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Turbulent transport of radiation in the solar convective zone

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ABSTRACT
A turbulent transport of radiation in the solar convective zone is investigated. The mean-field equation for the irradiation intensity is derived. It is shown that due to the turbulent effects, the effective penetration length of radiation can be increased several times in comparison with the mean penetration length of radiation (defined as an inverse mean absorption coefficient). Using the model of the solar convective zone based on mixing length theory, where the mean penetration length of radiation is usually much smaller than the turbulent correlation length, it is demonstrated that the ratio of the effective penetration length to the mean penetration length of radiation increases 2.5 times in the vicinity of the solar surface. The main reasons for this are the compressibility effects that become important in the vicinity of the solar surface where temperature and density fluctuations increase towards the solar surface, enhancing fluctuations of the radiation absorption coefficient and increasing the effective penetration length of radiation.

Key words: radiative transfer – turbulence – Sun: interior.

1 INTRODUCTION

The turbulent transport of temperature, particles and magnetic fields has been studied analytically, in laboratory and field experiments and in numerical simulations, for more than a century (see, e.g. Monin & Yaglom 1971, 1975; McComb 1990; Frisch 1995; Lesieur 2008; Davidson 2013; Rogachevskii 2021). However, some fundamental questions remain. This is particularly true in applications to astrophysics, where the governing parameter values are too extreme to be modelled, either experimentally or numerically.

In astrophysical turbulent flows, radiative transport can be affected by turbulence. This effect is different for optically thick and optically thin regimes of the radiative transport. For an optically thick regime, the mean free path of photons is much smaller in comparison with the typical scales of the flow. In contrast, for an optically thin regime, the mean free path of photons is much larger than the fluid motion scales, so the photons propagate over large distances before they are absorbed and re-emitted again (see, e.g. Chandrasekhar 1960; Mihalas & Mihalas 1984; Apresyan & Kravtsov 1996; Liou 2002; Howell, Menguc & Siegel 2010).

The relaxation time of small temperature perturbations by radiative diffusion has been determined by Spiegel (1957), where a time-dependent equation for the temperature field of a medium with deviations from radiative equilibrium has been derived, assuming that the medium is grey-like and there are no internal motions or compressional effects, and heat is exchanged only radiatively. It has been shown that perturbations of small amplitude imposed on a homogeneous medium decay exponentially in time, and the decay time depends on a characteristic length of the perturbations (Spiegel 1957).

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Turbulent diffusion can be increased by radiative diffusion (i.e. by a photon diffusion). For instance, the decay rates of sinusoidal large-scale temperature perturbations in the optically thick and thin regimes have been determined by Brandenburg & Das (2021) using radiative hydrodynamic direct numerical simulations of forced turbulence. They have shown that the rate of decay increases with the wavenumber. However, this effect is much weaker in comparison with the effect of the standard turbulent diffusion (Brandenburg & Das 2021).

In the present study, we investigate the turbulent transport of radiation in the solar convective zone. We derive a mean-field equation for the irradiation intensity, and show that the effective penetration length of radiation can be increased by turbulence several times in comparison with the mean penetration length of radiation, which is defined as an inverse mean absorption coefficient. This effect has been tested using a model of the solar convective zone (Spruit 1974) based on mixing length theory. According to this model, the mean penetration length of radiation is much less than the turbulent correlation length. We show that the ratio of the effective penetration length to the mean penetration length of radiation increases 2.5 times in the vicinity of the solar surface.

This paper is organized as follows. In Section 2, we discuss a general concept of the turbulent transport of radiation, and in Section 3 we derive the mean-field radiation transport equation. In Section 4, we derive an expression for the effective penetration length of radiation in turbulent flows, which depends on the ratio of fluctuations of the radiation absorption coefficient to the mean penetration length of radiation. To calculate the effective penetration length of radiation, in Section 4 we determine fluctuations of the radiation absorption coefficient, which are caused by fluctuations of fluid temperature and density, and the temperature–density correlations. In Section 5, we apply the results we obtain to the solar convective zone. Finally, in Section 6, we discuss our results and draw conclusions.

In Appendix A, we derive expressions for the level of temperature and density fluctuations as well as temperature–density correlations, which allow us to determine fluctuations of the radiation absorption coefficient.

2 GENERAL CONCEPT OF TURBULENT TRANSPORT OF RADIATION

In solar and stellar convective zones, the convective transport of energy is more effective than the radiative transfer. The Schwarzschild criterion for the onset of convection does not take into account the effect of turbulence on radiative transfer. However, the absorption coefficient of radiation depends on temperature and density, and there are strong fluctuations of the fluid temperature and density in solar and stellar convective zones. These fluctuations affect the absorption coefficient of radiation, and therefore they affect the turbulent transfer of radiation.

In solar and stellar convective zones, the characteristic times of turbulent motions are much larger than the radiation time, and the integral turbulent scales are much larger than the mean penetration length of radiation, which is defined as the inverse absorption coefficient of radiation. The latter implies that turbulent eddies are optically thick, and inhomogeneities in the fluid temperature and density can strongly affect the radiation transfer.

To describe the radiation transfer, we use the radiation transport equation. This equation represents a steady-state version of the equation for electromagnetic energy transfer, because the time of photon propagation (the radiation time) is very short. This equation is characterized by the absorption coefficient of radiation and the black-body radiation intensity of the gas. The radiative transport equation for the intensity $I(\boldsymbol{r}, \hat{\boldsymbol{s}}, \omega)$ is (see, e.g. Chandrasekhar 1960; Mihalas & Mihalas 1984; Apresyan & Kravtsov 1996; Liou 2002; Howell et al. 2010)

$$(\hat{\boldsymbol{s}}\boldsymbol{\cdot}\boldsymbol{\nabla})\, I(\boldsymbol{r}, \hat{\boldsymbol{s}}, \omega) = -\kappa(\boldsymbol{r}, \omega)\, [I - I_{\rm b}(T, \omega)]\,, \tag{1}$$

where $\boldsymbol{r}$ is the position vector, $\hat{\boldsymbol{s}} = \boldsymbol{k}/k$ is the unit vector in the direction of radiation, $\boldsymbol{k}$ is the wave vector, $\kappa(\boldsymbol{r}, \omega) = \rho\, \kappa_{\rm op}$ is the absorption coefficient of gas, $\kappa_{\rm op} = \kappa_0\, \rho^a\, T^b$ is the opacity of the gas, T and ρ are the gas temperature and density, $I_{\rm b}(T, \omega)$ is the black-body radiation intensity of the gas, and ω is the radiation frequency. Here we take into account the radiation absorption in gases and neglect the radiation scattering in gases. The function $I_{\rm b}(T, \omega)$ in a local thermodynamic equilibrium is given by

$$I_{\rm b}(T, \omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \left[\exp\left(\frac{\hbar\omega}{k_{\rm B} T}\right) - 1\right]^{-1}, \tag{2}$$

where $\hbar$ is Planck's constant, c is the speed of light and $k_{\rm B}$ is the Boltzmann constant. The integral $\int I_{\rm b}(T, \omega)\, {\rm d}\omega \propto \sigma T^4$ yields the Stefan–Boltzmann law.

Our goal is to derive the effective radiation transport equation with effective transport coefficients: the effective absorption coefficient of radiation and the effective source of the radiation intensity. To take into account the turbulence effects, we apply a mean-field approach and average the radiation transport equation (1) over an ensemble of fluctuations. In the framework of the mean-field approach, all quantities are decomposed into the mean and fluctuating parts: $I = \overline{I} + I'$, $I_{\rm b} = \overline{I}_{\rm b} + I_{\rm b}'$ and $\kappa = \overline{\kappa} + \kappa'$. We adopt the Reynolds averaging, where $\overline{I} = \langle I \rangle$, $\overline{I}_{\rm b} = \langle I_{\rm b} \rangle$ and $\overline{\kappa} = \langle \kappa \rangle$ are the mean fields, I', $I_{\rm b}'$ and κ' are the fluctuating fields with zero mean, and the angular brackets denote ensemble averaging. To derive the mean-field radiation transport equation, we adopt a method applied by Kliorin et al. (1989) and Liberman et al. (2017, 2018).

The obtained mean-field equation contains the correlation function for fluctuations of the absorption coefficient of radiation κ' and the radiation intensity I', i.e. $\langle \kappa'\, I' \rangle$. This correlation is due to fluctuations of temperature and density. This equation also contains the correlation function for fluctuations of the absorption coefficient of radiation κ' and the black-body radiation intensity of the gas $\langle \kappa'\, I_{\rm b}' \rangle$ due to fluctuations of temperature.

To determine the correlation functions, $\langle \kappa'\, I' \rangle$ and $\langle \kappa'\, I_{\rm b}' \rangle$, we derive the equation for fluctuations of the radiation intensity I' by subtracting the obtained mean-field equation from the radiation transport equation (1). As the equation for fluctuations of the radiation intensity I' is a linear equation, we solve this equation exactly. However, the solution of this equation is non-linear in fluctuations of κ'. This causes the appearance of the high-order moments in the expression for the correlation function $\langle \kappa'\, I' \rangle$. We assume that fluctuations of κ' are essentially less than the mean absorption coefficient of radiation. This allows us to obtain the closed results.

The main expected result of this study is that the derived mean-field equation for the radiation transfer with the effective transport coefficients yields the effective penetration length of radiation. When the effective penetration length of radiation is larger than the mean penetration length of radiation, the absorbtion coefficient decreases and an observer can see more deeper layers inside the stars. The reasons for this increase of the effective penetration length of radiation in turbulent flows are the correlation between fluctuations of the radiation absorption coefficient κ' and fluctuations of the irradiation intensity I'. We show below that this correlation function should be negative, because an increase of the absorption of radiation decreases the radiation intensity, and vice versa. We also demonstrate in this study that this effect is essential in the vicinity of the solar surface.

3 MEAN-FIELD RADIATION TRANSPORT EQUATION

In this section, we derive the mean-field radiation transport equation. Ensemble averaging of equation (1) yields the equation for the mean radiation intensity $\overline{I}$:

$$(\hat{\boldsymbol{s}}\boldsymbol{\cdot}\boldsymbol{\nabla})\, \bar{I} = -\bar{\kappa}\, \left(\bar{I} - \bar{I}_b\right) - \langle \kappa'\, I' \rangle + \langle \kappa'\, I_{\rm b}' \rangle. \tag{3}$$

This equation contains unknown correlation functions, $\langle \kappa'\, I' \rangle$ and $\langle \kappa'\, I_{\rm b}' \rangle$. To determine these correlation functions, we derive the equation for fluctuations of the radiation intensity I'. To this end, we subtract the mean-field radiation transport equation (3) from equation (1), so that the equation for fluctuations of I' is

$$\left(\hat{\boldsymbol{s}}\boldsymbol{\cdot}\boldsymbol{\nabla} + \overline{\kappa} + \kappa'\right) I'(\boldsymbol{r}, \hat{\boldsymbol{s}}) = I_{\rm source}, \tag{4}$$

where the source term $I_{\rm source}$ is given by

$$I_{\rm source} = -\kappa'\, \left(\overline{I} - \overline{I}_{\rm b}\right) + \langle \kappa'\, I' \rangle + \left(\overline{\kappa} + \kappa'\right) I_{\rm b}' - \langle \kappa'\, I_{\rm b}' \rangle. \tag{5}$$

The solution of equation (4) is

$$I'(\boldsymbol{r}, \hat{\boldsymbol{s}}) = \int_{-\infty}^{\infty} \exp\left[-\left|\int_{s'}^{s} \left[\overline{\kappa} + \kappa'(s'')\right]\, {\rm d}s''\right|\right] \times I_{\rm source}(s')\, {\rm d}s', \tag{6}$$

where $s = \boldsymbol{r}\boldsymbol{\cdot}\hat{\boldsymbol{s}}$. This solution is non-linear in fluctuations of κ'. The latter causes the appearance of the high-order moments in the expression for the correlation function $\langle \kappa'\, I' \rangle$. The high-order moments are much less than the lower-order moments, because $\kappa' \ll \overline{\kappa}$. This allows us to expand the function, $\exp\left[-\int_{s'}^{s} \kappa'(s'')\, {\rm d}s''\right]$, in equation (6) in the Taylor series:

$$\exp\left[-\int_{s'}^{s}\kappa'(s'')\,\mathrm{d}s''\right]=1-\int_{s'}^{s}\kappa'(s'')\,\mathrm{d}s''+O\left(\kappa'^2\right). \tag{7}$$

Therefore, equation (6) can be rewritten as

$$I'(\boldsymbol{r},\hat{\boldsymbol{s}})=\int_{-\infty}^{\infty}I_{\rm source}(s')\,\exp\left(-\overline{\kappa}|s-s'|\right)\times\left[1-\int_{s'}^{s}\kappa'(s'')\,\mathrm{d}s''\right]\,\mathrm{d}s'+O\left(\kappa'^2\right). \tag{8}$$

Multiplying equation (8) by κ' and averaging over the ensemble, we obtain the expression for the one-point correlation function $\langle\kappa'\,I'\rangle$:

$$\left\langle\kappa'\,I'\right\rangle\left\{1+\int_{-\infty}^{\infty}\left[\int_{s'}^{s}\langle\kappa'(s)\kappa'(s'')\rangle\,\mathrm{d}s''\right]\times\exp\left(-\overline{\kappa}|s-s'|\right)\,\mathrm{d}s'\right\}=-\left[\int_{-\infty}^{\infty}\langle\kappa'(s)\kappa'(s')\rangle\times\exp\left(-\overline{\kappa}|s-s'|\right)\,\mathrm{d}s'\right]\left(\overline{I}-\overline{I}_{\rm b}\right). \tag{9}$$

Here, we neglect the third-order and higher-order moments in fluctuations of κ'. Equation (9) can be rewritten as

$$\left\langle\kappa'\,I'\right\rangle=-\overline{\kappa}\left(\overline{I}-\overline{I}_{\rm b}\right)\frac{2\overline{\kappa}\,J_1}{1+2\overline{\kappa}\,J_2}, \tag{10}$$

where the integrals J_1 and J_2 in equation (10) are defined as

$$J_1=\int_0^{\infty}\Phi(Z)\exp(-\overline{\kappa}Z)\,\mathrm{d}Z, \tag{11}$$

$$J_2=\overline{\kappa}\int_0^{\infty}\left[\int_0^{Z}\Phi(Z')\,\mathrm{d}Z'\right]\exp(-\overline{\kappa}Z)\,\mathrm{d}Z, \tag{12}$$

the function $\Phi(Z)$ is defined as $\Phi(Z)=\langle\kappa'(s)\kappa'(s')\rangle$ and $Z=|s-s'|$.

Substituting equation (10) into the mean-field equation (3), we arrive at the mean-field radiation transport equation:

$$(\hat{\boldsymbol{s}}\cdot\nabla)\,\overline{I}=-\kappa_{\rm eff}\,\left(\overline{I}-I_{\rm b}^{\rm eff}\right). \tag{13}$$

Here, the effective absorption coefficient $\kappa_{\rm eff}$ is given by

$$\kappa_{\rm eff}=\overline{\kappa}\left(1-\frac{2\overline{\kappa}\,J_1}{1+2\overline{\kappa}\,J_2}\right), \tag{14}$$

and the effective radiation intensity is

$$I_{\rm b}^{\rm eff}=\overline{I}_{\rm b}+\frac{\langle\kappa'\,I_{\rm b}'\rangle}{\kappa_{\rm eff}}. \tag{15}$$

The function $\bar{I}_{\rm b}$ is expanded in the Taylor series as

$$\bar{I}_{\rm b}=\left[I_{\rm b}+\frac{\langle\theta^2\rangle}{2}\,\frac{\partial^2 I_{\rm b}}{\partial T^2}\right]_{T=\overline{T}}, \tag{16}$$

where the fluid temperature is decomposed into the mean $\overline{T}$ and fluctuating θ parts: $T=\overline{T}+\theta$. Solution of the mean-field radiation transport equation (13) for the mean irradiation intensity $\overline{I}(\boldsymbol{r},\hat{\boldsymbol{s}},\omega)$ is given by

$$\bar{I}(\boldsymbol{r},\hat{\boldsymbol{s}},\omega)=\int_{-\infty}^{\infty}I_{\rm b}^{\rm eff}(\boldsymbol{r}',\omega)\,\exp\left[-\left|\tau(\boldsymbol{r},\boldsymbol{r}',\hat{\boldsymbol{s}})\right|\right]\,\hat{\boldsymbol{s}}\cdot\mathrm{d}\boldsymbol{r}', \tag{17}$$

where $\tau(\boldsymbol{r},\boldsymbol{r}',\hat{\boldsymbol{s}})=\int_{\boldsymbol{r}}^{\boldsymbol{r}'}\kappa_{\rm eff}(\boldsymbol{r}'')\,\hat{\boldsymbol{s}}\cdot\mathrm{d}\boldsymbol{r}''$ is the optical depth.

4 EFFECTIVE PENETRATION LENGTH OF RADIATION AND FLUCTUATIONS OF ABSORPTION COEFFICIENT

In this section, we determine the effective penetration length of radiation in turbulent flows, defined as $L_{\rm eff}=\kappa_{\rm eff}^{-1}$. As the main contribution to the second moment $\langle\kappa'(s)\kappa'(s')\rangle$ of fluctuations of the absorption coefficient is from the integral scale of turbulence ℓ_0, we assume that this correlation function has the form:

$$\langle\kappa'(s)\kappa'(s')\rangle=\left\langle\kappa'^2\right\rangle\,\exp\left(-\frac{|s-s'|}{\ell_0}\right). \tag{18}$$

Using equations (11), (12) and (18), we calculate the integrals J_1 and J_2 as

$$J_1=J_2=\frac{\left\langle\kappa'^2\right\rangle}{\overline{\kappa}^3}\left(1+\frac{L_r}{\ell_0}\right)^{-1}, \tag{19}$$

where $L_r=\overline{\kappa}^{\,-1}$ characterizes the mean penetration length of radiation in a turbulent flow. Therefore, equations (14) and (19) allow us to determine the effective penetration length $L_{\rm eff}=\kappa_{\rm eff}^{-1}$ of radiation in a turbulent flow as

$$L_{\rm eff}=L_r\left[1+\frac{2\left\langle\kappa'^2\right\rangle}{\overline{\kappa}^2}\left(1+\frac{L_r}{\ell_0}\right)^{-1}\right]. \tag{20}$$

We consider two limiting cases:

(i) $\ell_0\ll L_r$, the effective penetration length $L_{\rm eff}$ is

$$L_{\rm eff}=L_r\left(1+\frac{2\left\langle\kappa'^2\right\rangle}{\overline{\kappa}^2}\,\frac{\ell_0}{L_r}\right); \tag{21}$$

(ii) $\ell_0\gg L_r$, the effective penetration length $L_{\rm eff}$ is

$$L_{\rm eff}=L_r\left(1+\frac{2\left\langle\kappa'^2\right\rangle}{\overline{\kappa}^2}\right). \tag{22}$$

Equation (22) implies that for $\ell_0\gg L_r$, the effective penetration length $L_{\rm eff}$ can increase three times in comparison with the mean penetration length L_r of radiation due to the turbulence effects when $\left\langle\kappa'^2\right\rangle\sim\overline{\kappa}^2$.

The mechanism of increase of the effective penetration length $L_{\rm eff}$ in turbulent flows is related to the correlation between fluctuations of the radiation absorption coefficient κ' and fluctuations of the irradiation intensity I'. The correlation $\langle\kappa'\,I'\rangle$ is negative because an increase of the absorption of radiation decreases the radiation intensity, and vice versa. Fluctuations of the radiation absorption coefficient are caused by fluctuations of fluid temperature and density in the turbulent flow.

Now we determine fluctuations of the radiation absorption coefficient. For simplicity, we assume that the opacity of gas is $\kappa_{\rm op}=\kappa_0\,\rho^a\,T^b$, so that the absorption coefficient of gas is $\kappa=\rho\,\kappa_{\rm op}=\kappa_0\,\rho^{a+1}\,T^b$. According to the Schwarzschild stability criterion, the case $a=1$ and $b=0$ corresponds to the marginally stable regime, while the case $a=1$ and $b=1$ corresponds to the unstable regime (Barekat & Brandenburg 2014). The equation $\kappa=\kappa_0\,\rho^{a+1}\,T^b$ allows us to determine the ratio of fluctuations of the absorption coefficient κ' to the mean value of $\overline{\kappa}$ as

$$\frac{\kappa'}{\overline{\kappa}}=(a+1)\,\frac{\rho'}{\overline{\rho}}+b\,\frac{\theta}{\overline{T}}, \tag{23}$$

where ρ' are density fluctuations and $\overline{\rho}$ is the mean fluid density.

Using equation (23), we determine the correlation function $\langle\kappa'\,I_{\rm b}'\rangle$ as

$$\langle\kappa'\,I_{\rm b}'\rangle=\overline{\kappa}\left[(a+1)\,\frac{\langle\rho'\theta\rangle}{\overline{\rho}}+b\,\frac{\langle\theta^2\rangle}{\overline{T}}\right]\left(\frac{\partial I_{\rm b}}{\partial T}\right)_{T=\overline{T}}, \tag{24}$$

where we take into account that $I_{\rm b}' = \theta\,(\partial I_{\rm b}/\partial T)_{T=\overline{T}}$. To find the effective penetration length of radiation, we determine the level of fluctuations $\langle \kappa'^2 \rangle$ of the absorption coefficient as

$$\frac{\langle \kappa'^2 \rangle}{\overline{\kappa}^2} = (a+1)^2\,\frac{\langle \rho'^2 \rangle}{\overline{\rho}^2} + b^2\,\frac{\langle \theta^2 \rangle}{\overline{T}^2} + 2b\,(a+1)\,\frac{\langle \theta\,\rho' \rangle}{\overline{\rho}\,\overline{T}}. \tag{25}$$

The intensity of temperature fluctuations in a developed compressible turbulence for large Péclet and Reynolds numbers is given by

$$\frac{\langle \theta^2 \rangle}{\overline{T}^2} = 8\,f_{\rm c}\,(\gamma-1)^2 \left(\frac{\sigma_{\rm c}}{1+\sigma_{\rm c}}\right)^3 \left[1-\frac{(\lambda\,\ell_0)^2}{9}\right] + \frac{8}{9}\,\ell_0^2 \left[\frac{\boldsymbol{\nabla}\overline{T}}{\overline{T}} + (\gamma-1)\,\boldsymbol{\lambda}\right]^2, \tag{26}$$

(see Appendix A), where $\gamma = c_{\rm p}/c_{\rm v}$ is the ratio of specific heats, $\boldsymbol{\lambda} = -\boldsymbol{\nabla}\ln\overline{\rho}$ characterizes the inhomogeneity of the mean fluid density, the parameter

$$\sigma_{\rm c} = \frac{\langle (\boldsymbol{\nabla}\cdot\boldsymbol{u})^2 \rangle}{\langle (\boldsymbol{\nabla}\times\boldsymbol{u})^2 \rangle} \tag{27}$$

is the degree of compressibility of the turbulent velocity field $\boldsymbol{u}$ and ℓ_0 is the integral scale of turbulence. The function $f_{\rm c}(q, q_{\rm c}, \sigma_{\rm c})$ depends on the degree of compressibility and the exponents of spectra q and $q_{\rm c}$ for the incompressible and compressible parts of velocity fluctuations (see Appendix A):

$$f_{\rm c} = \frac{q_{\rm c}-1}{3q_{\rm c}-5} + \frac{2(q_{\rm c}-1)}{\sigma_{\rm c}(q+2q_{\rm c}-5)} + \frac{q_{\rm c}-1}{\sigma_{\rm c}^2(2q+q_{\rm c}-5)}. \tag{28}$$

Equation (26) is different from that derived by Rogachevskii & Kleeorin (2021). In this study, we take into account a strong density stratification. The latter is important in view of applications to the solar convective zone, where the fluid density in the radial direction is changed by seven orders of magnitude. We also neglect the gradient of the turbulent diffusion that is changed very slowly in the solar convective zone. The first term on the right-hand side of equation (26) determines a compressibility contribution of velocity fluctuations to temperature fluctuations, while the second term in equation (26) is proportional to the squared gradient of the mean entropy.

The level of fluid density fluctuations in a developed compressible turbulence for large Péclet and Reynolds numbers is given by

$$\frac{\langle \rho'^2 \rangle}{\overline{\rho}^2} = 8\,f_{\rm c} \left(\frac{\sigma_{\rm c}}{1+\sigma_{\rm c}}\right)^3 \left[1-\frac{(\lambda\,\ell_0)^2}{9}\right], \tag{29}$$

and the cross-correlation $\langle \theta\,\rho' \rangle$ is

$$\frac{\langle \theta\,\rho' \rangle}{\overline{\rho}\,\overline{T}} = 8\,f_{\rm c}\,(\gamma-1) \left(\frac{\sigma_{\rm c}}{1+\sigma_{\rm c}}\right)^3 \left[1-\frac{(\lambda\,\ell_0)^2}{9}\right]. \tag{30}$$

Equations (26), (29) and (30) are valid for small $\sigma_{\rm c} < 1$. The latter condition is typical for developed turbulence and turbulent convection. Derivation of equations (26), (29) and (30) is given in Appendix A.

Therefore, the level of fluctuations $\langle \kappa'^2 \rangle$ of the absorption coefficient is

$$\frac{\langle \kappa'^2 \rangle}{\overline{\kappa}^2} = 8\,f_{\rm c} \left(\frac{\sigma_{\rm c}}{1+\sigma_{\rm c}}\right)^3 \left[1-\frac{(\lambda\,\ell_0)^2}{9}\right] \left[a+1+b\,(\gamma-1)\right]^2 + \frac{8}{9}\,\ell_0^2\,b^2 \left[\frac{\boldsymbol{\nabla}\overline{T}}{\overline{T}} + (\gamma-1)\,\boldsymbol{\lambda}\right]^2. \tag{31}$$

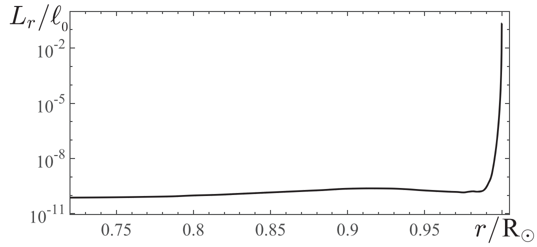

Figure 1. The radial profile of the ratio L_r/ℓ_0 for the solar convective zone.

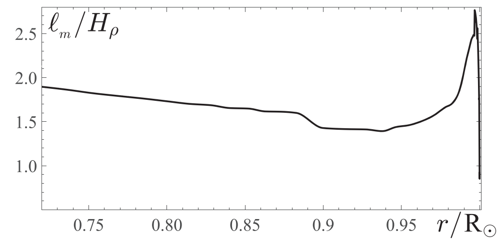

Figure 2. The profile of the ratio $\ell_{\rm m}/H_\rho$ of the mixing length $\ell_{\rm m}$ to the density stratification length H_ρ versus $r/R_\odot$ that is based on the model of the solar convective zone.

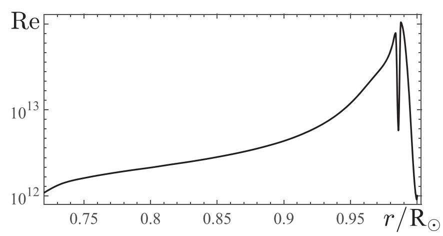

Figure 3. The radial profile of the Reynolds number $\mathrm{Re}(r) = u_0\,\ell_0/\nu$ for the solar convective zone.

For $a = 1$ and $b = 1$, the level of fluctuations $\langle \kappa'^2 \rangle$ of the absorption coefficient is given by

$$\frac{\langle \kappa'^2 \rangle}{\overline{\kappa}^2} = 8\,f_{\rm c} \left(\frac{\sigma_{\rm c}}{1+\sigma_{\rm c}}\right)^3 \left[1-\frac{(\lambda\,\ell_0)^2}{9}\right] (\gamma+1)^2 + \frac{8}{9}\,\ell_0^2\,b^2 \left[\frac{\boldsymbol{\nabla}\overline{T}}{\overline{T}} + (\gamma-1)\,\boldsymbol{\lambda}\right]^2. \tag{32}$$

Note that for nearly isentropic flows where $\boldsymbol{\nabla}\ln\overline{T} \approx (\gamma-1)\,\boldsymbol{\nabla}\ln\overline{\rho}$, the last term in equations (31) and (32) is small. This term is proportional to the gradient of the mean entropy $\boldsymbol{\nabla}\overline{S} = c_{\rm v}[\boldsymbol{\nabla}\ln\overline{T} - (\gamma-1)\,\boldsymbol{\nabla}\ln\overline{\rho}]$.

5 APPLICATION TO THE SOLAR CONVECTIVE ZONE

In this section, we apply the obtained results to the solar convective zone. We use the model of the solar convective zone by Spruit (1974) based on mixing length theory. According to this model, the mean penetration length of radiation is much less than the turbulent correlation length. Indeed, in Figs 1–3 we show the radial profiles of the ratio of the mean penetration length of radiation to the integral scale of turbulence $L_r(r)/\ell_0$, the ratio $\ell_{\rm m}/H_\rho$ of the mixing length $\ell_{\rm m}$ to the density stratification length $H_\rho = \lambda^{-1}$ and the Reynolds number $\mathrm{Re}(r) = u_0\,\ell_0/\nu$ for the solar convective zone based on the

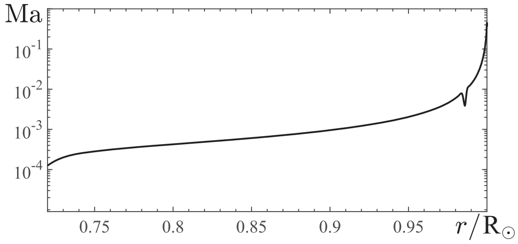

Figure 4. The radial profile of the Mach number $\mathrm{Ma}(r) = u_0/c_\mathrm{s}$ for the solar convective zone.

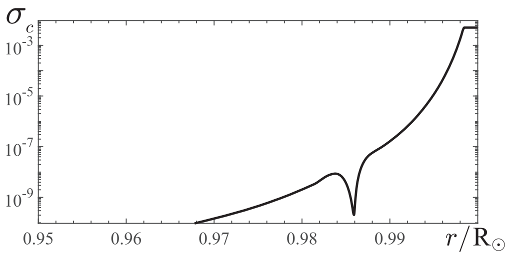

Figure 5. The radial profile of the degree of compressibility σ_c for the solar convective zone.

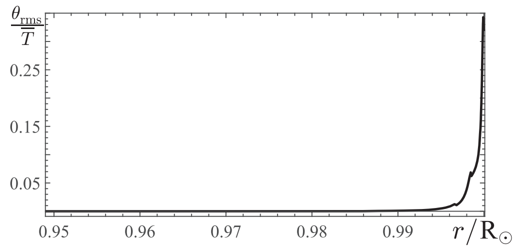

Figure 6. The radial profile of the rms of temperature fluctuations θ_rms measured in the units of mean temperature $\overline{T}$ for the solar convective zone.

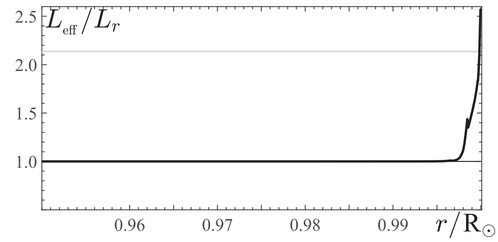

Figure 7. The radial profile of the ratio L_eff/L_r for $a = b = 1$ for the solar convective zone.

model by Spruit (1974). The radius r is measured in units of the solar radius $\mathrm{R}_\odot$. The mixing length ℓ_m is identified by the size of the solar granulations, while the ratio $\ell_\mathrm{m}/\ell_0 = 5{-}7$ is justified by the results of analytical study (Elperin et al. 2002, 2006) and laboratory experiments (Bukai et al. 2009), which show that the integral scale ℓ_0 of the turbulent convection is five to seven times smaller in comparison with the size of the large-scale circulations. These turbulent parameters increase towards the solar surface.

To determine the effective penetration length of radiation, we estimate the degree of compressibility of the turbulent fluid flow for small Mach numbers as (Rogachevskii & Kleeorin 2021)

$$\sigma_\mathrm{c} \sim \mathrm{Ma}^5\, \mathrm{Re}^{1/4}, \tag{33}$$

where $\mathrm{Ma} = u_0/c_\mathrm{s}$ is the Mach number, $u_0 = \langle \boldsymbol{u}^2 \rangle^{1/2}$, $c_\mathrm{s} = (\gamma\, \overline{P}/\overline{\rho})^{1/2}$ is the sound speed, $\mathrm{Re} = u_0\, \ell_0/\nu$ is the Reynolds number and ν is the kinematic viscosity. The estimate (33) is obtained assuming that the effect of compressibility on the viscous heating $\overline{J}_\nu^{(\mathrm{c})}$ is of the order of the radiative wave energy density E_w. In particular, turbulence can generate acoustic waves, and the rate of the energy radiated by the acoustic waves per unit mass for small Mach numbers is given by (Lighthill 1952, 1954; Proudman 1952)

$$E_\mathrm{w} = \frac{\langle \boldsymbol{u}^2 \rangle}{\tau_0}\, \mathrm{Ma}^5, \tag{34}$$

where $\tau_0 = \ell_0/u_0$ is the turbulent correlation time. The compressibility contribution $\overline{J}_\nu^{(\mathrm{c})}$ to the rate of the viscous heating is (Rogachevskii & Kleeorin 2021)

$$\overline{J}_\nu^{(\mathrm{c})} = \frac{\langle \boldsymbol{u}^2 \rangle}{\tau_0}\, \frac{\sigma_\mathrm{c}}{1+\sigma_\mathrm{c}}\, \mathrm{Re}^{-1/4}. \tag{35}$$

Equations (34) and (35) yield the estimate (33) for the degree of compressibility σ_c for small Mach numbers.

In Figs 4 and 5, we show the radial profiles of the Mach number $\mathrm{Ma}(r) = u_0/c_\mathrm{s}$ and the degree of compressibility σ_c of the fluid velocity field for the solar convective zone based on the model by Spruit (1974). The degree of compressibility σ_c increases to the surface because the decrease of the sound speed in the vicinity of the solar surface. In Fig. 6, we plot the radial profile of the rms of temperature fluctuations θ_rms measured in units of the mean temperature $\overline{T}$ using the above parameters for the solar convective zone. Temperature fluctuations increase towards to the solar surface due to the compressibility effects.

Similar behaviour is observed for fluid density fluctuations $\langle \rho'^2 \rangle$ and the temperature–density correlations $\langle \theta\, \rho' \rangle$ (see equations 29 and 30). In particular, these second moments enhance towards to the solar surface, resulting in an increase of fluctuations of the radiation absorption coefficient and the effective penetration length of radiation. This is seen in Fig. 7, where we show the radial profile of the ratio of the turbulence-induced effective penetration length of radiation to the mean radiation penetration length L_eff/L_r for the solar convective zone based on the model by Spruit (1974). The ratio L_eff/L_r increases 2.5 times in the vicinity of the solar surface.

6 CONCLUSIONS

We study a turbulent transport of radiation in the solar convective zone. To this end, we derive a mean-field equation for the irradiation intensity and show that, due to the turbulent effects, the effective penetration length of radiation is increased several times in comparison with the mean penetration length of radiation, which is defined as an inverse mean absorption coefficient. To demonstrate this effect, we adopt a model of the solar convective zone based on mixing length theory. The mean penetration length of radiation in this model is much smaller than the turbulent integral scale. We have shown that the ratio of the effective penetration length of radiation to the mean penetration length of radiation is increased 2.5 times in the vicinity of the solar surface.

This effect can be explained by the compressibility effects that become important in the vicinity of the solar surface, so that the level of temperature and density fluctuations is increased towards the solar surface. This causes an increase of fluctuations of the radiation absorption coefficient and the effective penetration length of radiation. Because the effective penetration length of radiation is changed only in the vicinity of the solar surface (at the depth

$\sim$2000 km), the effect of turbulence on the radiation transport is not strong for solar-type stars. However, this effect can be essential for cold stars (such as M3–M5 stars), for which the Mach number is larger than that for the Sun.

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DATA AVAILABILITY

There are no new data associated with this article.

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APPENDIX A: TEMPERATURE AND DENSITY FLUCTUATIONS

In this appendix, we derive the expression for the level of temperature fluctuations $\langle\theta^2\rangle$, density fluctuations $\left\langle\rho'^2\right\rangle$ and temperature–density correlations $\left\langle\theta\,\rho'\right\rangle$ using the method described by Rogachevskii, Kleeorin & Brandenburg (2018), Rogachevskii & Kleeorin (2021) and Rogachevskii (2021). The temperature field $T(t, \boldsymbol{r})$ in a compressible fluid velocity field $\mathbf{U}(t, \boldsymbol{r})$ is described by (Landau & Lifshits 1987)

$$\frac{\partial T}{\partial t} + (\boldsymbol{U}\cdot\nabla)T + (\gamma-1)T(\nabla\cdot\boldsymbol{U}) = D\Delta T + J_\nu, \tag{A1}$$

where D is the molecular thermal conductivity, $\gamma = c_{\rm p}/c_{\rm v}$ is the ratio of specific heats and J_ν is the heating source due to viscous dissipation.

We study turbulent flows with large Reynolds ($\mathrm{Re} = u_0\,\ell_0/\nu \gg 1$) and Péclet ($\mathrm{Pe} = u_0\,\ell_0/D \gg 1$) numbers, where u_0 is the characteristic turbulent velocity in the integral scale ℓ_0 of turbulence. Equations for the intensity of temperature fluctuations are derived by means of the mean-field approach, where the temperature $T = \overline{T} + \theta$, pressure $P = \overline{P} + p$, density $\rho = \overline{\rho} + \rho'$ and velocity $\boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u}$ are decomposed into mean and fluctuating parts, where $\overline{T} = \langle T\rangle$ is the mean fluid temperature, $\overline{P} = \langle P\rangle$ is the mean fluid pressure, $\overline{\rho} = \langle\rho\rangle$ is the mean fluid density and $\overline{\boldsymbol{U}} = \langle\boldsymbol{U}\rangle$ is the mean fluid velocity. Here, θ, p, ρ' and $\boldsymbol{u}$ are fluctuations of temperature, pressure, density and velocity, respectively, and the angular brackets denote an ensemble averaging. Application of the mean-field approach implies that there is a separation of spatial ($\ell_0 \ll H_T$) and temporal ($\tau_0 \ll t_T$) scales, where H_T and t_T are the characteristic spatial and temporal scales characterizing the variations of the mean temperature field, and $\tau_0 = \ell_0/u_0$.

Ensemble averaging of equation (A1) yields the mean temperature field,

$$\frac{\partial\overline{T}}{\partial t} + \nabla\boldsymbol{\cdot}\langle\theta\,\boldsymbol{u}\rangle = -(\gamma-2)\,\langle\theta\,(\nabla\cdot\boldsymbol{u})\rangle + D\,\Delta\overline{T} + \overline{J}_\nu, \tag{A2}$$

where $\langle\theta\,\boldsymbol{u}\rangle$ is the turbulent heat flux, and $\overline{J}_\nu$ is the mean heating source caused by the viscous dissipation of the turbulent kinetic energy. Here the case $\overline{\boldsymbol{U}} = 0$ is studied for simplicity. By means of equations (A1) and (A2), we obtain the equation for temperature fluctuations, $\theta(\boldsymbol{x}, t) = T - \overline{T}$,

$$\frac{\partial\theta}{\partial t} + \mathcal{Q} - D\Delta\theta = -(\boldsymbol{u}\cdot\nabla)\overline{T} - (\gamma-1)\,\overline{T}\,\nabla\cdot\boldsymbol{u}, \tag{A3}$$

where $\mathcal{Q} = \nabla\cdot(\theta\boldsymbol{u} - \langle\boldsymbol{u}\,\theta\rangle) + (\gamma-2)\,(\theta\,\nabla\cdot\boldsymbol{u} - \langle\theta\,\nabla\cdot\boldsymbol{u}\rangle)$ is the non-linear term and temperature fluctuations are caused by the source $-(\boldsymbol{u}\cdot\nabla)\overline{T} - (\gamma-1)\,\overline{T}\,\nabla\cdot\boldsymbol{u}$. For simplicity, we describe the effect of turbulence on the temperature field, and neglect the feedback effect of the temperature on the turbulence.

We use two-point second-order correlation functions taking into account small-scale properties of the turbulence, where the turbulent correlation time and the turbulent kinetic energy spectrum are related via the Kolmogorov scalings (Monin & Yaglom 1971, 1975; McComb 1990; Frisch 1995). We adopt the multiscale approach (Roberts & Soward 1975), and rewrite the two-point second-order correlation functions as

$$\langle\theta(\boldsymbol{x},t)\,\theta(\boldsymbol{y},t)\rangle = \int \mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\langle\theta(\boldsymbol{k}_1,t)\theta(\boldsymbol{k}_2,t)\rangle \times\exp\left[\mathrm{i}(\boldsymbol{k}_1\cdot\boldsymbol{x}+\boldsymbol{k}_2\cdot\boldsymbol{y})\right] = \int \Theta^{(\mathrm{II})}(\boldsymbol{k},\,\boldsymbol{R},\,t)\,\exp[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}]\,\mathrm{d}\boldsymbol{k}, \tag{A4}$$

where

$$\Theta^{(\mathrm{II})}(\boldsymbol{k},\,\boldsymbol{R},\,t) = \int \langle\theta(\boldsymbol{k}_1,t)\,\theta(\boldsymbol{k}_2,t)\rangle\,\exp[\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{R}]\,\mathrm{d}\boldsymbol{K}. \tag{A5}$$

Also, we use large-scale variables, $\boldsymbol{R} = (\boldsymbol{x}+\boldsymbol{y})/2$, $\boldsymbol{K} = \boldsymbol{k}_1 + \boldsymbol{k}_2$, as well as small-scale variables, $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$, $\boldsymbol{k} = (\boldsymbol{k}_1 - \boldsymbol{k}_2)/2$. Here $\boldsymbol{k}_1 = \boldsymbol{k} + \boldsymbol{K}/2$ and $\boldsymbol{k}_2 = -\boldsymbol{k} + \boldsymbol{K}/2$. Mean fields depend on the large-scale variables, while fluctuations depend on the small-scale variables.

The procedure of the derivations of the expressions for the intensity of temperature fluctuations implies the following steps:

(i) derivation of equations for the second-order moments in the Fourier space using the multiscale approach;

(ii) application of the spectral τ approach (see below), which relates the deviations of the third-order moments from those of the background turbulence with the corresponding deviations of the second-order moments;

(iii) solution of the obtained equations for the second-order moments in the Fourier space;

(iv) inverse transformation to the physical space to derive expressions for the intensity of temperature fluctuations.

Using equation (A3) for temperature fluctuations θ and the Navier–Stokes equation for velocity fluctuations $\boldsymbol{u}$ rewritten in Fourier space, we derive an equation for the second-order moment $\langle\theta(\boldsymbol{k}_1)\,\theta(\boldsymbol{k}_2)\rangle$ as

$$\frac{\partial}{\partial t}\,\langle\theta(\boldsymbol{k}_1)\,\theta(\boldsymbol{k}_2)\rangle = -\Big[\,\langle u_i(\boldsymbol{k}_1)\,\theta(\boldsymbol{k}_2)\rangle + \langle\theta(\boldsymbol{k}_1)\,u_i(\boldsymbol{k}_2)\rangle\,\Big]\,\nabla_i\overline{T} -(\gamma-1)\Big[\big\langle(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,\theta(\boldsymbol{k}_2)\big\rangle + \big\langle\theta(\boldsymbol{k}_1)\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\big\rangle\Big]\overline{T} + \hat{\mathcal{M}}\Theta^{(\mathrm{III})}, \tag{A6}$$

where $\hat{\mathcal{M}}\Theta^{(\mathrm{III})}$ are the third-order moment terms caused by the non-linear terms in the equation for temperature fluctuations. Temperature and velocity fluctuations $\theta(\boldsymbol{k}_{1,2},t)$ and $u_i(\boldsymbol{k}_{1,2},t)$ depend also on t, and the mean temperature $\overline{T}(t,\boldsymbol{R})$ depends on t and $\boldsymbol{R}$ as well. For brevity of notations, we do not show these dependences hereafter.

Equation (A6) for the second-order moments includes the third-order moments $\hat{\mathcal{M}}\Theta^{(\mathrm{III})}$, and the closure problem arises, that is, how to express the third-order moments $\hat{\mathcal{M}}\Theta^{(\mathrm{III})}$ through the lower-order moments (Monin & Yaglom 1971, 1975; McComb 1990). We adopt the spectral τ approach, which postulates that the deviations of the third-moment terms, $\hat{\mathcal{M}}\Theta^{(\mathrm{III})}(\boldsymbol{k})$, from those afforded by the background turbulence, $\hat{\mathcal{M}}\Theta^{(\mathrm{III},0)}(\boldsymbol{k})$, can be expressed through similar deviations of the second-order moments, $\Theta^{(\mathrm{II})}(\boldsymbol{k}) - \Theta^{(\mathrm{II},0)}(\boldsymbol{k})$ as (Orszag 1970; Pouquet, Frisch & Leorat 1976; Kleeorin, Rogachevskii & Ruzmaikin 1990)

$$\hat{\mathcal{M}}\Theta^{(\mathrm{III})}(\boldsymbol{k}) - \hat{\mathcal{M}}\Theta^{(\mathrm{III},0)}(\boldsymbol{k}) = -\frac{\Theta^{(\mathrm{II})}(\boldsymbol{k}) - \Theta^{(\mathrm{II},0)}(\boldsymbol{k})}{\tau_r(k)}. \tag{A7}$$

Here $\tau_r(k)$ is the scale-dependent relaxation time, which can be identified with the correlation time $\tau(k)$ of the turbulent velocity field for large Reynolds and Péclet numbers. Because the functions with superscript (0) describe the background turbulence with a zero turbulent heat flux, equation (A7) is reduced to $\hat{\mathcal{M}}\Theta^{(\mathrm{III})} = -\langle\theta(\boldsymbol{k}_1)\,\theta(\boldsymbol{k}_2)\rangle\,/\tau(k)$. We apply the τ approximation only for the deviations from the background turbulence, while the background turbulence is assumed to be known (see below). Validation of the τ approximation for different problems has been performed in various numerical simulations (Brandenburg, Käpylä & Mohammed 2004; Brandenburg & Subramanian 2005a, b, c; Brandenburg et al. 2008; Rogachevskii et al. 2011, 2018; Rädler et al. 2011; Haugen et al. 2012; Elperin et al. 2017).

Because the characteristic times of variation of the second-order moment $\Theta^{(\mathrm{II})}$ are much larger than the correlation time $\tau(k)$ in all turbulence scales, we use the steady-state solution of equation (A6) as

$$\langle\theta(\boldsymbol{k}_1)\,\theta(\boldsymbol{k}_2)\rangle = -\tau(k)\bigg\{\Big[\,\langle u_i(\boldsymbol{k}_1)\,\theta(\boldsymbol{k}_2)\rangle + \langle\theta(\boldsymbol{k}_1)\,u_i(\boldsymbol{k}_2)\rangle\,\Big]\nabla_i\overline{T} + (\gamma-1)\Big[\big\langle(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,\theta(\boldsymbol{k}_2)\big\rangle + \big\langle\theta(\boldsymbol{k}_1)\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\big\rangle\Big]\overline{T}\bigg\}. \tag{A8}$$

Similarly, we derive expression for the second moments entering in equation (A8), i.e. for $\langle u_i(\boldsymbol{k}_1)\,\theta(\boldsymbol{k}_2)\rangle$ and $\big\langle\theta(\boldsymbol{k}_1)\,u_j(\boldsymbol{k}_2)\big\rangle$, as

$$\langle u_i(\boldsymbol{k}_1)\,\theta(\boldsymbol{k}_2)\rangle = -\tau(k)\bigg[\big\langle u_i(\boldsymbol{k}_1)\,u_j(\boldsymbol{k}_2)\big\rangle\;\nabla_j\overline{T} +(\gamma-1)\;\big\langle u_i(\boldsymbol{k}_1)\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\big\rangle\,\overline{T}\bigg], \tag{A9}$$

$$\big\langle\theta(\boldsymbol{k}_1)\,u_j(\boldsymbol{k}_2)\big\rangle = -\tau(k)\bigg[\big\langle u_i(\boldsymbol{k}_1)\,u_j(\boldsymbol{k}_2)\big\rangle\;\nabla_i\overline{T} +(\gamma-1)\;\big\langle(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,u_j(\boldsymbol{k}_2)\big\rangle\,\overline{T}\bigg], \tag{A10}$$

and for the second moments $\big\langle(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,\theta(\boldsymbol{k}_2)\big\rangle$ and $\big\langle\theta(\boldsymbol{k}_1)\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\big\rangle$ as

$$\big\langle(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,\theta(\boldsymbol{k}_2)\big\rangle = -\tau(k)\bigg[\big\langle(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,u_j(\boldsymbol{k}_2)\big\rangle\;\nabla_j\overline{T} +(\gamma-1)\;\big\langle(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\big\rangle\,\overline{T}\bigg], \tag{A11}$$

$$\big\langle\theta(\boldsymbol{k}_1)\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\big\rangle = -\tau(k)\bigg[\big\langle u_i(\boldsymbol{k}_1)\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\big\rangle\;\nabla_i\overline{T} +(\gamma-1)\;\big\langle(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\big\rangle\,\overline{T}\bigg]. \tag{A12}$$

Substituting equations (A9)–(A12) into equation (A8), we obtain

$$\langle\theta(\boldsymbol{k}_1)\,\theta(\boldsymbol{k}_2)\rangle = 2\tau^2(k)\bigg\{\big\langle u_i(\boldsymbol{k}_1)\,u_j(\boldsymbol{k}_2)\big\rangle\;(\nabla_i\overline{T})(\nabla_j\overline{T}) +(\gamma-1)\Big[\big\langle(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,u_j(\boldsymbol{k}_2)\big\rangle + \big\langle u_j(\boldsymbol{k}_1)\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\big\rangle\Big] \times\overline{T}\,\nabla_j\overline{T} + (\gamma-1)^2\;\big\langle(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\big\rangle\,\overline{T}^2\bigg\}. \tag{A13}$$

Because all terms in equations (A8) and (A13) are proportional to either $(\nabla \overline{T})^2$, or $(\nabla \overline{\rho})^2$, or $(\nabla \overline{T})(\nabla \overline{\rho})$ (see below), and we consider homogeneous density stratified turbulence, we do not need to perform additional Taylor expansions over small parameters ℓ_0/H_T and ℓ_0/H_ρ in these terms (Rogachevskii & Kleeorin 2021; Rogachevskii 2021), where H_T and H_ρ are the characteristic scales of variations of the mean temperature and mean density, respectively. In particular, hereafter we neglect small terms $\sim O[(\ell_0/H_T)^3,\ (\ell_0/H_\rho)^3]$. This implies that we replace $\boldsymbol{k}_1$ by $\boldsymbol{k}$ and $\boldsymbol{k}_2$ by $-\boldsymbol{k}$ in all second moments in equation (A13).

In equation (A13) we take into account a one-way coupling (i.e. we neglect the feedback effect of the mean temperature gradients on the turbulent velocity field). This implies that we replace the correlation function $f_{ij} = \left\langle u_i(\boldsymbol{k})\, u_j(-\boldsymbol{k})\right\rangle$ in equation (A13) by $f_{ij}^{(0)} = \left\langle u_i(\boldsymbol{k})\, u_j(-\boldsymbol{k})\right\rangle^{(0)}$ for the background turbulence with a zero turbulent heat flux. Similarly, we replace $\left\langle (\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}}\, u_j(-\boldsymbol{k})\right\rangle$, $\left\langle u_j(\boldsymbol{k})\, (\mathrm{div}\boldsymbol{u})_{-\boldsymbol{k}}\right\rangle$ and $\langle (\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}}\, (\mathrm{div}\boldsymbol{u})_{-\boldsymbol{k}}\rangle$ in equation (A13) by the corresponding correlation functions for the background turbulence with a zero turbulent heat flux.

To find the intensity of temperature fluctuations $\langle\theta^2\rangle$ for large Péclet numbers, we adopt a model for the background turbulence, $f_{ij}^{(0)}(\boldsymbol{k}) = \left\langle u_i(\boldsymbol{k})\, u_j(-\boldsymbol{k})\right\rangle^{(0)}$, that is a statistically stationary density-stratified compressible turbulence given by (Elperin, Kleeorin & Rogachevskii 1995; Amir et al. 2017; Rogachevskii 2021):

$$
\begin{aligned}
f_{ij}^{(0)}(\boldsymbol{k}) &= \frac{1}{8\pi k^2\,(1+\sigma_{\rm c})} \\
&\times\Bigg\{ E(k)\left[(\delta_{ij}-k_{ij})\left(1-\frac{\lambda^2}{k^2}\right)+\frac{\lambda^2}{k^2}\,(\delta_{ij}-\lambda_{ij})\right] \\
&+\frac{i}{k^2}\,[E(k)+\sigma_c\,E_{\rm c}(k)]\,\Big(k_j\lambda_i-k_i\lambda_j\Big) \\
&+2\sigma_{\rm c}\,E_{\rm c}(k)\,k_{ij}\Bigg\}\left\langle \boldsymbol{u}^2\right\rangle, \qquad \text{(A14)}
\end{aligned}
$$

where $k_{ij} = k_i\,k_j/k^2$, $\lambda_{ij} = \lambda_i\,\lambda_j/\lambda^2$ and $\boldsymbol{\lambda} = -\boldsymbol{\nabla}\ln\overline{\rho}$ characterizes the fluid density stratification. This model is different from that derived by Rogachevskii et al. (2018) and Rogachevskii & Kleeorin (2021). In particular, this model takes into account a strong density stratification. In addition, the turbulent flux $\langle\rho'\boldsymbol{u}\rangle$ is very small $[\sim O(\lambda\,\ell_0)^3]$.

The background turbulence is of Kolmogorov type with a constant energy flux over the spectrum, that is, the turbulent kinetic energy spectrum for the incompressible part of turbulence in the inertial range $k_0 < k < k_\nu$ is $E(k) = -{\rm d}\tilde{\tau}(k)/{\rm d}k$. Here $\tilde{\tau}(k) = (k/k_0)^{1-q}$, with $1 < q < 3$, is the exponent of the turbulent kinetic energy spectrum. Similarly, the turbulent kinetic energy spectrum for the compressible part of turbulence is $E_{\rm c}(k) = -{\rm d}\tilde{\tau}_{\rm c}(k)/{\rm d}k$, where $\tilde{\tau}_{\rm c}(k) = (k/k_0)^{1-q_{\rm c}}$ with $1 < q_{\rm c} < 3$. For example, the exponent of the incompressible part of the spectrum, $q = 5/3$, corresponds to the Kolmogorov spectrum, while the exponent of the compressible part of the spectrum, $q_{\rm c} = 2$, describes the Burgers turbulence with shock waves. These exponents of the spectra are observed in numerical simulations in compressible turbulence (Kritsuk et al. 2007; Federrath 2013). The correlation time for a compressible turbulence in the Fourier space is (Rogachevskii & Kleeorin 2021)

$$
\tau(k) = \frac{2\tau_0}{1+\sigma_{\rm c}}\left[\tilde{\tau}(k)+\sigma_{\rm c}\,\tilde{\tau}_{\rm c}(k)\right]. \qquad \text{(A15)}
$$

To determine the level of temperature fluctuations $\left\langle\theta^2\right\rangle = \int \tau^2(k)\,\langle\theta(\boldsymbol{k})\,\theta(-\boldsymbol{k})\rangle\,{\rm d}\boldsymbol{k}$ for large Péclet numbers, we use equations (A13)–(A15). To this end, we calculate the following integrals:

$$
\int \tau^2(k)\,\left\langle u_i(\boldsymbol{k})\,u_j(-\boldsymbol{k})\right\rangle^{(0)}\,{\rm d}\boldsymbol{k} = \frac{4}{9}\,\ell_0^2\,\delta_{ij}, \qquad \text{(A16)}
$$

$$
\int \tau^2(k)\,\langle u_i(\boldsymbol{k})\,(\mathrm{div}\,\boldsymbol{u})_{-\boldsymbol{k}}\rangle^{(0)}\,{\rm d}\boldsymbol{k} = \frac{4}{9}\,\ell_0^2\,\lambda_i, \qquad \text{(A17)}
$$

$$
\begin{aligned}
&\int \tau^2(k)\,\langle(\mathrm{div}\,\boldsymbol{u})_{\boldsymbol{k}}\,(\mathrm{div}\,\boldsymbol{u})_{-\boldsymbol{k}}\rangle^{(0)}\,{\rm d}\boldsymbol{k} = \frac{4}{9}\,(\ell_0\,\lambda)^2 \\
&+4\,f_{\rm c}\left(\frac{\sigma_{\rm c}}{1+\sigma_{\rm c}}\right)^3\left[1-\frac{1}{9}\,(\ell_0\,\lambda)^2\right]. \qquad \text{(A18)}
\end{aligned}
$$

For the integration in $\boldsymbol{k}$ space in equations (A16)–(A18), we use the following identities:

$$
\int_{k_0}^{k_\nu} \tau^2(k)\,\left[E(k)+\sigma_{\rm c}\,E_{\rm c}(k)\right]\,{\rm d}k = \frac{4}{3}\,\tau_0^2\,(1+\sigma_{\rm c}), \qquad \text{(A19)}
$$

$$
\int_{k_0}^{k_\nu} \tau^2(k)\,k^2\,E_{\rm c}(k)\,{\rm d}k = 4\,f_{\rm c}\left(\frac{\tau_0}{\ell_0}\right)^2\left(\frac{\sigma_{\rm c}}{1+\sigma_{\rm c}}\right)^2. \qquad \text{(A20)}
$$

Therefore, equations (A13) and (A16)–(A18) yield the level of temperature fluctuations $\langle\theta^2\rangle$ for large Péclet numbers given by equation (26).

Derivation of equations (29) and (30) for $\left\langle\rho'^2\right\rangle$ and $\left\langle\theta\,\rho'\right\rangle$ is performed in a similar way. In particular, using the continuity equation for the fluid density fluctuations ρ' written in the Fourier space, we obtain the evolutionary equation for the second moment $\left\langle\rho'(\boldsymbol{k}_1)\,\rho'(\boldsymbol{k}_2)\right\rangle$ as

$$
\begin{aligned}
\frac{\partial}{\partial t}\left\langle\rho'(\boldsymbol{k}_1)\,\rho'(\boldsymbol{k}_2)\right\rangle &= -\Big[\left\langle u_i(\boldsymbol{k}_1)\,\rho'(\boldsymbol{k}_2)\right\rangle \\
&+\left\langle\rho'(\boldsymbol{k}_1)\,u_i(\boldsymbol{k}_2)\right\rangle\Big]\,\nabla_i\overline{\rho} - \Big[\left\langle(\mathrm{div}\,\boldsymbol{u})_{\boldsymbol{k}_1}\,\rho(\boldsymbol{k}_2)\right\rangle \\
&+\left\langle\rho(\boldsymbol{k}_1)\,(\mathrm{div}\,\boldsymbol{u})_{\boldsymbol{k}_2}\right\rangle\Big]\,\overline{\rho} + \hat{\mathcal{M}}\rho^{(\rm III)}, \qquad \text{(A21)}
\end{aligned}
$$

where $\hat{\mathcal{M}}\rho^{(\rm III)}$ are the third-order moment terms related to non-linear terms in the equation for density fluctuations.

Applying the spectral τ approach, we obtain the expression for the second moment $\left\langle\rho'(\boldsymbol{k}_1)\,\rho'(\boldsymbol{k}_2)\right\rangle$ as

$$
\begin{aligned}
\left\langle\rho'(\boldsymbol{k}_1)\,\rho'(\boldsymbol{k}_2)\right\rangle &= -\tau(k)\Big\{\Big[\left\langle u_i(\boldsymbol{k}_1)\,\rho'(\boldsymbol{k}_2)\right\rangle \\
&+\left\langle\rho'(\boldsymbol{k}_1)\,u_i(\boldsymbol{k}_2)\right\rangle\Big]\nabla_i\overline{\rho} + \Big[\left\langle(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,\rho'(\boldsymbol{k}_2)\right\rangle \\
&+\left\langle\rho'(\boldsymbol{k}_1)\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\right\rangle\Big]\overline{\rho}\Big\}. \qquad \text{(A22)}
\end{aligned}
$$

Similarly, we derive expression for the second moments $\left\langle u_i(\boldsymbol{k}_1)\,\rho'(\boldsymbol{k}_2)\right\rangle$ and $\left\langle\rho'(\boldsymbol{k}_1)\,u_j(\boldsymbol{k}_2)\right\rangle$ as

$$
\begin{aligned}
\left\langle u_i(\boldsymbol{k}_1)\,\rho'(\boldsymbol{k}_2)\right\rangle &= -\tau(k)\Big[\left\langle u_i(\boldsymbol{k}_1)\,u_j(\boldsymbol{k}_2)\right\rangle\,\nabla_j\overline{\rho} \\
&+\left\langle u_i(\boldsymbol{k}_1)\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\right\rangle\,\overline{\rho}\Big], \qquad \text{(A23)}
\end{aligned}
$$

$$
\begin{aligned}
\left\langle\rho'(\boldsymbol{k}_1)\,u_j(\boldsymbol{k}_2)\right\rangle &= -\tau(k)\Big[\left\langle u_i(\boldsymbol{k}_1)\,u_j(\boldsymbol{k}_2)\right\rangle\,\nabla_i\overline{\rho} \\
&+\left\langle(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,u_j(\boldsymbol{k}_2)\right\rangle\,\overline{\rho}\Big], \qquad \text{(A24)}
\end{aligned}
$$

and for $\left\langle(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,\rho'(\boldsymbol{k}_2)\right\rangle$ and $\left\langle\rho'(\boldsymbol{k}_1)\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\right\rangle$ as

$$
\begin{aligned}
\left\langle(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,\rho'(\boldsymbol{k}_2)\right\rangle &= -\tau(k)\Big[\left\langle(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,u_j(\boldsymbol{k}_2)\right\rangle\,\nabla_j\overline{\rho} \\
&+\left\langle(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\right\rangle\,\overline{\rho}\Big], \qquad \text{(A25)}
\end{aligned}
$$

$$\left\langle \rho'(\boldsymbol{k}_1)\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\right\rangle = -\tau(k)\left[\left\langle u_i(\boldsymbol{k}_1)\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\right\rangle \nabla_i\overline{\rho} + \left\langle (\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\right\rangle \overline{\rho}\right]. \tag{A26}$$

Substituting equations (A23)–(A26) into equation (A22) we obtain

$$\left\langle \rho'(\boldsymbol{k}_1)\,\rho'(\boldsymbol{k}_2)\right\rangle = 2\tau^2(k)\Big\{\left\langle u_i(\boldsymbol{k}_1)\,u_j(\boldsymbol{k}_2)\right\rangle (\nabla_i\overline{\rho})(\nabla_j\overline{\rho}) + \Big[\left\langle (\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,u_j(\boldsymbol{k}_2)\right\rangle + \left\langle u_j(\boldsymbol{k}_1)\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\right\rangle\Big]\overline{\rho}\,\nabla_j\overline{\rho} + \left\langle (\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\right\rangle \overline{\rho}^2\Big\}. \tag{A27}$$

We do not take into account small terms $\sim O[(\ell_0/H_\rho)^3]$ in equations (A22) and (A27). As all terms in equations (A22) and (A27) are proportional to $(\boldsymbol{\nabla}\overline{\rho})^2$ and we consider homogeneous density stratified turbulence, we do not need to perform additional Taylor expansions over the small parameter ℓ_0/H_ρ in these terms. Thus, we replace $\boldsymbol{k}_1$ by $\boldsymbol{k}$ and $\boldsymbol{k}_2$ by $-\boldsymbol{k}$ in all second moments in equation (A27). Using equations (A16)–(A18) and (A27), we determine the level of density fluctuations $\left\langle \rho'^2\right\rangle = \int \tau^2(k)\left\langle \rho'(\boldsymbol{k})\,\rho'(-\boldsymbol{k})\right\rangle \mathrm{d}\boldsymbol{k}$ for large Péclet numbers given by equation (29).

Now we derive the evolutionary equation for the second moment $\left\langle \theta(\boldsymbol{k}_1)\,\rho'(\boldsymbol{k}_2)\right\rangle$ using the continuity equation for the fluid density fluctuations and the equation for the temperature fluctuations written in the Fourier space:

$$\frac{\partial}{\partial t}\left\langle \theta(\boldsymbol{k}_1)\,\rho'(\boldsymbol{k}_2)\right\rangle = -\left\langle u_i(\boldsymbol{k}_1)\,\rho'(\boldsymbol{k}_2)\right\rangle \nabla_i\overline{T} - \left\langle \theta(\boldsymbol{k}_1)\,u_j(\boldsymbol{k}_2)\right\rangle \nabla_j\overline{\rho} - (\gamma-1)\left\langle (\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,\rho'(\boldsymbol{k}_2)\right\rangle \overline{T} - \left\langle \theta(\boldsymbol{k}_1)\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\right\rangle \overline{\rho} + \hat{\mathcal{M}}\Theta_\rho^{(\mathrm{III})}, \tag{A28}$$

where $\hat{\mathcal{M}}\Theta_\rho^{(\mathrm{III})}$ are the third-order moment terms related to the nonlinear terms in the equations for temperature and density fluctuations. Applying the spectral τ approach, we obtain the expression for the second moment $\left\langle \theta(\boldsymbol{k}_1)\,\rho'(\boldsymbol{k}_2)\right\rangle$ as

$$\left\langle \theta(\boldsymbol{k}_1)\,\rho'(\boldsymbol{k}_2)\right\rangle = -\tau(k)\Big[\left\langle u_i(\boldsymbol{k}_1)\,\rho'(\boldsymbol{k}_2)\right\rangle \nabla_i\overline{T} + \left\langle \theta(\boldsymbol{k}_1)\,u_j(\boldsymbol{k}_2)\right\rangle \nabla_j\overline{\rho} + (\gamma-1)\left\langle (\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,\rho'(\boldsymbol{k}_2)\right\rangle \overline{T} + \left\langle \theta(\boldsymbol{k}_1)\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\right\rangle \overline{\rho}\Big]. \tag{A29}$$

Substituting equations (A10), (A12), (A23) and (A25) into equation (A29), we obtain

$$\left\langle \theta(\boldsymbol{k}_1)\,\rho'(\boldsymbol{k}_2)\right\rangle = 2\tau^2(k)\Big\{\left\langle u_i(\boldsymbol{k}_1)\,u_j(\boldsymbol{k}_2)\right\rangle (\nabla_i\overline{T})(\nabla_j\overline{\rho}) + (\gamma-1)\Big[\left\langle (\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\right\rangle \overline{\rho} + \left\langle (\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_1}\,u_j(\boldsymbol{k}_2)\right\rangle \times(\nabla_j\overline{\rho})\Big]\overline{T} + \left\langle u_i(\boldsymbol{k}_1)\,(\mathrm{div}\boldsymbol{u})_{\boldsymbol{k}_2}\right\rangle \overline{\rho}\,(\nabla_i\overline{T})\Big\}. \tag{A30}$$

We neglect small terms $\sim O[(\ell_0/H_T)^3,\ (\ell_0/H_\rho)^3]$ in equations (A29) and (A30). Because all terms in equations (A29) and (A30) are proportional to either $(\boldsymbol{\nabla}\overline{\rho})^2$ or $(\boldsymbol{\nabla}\overline{T})(\boldsymbol{\nabla}\overline{\rho})$, and we consider homogeneous density stratified turbulence, we do not need to perform additional Taylor expansions over small parameters ℓ_0/H_T and ℓ_0/H_ρ in these terms. Therefore, we replace $\boldsymbol{k}_1$ by $\boldsymbol{k}$ and $\boldsymbol{k}_2$ by $-\boldsymbol{k}$ in all second moments in equation (A30). Using equations (A16)–(A18) and (A30), we determine $\left\langle \theta\,\rho'\right\rangle = \int \tau^2(k)\left\langle \theta(\boldsymbol{k})\,\rho'(-\boldsymbol{k})\right\rangle \mathrm{d}\boldsymbol{k}$ for large Péclet numbers given by equation (30).

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