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Budget equations and astrophysical non-linear mean-field dynamos

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ABSTRACT

Solar, stellar and galactic large-scale magnetic fields are originated due to a combined action of non-uniform (differential) rotation and helical motions of plasma via mean-field dynamos. Usually, non-linear mean-field dynamo theories take into account algebraic and dynamic quenching of alpha effect and algebraic quenching of turbulent magnetic diffusivity. However, the theories of the algebraic quenching do not take into account the effect of modification of the source of turbulence by the growing large-scale magnetic field. This phenomenon is due to the dissipation of the strong large-scale magnetic field resulting in an increase of the total turbulent energy. This effect has been studied using the budget equation for the total turbulent energy (which takes into account the feedback of the generated large-scale magnetic field on the background turbulence) for (i) a forced turbulence, (ii) a shear-produced turbulence, and (iii) a convective turbulence. As the result of this effect, a non-linear dynamo number decreases with increase of the large-scale magnetic field, so that that the mean-field $\alpha\Omega$, α^2, and $\alpha^2\Omega$ dynamo instabilities are always saturated by the strong large-scale magnetic field.

Key words: dynamo – MHD – turbulence – Sun: interior – activity – galaxies: magnetic fields.

1 INTRODUCTION

Large-scale magnetic fields in the Sun, stars, and galaxies are believed to be generated by a joint action of a differential rotation and helical motions of plasma (see e.g. Moffatt 1978; Parker 1979; Krause & Rädler 1980; Zeldovich, Ruzmaikin & Sokoloff 1983; Ruzmaikin, Shukurov & Sokoloff 1988; Rüdiger, Hollerbach & Kitchatinov 2013; Moffatt & Dormy 2019; Rogachevskii 2021; Shukurov & Subramanian 2021). This mechanism can be described by the $\alpha\Omega$ or $\alpha^2\Omega$ mean-field dynamos. In particular, the effect of turbulence in the mean-field induction equation is determined by the turbulent electromotive force (EMF), $\langle \boldsymbol{u} \times \boldsymbol{b} \rangle$, which can be written for a weak mean magnetic field $\overline{\boldsymbol{B}}$ as $\langle \boldsymbol{u} \times \boldsymbol{b} \rangle = \alpha_{_{\rm K}}\, \overline{\boldsymbol{B}} + \boldsymbol{V}^{\rm (eff)} \times \overline{\boldsymbol{B}} - \eta_{_{T}}\, (\nabla \times \overline{\boldsymbol{B}})$, where $\alpha_{_{\rm K}}$ is the kinetic α effect caused by helical motions of plasma, $\eta_{_{T}}$ is the turbulent magnetic diffusion coefficient, $\boldsymbol{V}^{\rm (eff)}$ is the effective pumping velocity caused by an inhomogeneity of turbulence. Here the angular brackets imply ensemble averaging, $\boldsymbol{u}$ and $\boldsymbol{b}$ are fluctuations of velocity and magnetic fields, respectively. The threshold of the $\alpha\Omega$ mean-field dynamo instability is described in terms of a dynamo number $D_{\rm L} = \alpha_{_{\rm K}}\, \delta\Omega\, L^3/\eta_{_{T}}^2$, where $\delta\Omega$ characterizes the non-uniform (differential) rotation and L is the stellar radius or the thickness of the galactic disc.

The mean-field dynamos are saturated by non-linear effects. In particular, a feedback of the growing large-scale magnetic field on plasma motions is described by algebraic quenching of the α effect, turbulent magnetic diffusion, and the effective pumping velocity. This implies that the turbulent transport coefficients, $\alpha_{_{\rm K}}\,(\overline{B})$, $\eta_{_{T}}\,(\overline{B})$, and $\boldsymbol{V}^{\rm (eff)}\,(\overline{B})$ depend on the mean magnetic field $\overline{\boldsymbol{B}}$ via algebraic decreasing functions. The quantitative theories of the algebraic non-linearities of the α effect, the turbulent magnetic diffusion and the effective pumping velocity have been developed using the quasi-linear approach for small fluid and magnetic Reynolds numbers (Rüdiger & Kichatinov 1993; Kitchatinov, Pipin & Rüdiger 1994; Rüdiger, Hollerbach & Kitchatinov 2013) and the tau approach for large fluid and magnetic Reynolds numbers (Field, Blackman & Chou 1999; Rogachevskii & Kleeorin 2000, 2001, 2004, 2006).

In addition to the algebraic non-linearity, there is also a dynamic non-linearity caused by an evolution of magnetic helicity density of a small-scale turbulent magnetic field during the non-linear stage of the mean-field dynamo. Indeed, the α effect has contributions from the kinetic α effect, $\alpha_{_{\rm K}}$, determined by the kinetic helicity and a magnetic α effect, $\alpha_{_{\rm M}}$, described by the current helicity of the small-scale magnetic field (Pouquet, Frisch & Léorat 1976). The dynamics of the current helicity are determined by the evolution of the small-scale magnetic helicity density $H_{\rm m} = \langle \boldsymbol{a}\cdot\boldsymbol{b} \rangle$, where $\boldsymbol{b} = \nabla\times\boldsymbol{a}$ and $\boldsymbol{a}$ are fluctuations of the magnetic vector potential. The total magnetic helicity, i.e. the sum of the magnetic helicity densities of the large-scale and small-scale magnetic fields, $H_{\rm M} + H_{\rm m}$, integrated over the volume, $\int (H_{\rm M} + H_{\rm m})\, dr^3$, is conserved for very small microscopic magnetic diffusivity η. Here $H_{\rm M} = \overline{\boldsymbol{A}}\cdot\overline{\boldsymbol{B}}$ is the magnetic helicity density of the large-scale magnetic field $\overline{\boldsymbol{B}} = \nabla\times\overline{\boldsymbol{A}}$ and $\overline{\boldsymbol{A}}$ is the mean magnetic vector potential.

As the mean-field dynamo instability amplifies the mean magnetic field, the large-scale magnetic helicity density $H_{\rm M}$ grows in time. Since the total magnetic helicity $\int (H_{\rm M} + H_{\rm m})\, dr^3$ is conserved for very small magnetic diffusivity, the magnetic helicity density $H_{\rm m}$ of the small-scale field changes during the dynamo action, and its evolution is determined by the dynamic equation (Kleeorin & Ruzmaikin 1982; Zeldovich, Ruzmaikin & Sokoloff 1983; Gruzi-

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nov & Diamond 1994; Kleeorin, Rogachevskii & Ruzmaikin 1995; Kleeorin & Rogachevskii 1999), which includes the source terms and turbulent fluxes of magnetic helicity (Kleeorin & Rogachevskii 1999; Blackman & Field 2000; Kleeorin et al. 2000; Vishniac & Cho 2001; Brandenburg & Subramanian 2005; Kleeorin & Rogachevskii 2022; Gopalakrishnan & Subramanian 2023).

Taking into account turbulent fluxes of the small-scale magnetic helicity, it has been shown by numerical simulations that a non-linear galactic dynamo governed by a dynamic equation for the magnetic helicity density $H_{\rm m}$ of a small-scale field (the dynamical non-linearity) saturates at a mean magnetic field comparable with the equipartition magnetic field (see e.g. Kleeorin et al. 2000; Blackman & Brandenburg 2002; Kleeorin et al. 2002, 2003a; Brandenburg & Subramanian 2005; Shukurov et al. 2006; Chamandy et al. 2014; Chamandy & Singh 2018). Numerical simulations demonstrate that the dynamics of magnetic helicity plays a crucial role in solar and stellar dynamos as well (see e.g. Kleeorin et al. 2003b; Sokoloff et al. 2006; Zhang et al. 2006; Käpylä, Korpi & Brandenburg 2010; Hubbard & Brandenburg 2012; Zhang et al. 2012; Del Sordo, Guerrero & Brandenburg 2013; Kleeorin et al. 2016; Safiullin et al. 2018; Kleeorin et al. 2020; Rincon 2021; Kleeorin et al. 2023). Different forms of magnetic helicity fluxes have been suggested in various studies using phenomenological arguments (Kleeorin & Rogachevskii 1999; Kleeorin et al. 2000; Vishniac & Cho 2001; Kleeorin et al. 2002; Subramanian & Brandenburg 2004; Brandenburg & Subramanian 2005). Recently, the turbulent magnetic helicity fluxes have been rigorously derived (Kleeorin & Rogachevskii 2022; Gopalakrishnan & Subramanian 2023). In particular, Kleeorin & Rogachevskii (2022) apply the mean-field theory, adopt the Coulomb gauge and consider a strongly density-stratified turbulence. They have found that the turbulent magnetic helicity fluxes depend on the mean magnetic field energy and include non-gradient and gradient contributions. In addition, Gopalakrishnan & Subramanian (2023) have recently shown that contributions to the turbulent magnetic helicity fluxes from the third-order moments can be described using the turbulent diffusion approximation.

In a non-linear $\alpha\Omega$ dynamo, one can define a non-linear dynamo number $D_{\rm N}\left(\overline{B}\right) = \alpha\left(\overline{B}\right)\,\delta\Omega\, L^3/\eta_{_T}^2\left(\overline{B}\right)$. If the non-linear dynamo number $D_{\rm N}\left(\overline{B}\right)$ decreases with the increase of the large-scale magnetic field, the mean-field dynamo instability is saturated by the non-linear effects. However, if the α effect and the turbulent magnetic diffusion are quenched as $(\overline{B}/\overline{B}_{\rm eq})^{-2}$ for strong mean magnetic fields, the non-linear dynamo number $D_{\rm N}\left(\overline{B}\right) \propto (\overline{B}/\overline{B}_{\rm eq})^2$ increases with the increase of the large-scale magnetic field, and the mean-field dynamo instability cannot be saturated for a strong mean magnetic field. Here, $\overline{B}_{\rm eq} = \left(\mu_0\,\overline{\rho}\,\langle \boldsymbol{u}^2\rangle\right)^{1/2}$ is the equipartition mean magnetic field and μ_0 is the magnetic permeability of the fluid. How is it possible to resolve this paradox?

The mean-field dynamo theories of the algebraic quenching imply that there is a background helical turbulence with a zero-mean magnetic field. The large-scale magnetic field is amplified by the mean-field dynamo instability. In a non-linear dynamo stage, the dissipation of the generated strong large-scale magnetic field results in an increase of the turbulent kinetic energy (TKE) of the background turbulence. The latter effect causes an increase of the turbulent magnetic diffusion coefficient and decrease of the non-linear dynamo number. This additional non-linear effect results in a saturation of the dynamo growth of a strong large-scale magnetic field.

However, this non-linear effect has not been yet taken into account in non-linear mean-field dynamo theories that derive the algebraic quenching of the turbulent magnetic diffusion. In the present study, we have taken into account this feedback effect of the mean magnetic field on the background turbulence using the budget equation for the total (kinetic plus magnetic) turbulent energy. We consider three different types of astrophysical turbulence:

(i) a forced turbulence (e.g. caused by supernova explosions in galaxies);
(ii) a shear-produced turbulence (e.g. in the atmosphere of the Earth or other planets); and
(iii) a convective turbulence (e.g. in a solar and stellar convective zones).

We have demonstrated that the non-linear dynamo number indeed decreases with the increase of the mean magnetic field for any strong values of the magnetic field, resulting in saturation of the mean-field dynamo instability.

This paper is organized as follows. In Section 2, we explain the essence of the algebraic and dynamic non-linearities, and discuss the procedure and assumptions for the derivation of the non-linear turbulent EMF. In Section 3, we consider the budget equations for the turbulent kinetic and magnetic energies which allow us to take into account the increase of TKE of the background turbulence by the dissipation of a strong mean magnetic field and to determine asymptotic properties of turbulent magnetic diffusion and non-linear dynamo numbers for a strong mean magnetic field for the mean-field $\alpha\,\Omega$ dynamo (see Section 4), the α^2 and $\alpha^2\,\Omega$ dynamos (see Sec. 5). In addition, in Sec. 5 we outline important asymptotic properties in the $\alpha^2\,\Omega$ dynamo. Finally, in Section 6, we discuss the obtained results.

2 NON-LINEAR TURBULENT EMF

To explain the essence of the algebraic and dynamic non-linearities, we discuss in this section the procedure and assumptions for the derivation of the non-linear turbulent EMF in a non-rotating and helical small-scale turbulence. In the framework of the mean-field approach, the mean magnetic field $\overline{\boldsymbol{B}}$ is determined by the induction equation

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \nabla \times \left[\overline{\boldsymbol{U}} \times \overline{\boldsymbol{B}} + \boldsymbol{\mathcal{E}}\left(\overline{\boldsymbol{B}}\right) - \eta \nabla \times \overline{\boldsymbol{B}}\right], \tag{1}$$

where $\overline{\boldsymbol{U}}$ is the mean velocity (differential rotation), η is the magnetic diffusion due to the electrical conductivity of plasma, and $\boldsymbol{\mathcal{E}}(\overline{\boldsymbol{B}}) = \langle \boldsymbol{u} \times \boldsymbol{b}\rangle$ is the the turbulent EMF. To derive equations for the non-linear coefficients defining the turbulent EMF, we use a mean-field approach in which the magnetic and velocity fields, the fluid pressure and density are separated into the mean and fluctuating parts, where the fluctuating parts have zero mean values. We consider the case of large hydrodynamic and magnetic Reynolds numbers. The momentum and induction equations for the turbulent fields are given by

$$\frac{\partial \boldsymbol{u}(t,\boldsymbol{x})}{\partial t} = -\frac{\nabla p_{\rm tot}}{\overline{\rho}} + \frac{1}{\mu_0 \overline{\rho}}\left[(\boldsymbol{b}\cdot\nabla)\overline{\boldsymbol{B}} + \left(\overline{\boldsymbol{B}}\cdot\nabla\right)\boldsymbol{b}\right] + \boldsymbol{u}^N + \boldsymbol{F}, \tag{2}$$

$$\frac{\partial \boldsymbol{b}(t,\boldsymbol{x})}{\partial t} = \left(\overline{\boldsymbol{B}}\cdot\nabla\right)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\overline{\boldsymbol{B}} + \boldsymbol{b}^N, \tag{3}$$

where $\overline{\rho}$ is the mean plasma density, μ_0 is the magnetic permeability of the plasma, $\boldsymbol{F}$ is a random external stirring force, $\boldsymbol{u}^N$ and $\boldsymbol{b}^N$ are the non-linear terms that include the molecular dissipative terms, $p_{\rm tot} = p + (\mu_0\,\overline{\rho})^{-1}\,(\overline{\boldsymbol{B}}\cdot\boldsymbol{b})$ are fluctuations of the total pressure and p are fluctuations of the plasma pressure. For simplicity, let us consider incompressible flow, so that the velocity $\boldsymbol{u}$ satisfies to the continuity equation, $\nabla\cdot\boldsymbol{u} = 0$ and the fluid density is constant. The assumptions and the procedure of the derivation of the non-linear turbulent EMF are as follows:

(i) We apply the multiscale approach (Roberts & Soward 1975), which allows us to introduce fast and slow variables, and separate small-scale effects corresponding to fluctuations and large-scale effects describing mean fields. The mean fields depend on slow variables, while fluctuations depend on fast variables. Separation into slow and fast variables is widely used in theoretical physics, and all calculations are reduced to the Taylor expansions of all functions assuming that characteristic turbulent spatial and time-scales are much smaller than the characteristic spatial and time-scales of the mean magnetic field variations.

(ii) Using equations (2)–(3) written in a Fourier space, we derive equations for the second-order moments for the velocity field $f_{ij} = \langle u_i u_j \rangle$, the magnetic field $h_{ij} = \langle b_i b_j \rangle$ and the cross-helicity $g_{ij} = \langle u_i b_j \rangle$.

(iii) We split the tensors $f_{\rm ij}$, $h_{\rm ij}$, and $g_{\rm ij}$ into non-helical h_{ij} and helical, $h_{ij}^{(H)}$ parts. The helical part of the tensor $h_{ij}^{(H)}$ for magnetic fluctuations depends on the small-scale magnetic helicity, and its evolution is determined by the dynamic equation that follows from the magnetic helicity conservation arguments (Kleeorin & Ruzmaikin 1982; Gruzinov & Diamond 1994; Kleeorin, Rogachevskii & Ruzmaikin 1995; Kleeorin & Rogachevskii 1999; Kleeorin et al. 2000; Blackman & Brandenburg 2002). The characteristic time of the evolution of the non-helical part of the magnetic tensor $h_{\rm ij}$ is of the order of the turbulent correlation time $\tau_0 = \ell_0/u_0$, while the relaxation time of the helical part of the magnetic tensor $h_{ij}^{(H)}$ is of the order of $\tau_0\,{\rm Rm}$, where ${\rm Rm} = \ell_0 u_0/\eta \gg 1$ is the magnetic Reynolds number, and u_0 is the characteristic turbulent velocity in the integral scale ℓ_0 of turbulent motions.

(iv) Equations for the second-order moments contain higher-order moments and a problem of closing the equations for the higher-order moments arises. Various approximate methods have been proposed for the solution of this closure problem (Monin & Yaglom 1971; McComb 1990; Monin & Yaglom 2013; Rogachevskii 2021). For small fluid and magnetic Reynolds numbers, the quasi-linear approach can be used (Rüdiger & Kichatinov 1993; Kitchatinov, Pipin & Rüdiger 1994; Rüdiger, Hollerbach & Kitchatinov 2013), while for large fluid and magnetic Reynolds numbers, the minimal tau approach (Field, Blackman & Chou 1999) or the spectral τ approach (Rogachevskii & Kleeorin 2000, 2001, 2004, 2006) are applied to derive the non-linear turbulent EMF. For instance, the spectral τ approach postulates that the deviations of the third-order moments, $\hat{\mathcal{M}} f_{ij}^{(III)}(\boldsymbol{k})$, from the contributions to these terms afforded by the background turbulence, $\hat{\mathcal{M}} f_{ij}^{(III,0)}(\boldsymbol{k})$, can be expressed through the similar deviations of the second-order moments, $f_{ij}^{(II)}(\boldsymbol{k}) - f_{ij}^{(II,0)}(\boldsymbol{k})$ (Orszag 1970; Pouquet, Frisch & Léorat 1976; Kleeorin, Rogachevskii & Ruzmaikin 1990):

$$\hat{\mathcal{M}} f_{ij}^{(III)}(\boldsymbol{k}) - \hat{\mathcal{M}} f_{ij}^{(III,0)}(\boldsymbol{k}) = - \frac{f_{ij}^{(II)}(\boldsymbol{k}) - f_{ij}^{(II,0)}(\boldsymbol{k})}{\tau_r(k)}, \tag{4}$$

where $\tau_r(k)$ is the scale-dependent relaxation time, which can be identified with the correlation time $\tilde{\tau}(k)$ of the turbulent velocity field for large fluid and magnetic Reynolds numbers. The superscript (0) corresponds to the background turbulence (with $\overline{\boldsymbol{B}} = 0$), and $\tau_r(k)$ is the characteristic relaxation time of the statistical moments. We apply the spectral τ approach only for the non-helical part $h_{\rm ij}$ of the tensor for magnetic fluctuations. The spectral τ approach is widely used in the theory of kinetic equations, in passive scalar turbulence and magnetohydrodynamic turbulence.

(v) We use the following model for the second-order moment $f_{ij}^{(0)}$ of isotropic inhomogeneous incompressible and helical background turbulence in a Fourier space:

$$f_{ij}^{(0)}(\boldsymbol{k}) = \frac{E(k)}{8\pi k^2} \Big\{ \Big[\delta_{ij} - k_{ij} + \frac{\rm i}{2k^2} \big(k_i \nabla_j - k_j \nabla_i\big) \Big] \left\langle \boldsymbol{u}^2 \right\rangle^{(0)} - \frac{\rm i}{k^2}\, \varepsilon_{ijp}\, k_p\, \langle \boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) \rangle \Big\}. \tag{5}$$

Here, $\delta_{\rm ij}$ is the Kronecker tensor, $k_{ij} = k_i\, k_j / k^2$ and $\langle \boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) \rangle$ is the kinetic helicity. The energy spectrum function is $E(k) = (2/3)\, k_0^{-1}\, (k/k_0)^{-5/3}$ in the inertial range of turbulence $k_0 \le k \le k_\nu$. Here the wave number $k_0 = 1/\ell_0$, the length ℓ_0 is the integral scale of turbulent motions, the wave number $k_\nu = \ell_\nu^{-1}$, the length $\ell_\nu = \ell_0 {\rm Re}^{-3/4}$ is the Kolmogorov (viscous) scale, and the expression for the turbulent correlation time is given by $\tilde{\tau}(k) = 2\, \tau_0\, (k/k_0)^{-2/3}$. The model for the second moment $h_{ij}^{(0)}$ for magnetic fluctuations in a Fourier space caused by the small-scale dynamo (with a zero mean magnetic field) is

$$h_{ij}^{(0)}(\boldsymbol{k}) = \frac{E(k)}{8\pi k^2} \Big(\delta_{ij} - k_{ij} \Big) \left\langle \boldsymbol{b}^2 \right\rangle^{(0)}. \tag{6}$$

We also take into account that the turbulent EMF is produced in a turbulence with a non-zero mean magnetic field, so that the cross-helicity tensor in the background turbulence vanishes, i.e. $g_{ij}^{(0)} = 0$.

(vi) We assume that the characteristic time of variation of the mean magnetic field $\overline{\boldsymbol{B}}$ is substantially larger than the correlation time $\tilde{\tau}(k)$ for all turbulence scales (which corresponds to the mean-field approach). This allows us to get a stationary solution for the equations for the second moments. Using the derived equations for the second moments $f_{\rm ij}$, $h_{\rm ij}$, and $g_{\rm ij}$, we determine the non-linear turbulent EMF $\mathcal{E}_i = \varepsilon_{imn} \int g_{mn}(\boldsymbol{k})\, d\boldsymbol{k}$. The details of the derivation of the non-linear turbulent EMF are given by Rogachevskii & Kleeorin (2004).

For illustration of these results, we consider a small-scale homogeneous turbulence with a mean velocity shear, $\overline{\boldsymbol{U}} = S\, z\, \boldsymbol{e}_y$. We also consider an axi-symmetric $\alpha\Omega$ dynamo problem in the Cartesian coordinates, so the mean magnetic field, $\overline{\boldsymbol{B}} = \overline{B}_y(x, z)\, \boldsymbol{e}_y + \boldsymbol{\nabla} \times [\overline{A}(x, z)\, \boldsymbol{e}_y]$, is determined by the following non-linear dynamo equations (Rogachevskii & Kleeorin 2004):

$$\frac{\partial \overline{A}}{\partial t} = \left[\alpha_{_{\rm K}}(\overline{B}) + \alpha_{_{\rm M}}(\overline{\boldsymbol{B}}) \right]\ \overline{B}_y + \eta_{_T}^{(A)} \left(\overline{\boldsymbol{B}}\right)\ \Delta \overline{A}, \tag{7}$$

$$\frac{\partial \overline{B}_y}{\partial t} = S\, \nabla_x\, \overline{A} + \nabla_j \left[\eta_{_T}^{(B)} \left(\overline{\boldsymbol{B}}\right) \nabla_j \right] \overline{B}_y. \tag{8}$$

Here, the non-linear α effect is given by

$$\alpha(\overline{\boldsymbol{B}}) = \alpha_{_{\rm K}}(\overline{B}) + \alpha_{_{\rm M}}(\overline{\boldsymbol{B}}), \tag{9}$$

where $\alpha^{\rm (K)}(\overline{B})$ is the kinetic α effect, and $\alpha^{\rm (M)}\left(\overline{\boldsymbol{B}}\right)$ is the magnetic α effect, which are given by

$$\alpha_{_{\rm K}}\left(\overline{B}\right) = \alpha_{_{\rm K}}^{(0)}\, \phi_{_{\rm K}}(\beta)\, (1 - \epsilon), \tag{10}$$

$$\alpha_{_{\rm M}}\left(\overline{B}\right) = \frac{\tau_0}{3 \mu_0\, \overline{\rho}}\, H_{\rm c} \left(\overline{\boldsymbol{B}}\right)\, \phi_{_{\rm M}}(\beta). \tag{11}$$

Here, $\alpha_{_{\rm K}}^{(0)} = -\tau_0 H_{\rm u}/3$ with $H_{\rm u} = \langle \boldsymbol{u} {\cdot} (\boldsymbol{\nabla} {\times} \boldsymbol{u}) \rangle$ being the kinetic helicity, $\beta = \sqrt{8}\, \overline{B}/\overline{B}_{\rm eq}$, the parameter $\epsilon = \langle \boldsymbol{b}^2 \rangle^{(0)}\, \ell_b / (\langle \boldsymbol{u}^2 \rangle^{(0)} \ell_0)$ characterized the small-scale dynamo is varied in the range $0 \le \epsilon \le 1$, where $\langle \boldsymbol{b}^2 \rangle^{(0)} / 2\mu_0$ and $\langle \boldsymbol{u}^2 \rangle^{(0)}/2$ are turbulent magnetic and kinetic energies of the background turbulence, ℓ_b is the characteristic scale of the localization of the magnetic energy due to the small-scale dynamo, and $H_{\rm c}\left(\overline{\boldsymbol{B}}\right) = \langle \boldsymbol{b} {\cdot} (\boldsymbol{\nabla} {\times} \boldsymbol{b}) \rangle$ is the current helicity of the small-scale magnetic field $\boldsymbol{b}$.

The quenching functions $\phi_{_{\rm K}}(\beta)$ and $\phi_{_{\rm M}}(\beta)$ of the kinetic and magnetic α effects are given by equations (A1)–(A2) in Appendix A.

Here, $\phi_{\rm M}(\beta)$ is the quenching function of the magnetic α effect derived by Field, Blackman & Chou (1999) using the minimal τ approximation (the τ approach applied in a physical space) and Rogachevskii & Kleeorin (2000) using the spectral τ approach.

The non-linear turbulent magnetic diffusion coefficients for the poloidal $\eta_{_{T}}^{(A)}\left(\overline{B}\right)$ and toroidal $\eta_{_{T}}^{(B)}\left(\overline{B}\right)$ mean magnetic field are given by

$$\eta_{_{T}}^{(A)}(\beta) = \eta_{_{T}}^{(0)}\, \phi_{\rm K}(\beta), \quad \eta_{_{T}}^{(B)}(\beta) = \eta_{_{T}}^{(0)}\, \phi_{\eta}^{(B)}(\beta), \tag{12}$$

where $\eta_{_{T}}^{(0)} = \tau_0 \left\langle \boldsymbol{u}^2 \right\rangle/3$ is the characteristic value of the turbulent magnetic diffusivity. The quenching function $\phi_{\eta}^{(B)}(\beta) = \phi_{\rm K}(\beta) + \phi(\beta)$ and the functions $\phi_{\rm K}(\beta)$ and $\phi(\beta)$ are given by equations (A1) and (A3) in Appendix A. Here, for simplicity, we consider a homogeneous background turbulence, so the effective pumping velocity of the large-scale magnetic field vanishes.

The asymptotic formulas for the kinetic and magnetic α effects, and the non-linear turbulent magnetic diffusion coefficients of the mean magnetic field for a weak field $\overline{B} \ll \overline{B}_{\rm eq}/4$ are given by

$$\alpha^{(\rm K)}(\beta) = \alpha_{_{\rm K}}^{(0)}\,(1-\epsilon)\left(1 - \frac{12\beta^2}{5}\right), \tag{13}$$

$$\alpha^{(\rm M)}\left(\overline{\boldsymbol{B}}\right) = \frac{\tau_0}{3\mu_0\,\overline{\rho}}\, H_{\rm c}\left(\overline{\boldsymbol{B}}\right)\left(1 - \frac{3\beta^2}{5}\right), \tag{14}$$

$$\eta_{_{T}}^{(A)}(\beta) = \eta_{_{T}}^{(0)}\left(1 - \frac{12}{5}\,\beta^2\right), \tag{15}$$

$$\eta_{_{T}}^{(B)}(\beta) = \eta_{_{T}}^{(0)}\left(1 - \frac{4}{5}\,(5-4\epsilon)\,\beta^2\right), \tag{16}$$

and for a strong field $\overline{B} \gg \overline{B}_{\rm eq}/4$ they are given by

$$\alpha^{(\rm K)}(\beta) = \frac{\alpha_{_{\rm K}}^{(0)}}{\beta^2}\,(1-\epsilon), \tag{17}$$

$$\alpha^{(\rm M)}\left(\overline{\boldsymbol{B}}\right) = \frac{\tau_0}{\mu_0\,\overline{\rho}}\,\frac{H_{\rm c}\left(\overline{\boldsymbol{B}}\right)}{\beta^2}, \tag{18}$$

$$\eta_{_{T}}^{(A)}(\beta) = \frac{\eta_{_{T}}^{(0)}}{\beta^2}, \quad \eta_{_{T}}^{(B)}(\beta) = \frac{2\eta_{_{T}}^{(0)}}{3\beta}\,(1+\epsilon). \tag{19}$$

It follows from equations (13)–(19) that small-scale dynamo decreases the kinetic α effect, and it increases the turbulent magnetic diffusion of the toroidal mean magnetic field.

As follows from equation (11), the magnetic α effect is proportional to the current helicity $H_{\rm c}\left(\overline{\boldsymbol{B}}\right)$ of the small-scale magnetic field (Pouquet, Frisch & Léorat 1976), which describes the dynamical quenching of the α effect. Note that the dynamical quenching related to evolution of the magnetic α effect is derived only from the induction equation, and it is a contribution from small-scale current helicity $\langle \boldsymbol{b}\cdot(\nabla\times\boldsymbol{b})\rangle$, which is related to the small-scale magnetic helicity density. On the other hand, the algebraic quenching of the kinetic and magnetic alpha effects and turbulent magnetic diffusion coefficients of the large-scale magnetic field are derived from both, the Navier–Stokes equation for velocity fluctuations and the induction equation for magnetic fluctuations. In particular, the algebraic quenching is a contribution from the correlation functions for velocity fluctuations $\langle u_i u_j\rangle$, magnetic fluctuations $\langle b_i b_j\rangle$ and the cross-helicity correlation function $\langle u_i b_j\rangle$. The algebraic quenching is a physical effect related to a feedback of the growing large-scale magnetic field on plasma motions. If the algebraic quenching of the α effect is taken into account, the algebraic quenching of the turbulent magnetic diffusion should be taken into account as well. For instance, many studies related to the mean-field numerical simulations of the evolution of the solar and galactic magnetic fields take into account algebraic and dynamic quenching of the α effect, but ignore the algebraic quenching of the turbulent magnetic diffusion (see e.g. Covas et al. 1997, 1998; Kleeorin et al. 2000; Blackman & Brandenburg 2002; Kleeorin et al. 2002, 2003b; Brandenburg & Subramanian 2005; Shukurov et al. 2006; Guerrero, Chatterjee & Brandenburg 2010; Chamandy et al. 2014; Kleeorin et al. 2016; Safiullin et al. 2018; Kleeorin et al. 2020, 2023).

The approach discussed in this section allows us to derive the non-linear turbulent EMF for an intermediate non-linearity. This means that the mean magnetic field is not enough strong to affect the background turbulence. The theory for a strong mean magnetic field should take into account a modification of the background turbulence by the mean magnetic field. In the next sections we take into account this effect. In particular, we obtain the dependence of the TKE $\overline{\rho}\,\langle \boldsymbol{u}^2\rangle^{(0)}/2$ on the mean magnetic field using the budget equations for the turbulent kinetic and magnetic energies. This describes an additional non-linear effect of the increase of the TKE of the background turbulence by the dissipation of a strong mean magnetic field. The latter increases turbulent magnetic diffusion and decreases the non-linear dynamo number for a strong field, resulting in a saturation of the dynamo growth of the large-scale magnetic field.

3 BUDGET EQUATIONS

Using the Navier–Stokes equation for velocity fluctuations, we derive the budget equation for the density of TKE, $E_{\rm K} = \overline{\rho}\,\langle \boldsymbol{u}^2\rangle/2$ as

$$\frac{\partial E_{\rm K}}{\partial t} + {\rm div}\, \boldsymbol{\Phi}_{\rm K} = \Pi_{\rm K} - \varepsilon_{\rm K}, \tag{20}$$

where $\boldsymbol{\Phi}_{\rm K} = \left\langle \boldsymbol{u}\left(\rho\, \boldsymbol{u}^2/2 + p\right)\right\rangle - \nu\,\overline{\rho}\,\nabla E_{\rm K}$ is the flux of TKE, $\varepsilon_{\rm K} = \nu\,\overline{\rho}\,\left\langle \left(\nabla_j u_i\right)^2\right\rangle$ is the dissipation rate of TKE, and

$$\Pi_{\rm K} = -\frac{1}{\mu_0}\Big[\langle \boldsymbol{u}\cdot[\boldsymbol{b}\times(\nabla\times\boldsymbol{b})]\rangle - \langle \boldsymbol{u}\times(\nabla\times\boldsymbol{b})\rangle\cdot\overline{\boldsymbol{B}} + \langle \boldsymbol{u}\times\boldsymbol{b}\rangle\cdot(\nabla\times\overline{\boldsymbol{B}})\Big] + \overline{\rho}\,\Big[g\,F_z - \left\langle u_i u_j\right\rangle \nabla_j \overline{U}_i + \langle \boldsymbol{u}\cdot\boldsymbol{f}\rangle\Big] \tag{21}$$

is the production rate of TKE. Here, $\overline{\boldsymbol{U}}$ is the mean velocity, ν is the kinematic viscosity and the angular brackets imply ensemble averaging, $\boldsymbol{F} = \langle s\, \boldsymbol{u}\rangle$ is the turbulent flux of the entropy, $s = \theta/\overline{T} + (\gamma^{-1} - 1)\,p/\overline{P}$ are entropy fluctuations, θ and $\overline{T}$ are fluctuations and mean fluid temperature, ρ and $\overline{\rho}$ are fluctuations and mean fluid density, p and $\overline{P}$ are fluctuations and mean fluid pressure, $\gamma = c_{\rm p}/c_{\rm v}$ is the ratio of specific heats, $\boldsymbol{g}$ is the acceleration due to the gravity, and $\overline{\rho}\,\boldsymbol{f}$ is the external steering force with a zero mean.

We consider three different cases when turbulence is produced either by convection, or by large-scale shear motions or by an external steering force, see the last three terms in the right-hand side (RHS) of equation (21). The first two terms in the RHS of equation (21) describe an energy exchange between the turbulent kinetic and magnetic energies (see below), and the third term in the RHS of equation (21) are due to the work of the Lorentz force in a non-uniform mean magnetic field. The estimate for the dissipation rate of the TKE density in homogeneous isotropic and incompressible turbulence with a Kolmogorov spectrum is $\varepsilon_{\rm K} = E_{\rm K}/\tau_0$, where τ_0 is the characteristic turbulent time at the integral scale.

Using the induction equation for magnetic fluctuations, we derive the budget equation for the density of turbulent magnetic energy

(TME), $E_{\rm M} = \langle \boldsymbol{b}^2\rangle/2\mu_0$ as

$$\frac{\partial E_{\rm M}}{\partial t} + {\rm div}\, \boldsymbol{\Phi}_{\rm M} = \Pi_{\rm M} - \varepsilon_{\rm M}, \tag{22}$$

where

$$\boldsymbol{\Phi}_{\rm M} = \frac{1}{\mu_0}\biggl[\langle \boldsymbol{b}\times(\boldsymbol{u}\times\boldsymbol{b})\rangle + \left\langle \boldsymbol{u}\, b_j\right\rangle \overline{B}_j - \langle \boldsymbol{u}\cdot\boldsymbol{b}\rangle\, \overline{\boldsymbol{B}} + \left\langle \boldsymbol{b}^2\right\rangle \overline{\boldsymbol{U}} - \left\langle \boldsymbol{b}\, b_j\right\rangle \overline{U}_j - \eta\, \langle \boldsymbol{b}\times(\nabla\times\boldsymbol{b})\rangle\biggr] \tag{23}$$

is the flux of TME, $\varepsilon_{\rm M} = \eta\, \left\langle(\nabla\times\boldsymbol{b})^2\right\rangle/\mu_0$ is the dissipation rate of TME, and

$$\Pi_{\rm M} = \frac{1}{\mu_0}\biggl[\langle \boldsymbol{u}\cdot[\boldsymbol{b}\times(\nabla\times\boldsymbol{b})]\rangle - \langle \boldsymbol{u}\times(\nabla\times\boldsymbol{b})\rangle\cdot\overline{\boldsymbol{B}} + \left\langle b_i\, b_j\right\rangle \nabla_j \overline{U}_i - \left\langle \boldsymbol{b}^2\right\rangle\, (\nabla\cdot\overline{\boldsymbol{U}})\biggr] \tag{24}$$

is the production rate of TME. Here, η is the magnetic diffusion due to electrical conductivity of the fluid. The first two terms in the RHS of equation (24) describe an energy exchange between the turbulent magnetic and kinetic energies. The estimate for the dissipation rate of the TME density is $\varepsilon_{\rm M} = E_{\rm M}/\tau_0$.

The density of total turbulent energy (TTE), $E_{\rm T} = E_{\rm K} + E_{\rm M}$, is determined by the following budget equation:

$$\frac{\partial E_{\rm T}}{\partial t} + {\rm div}\, \boldsymbol{\Phi}_{\rm T} = \Pi_{\rm T} - \varepsilon_{\rm T}, \tag{25}$$

where

$$\Pi_{\rm T} = \biggl[\Bigl(\left\langle b_i\, b_j\right\rangle - \mu_0\,\overline{\rho}\, \left\langle u_i u_j\right\rangle\Bigr) \nabla_j \overline{U}_i - \left\langle \boldsymbol{b}^2\right\rangle\, (\nabla\cdot\overline{\boldsymbol{U}}) - \langle \boldsymbol{u}\times\boldsymbol{b}\rangle\cdot\left(\nabla\times\overline{\boldsymbol{B}}\right)\biggr]\mu_0^{-1} + \overline{\rho}\,\Bigl(g\, F_z + \langle \boldsymbol{u}\cdot\boldsymbol{f}\rangle\Bigr). \tag{26}$$

is the production rate of $E_{\rm T}$, $\varepsilon_{\rm T} = \varepsilon_{\rm K} + \varepsilon_{\rm M}$ is the dissipation rate of $E_{\rm T}$ and $\boldsymbol{\Phi}_{\rm T} = \boldsymbol{\Phi}_{\rm K} + \boldsymbol{\Phi}_{\rm M}$ is the flux of $E_{\rm T}$.

To determine the production rate of TTE, we use the following second moments for magnetic fluctuations (Rogachevskii & Kleeorin 2007),

$$\left\langle b_i\, b_j\right\rangle = \frac{\overline{\boldsymbol{B}}^2}{2}\left[2q_{\rm p}\left(\overline{B}\right)\, \delta_{ij} - q_{\rm s}\left(\overline{B}\right)\left(\delta_{ij} + \beta_{ij}\right)\right], \tag{27}$$

and velocity fluctuations,

$$\overline{\rho}\, \left\langle u_i\, u_j\right\rangle = -\frac{\overline{\boldsymbol{B}}^2}{2\mu_0}\left[2q_{\rm p}\left(\overline{B}\right)\, \delta_{ij} - q_{\rm s}\left(\overline{B}\right)\left(\delta_{ij} + \beta_{ij}\right)\right] + \overline{\rho}\, \left\langle u_i\, u_j\right\rangle^{(0)}, \tag{28}$$

(see Appendix B), where $\beta_{ij} = \overline{B}_i\overline{B}_j/\overline{\boldsymbol{B}}^2$. The tensor $\left\langle u_i\, u_j\right\rangle^{(0)}$ for a background turbulence (with a zero-mean magnetic field) in equation (28) has two contributions caused by background isotropic velocity fluctuations and tangling anisotropic velocity fluctuations due to the mean velocity shear (Elperin et al. 2002):

$$\left\langle u_i\, u_j\right\rangle^{(0)} = \frac{1}{3}\left\langle \boldsymbol{u}^2\right\rangle^{(0)} \delta_{ij} - 2\nu_{_T}^{(0)}\, \left(\partial\overline{U}\right)_{ij}, \tag{29}$$

where $\left(\partial\overline{U}\right)_{ij} = (\nabla_i\overline{U}_j + \nabla_j\overline{U}_i)/2$ and $\nu_{_T}^{(0)} = \tau_0\langle\boldsymbol{u}^2\rangle^{(0)}/3$ is the turbulent viscosity. For simplicity, in equation (27), we do not take into account a small-scale dynamo with a zero-mean magnetic field.

The non-linear functions $q_{\rm p}(\overline{B})$ and $q_{\rm s}(\overline{B})$ entering in equation (27)–(28) are given by equations (B6)–(B7) in Appendix B. The asymptotic formulae for the non-linear functions $q_{\rm p}(\overline{B})$ and $q_{\rm s}(\overline{B})$ are as follows. For a very weak mean magnetic field, $\overline{B} \ll \overline{B}_{\rm eq}/4{\rm Rm}^{1/4}$, the non-linear functions are given by

$$q_{\rm p}(\overline{B}) = \frac{2}{5}\left[\ln {\rm Rm} + \frac{4}{45}\right] + {\rm O}(\beta^2), \tag{30}$$

$$q_{\rm s}(\overline{B}) = \frac{8}{15}\left[\ln {\rm Rm} + \frac{2}{15}\right] + {\rm O}(\beta^2), \tag{31}$$

where $\overline{B}^2_{\rm eq} = \mu_0\,\overline{\rho}\,\langle\boldsymbol{u}^2\rangle$. For $\overline{B}_{\rm eq}/4{\rm Rm}^{1/4} \ll \overline{B} \ll \overline{B}_{\rm eq}/4$, these non-linear functions are given by

$$q_{\rm p}(\overline{B}) = \frac{16}{25}\left[5|\ln(\sqrt{2}\beta)| + 1 + 4\beta^2\right], \tag{32}$$

$$q_{\rm s}(\overline{B}) = \frac{32}{15}\left[|\ln(\sqrt{2}\beta)| + \frac{1}{30} + \frac{3}{2}\beta^2\right], \tag{33}$$

and for $\overline{B} \gg \overline{B}_{\rm eq}/4$ they are given by

$$q_{\rm p}(\overline{B}) = \frac{4}{3\beta^2}, \quad q_{\rm s}(\overline{B}) = \frac{\pi\sqrt{2}}{3\beta^3}. \tag{34}$$

where $\beta = \sqrt{8}\,\overline{B}/\overline{B}_{\rm eq}$.

Substituting equations (27)–(29) into equation (26), we obtain the production rate of the TTE as

$$\Pi_{\rm T} = \left[\frac{\overline{\boldsymbol{B}}^2}{2\mu_0}\Bigl(3q_{\rm p}\left(\overline{B}\right) - q_{\rm s}\left(\overline{B}\right)\Bigr) - \frac{\overline{\rho}\left\langle \boldsymbol{u}^2\right\rangle^{(0)}}{3}\right](\nabla\cdot\overline{\boldsymbol{U}}) + \left[2\nu_{_T}\left(\overline{B}\right)\overline{\rho}\,\left(\partial\overline{U}\right)_{ij} - \frac{1}{\mu_0}q_{\rm s}\left(\overline{B}\right)\,\overline{B}_i\overline{B}_j\right]\left(\partial\overline{U}\right)_{ij} - \frac{1}{\mu_0}\boldsymbol{\mathcal{E}}\left(\overline{\boldsymbol{B}}\right)\cdot(\nabla\times\overline{\boldsymbol{B}}) + \overline{\rho}\,\Bigl(g\, F_z + \langle \boldsymbol{u}\cdot\boldsymbol{f}\rangle\Bigr), \tag{35}$$

where $\boldsymbol{\mathcal{E}}\left(\overline{\boldsymbol{B}}\right) = \langle \boldsymbol{u}\times\boldsymbol{b}\rangle$ is the turbulent non-linear EMF. The turbulent viscosity $\nu_{_T}\left(\overline{B}\right)$ depends on the mean magnetic field. In particular, for weak field $\overline{B} \ll \overline{B}_{\rm eq}/4$, the turbulent viscosity $\nu_{_T}\left(\overline{B}\right) \sim \nu_{_T}^{(0)} = \tau_0\langle\boldsymbol{u}^2\rangle^{(0)}/3$, and for strong field $\overline{B} \gg \overline{B}_{\rm eq}/4$, it is $\nu_{_T}\left(\overline{B}\right) \sim \nu_{_T}^{(0)}(1+\epsilon)/(4\overline{B}/\overline{B}_{\rm eq})$ (Rogachevskii & Kleeorin 2007). Using the steady-state solution of equation (25), we estimate the TTE density as $E_{\rm K} + E_{\rm M} \sim \tau\, \Pi_{\rm T}$, where τ is of the order of the turbulent time. Equation (27) yields the density of TME $E_{\rm M} = \langle\boldsymbol{b}^2\rangle/2\mu_0$ as

$$E_{\rm M} = \left[3q_{\rm p}\left(\overline{B}\right) - 2q_{\rm s}\left(\overline{B}\right)\right]\frac{\overline{\boldsymbol{B}}^2}{2\mu_0}. \tag{36}$$

In the next sections, we apply the budget equations for analysis of non-linear mean-field $\alpha\Omega$, α^2, and $\alpha^2\Omega$ dynamos.

4 MEAN-FIELD $\alpha\Omega$ DYNAMO

In this section, we consider the axisymmetric mean-field $\alpha\Omega$ dynamo, so that the mean magnetic field can be decomposed as

$$\overline{\boldsymbol{B}} = \overline{B}_y(t,x,z)\boldsymbol{e}_y + {\rm rot}[\overline{A}(t,x,z)\boldsymbol{e}_y], \tag{37}$$

and non-linear mean-field induction equation reads

$$\frac{\partial}{\partial t}\begin{pmatrix}\overline{A}\\ \overline{B}_y\end{pmatrix} = \hat{N}\begin{pmatrix}\overline{A}\\ \overline{B}_y\end{pmatrix}, \tag{38}$$

where the operator $\hat{N}$ is given by

$$\hat{N} = \begin{pmatrix}\eta_{_T}^{(A)}\left(\overline{B}\right)\Delta & \alpha\left(\overline{\boldsymbol{B}}\right)\\ R_\alpha R_\omega\,\hat{\Omega} & \nabla_j\,\eta_{_T}^{(B)}\left(\overline{B}\right)\nabla_j\end{pmatrix}, \tag{39}$$

and the operator

$$\hat{\Omega}\,\overline{A} = \frac{\partial(\delta\Omega\,\sin\vartheta,\ \overline{A})}{\partial(z,\ x)} \tag{40}$$

describes differential rotation. Here, ϑ is the angle between $\delta\boldsymbol{\Omega}$ and the vertical coordinate z and L is the characteristic scale (e.g. the radius of a star or the thickness of a galactic disc). The total α effect is the sum of the kinetic α effect, $\alpha_{\rm K}(\overline{B})$, and the magnetic α effect, $\alpha_{\rm M}(\overline{\boldsymbol{B}})$, $\alpha(\overline{\boldsymbol{B}}) = \alpha_{\rm K}(\overline{B}) + \alpha_{\rm M}(\overline{\boldsymbol{B}})$, where the kinetic α effect is proportional to the kinetic helicity $H_{\rm u} = \langle \boldsymbol{u}\cdot(\nabla\times\boldsymbol{u})\rangle$, and the magnetic α effect is proportional to the current helicity $H_{\rm c}\left(\overline{\boldsymbol{B}}\right) = \langle \boldsymbol{b}\cdot(\nabla\times\boldsymbol{b})\rangle$ of the small-scale magnetic field $\boldsymbol{b}$. Equations (38)–(40) are written in dimensionless variables: the coordinate is measured in the units of L; the time t is measured in the units of turbulent magnetic diffusion time $L^2/\eta_{_T}^{(0)}$; the mean magnetic field is measured in the units of $\overline{B}_*$, where $\overline{B}_* \equiv \sigma\ \overline{B}_*^{\rm eq}$, $\sigma = \ell_0/\sqrt{2}L$, $\overline{B}_*^{\rm eq} = u_0\sqrt{\mu_0\overline{\rho}_*}$; and the magnetic potential $\overline{A}$ is measured in the units of $R_\alpha L \overline{B}_*$. Here, $R_\alpha = \alpha_* L/\eta_{_T}^{(0)}$, the fluid density $\overline{\rho}$ is measured in the units $\overline{\rho}_*$; the differential rotation $\delta\Omega$ is measured in units of the maximal value of the angular velocity Ω; the α effect is measured in units of the maximum value of the kinetic α effect, α_*; the integral scale of the turbulent motions $\ell_0 = \tau_0 u_0$ and the characteristic turbulent velocity $u_0 = \sqrt{\langle \boldsymbol{u}^2\rangle^{(0)}}$ at the scale ℓ_0 are measured in units of their maximum values in the turbulent region; and the turbulent magnetic diffusion coefficients are measured in units of their maximum values. The magnetic Reynolds number $\mathrm{Rm} = \ell_0 u_0/\eta$ is defined using the maximal values of the integral scale ℓ_0 and the characteristic turbulent velocity u_0. The dynamo number for the linear $\alpha\Omega$ dynamo is defined as $D_{\rm L} = R_\alpha R_\omega$, where $R_\omega = (\delta\Omega)\, L^2/\eta_{_T}^{(0)}$.

Now we define the non-linear dynamo number $D_{\rm N}\left(\overline{B}\right)$ for the $\alpha\Omega$ dynamo as

$$D_{\rm N}\left(\overline{B}\right) = \frac{\alpha\left(\overline{B}\right)\ \delta\Omega\, L^3}{\eta_{_T}^{(B)}\left(\overline{B}\right)\ \eta_{_T}^{(A)}\left(\overline{B}\right)}, \tag{41}$$

where we take into account that the non-linear turbulent magnetic diffusion coefficients of the poloidal and toroidal components of the mean magnetic field are different (Rogachevskii & Kleeorin 2004).

Next, we take into account the feedback of the mean magnetic field on the background turbulence using the budget equation for the TTE. In a shear-produced non-convective turbulence, the leading-order contributions to the production rate of the TKE for a strong large-scale magnetic field ($\overline{B} \gg \overline{B}_{\rm eq}/4$) is due to the term $-\boldsymbol{\mathcal{E}}\left(\overline{\boldsymbol{B}}\right)\cdot(\nabla\times\overline{\boldsymbol{B}})/\mu_0$, so that the leading-order contribution to the TKE density for a strong large-scale magnetic field is estimated as

$$E_{\rm K} = -\frac{\tau}{\mu_0}\,\boldsymbol{\mathcal{E}}\left(\overline{\boldsymbol{B}}\right)\cdot(\nabla\times\overline{\boldsymbol{B}}). \tag{42}$$

Indeed, let us estimate the leading-order contributions to the production rate of the TTE given by (35). Using equations (7)–(8), we can rewrite the turbulent EMF as $\mathcal{E}_i = \alpha\overline{B}_i - \eta_{ij}^{(\rm T)}(\nabla\times\overline{\boldsymbol{B}})_j$, where $\eta_{ij}^{(\rm T)}$ is the diagonal tensor with components $\eta_{11}^{(\rm T)} = \eta_{_T}^{(A)}$ and $\eta_{22}^{(\rm T)} = \eta_{_T}^{(B)}$. Now we estimate:

$$\eta_{ij}^{(\rm T)}(\nabla\times\overline{\boldsymbol{B}})_j(\nabla\times\overline{\boldsymbol{B}})_i = \eta_{_T}^{(A)}(\nabla\times\overline{\boldsymbol{B}})_\varphi^2 + \eta_{_T}^{(B)}(\nabla\times\overline{\boldsymbol{B}})_{\rm p}^2,$$

where $(\nabla\times\overline{\boldsymbol{B}})_\varphi$ and $(\nabla\times\overline{\boldsymbol{B}})_{\rm p}$ are the toroidal and poloidal components of the electric current, which can be estimated as $|(\nabla\times\overline{\boldsymbol{B}})_\varphi| \sim |\overline{B}_{\rm p}|/L_B$ and $|(\nabla\times\overline{\boldsymbol{B}})_{\rm p}| \sim |\overline{B}_\varphi|/L_B$. Here, the characteristic scale of the mean magnetic field variations L_B is defined as $L_B = \overline{B}/|\nabla\times\overline{\boldsymbol{B}}|$. We also take into account that for a strong field ($\overline{B} \gg \overline{B}_{\rm eq}/4$), $\eta_{_T}^{(A)}(\beta) \sim \eta_{_T}^{(0)}/\beta^2$, while $\eta_{_T}^{(B)}(\beta) \sim \eta_{_T}^{(0)}/\beta$, where $\overline{B}_\varphi$ and $\overline{B}_{\rm p}$ are the toroidal and poloidal components of the mean magnetic field. For the $\alpha\Omega$ dynamo, the toroidal component of the mean magnetic field is much larger than the poloidal component, i.e. $|\overline{B}_{\rm p}| \ll |\overline{B}_\varphi|$. This yields

$$-\boldsymbol{\mathcal{E}}\left(\overline{\boldsymbol{B}}\right)\cdot(\nabla\times\overline{\boldsymbol{B}}) \sim \frac{\eta_{_T}^{(B)}}{L_B^2}\,\overline{B}_\varphi^2 \sim \frac{\eta_{_T}^{(0)}}{4L_B^2}\,\overline{B}_\varphi\,\overline{B}_{\rm eq}, \tag{43}$$

where the magnetic energy of the equipartition field $\overline{B}_{\rm eq}$ is defined as $\overline{B}_{\rm eq}^2/2\mu_0 = E_{\rm K}^{(0)}$. For a shear-produced turbulence, $E_{\rm K}^{(0)} \approx \overline{\rho}\,\ell_0^2\, S^2$ with the squared shear $S^2 = \left(\partial\overline{U}\right)_{ij}^2$ and $\ell_0 = \tau[\langle \boldsymbol{u}^2\rangle^{(0)}]^{1/2}$ being the integral scale of turbulence at vanishing mean magnetic field. We assume also that the correlation time is independent of the mean magnetic field.

Contributions of other terms to the production rate of TTE and TKE for a strong large-scale magnetic field are much smaller than that described by equation (43). For instance, the contribution $\alpha\overline{\boldsymbol{B}}\cdot(\nabla\times\overline{\boldsymbol{B}})$ to $-\boldsymbol{\mathcal{E}}\left(\overline{\boldsymbol{B}}\right)\cdot(\nabla\times\overline{\boldsymbol{B}})$ is much smaller for a strong field, because

$$\overline{\boldsymbol{B}}\cdot(\nabla\times\overline{\boldsymbol{B}}) = \overline{B}_{\rm p}(\nabla\times\overline{\boldsymbol{B}})_{\rm p} + \overline{B}_\varphi(\nabla\times\overline{\boldsymbol{B}})_\varphi \sim \frac{\overline{B}_{\rm p}\overline{B}_\varphi}{L_B},$$

and for a strong field $\alpha(\beta) \sim \alpha^{(0)}/\beta^2$. Similarly, the checking of the contributions of the remaining terms to the production rate of TTE and TKE for a strong large-scale magnetic field shows that they are much smaller than that described by equation (43). Therefore, the leading-order contribution to the TKE density $E_{\rm K}\left(\overline{B}\right)$ for strong mean magnetic fields is

$$E_{\rm K}\left(\overline{B}\right) \sim \frac{E_{\rm K}^{(0)}}{6}\left(\frac{\ell_0}{L_B}\right)^2\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right). \tag{44}$$

Equation (44) implies that the TKE increases due to the dissipation of the strong large-scale magnetic field.

This yields the estimate for the turbulent magnetic diffusion coefficient of toroidal magnetic field $\eta_{_T}^{(B)}\left(\overline{B}\right) = \eta_{_T}^{(0)}\,\phi_\eta^{(B)}\,E_{\rm K}\left(\overline{B}\right)/E_{\rm K}^{(0)}$ in the limit of a strong field as

$$\frac{\eta_{_T}^{(B)}\left(\overline{B}\right)}{\eta_{_T}^{(0)}} \approx \frac{1}{24}\left(\frac{\ell_0}{L_B}\right)^2 = const, \tag{45}$$

where $\eta_{_T}^{(0)} = 2\tau\,E_{\rm K}^{(0)}/3\overline{\rho}$ and we take into account the increase of the TKE caused by the dissipation of the strong large-scale magnetic field [see equation (44)]. As follows from equation (19), the ratio of turbulent magnetic diffusion coefficients of poloidal and toroidal fields $\eta_{_T}^{(A)}\left(\overline{B}\right)/\eta_{_T}^{(B)}\left(\overline{B}\right)$ is given by

$$\frac{\eta_{_T}^{(A)}\left(\overline{B}\right)}{\eta_{_T}^{(B)}\left(\overline{B}\right)} \approx \frac{1}{2}\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^{-1}. \tag{46}$$

The dependence of the total α effect on the mean magnetic field, $\alpha\left(\overline{B}\right)$, is caused by the algebraic and dynamic quenching. The algebraic quenching describes the feedback of the mean magnetic field on the plasma motions, while the dynamic quenching of the total α effect is caused by the evolution of the magnetic α effect related to the small-scale current and magnetic helicities. In particular, the dynamic equation for the small-scale current helicity (which determines the evolution of the magnetic α effect) in a steady state yields the estimate for the total α effect in the limit of a strong mean field as $\alpha\left(\overline{B}\right) \propto -{\rm div}\boldsymbol{F}_{\rm M}/\overline{B}^2$, where $\boldsymbol{F}_{\rm M}$ is the magnetic helicity flux of the small-scale magnetic field. This implies that the total α effect for strong magnetic fields behaves as

$$\frac{\alpha\left(\overline{B}\right)}{\alpha_{\rm K}^{(0)}} \propto \left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^{-2}. \tag{47}$$

Note that the algebraic and dynamic quenching of the alpha effect yield similar behaviour for a strong large-scale magnetic field [see equations (17)–(18) and (47) and paper by Chamandy et al. (2014)].

Therefore, the ratio $D_{\rm N}\left(\overline{B}\right)/D_{\rm L}$ of the non-linear and linear dynamo numbers in a shear-produced turbulence for strong mean magnetic fields is estimated as [see equations (41) and (45)–(47)]:

$$\frac{D_{\rm N}\left(\overline{B}\right)}{D_{\rm L}} \approx 2\left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^{-1}\left(\frac{\eta_{_{T}}^{(B)}}{\eta_{_{T}}^{(0)}}\right)^{-2} \propto \left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^{-1}. \qquad (48)$$

Equation (48) implies that the non-linear dynamo number decreases with the increase of the mean magnetic field for any strong values of the field for a shear-produced turbulence. This results in saturation of the mean-field dynamo instability.

In a convective turbulence, the largest contributions to the production rate of TTE for a strong mean magnetic fields is due to the buoyancy term $\overline{\rho}\, g\, F_z$ and the term $\eta_{_{T}}^{(B)}\left(\overline{B}\right)\,(\nabla\times\overline{\boldsymbol{B}})^2/\mu_0$ [see equation (35)]. This implies that the leading-order contribution to the TKE density $E_{_{\rm K}}\left(\overline{B}\right)$ in a convective turbulence for strong mean magnetic fields is given by equation (44), where $E_{_{\rm K}}^{(0)}=(\overline{\rho}/2)\,(2g\,F_z\,\ell_0)^{2/3}$. Therefore, equations for the ratios $\eta_{_{T}}^{(B)}\left(\overline{B}\right)/\eta_{_{T}}^{(0)}$, $\eta_{_{T}}^{(A)}\left(\overline{B}\right)/\eta_{_{T}}^{(B)}\left(\overline{B}\right)$, and $D_{\rm N}\left(\overline{B}\right)/D_{\rm L}$ in a convective turbulence for strong mean magnetic fields are similar to equations (45)–(48), respectively. The difference is only in equation for $E_{_{\rm K}}^{(0)}$ that for a convective turbulence is given by $E_{_{\rm K}}^{(0)}=(\overline{\rho}/2)\,(2g\,F_z\,\ell_0)^{2/3}$ and for a shear-produced turbulence is $E_{_{\rm K}}^{(0)}=(2/3)\,\overline{\rho}\,\ell_0^2\,S^2$. The similar situation is also for a forced turbulence except for the expression for $E_{_{\rm K}}^{(0)}$ for a forced turbulence reads $E_{_{\rm K}}^{(0)}=\overline{\rho}\,\tau_0\,\langle\boldsymbol{u}\cdot\boldsymbol{f}\rangle$.

This implies that for the $\alpha\Omega$ dynamo, the non-linear dynamo number decreases with increase of the mean magnetic field for a forced turbulence, and a shear-produced turbulence and a convective turbulence. This causes saturation of the mean-field $\alpha\Omega$ dynamo instability for a strong mean magnetic field.

5 MEAN-FIELD α^2 AND $\alpha^2\Omega$ DYNAMOS

First, we start with the non-linear axisymmetric mean-field α^2 dynamo, so that non-linear mean-field induction equation reads

$$\frac{\partial}{\partial t}\begin{pmatrix}\overline{A}\\ \overline{B}_y\end{pmatrix} = \hat{N}\begin{pmatrix}\overline{A}\\ \overline{B}_y\end{pmatrix}, \qquad (49)$$

where the mean magnetic field is $\overline{\boldsymbol{B}}=\overline{B}_y(t,x,z)\boldsymbol{e}_y+{\rm rot}[\overline{A}(t,x,z)\boldsymbol{e}_y]$, the operator $\hat{N}$ is given by

$$\hat{N} = \begin{pmatrix}\eta_{_{T}}^{(A)}\left(\overline{B}\right)\Delta & \alpha\left(\overline{B}\right)\\ -R_\alpha^2\nabla_j\alpha\left(\overline{B}\right)\nabla_j & \nabla_j\eta_{_{T}}^{(B)}\left(\overline{B}\right)\nabla_j\end{pmatrix}, \qquad (50)$$

and the total α effect is given by $\alpha\left(\overline{B}\right)=\alpha_{_{\rm K}}\left(\overline{B}\right)+\alpha_{_{\rm M}}\left(\overline{\boldsymbol{B}}\right)$. Now we introduce the effective dynamo number $D_{\rm N}^{(\alpha)}\left(\overline{B}\right)$ in the non-linear α^2 dynamo defined as $D_{\rm N}^{(\alpha)}\left(\overline{B}\right)=\alpha^2\left(\overline{B}\right)\,L^2/[\eta_{_{T}}^{(B)}\left(\overline{B}\right)\eta_{_{T}}^{(A)}\left(\overline{B}\right)]$. Similarly, the effective dynamo number for a linear α^2 dynamo is defined as $D_{\rm L}^{(\alpha)}=R_\alpha^2$, where $R_\alpha=\alpha_*L/\eta_{_{T}}^{(0)}$, α_* is the maximum value of the kinetic α effect and L is the stellar radius or the thickness of the galactic disc.

The poloidal and toroidal components of the mean magnetic field in the non-linear α^2 mean-field dynamo are of the same order of magnitude. Equations (44)–(47) obtained in Section 4 can be used for the non-linear α^2 mean-field dynamo as well. Therefore, the ratio $D_{\rm N}^{(\alpha)}\left(\overline{B}\right)/D_{\rm L}^{(\alpha)}$ for strong mean magnetic fields is given by

$$\frac{D_{\rm N}^{(\alpha)}}{D_{\rm L}^{(\alpha)}} \approx \left(\frac{\overline{B}}{\overline{B}_{\rm eq}}\right)^{-3}. \qquad (51)$$

These equations take into account the feedback of the mean magnetic field on the background turbulence by means of the budget equation for the TTE. Thus, equation (51) implies that for the α^2 dynamo, the non-linear dynamo number decreases with increase of the mean magnetic field. This causes a saturation of the mean-field α^2 dynamo instability for a strong mean magnetic field. The discussion on the possibility of an oscillatory α^2 dynamo is given in Appendix C.

Next, we consider the axisymmetric mean-field $\alpha^2\Omega$ dynamo, so that and non-linear mean-field induction equation reads

$$\frac{\partial}{\partial t}\begin{pmatrix}\overline{A}\\ \overline{B}_y\end{pmatrix} = \hat{N}\begin{pmatrix}\overline{A}\\ \overline{B}_y\end{pmatrix}, \qquad (52)$$

where the mean magnetic field is $\overline{\boldsymbol{B}}=\overline{B}_y(t,x,z)\boldsymbol{e}_y+{\rm rot}[\overline{A}(t,x,z)\boldsymbol{e}_y]$, the operator $\hat{N}$ is

$$\hat{N} = \begin{pmatrix}\eta_{_{T}}^{(A)}\left(\overline{B}\right)\Delta & \alpha\left(\overline{\boldsymbol{B}}\right)\\ R_\alpha\left[R_\omega\hat{\Omega}-R_\alpha\nabla_j\alpha\left(\overline{B}\right)\nabla_j\right] & \nabla_j\eta_{_{T}}^{(B)}\left(\overline{B}\right)\nabla_j\end{pmatrix}, \qquad (53)$$

and $R_\alpha=\alpha_*L/\eta_{_{T}}^{(0)}$ and $R_\omega=(\delta\Omega)\,L^2/\eta_{_{T}}^{(0)}$. The kinematic and weakly non-linear $\alpha^2\,\Omega$ dynamos have been studied using asymptotic analysis (Meunier, Nesme-Ribes & Sokoloff 1996; Griffiths et al. 2001; Bassom et al. 2005).

We consider a kinematic dynamo problem, assuming for simplicity that the kinetic α effect is a constant, and the mean velocity $\overline{\boldsymbol{U}}=(0,Sz,0)$, where $S\equiv\delta\Omega$. We seek a solution for the linearized equation (52) as a real part of the following functions:

$$\overline{A}=A_0\exp[\tilde{\gamma}t-{\rm i}\,(k_xx+k_zz)], \qquad (54)$$

$$\overline{B}_\varphi=B_0\exp[\tilde{\gamma}t-{\rm i}\,(k_xx+k_zz)], \qquad (55)$$

where $\tilde{\gamma}=\gamma+{\rm i}\,\omega$. Equations (52)–(55) yield the growth rate of the dynamo instability and the frequency of the dynamo waves as

$$\gamma = \frac{R_\alpha R_\alpha^{\rm cr}}{\sqrt{2}}\left[\left[1+\left(\frac{\zeta\,R_\omega}{R_\alpha\,R_\alpha^{\rm cr}}\right)^2\right]^{1/2}+1\right]^{1/2}-\left(R_\alpha^{\rm cr}\right)^2, \qquad (56)$$

$$\omega = -{\rm sgn}(R_\omega)\,\frac{R_\alpha R_\alpha^{\rm cr}}{\sqrt{2}}\left[\left[1+\left(\frac{\zeta\,R_\omega}{R_\alpha\,R_\alpha^{\rm cr}}\right)^2\right]^{1/2}-1\right]^{1/2}, \qquad (57)$$

where $\zeta^2=1-\left(k_x/R_\alpha^{\rm cr}\right)^2$. Here, we took into account that $(x+{\rm i}y)^{1/2}=\pm(X+{\rm i}Y)$, where $X=2^{-1/2}\,[(x^2+y^2)^{1/2}+x]^{1/2}$ and $Y={\rm sgn}(y)\,2^{-1/2}\,[(x^2+y^2)^{1/2}-x]^{1/2}$. Here the threshold $R_\alpha^{\rm cr}$ for the mean-field dynamo instability, defined by the conditions $\gamma=0$ and $R_\omega=0$, is given by $R_\alpha^{\rm cr}=(k_x^2+k_z^2)^{1/2}$. Equations (52)–(55) also yield the squared ratio of amplitudes $|A_0/B_0|^2$,

$$\left|\frac{A_0}{B_0}\right|^2 = \left(R_\alpha R_\alpha^{\rm cr}\right)^{-2}\left[1+\left(\frac{\zeta\,R_\omega}{R_\alpha\,R_\alpha^{\rm cr}}\right)^2\right]^{-1/2}, \qquad (58)$$

and the phase shift δ between the toroidal field $\overline{B}_\varphi$ and the magnetic vector potential $\overline{A}$ is given by

$$\sin(2\delta) = -\zeta\,R_\omega\,\left[\left(R_\alpha R_\alpha^{\rm cr}\right)^2+\zeta^2R_\omega^2\right]^{-1/2}. \qquad (59)$$

Equation (58) yields the energy ratio of poloidal $\overline{B}_{\rm pol}$ and toroidal $\overline{B}_\varphi$ mean magnetic field components as

$$\frac{\overline{B}_{\rm pol}^2}{\overline{B}_\varphi^2} = \left[1 + \left(\frac{\zeta\, R_\omega}{R_\alpha\, R_\alpha^{\rm cr}}\right)^2\right]^{-1/2}, \tag{60}$$

where $\overline{B}_{\rm pol}^2 = \overline{B}_x^2 + \overline{B}_z^2 = (R_\alpha R_\alpha^{\rm cr}\, \overline{A})^2$.

Asymptotic formulas for the growth rate of the dynamo instability and the frequency of the dynamo waves for a weak differential rotation, $\zeta\, R_\omega \ll R_\alpha R_\alpha^{\rm cr}$, are given by

$$\gamma = R_\alpha R_\alpha^{\rm cr} \left[1 + \frac{1}{8}\left(\frac{\zeta\, R_\omega}{R_\alpha\, R_\alpha^{\rm cr}}\right)^2\right] - \left(R_\alpha^{\rm cr}\right)^2, \tag{61}$$

$$\omega = -\frac{\zeta\, R_\omega}{\sqrt{2}}. \tag{62}$$

In this case, the mean-field α^2 dynamo is slightly modified by a weak differential rotation, and the phase shift between the fields $\overline{B}_\varphi$ and $\overline{B}_{\rm pol}$ vanishes, while $\overline{B}_{\rm pol}/\overline{B}_\varphi \sim 1$ [see equations (59)–(60)]. In the opposite case, for a strong differential rotation, $\zeta\, R_\omega \gg R_\alpha R_\alpha^{\rm cr}$, the growth rate of the dynamo instability and the frequency of the dynamo waves are given by

$$\gamma = \left[\frac{1}{2}\, \zeta\, R_\alpha^{\rm cr}\, R_\alpha |R_\omega|\right]^{1/2} - \left(R_\alpha^{\rm cr}\right)^2, \tag{63}$$

$$\omega = -{\rm sgn}(R_\omega) \left[\frac{1}{2}\, \zeta\, R_\alpha^{\rm cr}\, R_\alpha |R_\omega|\right]^{1/2}. \tag{64}$$

In this case, the mean-field $\alpha\Omega$ dynamo is slightly modified by a weak α^2 effect, and the phase shift between the fields $\overline{B}_\varphi$ and $\overline{B}_{\rm pol}$ tends to $-\pi/4$, while $\overline{B}_{\rm pol}/\overline{B}_\varphi \ll 1$ [see equations (59)–(60)]. The necessary condition for the dynamo ($\gamma > 0$) in this case reads:

(i) when $R_\alpha/R_\alpha^{\rm cr} < \sqrt{2}$, the mean-field $\alpha^2\,\Omega$ dynamo is excited when

$$D_{\rm L} > \frac{2}{\zeta}\, \left(R_\alpha^{\rm cr}\right)^3; \tag{65}$$

(ii) when $R_\alpha/R_\alpha^{\rm cr} > \sqrt{2}$, the mean-field $\alpha^2\,\Omega$ dynamo is excited for any differential rotation, R_ω. Here $D_{\rm L} = R_\alpha\, R_\omega$.

Analysis which is similar to that performed in Section 4 [see equations (44)–(47)] yields the ratio of the non-linear and linear dynamo numbers $D_{\rm N}\left(\overline{B}\right)/D_{\rm L}$ in the non-linear $\alpha^2\Omega$ dynamo for strong mean magnetic fields that is coincided with equation (51). The latter implies that for the $\alpha^2\Omega$ dynamo, the non-linear dynamo number decreases with increase of the mean magnetic field, so that the non-linear mean-field dynamo instability is always saturated for strong mean magnetic fields.

6 DISCUSSION AND CONCLUSIONS

In the sun, stars and galaxies, the large-scale magnetic fields are originated due to the mean-field dynamo instabilities. The saturation of the dynamo generated large-scale magnetic fields is caused by algebraic and dynamic non-linearities. A key parameter that controls the saturation of the $\alpha\Omega$ dynamo instability is the non-linear dynamo number $D_{\rm N}\left(\overline{B}\right) = \alpha\left(\overline{B}\right)\, \delta\Omega\, L^3/\eta_{_T}^2\left(\overline{B}\right)$. When the total α effect and the turbulent magnetic diffusion are quenched as $(\overline{B}/\overline{B}_{\rm eq})^{-2}$ for strong mean magnetic fields, the non-linear dynamo number $D_{\rm N}\left(\overline{B}\right)$ increases with the increase of the large-scale magnetic field. The latter implies that the mean-field dynamo instability cannot be saturated for a strong field.

In the present study, we have shown that the dissipation of the generated strong large-scale magnetic field increases both, the TKE of the background turbulence and the turbulent magnetic diffusion coefficient. This non-linear effect is taken into account by means of the budget equation (25) for the TTE. As the result for a strong mean magnetic field, the product of the turbulent diffusion coefficients of the poloidal and toroidal fields behaves as $\eta_{_T}^{(A)}\, \eta_{_T}^{(B)} \propto \left(\overline{B}/\overline{B}_{\rm eq}\right)^{-1}$. This additional non-linear effect decreases the non-linear dynamo number for a strong field and causes a saturation of the dynamo growth of large-scale magnetic field.

Using this approach and considering various origins of turbulence (e.g. a forced turbulence, a shear-produced turbulence and a convective turbulence), we have demonstrated that the mean-field $\alpha\Omega$, α^2, and $\alpha^2\Omega$ dynamo instabilities can be always saturated for any strong mean magnetic field. Indeed, the ratio of the non-linear and linear dynamo numbers for the $\alpha\Omega$ dynamo is $D_{\rm N}\left(\overline{B}\right)/D_{\rm L} \propto \alpha/[\eta_{_T}^{(A)}\, \eta_{_T}^{(B)}] \propto \left(\overline{B}/\overline{B}_{\rm eq}\right)^{-1}$, i.e. it decreases with the growth of the mean magnetic field. On the other hand, for the α^2 dynamo, the ratio of the non-linear and linear dynamo numbers $D_{\rm N}\left(\overline{B}\right)/D_{\rm L} \propto \alpha^2/[\eta_{_T}^{(A)}\, \eta_{_T}^{(B)}] \propto \left(\overline{B}/\overline{B}_{\rm eq}\right)^{-3}$. Here we took into account that the scaling for the α effect for a strong mean magnetic field is $\alpha \propto \left(\overline{B}/\overline{B}_{\rm eq}\right)^{-2}$. These results have very important applications for astrophysical magnetic fields.

For validation of these results in direct numerical simulations, the test-field method (Karak et al. 2014) can be applied to determine the quenching of the turbulent magnetic diffusion coefficients of toroidal and poloidal components of the mean magnetic field as well as the quenching of the total α effect. Note that various mean-field numerical simulations (Elstner, Rüdiger & Schultz 1996; Kleeorin et al. 2003a), which took into account simultaneously both, the algebraic quenching of the α effect and the turbulent magnetic diffusion coefficients, were unable to find steady solutions of the non-linear mean-field dynamo equations. To obtain such solutions in mean-field numerical simulations, the budget equation (25) for the TTE should be used.

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DATA AVAILABILITY

There are no new data associated with this article.

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APPENDIX A: QUENCHING FUNCTIONS

The quenching functions $\phi_{_{\rm K}}(\beta)$ and $\phi_{_{\rm M}}(\beta)$ are

$$\phi_{_{\rm K}}(\beta) = A_1^{(1)}\left(\sqrt{2}\beta\right) + A_2^{(1)}\left(\sqrt{2}\beta\right) = \frac{1}{7}\left\{3\left[1 - 4\beta^2 + 8\beta^4 \ln\left(1 + (2\beta^2)^{-1}\right)\right] + 4\phi_{_{\rm M}}\left(\sqrt{2}\,\beta\right)\right\}, \tag{A1}$$

$$\phi_{_{\rm M}}(\beta) = \frac{3}{4\pi}\left[\tilde{A}_1\left(\beta^2\right) + \tilde{A}_2\left(\beta^2\right)\right] = \frac{3}{\beta^2}\left(1 - \frac{\arctan\beta}{\beta}\right). \tag{A2}$$

The quenching function $\phi_\eta^{(B)}(\beta)$ is given by $\phi_\eta^{(B)}(\beta) = \phi_{_{\rm K}}(\beta) + \phi(\beta)$ and

$$\phi(\beta) = (2 - 3\epsilon)\, A_2^{(1)}\left(\sqrt{2}\beta\right) - \frac{3}{2\pi}\,(1 - \epsilon)\,\tilde{A}_2\left(2\beta^2\right), \tag{A3}$$

where the functions $A_1^{(1)}(\beta)$ and $A_2^{(1)}(\beta)$ are given by

$$A_1^{(1)}(\beta) = \frac{6}{5}\left[\frac{\arctan\beta}{\beta}\left(1 + \frac{5}{7\beta^2}\right) + \frac{1}{14}L(\beta) - \frac{5}{7\beta^2}\right], \tag{A4}$$

$$A_2^{(1)}(\beta) = -\frac{6}{5}\left[\frac{\arctan\beta}{\beta}\left(1 + \frac{15}{7\beta^2}\right) - \frac{2}{7}L(\beta) - \frac{15}{7\beta^2}\right], \tag{A5}$$

and $L(\beta) = 1 - 2\beta^2 + 2\beta^4 \ln(1 + \beta^{-2})$. For $\beta \ll 1$, these functions are given by

$$A_1^{(1)}(\beta) \sim 1 - \frac{2}{5}\beta^2, \quad A_2^{(1)}(\beta) \sim -\frac{4}{5}\beta^2,$$

and for $\beta \gg 1$, they are given by

$$A_1^{(1)}(\beta) \sim \frac{3\pi}{5\beta} - \frac{2}{\beta^2}, \quad A_2^{(1)}(\beta) \sim -\frac{3\pi}{5\beta} + \frac{4}{\beta^2}.$$

The functions $\tilde{A}_1(x)$ and $\tilde{A}_2(x)$ are given by

$$\tilde{A}_1(x) = \frac{2\pi}{x}\left[(x + 1)\frac{\arctan(\sqrt{x})}{\sqrt{x}} - 1\right], \tag{A6}$$

$$\tilde{A}_2(x) = -\frac{2\pi}{x}\left[(x + 3)\frac{\arctan(\sqrt{x})}{\sqrt{x}} - 3\right]. \tag{A7}$$

For $x \ll 1$, these functions are given by

$$\tilde{A}_1(x) \sim \frac{4\pi}{3}\left(1 - \frac{1}{5}x\right), \quad \tilde{A}_2(x) \sim -\frac{8\pi}{15}x,$$

In the case of $x \gg 1$, these functions are given by

$$\tilde{A}_1(x) \sim \frac{\pi^2}{\sqrt{x}} - \frac{4\pi}{x} , \quad \tilde{A}_2(x) \sim -\frac{\pi^2}{\sqrt{x}} + \frac{8\pi}{x}.$$

APPENDIX B: DERIVATION OF EQUATIONS (27)–(28)

In this appendix, we derive equations (27)–(28) [for more details see paper by Rogachevskii & Kleeorin (2007)]. Using procedure described in Section 2, we derive equations for the correlation functions of the velocity fluctuations $f_{ij} = \langle u_i\, u_j \rangle$, the magnetic fluctuations $h_{ij} = \langle b_i\, b_j \rangle$, and the cross-helicity $g_{ij} = \langle u_i\, b_j \rangle$ in the Fourier space:

$$\frac{\partial f_{ij}(\boldsymbol{k})}{\partial t} = -\mathrm{i}\, (\boldsymbol{k}\boldsymbol{\cdot}\overline{\boldsymbol{B}})\, \Phi_{ij}(\boldsymbol{k}) + \hat{\mathcal{M}} f_{ij}^{(III)}(\boldsymbol{k}), \tag{B1}$$

$$\frac{\partial h_{ij}(\boldsymbol{k})}{\partial t} = \mathrm{i}\, (\boldsymbol{k}\boldsymbol{\cdot}\overline{\boldsymbol{B}}) \Phi_{ij}(\boldsymbol{k}) + \hat{\mathcal{M}} h_{ij}^{(III)}(\boldsymbol{k}), \tag{B2}$$

$$\frac{\partial g_{ij}(\boldsymbol{k})}{\partial t} = -\mathrm{i}\, \left(\boldsymbol{k}\boldsymbol{\cdot}\overline{\boldsymbol{B}}\right)\, \left[f_{ij}(\boldsymbol{k}) - h_{ij}(\boldsymbol{k})\right] + \hat{\mathcal{M}} g_{ij}^{(III)}(\boldsymbol{k}), \tag{B3}$$

where $\Phi_{ij}(\boldsymbol{k}) = g_{ij}(\boldsymbol{k}) - g_{ji}(-\boldsymbol{k})$, and $\hat{\mathcal{M}} f_{ij}^{(III)}$, $\hat{\mathcal{M}} h_{ij}^{(III)}$, and $\hat{\mathcal{M}} g_{ij}^{(III)}$ are the third-order moment terms appearing due to the non-linear terms. We split the tensor $\langle b_i\, b_j \rangle$ of magnetic fluctuations into non-helical h_{ij} and helical $h_{ij}^{(H)}$ parts. The helical part $h_{ij}^{(H)}$ depends on the magnetic helicity, and it is determined by the dynamic equation which follows from the magnetic helicity conservation arguments. We also split the second-order correlation functions into symmetric and antisymmetric parts with respect to the wave vector $\boldsymbol{k}$, e.g. $f_{ij} = f_{ij}^{(s)} + f_{ij}^{(a)}$, where the tensors $f_{ij}^{(s)} = [f_{ij}(\boldsymbol{k}) + f_{ij}(-\boldsymbol{k})]/2$ describes the symmetric part of the tensor and $f_{ij}^{(a)} = [f_{ij}(\boldsymbol{k}) - f_{ij}(-\boldsymbol{k})]/2$ determines the antisymmetric part of the tensor. We apply the spectral τ approximation [see equation (4)] for the non-helical parts of the tensors. We assume that the characteristic time of variation of the mean magnetic field $\overline{\boldsymbol{B}}$ is substantially larger than the correlation time $\tilde{\tau}(k)$ for all turbulence scales. This allows us to get a stationary solution for the equations for the second-order moments

$$f_{ij}^{(s)}(\boldsymbol{k}) = \frac{1}{1+2\psi}\, \left[(1+\psi)\, f_{ij}^{(0s)}(\boldsymbol{k}) + \psi\, h_{ij}^{(0s)}(\boldsymbol{k})\right], \tag{B4}$$

$$h_{ij}^{(s)}(\boldsymbol{k}) = \frac{1}{1+2\psi}\, \left[\psi\, f_{ij}^{(0s)}(\boldsymbol{k}) + (1+\psi)\, h_{ij}^{(0s)}(\boldsymbol{k})\right], \tag{B5}$$

where $\psi(\boldsymbol{k}) = 2(\tilde{\tau}\, \boldsymbol{k}\boldsymbol{\cdot}\overline{\boldsymbol{B}})^2$. Next, we specify a model for the background turbulence (with zero mean magnetic field $\overline{\boldsymbol{B}} = 0$) [denoted with the superscript (0)], see equations (5)–(6). The background turbulence here is assumed to be homogeneous, isotropic, and non-helical. Integration in $\boldsymbol{k}$ space in equations (B4)–(B5) yields equations (27)–(28), where the non-linear functions $q_{\rm p}(\beta)$ and $q_{\rm s}(\beta)$ are given by

$$q_{\rm p}(\beta) = \frac{2}{3\beta^2}\, \left[A_1^{(0)}(0) - A_1^{(0)}(\sqrt{2}\beta) - A_2^{(0)}(\sqrt{2}\beta)\right], \tag{B6}$$

$$q_{\rm s}(\beta) = -\frac{2}{3\beta^2}\, A_2^{(0)}(\sqrt{2}\beta), \tag{B7}$$

and $\beta = \sqrt{8}\, \overline{B}/\overline{B}_{\rm eq}$. The functions $A_1^{(0)}(\beta)$ and $A_2^{(0)}(\beta)$ are given by

$$A_1^{(0)}(\beta) = \frac{1}{5}\Bigg[2 + 2\frac{\arctan\beta}{\beta^3}(3+5\beta^2) - \frac{6}{\beta^2} - \beta^2\ln \mathrm{Rm} -2\beta^2\ln\left(\frac{1+\beta^2}{1+\beta^2\sqrt{\mathrm{Rm}}}\right)\Bigg], \tag{B8}$$

$$A_2^{(0)}(\beta) = \frac{2}{5}\Bigg[2 - \frac{\arctan\beta}{\beta^3}(9+5\beta^2) + \frac{9}{\beta^2} - \beta^2\ln \mathrm{Rm} -2\beta^2\ln\left(\frac{1+\beta^2}{1+\beta^2\sqrt{\mathrm{Rm}}}\right)\Bigg]. \tag{B9}$$

For $\overline{B} \ll \overline{B}_{\rm eq}/4\mathrm{Rm}^{1/4}$, these functions are given by

$$A_1^{(0)}(\beta) \sim 2 - \frac{1}{5}\beta^2\ln \mathrm{Rm},$$

$$A_2^{(0)}(\beta) \sim -\frac{2}{5}\beta^2\left[\ln \mathrm{Rm} + \frac{2}{15}\right].$$

For $\overline{B}_{\rm eq}/4\mathrm{Rm}^{1/4} \ll \overline{B} \ll \overline{B}_{\rm eq}/4$, these functions are given by

$$A_1^{(0)}(\beta) \sim 2 + \frac{2}{5}\beta^2\left[2\ln\beta - \frac{16}{15} + \frac{4}{7}\beta^2\right],$$

$$A_2^{(0)}(\beta) \sim \frac{2}{5}\beta^2\left[4\ln\beta - \frac{2}{15} - 3\beta^2\right],$$

and for $\overline{B} \gg \overline{B}_{\rm eq}/4$, they are given by

$$A_1^{(0)}(\beta) \sim \frac{\pi}{\beta} - \frac{3}{\beta^2} , \quad A_2^{(0)}(\beta) \sim -\frac{\pi}{\beta} + \frac{6}{\beta^2}.$$

APPENDIX C: OSCILLATORY α^2 DYNAMO

In this appendix, we discuss a long-standing question: 'When can a one-dimensional kinematic α^2 dynamo be oscillatory?' The mean magnetic field $\overline{\boldsymbol{B}}(t, z) = \boldsymbol{\nabla} \times \overline{\boldsymbol{A}} = (-\nabla_z \overline{A}_y, \nabla_z \overline{A}_x, 0)$ is determined by the following equation

$$\frac{\partial \Psi}{\partial t} = \hat{L}\, \Psi, \tag{C1}$$

where $\overline{\boldsymbol{A}}$ is the mean magnetic vector potential in the Weyl gauge. The linear operator $\hat{L}$ and the function $\Psi(t, z)$ are given by

$$\hat{L} = \begin{pmatrix} \eta_{_T}^{(0)}\nabla_z^2 & -\alpha_{_{\rm K}}^{(0)}\nabla_z \\ \alpha_{_{\rm K}}^{(0)}\nabla_z & \eta_{_T}^{(0)}\nabla_z^2 \end{pmatrix}, \quad \Psi = \begin{pmatrix} A_x \\ A_y \end{pmatrix}, \tag{C2}$$

where $\eta_{_T}^{(0)}$ is the turbulent magnetic diffusion coefficient, and $\alpha_{_{\rm K}}^{(0)}$ is the kinetic α effect caused by the helical turbulent motions in plasma. If the linear operator $\hat{L}$ is not self-adjoint, it has complex eigenvalues. This case corresponds to the oscillatory growing solution, i.e. the dynamo is oscillatory. On the other hand, any self-adjoint operator, $\hat{M}$, defining by the following condition,

$$\int \Psi^* \hat{M} \tilde{\Psi}\, dz = \int \tilde{\Psi} \hat{M}^* \Psi^*\, dz, \tag{C3}$$

has real eigenvalues, where the asterisk denotes complex conjugation. Now we determine conditions when the linear operator $\hat{L}$ is not self-adjoint, i.e. it has complex eigenvalues. To this end, we determine the integrals $\int \Psi^* \hat{L} \tilde{\Psi}\, dz$ and $\int \tilde{\Psi} \hat{L}^* \Psi^*\, dz$ as:

$$\int \Psi^* \hat{L} \tilde{\Psi}\, dz = \int \alpha_{_{\rm K}}^{(0)} \left(A_y^* \nabla_z \tilde{A}_x - A_x^* \nabla_z \tilde{A}_y\right)\, dz - \int \eta_{_T}^{(0)} \left[\left(\nabla_z A_x^*\right)\, \nabla_z \tilde{A}_x + \left(\nabla_z A_y^*\right)\, \nabla_z \tilde{A}_y\right]\, dz + \left[\eta_{_T}^{(0)} \left(A_x^*\, \nabla_z \tilde{A}_x + A_y^*\, \nabla_z \tilde{A}_y\right)\right]_{z=L_{\rm bott}}^{z=L_{\rm top}}, \tag{C4}$$

$$\int \tilde{\Psi} \hat{L}^* \Psi^* \, dz = \int \alpha_{\rm K}^{(0)} \left(A_y^* \nabla_z \tilde{A}_x - A_x^* \nabla_z \tilde{A}_y \right) \, dz$$
$$- \int \eta_{T}^{(0)} \left[\left(\nabla_z A_x^* \right) \, \nabla_z \tilde{A}_x + \left(\nabla_z A_y^* \right) \, \nabla_z \tilde{A}_y \right] \, dz$$
$$+ \left[\eta_{T}^{(0)} \left(\tilde{A}_x \, \nabla_z A_x^* + \tilde{A}_y \, \nabla_z A_y^* \right) + \alpha_{\rm k} \left(A_x^* \tilde{A}_y - A_y^* \, \tilde{A}_x \right) \right]_{z=L_{\rm bott}}^{z=L_{\rm top}} , \qquad \text{(C5)}$$

where $z = L_{\rm bott}$ and $z = L_{\rm top}$ are the bottom and upper boundaries, respectively. When $\eta_{T}^{(0)}$ and $\alpha_{\rm K}^{(0)}$ vanish at the boundaries where the turbulence is very weak, the operator $\hat{L}$ satisfies condition (C3) and the α^2 dynamo is not oscillatory. On the other hand, when $\alpha_{\rm K}^{(0)}$ vanishes only at one boundary, while it is non-zero at the other boundary, the operator $\hat{L}$ does not satisfy condition (C3), and the α^2 dynamo is oscillatory. The latter case has been considered in analytical study by Shukurov, Sokoloff & Ruzmaikin (1985), Rädler & Bräuer (1987), and in numerical study by Baryshnikova & Shukurov (1987). Brandenburg (2017) has recently considered the one-dimensional kinematic α^2 dynamo with different conditions at two boundaries: $\boldsymbol{A} = 0$ at $z = L_{\rm bott}$ and $\nabla_z \boldsymbol{A} = 0$ at $z = L_{\rm top}$, so that the operator $\hat{L}$ may not satisfy condition (C3), and the α^2 dynamo may be oscillatory.

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