, the minor outer part of the merged cloud matter of the condensation would have very rapid rotation due to the initial motions of that portions of cloud materials which would not stop in this zone because they do not undergo any direct cloud collision (A. D. Chernin 1991). This material could keep its motion on gravitationally bound orbits around the major internal body condensation. The formed large-scale sheared motions are superimposed on small-scale interstellar turbulence.

In the supersonic and inelastic collision of the PSC, an essential part of the initial kinetic energy that is lost during the mass lost, is due to dissipation and subsequent radiative emission. The cooling time scale for the material compressed in the collision would be less than the time scale of the hydrodynamic processes.

The parameters of PSC are as follows (see e.g. A. D. Chernin 1991; I. Rogachevskii et al. 2006): a mass is $M \leq M_{\odot}$, the radius is $R \sim 10^{17}$ cm, the internal temperature is $\bar{T} \sim 10$ K, the mean radial velocity of the cloud is $\bar{U} \sim 10^5$ – 10^6 cm s $^{-1}$. Some other

parameters for the PSC are given in Table 1. Using the parameters given in Table 1, we find the α effect as $\alpha \equiv |\alpha_{\varphi\varphi}| \sim 10^2$ – 10^4 cm s $^{-1}$ and the effective pumping velocity of the mean magnetic field is $|V_\varphi^{\text{eff}}| \sim 10^2$ – 10^4 cm s $^{-1}$.

7 CONCLUSIONS

In the present study we determine the α effect and the effective pumping velocity of a large-scale magnetic field caused by a combined effect of the density-stratified turbulence and a large-scale shear for arbitrary Mach numbers. These phenomena are derived applying the spectral τ approach that is valid for large fluid and magnetic Reynolds numbers.

We demonstrate that the finite Mach number effects does not affect the contributions caused by the inhomogeneity of turbulence to the α tensor, but they influence the effective pumping velocity of the mean magnetic field. In addition, the isotropic part of the α tensor is independent of the exponent of the turbulent kinetic energy spectrum for this system. On the other hand, the anisotropic part of the α tensor depends on this exponent, and the latitudinal profile of differential rotation also contributes to this anisotropic part of the α tensor. The latter may be important for the dynamo operation in the upper parts of the solar and stellar convection zones. We also find an additional contribution to the effective pumping velocity of the mean magnetic field that is proportional to the product of the fluid density stratification and the divergence of the mean fluid velocity caused by collapsing (or expanding) astrophysical clouds. On the other hand, we show that the α tensor is independent of the effects of collapsing (or expanding) clouds.

These effects may be the reasons for generation of the large-scale magnetic field in protoplanetary discs, colliding PGC, merging PSC, solar and stellar convective zones. In particular, the theoretical results of Section 6.2 are directly applicable to the solar and stellar convective zones (which is a subject of a separate study).

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DATA AVAILABILITY

There are no new data associated with this article.

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APPENDIX A: IDENTITIES USED FOR DERIVATION OF EMF

The tensors $I_{ijmn}(\bar{\mathbf{U}})$, $J_{ijmn}(\bar{\mathbf{U}})$, and $E_{ijmn}(\bar{\mathbf{U}})$ in equations (4)–(6) are given by (N. Kleeorin & I. Rogachevskii 2022)

$$I_{ijmn}(\bar{\mathbf{U}}) = \left\{ 2k_{iq}\delta_{mp}\delta_{jn} + 2k_{jq}\delta_{im}\delta_{pn} - \delta_{im}\delta_{jq}\delta_{np} - \delta_{iq}\delta_{jn}\delta_{mp} + 4k_{pq}\delta_{im}\delta_{jn} + \delta_{im}\delta_{jn}k_q \frac{\partial}{\partial k_p} - \frac{i\lambda_r}{2k^2} \left[(k_i\delta_{jn}\delta_{pm} - k_j\delta_{im}\delta_{pn}) (2k_{rq} - \delta_{rq}) + k_q (\delta_{ip}\delta_{jn}\delta_{rm} - \delta_{im}\delta_{jp}\delta_{rn}) - 2k_{pq} (k_i\delta_{jn}\delta_{rm} - k_j\delta_{im}\delta_{rn}) \right] \right\} \nabla_p \bar{U}_q, \quad (\text{A1})$$

$$J_{ijmn}(\bar{\mathbf{U}}) = \left\{ 2k_{iq}\delta_{jn}\delta_{pm} - \delta_{iq}\delta_{jn}\delta_{pm} + \delta_{im}\delta_{jq}\delta_{pn} + 2k_{pq}\delta_{im}\delta_{jn} + \delta_{im}\delta_{jn}k_q \frac{\partial}{\partial k_p} - \frac{i\lambda_r}{2k^2} \left[k_i\delta_{jn}\delta_{pm} \times (2k_{rq} - \delta_{rq}) + \delta_{jn}\delta_{rm} (k_q\delta_{ip} - 2k_i k_{pq}) \right] \right\} \nabla_p \bar{U}_q, \quad (\text{A2})$$

$$E_{ijmn}(\bar{\mathbf{U}}) = \left[\delta_{im}\delta_{jq}\delta_{pn} + \delta_{iq}\delta_{jn}\delta_{pm} + \delta_{im}\delta_{jn}k_q \frac{\partial}{\partial k_p} \right] \nabla_p \bar{U}_q. \quad (\text{A3})$$

Equations (A1)–(A3) are valid for weak large-scale shear ($\bar{\mathbf{W}}\tau_0 \ll 1$), where $\bar{\mathbf{W}}$ is the mean vorticity, and we neglected the second-order derivatives of the mean velocity $\bar{\mathbf{U}}$. The reason of the appearance of the stratification parameter λ_i in the tensors I_{ijmn} and J_{ijmn} is caused by the exclusion of the gradient of pressure fluctuations from the Navier–Stokes equation (1) by taking twice curl from this equation. On the other hand, the parameter Λ_i that characterizes the inhomogeneity of turbulence cannot enter in the tensors I_{ijmn} and J_{ijmn} . It appears only in the tensor $f_{ij}^{(0)}$.

To derive expression for the contributions to the turbulent electromotive force caused by a density-stratified and inhomogeneous turbulence with a non-uniform large-scale flow and low Mach numbers, we use the following identities:

$$\mathcal{E}_f^{(1a)} = \langle \mathbf{u}^2 \rangle^{(0)} \frac{\bar{B}_s \varepsilon_{fij}}{8\pi} \int \tau^2(k) k_s I_{ijmn} E(k) (\tilde{\lambda}_m k_n - \tilde{\lambda}_n k_m) k^{-4} d\mathbf{k} = \frac{4}{45} \ell_0^2 \bar{B}_j \tilde{\lambda}_r \left[(2q-1) \varepsilon_{frn} \Delta_{pqjn} + 5(\varepsilon_{fjq} \delta_{rp} + \varepsilon_{frp} \delta_{qj} + \varepsilon_{fqr} \delta_{jp}) \right] \nabla_p \bar{U}_q, \quad (\text{A4})$$

$$\mathcal{E}_f^{(1b)} = \langle \mathbf{u}^2 \rangle^{(0)} \frac{\bar{B}_s \varepsilon_{fij}}{8\pi} \int \tau^2(k) k_s J_{ijmn} E(k) (\tilde{\lambda}_m k_n - \tilde{\lambda}_n k_m) k^{-4} d\mathbf{k} = \frac{4}{45} \ell_0^2 \bar{B}_j \tilde{\lambda}_r \left[2(q+3) \varepsilon_{frn} \Delta_{pqjn} + 5\varepsilon_{fpr} \delta_{qj} \right] \nabla_p \bar{U}_q, \quad (\text{A5})$$

where $\tilde{\lambda}_i = \lambda_i - \Lambda_i/2$, and

$$\mathcal{E}_f^{(2a)} = \langle \mathbf{u}^2 \rangle^{(0)} \frac{\bar{B}_s \varepsilon_{fij}}{8\pi} \int \frac{E(k)}{k^2} \tau^2(k) (-ik_s) I_{ijmn} \times P_{mn} d\mathbf{k} = \frac{2}{45} \ell_0^2 \bar{B}_j \tilde{\lambda}_r \left[3\varepsilon_{fpm} \Delta_{rmjq} + 2\varepsilon_{fmr} \Delta_{pqmj} + 5(\varepsilon_{fjp} \delta_{rq} + \varepsilon_{frp} \delta_{jq}) \right] \nabla_p \bar{U}_q, \quad (\text{A6})$$

$$\begin{aligned}\mathcal{E}_f^{(2b)} &= \frac{\langle \mathbf{u}^2 \rangle^{(0)}}{8\pi} \bar{B}_s \varepsilon_{fij} \int \frac{E(k)}{k^2} \tau^2(k) (-ik_s) J_{ijmn} P_{mn} d\mathbf{k} \\ &= \frac{1}{45} \ell_0^2 \bar{B}_j \lambda_r \left[3\varepsilon_{fpm} \Delta_{rmjq} + 2\varepsilon_{fmr} \Delta_{pqmj} \right. \\ &\quad \left. + 5(\varepsilon_{fjp} \delta_{rq} + \varepsilon_{frp} \delta_{jq}) \right] \nabla_p \bar{U}_q, \quad (\text{A7})\end{aligned}$$

$$\begin{aligned}\mathcal{E}_f^{(2c)} &= -\frac{\langle \mathbf{u}^2 \rangle^{(0)}}{8\pi} \bar{B}_j \lambda_r \varepsilon_{fij} \int \frac{E(k)}{k^2} \tau^2(k) I_{irmn} P_{mn} d\mathbf{k} \\ &= \frac{4}{45} \ell_0^2 \bar{B}_j \lambda_r \left[(3+q) \varepsilon_{fnj} \Delta_{pqnr} - 5q \varepsilon_{frj} \delta_{pq} \right] \nabla_p \bar{U}_q, \quad (\text{A8})\end{aligned}$$

where $\ell_0^2 = \langle \mathbf{u}^2 \rangle^{(0)} \tau_0^2$ and $\Delta_{ijmn} = \delta_{ij} \delta_{mn} + \delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}$.

For the integration over angles in the $\mathbf{k}$ space, we used the following integrals:

$$\int_0^{2\pi} d\varphi \int_0^\pi k_{ij} \sin \vartheta d\vartheta = \frac{4\pi}{3} \delta_{ij}, \quad (\text{A9})$$

$$\int_0^{2\pi} d\varphi \int_0^\pi k_{ijmn} \sin \vartheta d\vartheta = \frac{4\pi}{15} \Delta_{ijmn}, \quad (\text{A10})$$

where $k_{ijmn} = k_{ij} k_{mn}$. To integrate over k , we used the following integral: $\int_{k_0}^{k_v} \tau^2(k) E(k) dk = 4\tau_0^2/3$, and

$$\begin{aligned}\int \tau(k) f_{ij}^{(S)}(\mathbf{k}) d\mathbf{k} &= \frac{4\ell_0^2}{45} \left[(4q-3) \delta_{ij} \text{div} \bar{\mathbf{U}} \right. \\ &\quad \left. - 2(q+3) (\partial \bar{\mathbf{U}})_{ij} \right]. \quad (\text{A11})\end{aligned}$$

We take into account that in anelastic approximation, $(ik_n - \nabla_n/2) f_{in}(\mathbf{k}) = -\lambda_n f_{in}(\mathbf{k})$. This implies that the contributions to the turbulent electromotive force caused by the last three terms in equation (4) is given by

$$\begin{aligned}\tilde{\mathcal{E}}_f &= \varepsilon_{fij} \bar{B}_j \int \tau(k) \left[ik_n - \lambda_n - \frac{1}{2} \nabla_n \right] f_{in}^{(S)}(\mathbf{k}) d\mathbf{k} \\ &= -2\lambda_n \varepsilon_{fij} \bar{B}_j \int \tau(k) f_{in}^{(S)}(\mathbf{k}) d\mathbf{k} = 2\mathcal{E}_f^{(2c)}. \quad (\text{A12})\end{aligned}$$

The contributions of the small-scale dynamo in the background turbulence to the turbulent electromotive force are given by

$$\begin{aligned}\mathcal{E}_f^{(M,\lambda)} &= \frac{\langle \mathbf{b}^2 \rangle^{(0)}}{8\pi} \bar{B}_s \varepsilon_{fij} \int \frac{E(k)}{k^2} \tau^2(k) (ik_s) J_{ijmn} P_{mn} d\mathbf{k} \\ &= \frac{\ell_0^2}{45} \left[\frac{\ell_M}{\ell_0} \right]^{3(q-1)} \left[\frac{\langle \mathbf{b}^2 \rangle^{(0)}}{\mu_0 \bar{\rho} \langle \mathbf{u}^2 \rangle^{(0)}} \right] \bar{B}_j \lambda_r \left[3\varepsilon_{fnp} \Delta_{rnjq} \right. \\ &\quad \left. - 2\varepsilon_{fnr} \Delta_{pqnj} - 5(\varepsilon_{fjp} \delta_{rq} + \varepsilon_{frp} \delta_{jq}) \right] \nabla_p \bar{U}_q, \quad (\text{A13})\end{aligned}$$

$$\begin{aligned}\mathcal{E}_f^{(\Lambda_M)} &= -\langle \mathbf{b}^2 \rangle^{(0)} \frac{\bar{B}_s \varepsilon_{fij}}{16\pi} \int \tau^2(k) k_s E_{ijmn} E(k) k^{-4} \\ &\quad \times (\Lambda_m^{(M)} k_n - \Lambda_n^{(M)} k_m) d\mathbf{k} = -\frac{2\ell_0^2}{45} \left[\frac{\ell_M}{\ell_0} \right]^{3(q-1)} \bar{B}_j \Lambda_r^{(M)} \\ &\quad \times \left[\frac{\langle \mathbf{b}^2 \rangle^{(0)}}{\mu_0 \bar{\rho} \langle \mathbf{u}^2 \rangle^{(0)}} \right] \left[2(q-1) \varepsilon_{frn} \Delta_{pqjn} + 5(\varepsilon_{fqj} \delta_{rp} \right. \\ &\quad \left. + \varepsilon_{fjr} \delta_{qp}) \right] \nabla_p \bar{U}_q, \quad (\text{A14})\end{aligned}$$

where we take into account that $\int_{k_M}^{k_v} \dots E(k) dk = \int_0^{\tau_M} \dots d\tilde{\tau}$, $\tau_M = (\ell_M/\ell_0)^{q-1}$, $\tilde{\tau}(k) = (k/k_0)^{1-q}$, $k_M = \ell_M^{-1}$, and $\ell_v = k_v^{-1} \rightarrow 0$.

Now we take into account that $\mathcal{E}_i = a_{ij} \bar{B}_j$, so that the corresponding contributions to the tensor a_{ij} are given by

$$\begin{aligned}a_{ij}^{(1a)} &= \frac{2\ell_0^2}{45} \left[10 (\tilde{\lambda} \cdot \bar{\mathbf{W}}) \delta_{ij} - 5(\tilde{\lambda}_i \bar{W}_j + \tilde{\lambda}_j \bar{W}_i) \right. \\ &\quad \left. - 2\tilde{\lambda}_m \left((4q-2) \varepsilon_{inm} (\partial \bar{\mathbf{U}})_{nj} + (2q-1) \varepsilon_{ijm} \text{div} \bar{\mathbf{U}} \right. \right. \\ &\quad \left. \left. - 5\varepsilon_{ijn} (\partial \bar{\mathbf{U}})_{nm} \right) \right], \quad (\text{A15})\end{aligned}$$

$$\begin{aligned}a_{ij}^{(1b)} &= -\frac{4\ell_0^2}{45} \tilde{\lambda}_m \left[(4q+7) \varepsilon_{inm} (\partial \bar{\mathbf{U}})_{nj} + \frac{5}{2} \left[(\tilde{\lambda} \cdot \bar{\mathbf{W}}) \delta_{ij} \right. \right. \\ &\quad \left. \left. - \tilde{\lambda}_j \bar{W}_i \right] + (2q+6) \varepsilon_{ijm} \text{div} \bar{\mathbf{U}} \right], \quad (\text{A16})\end{aligned}$$

$$\begin{aligned}a_{ij}^{(2a)} &= 2a_{ij}^{(2b)} = \frac{2\ell_0^2}{45} \left[(\lambda \cdot \bar{\mathbf{W}}) \delta_{ij} + \lambda_i \bar{W}_j + \lambda_j \bar{W}_i \right. \\ &\quad \left. + 2\lambda_m \left(\varepsilon_{inm} (\partial \bar{\mathbf{U}})_{nj} + \varepsilon_{ijn} (\partial \bar{\mathbf{U}})_{mn} + \varepsilon_{ijm} \text{div} \bar{\mathbf{U}} \right) \right], \quad (\text{A17})\end{aligned}$$

$$a_{ij}^{(2c)} = \frac{4}{45} \ell_0^2 \varepsilon_{ijm} \lambda_n \left[(4q-3) \delta_{mn} \text{div} \bar{\mathbf{U}} - 2(q+3) (\partial \bar{\mathbf{U}})_{mn} \right]. \quad (\text{A18})$$

The total contribution to the turbulent electromotive force caused by a density-stratified homogeneous ($\Lambda_i = 0$) turbulence with a large-scale shear for a low Mach numbers is

$$\begin{aligned}a_{ij}^{(\lambda)} &= a_{ij}^{(1a)} + a_{ij}^{(1b)} + a_{ij}^{(2a)} + a_{ij}^{(2b)} + 2a_{ij}^{(2c)} \\ &= \frac{\ell_0^2}{45} \left[13 (\lambda \cdot \bar{\mathbf{W}}) \delta_{ij} + 3 \bar{W}_i \lambda_j - 7 \bar{W}_j \lambda_i \right. \\ &\quad \left. - 2\lambda_m \left((4q-7) \varepsilon_{inm} (\partial \bar{\mathbf{U}})_{nj} - (14q-22) \varepsilon_{ijm} \text{div} \bar{\mathbf{U}} \right. \right. \\ &\quad \left. \left. + (8q+11) \varepsilon_{ijn} (\partial \bar{\mathbf{U}})_{mn} \right) \right]. \quad (\text{A19})\end{aligned}$$

The total contribution to the turbulent electromotive force caused by an inhomogeneous turbulence with a non-uniform large-scale flow for a low Mach numbers is $\mathcal{E}_i^{(\Lambda)} = a_{ij}^{(\Lambda)} \bar{B}_j$ and $a_{ij}^{(\Lambda)} = a_{ij}^{(1a)}(\lambda=0) + a_{ij}^{(1b)}(\lambda=0)$, i.e.

$$\begin{aligned}a_{ij}^{(\Lambda)} &= \frac{\ell_0^2}{45} \left[5 \bar{W}_j \Lambda_i - 5 (\Lambda \cdot \bar{\mathbf{W}}) \delta_{ij} \right. \\ &\quad \left. + 2\Lambda_m \left((4q-2) \varepsilon_{inm} (\partial \bar{\mathbf{U}})_{nj} + (4q+5) \varepsilon_{ijm} \text{div} \bar{\mathbf{U}} \right. \right. \\ &\quad \left. \left. - 5 \varepsilon_{ijn} (\partial \bar{\mathbf{U}})_{nm} \right) \right]. \quad (\text{A20})\end{aligned}$$

The contributions of the small-scale dynamo in the background turbulence to the tensor a_{ij} are given by

$$\begin{aligned}a_{ij}^{(M,\lambda)} &= -\frac{\ell_0^2}{45} \left(\frac{\ell_M}{\ell_0} \right)^{3(q-1)} \left[\frac{\langle \mathbf{b}^2 \rangle^{(0)}}{\mu_0 \bar{\rho} \langle \mathbf{u}^2 \rangle^{(0)}} \right] \left[(\lambda \cdot \bar{\mathbf{W}}) \delta_{ij} \right. \\ &\quad \left. + \lambda_i \bar{W}_j + \lambda_j \bar{W}_i + 2\lambda_m \left(\varepsilon_{inm} (\partial \bar{\mathbf{U}})_{nj} \right. \right. \\ &\quad \left. \left. + \varepsilon_{ijn} (\partial \bar{\mathbf{U}})_{mn} + \varepsilon_{ijm} \text{div} \bar{\mathbf{U}} \right) \right], \quad (\text{A21})\end{aligned}$$

$$\begin{aligned}a_{ij}^{(\Lambda_M)} &= \frac{2\ell_0^2}{45} \left(\frac{\ell_M}{\ell_0} \right)^{3(q-1)} \left[\frac{\langle \mathbf{b}^2 \rangle^{(0)}}{\mu_0 \bar{\rho} \langle \mathbf{u}^2 \rangle^{(0)}} \right] \left[\frac{5}{2} (\Lambda_j^{(M)} \bar{W}_i \right. \\ &\quad \left. - \Lambda_i^{(M)} \bar{W}_j) + \Lambda_m^{(M)} \left(4(q-1) \varepsilon_{inm} (\partial \bar{\mathbf{U}})_{nj} \right. \right. \\ &\quad \left. \left. + 5\varepsilon_{ijn} (\partial \bar{\mathbf{U}})_{mn} + (2q-7) \varepsilon_{ijm} \text{div} \bar{\mathbf{U}} \right) \right]. \quad (\text{A22})\end{aligned}$$

To derive equations (A15)–(A22), we use the following identities:

$$\varepsilon_{inp} \Delta_{jnqr} (\nabla_p \bar{U}_q) \lambda_r = \frac{1}{2} \left[(\lambda \cdot \bar{W}) \delta_{ij} + \bar{W}_j \lambda_i - 4\bar{W}_i \lambda_j \right] + \left[\varepsilon_{ijn} (\partial \bar{U})_{mn} + \varepsilon_{imn} (\partial \bar{U})_{nj} \right] \lambda_m, \quad (A23)$$

$$\varepsilon_{inr} \Delta_{pqnr} (\nabla_p \bar{U}_q) \lambda_r = \lambda_m \left[2\varepsilon_{imn} (\partial \bar{U})_{nj} + \varepsilon_{ijm} \operatorname{div} \bar{U} \right], \quad (A24)$$

$$\varepsilon_{ijn} \Delta_{pqnr} (\nabla_p \bar{U}_q) \lambda_r = \lambda_m \left[2\varepsilon_{ijn} (\partial \bar{U})_{nm} + \varepsilon_{ijm} \operatorname{div} \bar{U} \right], \quad (A25)$$

$$\varepsilon_{ijp} (\nabla_p \bar{U}_q) \lambda_q = \frac{1}{2} \left[\bar{W}_j \lambda_i - \bar{W}_i \lambda_j \right] + \varepsilon_{ijp} (\partial \bar{U})_{pq} \lambda_q. \quad (A26)$$

$$\varepsilon_{irp} (\nabla_p \bar{U}_j) \lambda_r = \frac{1}{2} \left[(\lambda \cdot \bar{W}) \delta_{ij} - \bar{W}_i \lambda_j \right] + \varepsilon_{imn} (\partial \bar{U})_{nj} \lambda_m, \quad (A27)$$

$$\varepsilon_{iqp} (\nabla_j \bar{U}_q) \lambda_r = \frac{1}{2} \left[(\bar{W} \cdot \lambda) \delta_{ij} - \bar{W}_i \lambda_j \right] + \varepsilon_{iqm} (\partial \bar{U})_{qj} \lambda_m, \quad (A28)$$

$$\varepsilon_{ijq} (\nabla_p \bar{U}_q) \lambda_p = \frac{1}{2} \left[\bar{W}_i \lambda_j - \bar{W}_j \lambda_i \right] + \varepsilon_{ijq} (\partial \bar{U})_{pq} \lambda_p. \quad (A29)$$

To take into account the compressible contributions (for arbitrary Mach numbers) to the turbulent electromotive force, we use the following identities:

$$\begin{aligned} \mathcal{E}_f^{(3a)} &= \frac{\langle \mathbf{u}^2 \rangle^{(0)} \sigma_c}{4\pi(1+\sigma_c)} \bar{B}_s \varepsilon_{fij} \int \frac{E(k)}{k^2} \tau^2(k) (-ik_s) I_{ijmn} \\ &\times k_{mn} d\mathbf{k} = \frac{4\sigma_c}{45(1+\sigma_c)} \ell_0^2 \bar{B}_j \lambda_r \varepsilon_{fnp} \Delta_{jnqr} \nabla_p \bar{U}_q, \end{aligned} \quad (A30)$$

$$\begin{aligned} \mathcal{E}_f^{(3b)} &= \frac{\langle \mathbf{u}^2 \rangle^{(0)} \sigma_c}{4\pi(1+\sigma_c)} \bar{B}_s \varepsilon_{fij} \int \frac{E(k)}{k^2} \tau^2(k) (-ik_s) J_{ijmn} \\ &\times k_{mn} d\mathbf{k} = \frac{2\sigma_c}{45(1+\sigma_c)} \ell_0^2 \bar{B}_j \lambda_r \varepsilon_{fnp} \Delta_{jnqr} \nabla_p \bar{U}_q, \end{aligned} \quad (A31)$$

$$\begin{aligned} \mathcal{E}_f^{(3c)} &= -\frac{\langle \mathbf{u}^2 \rangle^{(0)} \sigma_c}{4\pi(1+\sigma_c)} \bar{B}_j \lambda_s \varepsilon_{fij} \int \frac{E(k)}{k^2} \tau^2(k) I_{ismn} \\ &\times k_{mn} d\mathbf{k} = \frac{4(2q+1)\sigma_c}{45(1+\sigma_c)} \ell_0^2 \bar{B}_j \lambda_s \varepsilon_{fjn} \Delta_{pqns} \nabla_p \bar{U}_q, \end{aligned} \quad (A32)$$

$$\begin{aligned} \mathcal{E}_f^{(3d)} &= \frac{\sigma_c \langle \mathbf{u}^2 \rangle^{(0)}}{4\pi(1+\sigma_c)} \bar{B}_j \varepsilon_{fij} \int \frac{E(k)}{k^2} \tau^2(k) \left[ik_s - \frac{1}{2} \Lambda_s \right] \\ &\times I_{ismn} k_{mn} d\mathbf{k} = \frac{2\sigma_c \ell_0^2}{45(1+\sigma_c)} \bar{B}_j \varepsilon_{fjn} \left[\lambda_r (5\delta_{np} \delta_{qr} \right. \\ &\left. - \Delta_{npqr}) - (2q+1) \Lambda_r \Delta_{npqr} \right] \nabla_p \bar{U}_q, \end{aligned} \quad (A33)$$

so that

$$\begin{aligned} a_{ij}^{(3a)} &= 2a_{ij}^{(3b)} = \frac{2\ell_0^2}{45} \left(\frac{\sigma_c}{1+\sigma_c} \right) \left[(\lambda \cdot \bar{W}) \delta_{ij} + \lambda_i \bar{W}_j \right. \\ &\left. - 4\lambda_j \bar{W}_i + 2\lambda_m (\varepsilon_{imn} (\partial \bar{U})_{nj} + \varepsilon_{ijn} (\partial \bar{U})_{mn}) \right], \end{aligned} \quad (A34)$$

$$a_{ij}^{(3c)} = \frac{4(2q+1)}{45} \ell_0^2 \left(\frac{\sigma_c}{1+\sigma_c} \right) \varepsilon_{ijn} \left[2\lambda_m (\partial \bar{U})_{mn} + \lambda_n \operatorname{div} \bar{U} \right]. \quad (A35)$$

$$\begin{aligned} a_{ij}^{(3d)} &= -\frac{2\ell_0^2}{45} \left(\frac{\sigma_c}{1+\sigma_c} \right) \left[\frac{5}{2} (\lambda_i \bar{W}_j - \lambda_j \bar{W}_i) \right. \\ &\left. - \lambda_m \varepsilon_{ijm} \operatorname{div} \bar{U} + 3\lambda_m \varepsilon_{ijn} (\partial \bar{U})_{mn} \right. \\ &\left. - (2q+1) \varepsilon_{ijn} (2\Lambda_m (\partial \bar{U})_{nj} + \Lambda_n \operatorname{div} \bar{U}) \right]. \end{aligned} \quad (A36)$$

Therefore,

$$a_{ij}^{(\text{tot})} = a_{ij}(\sigma_c = 0) + a_{ij}^{(\sigma_c)}, \quad (A37)$$

where

$$a_{ij}(\sigma_c = 0) = a_{ij}^{(1a)} + a_{ij}^{(1b)} + a_{ij}^{(2a)} + a_{ij}^{(2b)} + 2a_{ij}^{(2c)}, \quad (A38)$$

and

$$\begin{aligned} a_{ij}^{(\sigma_c)} &= -\frac{\sigma_c}{1+\sigma_c} \left(a_{ij}^{(2a)} + a_{ij}^{(2b)} + 2a_{ij}^{(2c)} \right) + a_{ij}^{(3a)} \\ &+ a_{ij}^{(3b)} + a_{ij}^{(3c)} + a_{ij}^{(3d)} = -\frac{\ell_0^2}{45} \left(\frac{\sigma_c}{1+\sigma_c} \right) \left[5\lambda_i \bar{W}_j \right. \\ &+ 10\lambda_j \bar{W}_i + 2\lambda_m \left[2(2q-3) \varepsilon_{ijm} \operatorname{div} \bar{U} \right. \\ &+ 6\varepsilon_{imn} (\partial \bar{U})_{nj} - (12q+13) \varepsilon_{ijn} (\partial \bar{U})_{mn} \left. \right] \\ &\left. - 2(2q+1) \varepsilon_{ijn} (2\Lambda_m (\partial \bar{U})_{nm} + \Lambda_n \operatorname{div} \bar{U}) \right]. \end{aligned} \quad (A39)$$

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