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Mean-field theory of differential rotation in density stratified turbulent convection

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A mean-field theory of differential rotation in a density stratified turbulent convection has been developed. This theory is based on the combined effects of the turbulent heat flux and anisotropy of turbulent convection on the Reynolds stress. A coupled system of dynamical budget equations consisting in the equations for the Reynolds stress, the entropy fluctuations and the turbulent heat flux has been solved. To close the system of these equations, the spectral τ approach, which is valid for large Reynolds and Péclet numbers, has been applied. The adopted model of the background turbulent convection takes into account an increase of the turbulence anisotropy and a decrease of the turbulent correlation time with the rotation rate. This theory yields the radial profile of the differential rotation which is in agreement with that for the solar differential rotation.

Key words: astrophysical plasmas, plasma nonlinear phenomena

1. Introduction

The origin of the solar and stellar magnetic fields is associated with a mean-field dynamo (referred to as the $\alpha\Omega$ or $\alpha^2\Omega$ dynamo) that is based on the combined effect of helical turbulent motions and a differential rotation (see, e.g. Moffatt 1978; Parker 1979; Krause & Rädler 1980; Zeldovich, Ruzmaikin & Sokolov 1983; Rüdiger, Kitchatinov & Hollerbach 2013). A non-zero mean kinetic helicity produced by a rotating density stratified turbulent convection, causes the α effect in the solar convective zone. One potential origin of the solar differential rotation is related to an anisotropic eddy viscosity (Kippenhahn 1963; Rüdiger 1980; Durney 1985; Rüdiger 1989). This idea has been applied in developing a theory of the differential rotation (Durney 1993; Kichatinov & Rüdiger 1993; Kitchatinov & Rüdiger 2005). The turbulent heat flux in these theories has been introduced phenomenologically using the mixing-length theory relation: $\langle \boldsymbol{u}'^2 \rangle \propto g\tau_0 \langle u'_z s' \rangle$, where $\langle u'_z s' \rangle$ is the vertical turbulent heat flux, $\boldsymbol{u}'$ and s' are fluctuations of fluid velocity and entropy, $\boldsymbol{g}$ is the gravity acceleration and τ_0 is the characteristic turbulent time. Also a quasi-linear approach that is valid for small fluid Reynolds numbers has been applied in these studies.

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An additional possibility for the production of the solar differential rotation is associated with an effect of the turbulent heat flux on the Reynolds stress in a rotating density stratified turbulent convection. Based on this idea, Kleeorin & Rogachevskii (2006) develop a mean-field theory of the differential rotation, where a coupled system of dynamical equations for the Reynolds stress, the entropy fluctuations and the turbulent heat flux has been solved adopting a spectral τ approach. It was demonstrated (Kleeorin & Rogachevskii 2006) that the ratio of the contributions to the Reynolds stress caused by the turbulent heat flux and the anisotropic eddy viscosity is of the order of $\sim 10(H_\rho/\ell_0)^2$, where ℓ_0 is the maximum scale of turbulent motions and H_ρ is the fluid density variation scale. This theory allows us to determine the profiles of the differential rotation in the upper part of the solar convection zone where the rotation is slow in comparison with the turbulent time.

In the lower part of the solar convective zone, the rotation is fast in comparison with the turbulent time. This causes a strong anisotropy of the turbulent convection that is an additional source of solar differential rotation. Key theoretical questions are how can turbulent convection be modified by the fast rotation, and how can it affect the production of the differential rotation? These issues remain open unresolved problems in solar physics and astrophysics. Note that different theories of the solar differential rotation can be validated using data from the surface measurements of the solar angular velocity (see, e.g. Howard & Harvey 1970; Snodgrass, Howard & Webster 1984) and helioseismology based on measurements of the frequency of p-mode oscillations (see, e.g. Duvall, Harvey & Pomerantz 1986; Dziembowski, Goode & Libbrecht 1989; Thompson 1990; Kosovichev *et al.* 1997; Schou, Antia & Basu 1998).

In the present study, the combined effects of the turbulent heat flux and the turbulence anisotropy increasing with the rotation rate on the Reynolds stress have been studied for a rotating density stratified turbulent convection. The spectral τ approach, which is valid for large Reynolds and Péclet numbers, has been used in this study. This allows us to advance the mean-field theory of the solar differential rotation and obtain profiles of the differential rotation versus radius which are in agreement with the measured profiles of the solar differential rotation.

2. Effect of rotation on the Reynolds stress, entropy fluctuations and turbulent heat flux

To develop the theory of differential rotation in small-scale density stratified turbulent convection, we use a mean-field approach whereby the velocity, pressure and entropy are decomposed into mean and fluctuating parts. This approach implies that there is a separation of temporal and spatial scales, so that the mean fields are varied on much larger scales in comparison with those for the fluctuations.

Let us determine the dependencies of the Reynolds stresses $\langle u_i'(t,\boldsymbol{x})u_j'(t,\boldsymbol{x})\rangle$ and the turbulent heat flux $\langle s'(t,\boldsymbol{x})u_i'(t,\boldsymbol{x})\rangle$ on the mean fields, where angular brackets denote ensemble averaging. To this end we use equations for the fluctuations of velocity and entropy in rotating turbulent convection, which are obtained by subtracting equations for the mean fields from the corresponding equations for the total fields. The equations for fluctuations of velocity $\boldsymbol{u}'$ and entropy s' are given by

$$\frac{\partial \boldsymbol{u}'}{\partial t} = -(\boldsymbol{U}\boldsymbol{\cdot}\nabla)\boldsymbol{u}' - (\boldsymbol{u}'\boldsymbol{\cdot}\nabla)\boldsymbol{U} - \nabla\left(\frac{p'}{\rho_0}\right) - \boldsymbol{g}s' + 2\boldsymbol{u}'\times\boldsymbol{\varOmega} + \boldsymbol{U}^N, \tag{2.1}$$

$$\frac{\partial s'}{\partial t} = -\frac{\varOmega_b^2}{g}(\boldsymbol{u}'\boldsymbol{\cdot}\boldsymbol{e}) - (\boldsymbol{U}\boldsymbol{\cdot}\nabla)s' + S^N. \tag{2.2}$$

Equations (2.1) and (2.2) are written in the reference frame rotating with angular velocity $\boldsymbol{\Omega}$. Here p' are fluctuations of fluid pressure, the entropy fluctuations are determined by $s' = (\gamma P_0)^{-1} p' - \rho_0^{-1} \rho'$, the mean fields $\boldsymbol{U}$ and S are the mean velocity and entropy, $\boldsymbol{e}$ is the unit vector directed opposite to $\boldsymbol{g}$ and $\Omega_b^2 = -\boldsymbol{g} \boldsymbol{\cdot} \boldsymbol{\nabla} S$. The fluid velocity for a low Mach number flows satisfies the continuity equation written in the anelastic approximation, $\operatorname{div}(\rho_0 \boldsymbol{U}) = 0$ and $\operatorname{div}(\rho_0 \boldsymbol{u}') = 0$. The variables with the subscript '0' correspond to the hydrostatic nearly isentropic basic reference state, i.e. $\boldsymbol{\nabla} P_0 = \rho_0 \boldsymbol{g}$ and $\boldsymbol{g} \boldsymbol{\cdot} [(\gamma P_0)^{-1} \boldsymbol{\nabla} P_0 - \rho_0^{-1} \boldsymbol{\nabla} \rho_0] \approx 0$, where γ is the ratio of specific heats. The turbulent convection is regarded as a small deviation from a well-mixed adiabatic reference state. The nonlinear terms $\boldsymbol{U}^N$ and S^N in (2.1) and (2.2) which include the molecular dissipative terms, are given by

$$U^N = \langle (\boldsymbol{u}' \boldsymbol{\cdot} \boldsymbol{\nabla}) \boldsymbol{u}' \rangle - (\boldsymbol{u}' \boldsymbol{\cdot} \boldsymbol{\nabla}) \boldsymbol{u}' + \boldsymbol{f}_\nu(\boldsymbol{u}'), \tag{2.3}$$

$$S^N = \langle (\boldsymbol{u}' \boldsymbol{\cdot} \boldsymbol{\nabla}) s' \rangle - (\boldsymbol{u}' \boldsymbol{\cdot} \boldsymbol{\nabla}) s' - (1/T_0) \boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{F}_\kappa(\boldsymbol{u}', s'), \tag{2.4}$$

where $\rho_0 \boldsymbol{f}_\nu(\boldsymbol{u}')$ is the molecular viscous force, $\boldsymbol{F}_\kappa(\boldsymbol{u}', s')$ is the heat flux associated with the molecular thermal conductivity.

To study the rotating turbulent convection we perform the derivations which include the following steps: (i) adopting new variables for fluctuations of velocity $\boldsymbol{v} = \sqrt{\rho_0} \boldsymbol{u}'$ and entropy $s = \sqrt{\rho_0} s'$; (ii) derivation of the equations for the second moments of the velocity fluctuations $\langle v_i v_j \rangle$, the entropy fluctuations $\langle s^2 \rangle$ and the turbulent heat flux $\langle v_i s \rangle$ in the $\boldsymbol{k}$ space, where we apply a multi-scale approach (Roberts & Soward 1975), which separates the mean fields varied on large scales from fluctuations varied on small scales; (iii) application of the spectral τ approximation and solution of the derived second-moment equations in the $\boldsymbol{k}$ space; (iv) returning to the physical space to obtain formulae for the Reynolds stress and the turbulent heat flux as the functions of the rotation rate.

Using (A 3)–(A 4) for the fluctuations of velocity and entropy in $\boldsymbol{k}$ space derived in appendix A, we obtain equations for the following correlation functions: $f_{ij}(\boldsymbol{k}, \boldsymbol{K}) = \langle v_i(t, \boldsymbol{k}_1) v_j(t, \boldsymbol{k}_2) \rangle$, $F_i(\boldsymbol{k}, \boldsymbol{K}) = \langle s(t, \boldsymbol{k}_1) v_i(t, \boldsymbol{k}_2) \rangle$ and $\Theta(\boldsymbol{k}, \boldsymbol{K}) = \langle s(t, \boldsymbol{k}_1) s(t, \boldsymbol{k}_2) \rangle$, where $\boldsymbol{k}_1 = \boldsymbol{k} + \boldsymbol{K}/2$ and $\boldsymbol{k}_2 = -\boldsymbol{k} + \boldsymbol{K}/2$. Here the wave vectors $\boldsymbol{K}$ and $\boldsymbol{k}$ are related to the large and small scales, respectively. Hereafter we omit the argument t in the correlation functions to simplify notations. The equations for these second moments are given by

$$\frac{\partial f_{ij}(\boldsymbol{k}, \boldsymbol{K})}{\partial t} = (I^U_{ijmn} + L^\Omega_{ijmn}) f_{mn} + M^F_{ij} + \hat{\mathcal{N}} \tilde{f}_{ij}, \tag{2.5}$$

$$\frac{\partial F_i(\boldsymbol{k}, \boldsymbol{K})}{\partial t} = (J^U_{im} + D^\Omega_{im}) F_m + g e_m P_{im}(\boldsymbol{k}_1) \Theta + \hat{\mathcal{N}} \tilde{F}_i, \tag{2.6}$$

$$\frac{\partial \Theta(\boldsymbol{k}, \boldsymbol{K})}{\partial t} = -\operatorname{div}(\boldsymbol{U} \Theta) + \hat{\mathcal{N}} \Theta, \tag{2.7}$$

where

$$I^U_{ijmn} = J^U_{im}(\boldsymbol{k}_1)\delta_{jn} + J^U_{jn}(\boldsymbol{k}_2)\delta_{im} = \Big[2k_{iq}\delta_{mp}\delta_{jn} + 2k_{jq}\delta_{im}\delta_{pn} - \delta_{im}\delta_{jq}\delta_{np} - \delta_{iq}\delta_{jn}\delta_{mp} + \delta_{im}\delta_{jn}k_q\frac{\partial}{\partial k_p}\Big]\nabla_p U_q - \delta_{im}\delta_{jn}(\operatorname{div}\boldsymbol{U} + \boldsymbol{U}\boldsymbol{\cdot}\nabla), \tag{2.8}$$

$$M^F_{ij} = ge_m[P_{im}(\boldsymbol{k}_1)F_j(\boldsymbol{k},\boldsymbol{K}) + P_{jm}(\boldsymbol{k}_2)F_i(-\boldsymbol{k},\boldsymbol{K})], \tag{2.9}$$

and $L^\Omega_{ijmn} = D^\Omega_{im}(\boldsymbol{k}_1)\delta_{jn} + D^\Omega_{jn}(\boldsymbol{k}_2)\delta_{im}$, $J^U_{ij}(\boldsymbol{k}) = 2k_{in}\nabla_j U_n - \nabla_j U_i - (1/2)\operatorname{div}\boldsymbol{U}\delta_{ij}$ and $D^\Omega_{ij}(\boldsymbol{k}) = 2\varepsilon_{ijm}\Omega_n k_{mn}$. Here δ_{ij} is the Kronecker tensor, $k_{ij} = k_i k_j/k^2$, ε_{ijk} is the Levi-Civita tensor and $F_i(-\boldsymbol{k},\boldsymbol{K}) = \langle s(\boldsymbol{k}_2)v_i(\boldsymbol{k}_1)\rangle$. The correlation functions f_{ij}, F_i and Θ are proportional to the non-uniform fluid density ρ_0. Here $\hat{\mathcal{N}}\tilde{f}_{ij}$, $\hat{\mathcal{N}}\tilde{F}_i$ and $\hat{\mathcal{N}}\Theta$ are the terms which are related to the third-order moments appearing due to the nonlinear terms. In particular,

$$\hat{\mathcal{N}}\tilde{f}_{ij} = \langle P_{im}(\boldsymbol{k}_1)v^N_m(\boldsymbol{k}_1)v_j(\boldsymbol{k}_2)\rangle + \langle v_i(\boldsymbol{k}_1)P_{jm}(\boldsymbol{k}_2)v^N_m(\boldsymbol{k}_2)\rangle, \tag{2.10}$$

$$\hat{\mathcal{N}}\tilde{F}_i = \langle s^N(\boldsymbol{k}_1)u_j(\boldsymbol{k}_2)\rangle + \langle s(\boldsymbol{k}_1)P_{im}(\boldsymbol{k}_2)v^N_m(\boldsymbol{k}_2)\rangle, \tag{2.11}$$

$$\hat{\mathcal{N}}\Theta = \langle s^N(\boldsymbol{k}_1)s(\boldsymbol{k}_2)\rangle + \langle s(\boldsymbol{k}_1)s^N(\boldsymbol{k}_2)\rangle. \tag{2.12}$$

The equations for the second-order moments contain high-order moments and a closure problem arises (see, e.g. McComb 1990; Monin & Yaglom 2013). We apply the spectral τ approximation that is a sort of third-order closure procedure (see, e.g. Orszag 1970; Pouquet, Frisch & Léorat 1976; Kleeorin, Rogachevskii & Ruzmaikin 1990; Rogachevskii & Kleeorin 2004). The spectral τ approximation postulates that the deviations of the third-order-moment terms, $\hat{\mathcal{N}}f_{ij}(\boldsymbol{k})$, from the contributions to these terms afforded by the background turbulent convection, $\hat{\mathcal{N}}f^{(0)}_{ij}(\boldsymbol{k})$, are expressed through the similar deviations of the second moments, $f_{ij}(\boldsymbol{k}) - f^{(0)}_{ij}(\boldsymbol{k})$, i.e.

$$\hat{\mathcal{N}}f_{ij}(\boldsymbol{k}) - \hat{\mathcal{N}}f^{(0)}_{ij}(\boldsymbol{k}) = -\frac{f_{ij}(\boldsymbol{k}) - f^{(0)}_{ij}(\boldsymbol{k})}{\tau_r(k)}, \tag{2.13}$$

and similarly for other tensors, where $\hat{\mathcal{N}}f_{ij} = \hat{\mathcal{N}}\tilde{f}_{ij} + M^F_{ij}(F^{\Omega=0})$ and $\hat{\mathcal{N}}F_i = \hat{\mathcal{N}}\tilde{F}_i + ge_nP_{in}(k)\Theta^{\Omega=0}$, the superscript (0) corresponds to the background turbulent convection (i.e. a turbulent convection with $\nabla_i U_j = 0$), $\tau_r(k)$ is the characteristic relaxation time of the statistical moments, which can be identified with the correlation time $\tau(k)$ of the turbulent velocity field for large Reynolds numbers. The quantities $F^{\Omega=0}$ and $\Theta^{\Omega=0}$ are for a non-rotating turbulent convection with non-zero spatial derivatives of the mean velocity. Validation of the τ approximation has been done in various numerical simulations and analytical studies (see, e.g. Brandenburg, Käpylä & Mohammed 2004; Brandenburg & Subramanian 2005; Rogachevskii & Kleeorin 2007; Rogachevskii *et al.* 2011; Brandenburg *et al.* 2012; Käpylä *et al.* 2012; Rogachevskii *et al.* 2012). Note that we apply the τ approximation (2.13) only to study the deviations from the background turbulent convection which are caused by the spatial derivatives of the mean velocity. The background turbulent convection is assumed to be known (see below).

We use the following model of the background turbulent convection which takes into account an increase of the anisotropy of turbulence with increase of the rate of rotation:

$$
\begin{aligned}
f_{ij}^{(0)} \equiv \langle v_i(\boldsymbol{k}_1) v_j(\boldsymbol{k}_2)\rangle^{(0)} &= \frac{E(k)[1+2k\varepsilon_u \delta(k_z)]}{8\pi\, k^2(k^2+\tilde{\lambda}^2)(1+\varepsilon_u)} \big[\delta_{ij}(k^2+\tilde{\lambda}^2) - k_i k_j - \tilde{\lambda}_i\tilde{\lambda}_j \\
&\quad + \mathrm{i}(\tilde{\lambda}_i k_j - \tilde{\lambda}_j k_i)\big]\, \rho_0 \langle \boldsymbol{u}'^2\rangle^{(0)}, \qquad (2.14)\\
F_i^{(0)} \equiv \langle v_i(\boldsymbol{k}_1) s(\boldsymbol{k}_2)\rangle^{(0)} &= \frac{3E(k)}{8\pi k^4}[k^2 e_j P_{ij}(\boldsymbol{k}) - \mathrm{i}\tilde{\lambda} k_j P_{ij}(\boldsymbol{e})]\rho_0 \langle s' u_z'\rangle^{(0)}, \qquad (2.15)
\end{aligned}
$$

and $\Theta^{(0)} \equiv \langle s(\boldsymbol{k}_1) s(\boldsymbol{k}_2)\rangle^{(0)} = \rho_0 \langle (s')^2\rangle^{(0)} E(k)/4\pi k^2$, where $\tilde{\boldsymbol{\lambda}} = (\boldsymbol{\lambda} - \boldsymbol{\nabla})/2$, $\boldsymbol{\lambda} = -(\boldsymbol{\nabla}\rho_0)/\rho_0$. We assume that the background turbulent convection is Kolmogorov-type turbulence with a constant flux of energy over the spectrum, i.e. the kinetic energy spectrum $E(k) = -\mathrm{d}\bar{\tau}(k)/\mathrm{d}k$, $\bar{\tau}(k) = (k/k_0)^{1-q}$ with the exponent of the kinetic energy spectrum $1 < q < 3$, e.g. $q = 5/3$ for the Kolmogorov spectrum. The turbulent correlation time $\tau(k) = 2\tau_\Omega\, \bar{\tau}(k)$, where $\tau_\Omega = \ell_0/u_0$, and ℓ_0 is the energy containing scale of turbulent motions, $u_0 = \sqrt{\langle \boldsymbol{u}'^2\rangle^{(0)}}$ is the characteristic turbulent velocity in the scale ℓ_0 and $k_0 = 1/\ell_0$. We consider anisotropic turbulent convection as a combination of a three-dimensional isotropic turbulence and two-dimensional turbulence in the plane perpendicular to the rotational axis. The degree of anisotropy ε_u is defined as the ratio of turbulent kinetic energies of two-dimensional to three-dimensional motions. In this model we neglect effects which are $O(\lambda^3, \nabla^3\langle \boldsymbol{v}^2\rangle^{(0)})$.

The effect of rotation on the turbulent correlation time is described just by a heuristic argument, i.e. we assume that $\tau_\Omega^{-2} = \tau_0^{-2} + \Omega^2/C_\Omega^2$, that yields:

$$
\tau_\Omega = \frac{\tau_0}{[1 + (C_\Omega^{-1}\Omega\tau_0)^2]^{1/2}}. \qquad (2.16)
$$

This implies that for fast rotation, $\Omega\tau_0 \gg 1$, the parameter $\omega = 8\Omega\tau_\Omega$ tends to the limiting value $\omega_m = 8C_\Omega$, where the dimensionless constant $C_\Omega \sim 1$.

The solution of (2.5)–(2.7) after application of the spectral τ approximation, and the integration over the $\boldsymbol{k}$ space (see appendix B) allow us to determine the Reynolds stress and the effective force versus angular velocity. The latter yields the mean-field equation for the differential rotation (see next section), which takes into account the effects of rotating density stratified turbulent convection.

3. Mean-field equation for differential rotation

The differential rotation in the axisymmetric fluid flow is determined by the linearized Navier–Stokes equation for the toroidal component $U_\varphi(r,\theta) \equiv r\sin\theta\,\delta\Omega$ of the mean velocity:

$$
\rho_0 \frac{\partial U_\varphi}{\partial t} = \frac{1}{r^3}\frac{\partial}{\partial r}(r^3\sigma_{r\varphi}) + \frac{1}{r\sin^2\theta}\frac{\partial}{\partial\theta}(\sin^2\theta\,\sigma_{\theta\varphi}) + 2\rho_0(\boldsymbol{U}\times\boldsymbol{\Omega})_\varphi, \qquad (3.1)
$$

where the tensor $\sigma_{ij} = -\langle v_i v_j\rangle$ is determined by the Reynolds stress:

$$
\sigma_{r\varphi} \equiv -e_j^\varphi e_i^r \langle v_i v_j\rangle = \sigma_{r\varphi}^{\nu_T} + \sigma_{r\varphi}^{F} + \sigma_{r\varphi}^{u}, \qquad (3.2)
$$

$$
\sigma_{\theta\varphi} \equiv -e_j^\varphi e_i^\theta \langle v_i v_j\rangle = \sigma_{\theta\varphi}^{\nu_T} + \sigma_{\theta\varphi}^{F} + \sigma_{\theta\varphi}^{u}. \qquad (3.3)
$$

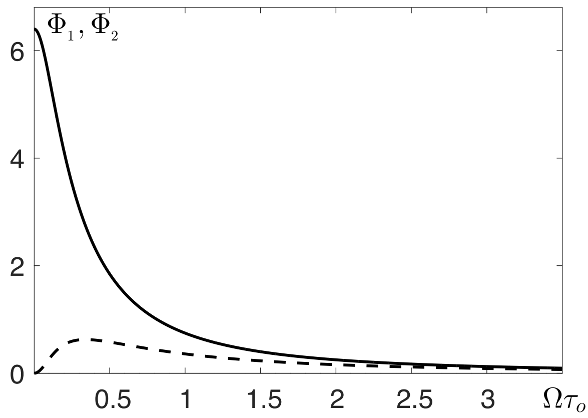

FIGURE 1. The functions $\varPhi_1(\varOmega\tau_0)$ (solid) and $\varPhi_2(\varOmega\tau_0)$ (dashed) versus $\varOmega\tau_0$.

and $\boldsymbol{e}^r$, $\boldsymbol{e}^\theta$ and $\boldsymbol{e}^\varphi$ are the unit vectors along the radial, meridional and toroidal directions of the spherical coordinates r, θ, φ. There are three contributions to the tensor $\sigma_{ij}=-\langle v_i v_j\rangle$ in (3.2) and (3.3). The first terms on the right-hand sides of (3.2) and (3.3) describe the contribution $\sigma_{ij}^{\nu_T}$ to the Reynolds stress caused by turbulent viscosity ν_T:

$$\sigma_{r\varphi}^{\nu_T}=\rho_0\nu_T r\frac{\partial}{\partial r}\left(\frac{U_\varphi}{r}\right), \tag{3.4}$$

$$\sigma_{\theta\varphi}^{\nu_T}=\rho_0\nu_T\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{U_\varphi}{\sin\theta}\right). \tag{3.5}$$

The second terms in (3.2) and (3.3) determine the contribution $\boldsymbol{\sigma}^F$ to the Reynolds stress caused by the turbulent heat flux:

$$\sigma_{r\varphi}^{F}=\tfrac{1}{6}\rho_0\tau_\varOmega^2 g\langle s' u_z'\rangle^{(0)}\varOmega\,\sin\theta[\varPhi_1(\omega)+\cos^2\theta\varPhi_2(\omega)]\,, \tag{3.6}$$

$$\sigma_{\theta\varphi}^{F}=\tfrac{1}{3}\rho_0\tau_\varOmega^2 g\langle s' u_z'\rangle^{(0)}\varOmega\,\sin^2\theta\cos\theta\varPhi_2(\omega), \tag{3.7}$$

where the parameter $\omega=8\varOmega\tau_\varOmega$. The functions $\varPhi_1(\omega)$ and $\varPhi_2(\omega)$ are given by (B 15*a*,*b*)–(B 17) in appendix B and are shown in figure 1. When the turbulent correlation time is independent of the rotation rate, equations (3.6) and (3.7) coincide with those obtained by Kleeorin & Rogachevskii (2006).

The third terms in (3.2) and (3.3) determine the contribution $\boldsymbol{\sigma}^u$ to the Reynolds stress caused by the anisotropy of turbulence due to the non-uniform fluid density and fast uniform rotation (see (B 13) in appendix B):

$$\sigma_{r\varphi}^{u}=-\frac{\lambda^2\ell_0^2}{20}\rho_0\langle\boldsymbol{u}'^2\rangle^{(0)}\tau_\varOmega\varOmega\,\sin\theta(1+\cos^2\theta), \tag{3.8}$$

$$\sigma_{\theta\varphi}^{u}=\frac{\lambda^2\ell_0^2}{20}\rho_0\langle\boldsymbol{u}'^2\rangle^{(0)}\tau_\varOmega\varOmega\,\sin^2\theta\cos\theta. \tag{3.9}$$

Equation (3.1) in a steady state that determines the profiles of the differential rotation, reads:

$$\hat{\mathcal{W}}(r)\rho_0\nu_T\left\{\frac{\partial}{\partial r}\frac{\delta\Omega}{\Omega}+\frac{1}{r}\left[a_F(\Phi_1(\omega)+\Phi_2(\omega)X^2)-2a_u\lambda^2\ell_0^2(1+X^2)\right]\right\}$$
$$-\frac{\rho_0\nu_T}{r^2}\hat{\mathcal{M}}(X)\left\{\frac{\delta\Omega}{\Omega}-[a_F\Phi_2(\omega)+a_u\lambda^2\ell_0^2]X^2\right\}=0, \qquad (3.10)$$

where the operators $\hat{\mathcal{W}}(r)$ and $\hat{\mathcal{M}}(X)$ are defined as

$$\hat{\mathcal{W}}(r)f(r)=\frac{1}{r^4}\frac{\partial}{\partial r}[r^4 f(r)], \quad \hat{\mathcal{M}}(X)\phi(X)=\left[(X^2-1)\frac{\partial^2}{\partial X^2}+4X\frac{\partial}{\partial X}\right]\phi(X), \qquad (3.11a,b)$$

$X=\cos\theta$, and the parameters a_F and a_u are given by $a_F=\tau_\Omega^2 g\langle s'u_z'\rangle^{(0)}/6\nu_T$ and $a_u=\tau_\Omega\langle \boldsymbol{u}'^2\rangle^{(0)}/40\nu_T$. We seek a solution of (3.10) in the form:

$$\frac{\delta\Omega}{\Omega}=\sum_{n=0}^{\infty}C_{2n}^{3/2}(X)\tilde{\Omega}_{2n}(r), \qquad (3.12)$$

where the radius r is measured in units of the solar radius $R_\odot$, and the function $C_n^{3/2}(X)$ satisfies the equation for the ultra-spherical polynomials:

$$[\hat{\mathcal{M}}(X)-n(n+3)]C_n^{3/2}(X)=0. \qquad (3.13)$$

The function $C_n^{3/2}(X)$ has the following properties:

$$\int_{-1}^{1}(1-X^2)C_n^{3/2}(X)C_m^{3/2}(X)\,\mathrm{d}X=\frac{(n+1)(n+2)}{n+3/2}\delta_{nm}, \qquad (3.14)$$

$C_0^{3/2}(X)=1$ and $C_2^{3/2}(X)=(3/2)(5X^2-1)$. Substituting (3.12) into (3.10), we obtain equations for the functions $\tilde{\Omega}_0(r)$:

$$\tilde{\Omega}_0(r)=\tilde{\Omega}_*-\frac{1}{5}\int_{r/R_\odot}^{1}\{12a_u\lambda^2\ell_0^2-a_F[5\Phi_1(\omega)+\Phi_2(\omega)]\}\frac{\mathrm{d}r}{r}, \qquad (3.15)$$

and $\tilde{\Omega}_2(r)$:

$$\hat{\mathcal{W}}(r)\rho_0\nu_T\left[\frac{\partial\tilde{\Omega}_2(r)}{\partial r}+\frac{2}{15r}(a_F\Phi_2(\omega)-2a_u\lambda^2\ell_0^2)\right]$$
$$-\frac{10\rho_0\nu_T}{r^2}\left[\tilde{\Omega}_2(r)-\frac{2}{15}(a_F\Phi_2(\omega)+a_u\lambda^2\ell_0^2)\right]=0, \qquad (3.16)$$

where $\tilde{\Omega}_*$ is the free constant determined by the surface boundary condition.

In figure 2 we show the total angular velocity $\tilde{\Omega}_{tot}=\tilde{\Omega}_0+1$ that includes the uniform rotation Ω versus the radius $r/R_\odot$. This theoretical profile is compared with the radial profile of the solar angular velocity obtained from the helioseismology observational data (Kosovichev *et al.* 1997) specified for the latitude $\phi=30^\circ$ and

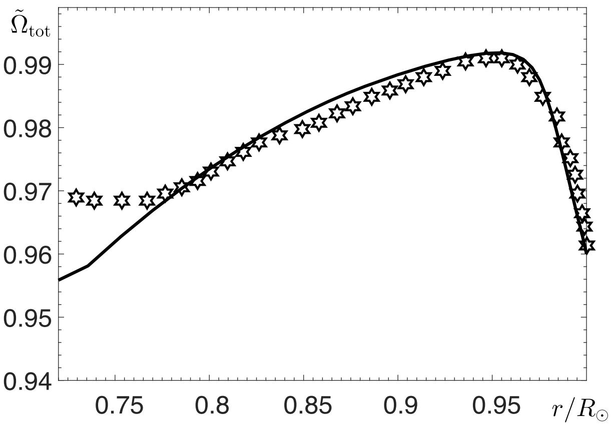

FIGURE 2. The total angular velocity $\tilde{\Omega}_{tot} = \tilde{\Omega}_0 + 1$ that includes the uniform rotation Ω versus the radius $r/R_\odot$ (solid). This theoretical profile is compared with the radial profile of the solar angular velocity obtained from helioseismology observational data (stars) at the latitude $\phi = 30°$ and normalized by the solar rotation frequency $\Omega_\odot(\phi = 0)$ at the equator, where $R_\odot$ is the solar radius.

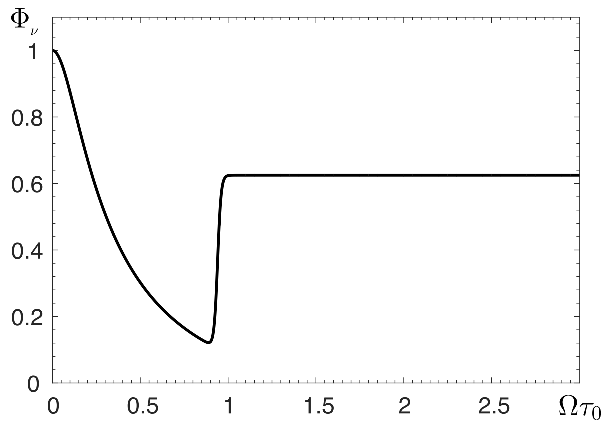

FIGURE 3. The rotation rate dependence of the functions $\Phi_\nu(\Omega\tau_0)$, where $\nu_T(\Omega\tau_0) = \nu_T^*\Phi_\nu(\Omega\tau_0)$.

normalized by the solar angular velocity $\Omega_\odot(\phi = 0)$ at the equator. Note that at $\phi = 30°$ the contribution from the term $C_2^{3/2}(X)\tilde{\Omega}_2(r)$ to the differential rotation vanishes, because the function $C_2^{3/2}(X) = (3/2)(5X^2 - 1)$ at an angle of approximately $\phi = 30°$ vanishes. To determine $\tilde{\Omega}_{tot}$ we use the rotation rate dependence of the turbulent viscosity $\nu_T(\omega) = \nu_T^*\Phi_\nu(\omega)$, where $\nu_T^* = \tau_0\langle \boldsymbol{u}'^2\rangle^{(0)}/6$, the functions $\Phi_\nu(\omega)$ is given by (B 18) in appendix B and is shown in figure 3. Strong change of the turbulent viscosity is caused by the fast rotation during the transition from isotropic three-dimensional turbulence to strongly anisotropic quasi two-dimensional turbulence.

For the comparison of the theoretical profiles of the differential rotation and observational data, we use the radial profiles of $\Omega\tau_0(r)$ (see figure 4) and the ratio $\ell_M(r)/H_\rho(r)$ (see figure 5) of the mixing length ℓ_M to the density stratification length

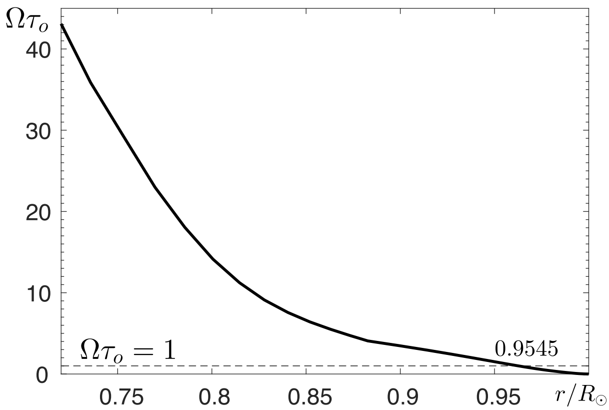

FIGURE 4. The profile of $\varOmega\tau_0$ versus $r/R_\odot$ based on the model of the solar convective zone by Spruit (1974).

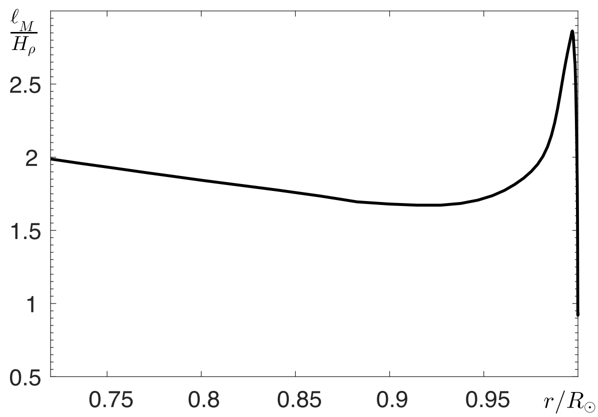

FIGURE 5. The profile of the ratio ℓ_M/H_ρ of the mixing length ℓ_M to the density stratification length H_ρ versus $r/R_\odot$ that is based on the model of the solar convective zone by Spruit (1974).

H_ρ based on the model of the solar convective zone by Spruit (1974). Inspection of figure 2 demonstrates that the theoretical profile of the differential rotation is in agreement with the profile of the solar differential rotation when $\ell_M/\ell_0 = 5$. The latter is justified by the results of analytical study (Elperin *et al.* 2002, 2006) and laboratory experiments (Bukai *et al.* 2009), which show that the integral scale ℓ_0 of the turbulent convection is smaller by a factor five than the size of the coherent structures (the large-scale circulations). We compare the theoretical and observation profiles of the differential rotation for the latitude $\phi = 30^\circ$ because for the latitudes which are far from $\phi = 30^\circ$, the contribution of the term $\propto \tilde{\varOmega}_2$ (determined by (3.16)) to the differential rotation cannot be ignored. A more detailed comparison of the theoretical and observation profiles of the differential rotation for different latitudes requires mean-field numerical modelling that is a subject of a separate study.

4. Conclusions

We discuss a new theory of differential rotation based on the combined effects of the turbulent heat flux and the turbulence anisotropy increasing with the rate of rotation on the Reynolds stress in a density stratified turbulent convection. We solve a coupled system of dynamical budget equations which includes the equations for the Reynolds stress, the entropy fluctuations and the turbulent heat flux, applying a spectral τ approach to close this system of equations. The model of the background turbulent convection takes into account an increase of the turbulence anisotropy and a decrease of the turbulent correlation time with the rotation rate. This theory allows us to obtain the profile of the differential rotation versus radius which is in agreement with the profile of the solar differential rotation.

The mechanism of the differential rotation that is related to the effect of the turbulent heat flux on Reynolds stress in a rotating turbulent convection is as follows. The total angular velocity includes the uniform rotation $\varOmega$ and the differential rotation $\delta\varOmega$. The uniform rotation results in the counter-rotation turbulent heat flux $\langle s' u'_\varphi \rangle$ that is directed oppositely to the uniform rotation $\varOmega$. The counter-rotation turbulent heat flux is similar to the counter-wind turbulent heat flux that is directed oppositely to the mean wind known in atmospheric physics (Elperin *et al.* 2002, 2006). In turbulent convection an ascending fluid element has a larger temperature than the temperature of the surrounding fluid and smaller toroidal fluid velocity, while a descending fluid element has a smaller temperature and larger toroidal fluid velocity. This results in a turbulent heat flux in the direction opposite to the uniform rotation. The entropy fluctuations produce fluctuations of the buoyancy force, that increases the fluctuations of the vertical and meridional components of the velocity which are correlated with the fluctuations of the toroidal component of the velocity. This implies that the off-diagonal components of the Reynolds stress, $\langle u'_r u'_\varphi \rangle$ and $\langle u'_\theta u'_\varphi \rangle$ are non-zero, producing the toroidal component of the effective force. The latter results in the formation of the differential rotation $\delta\varOmega$ in turbulent convection.

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Appendix A. Derivation of equations for the second moments

Equations (2.1) and (2.2), in the new variables, for fluctuations of velocity $\boldsymbol{v} = \sqrt{\rho_0}\boldsymbol{u}'$ and entropy $s = \sqrt{\rho_0} s'$ are given by

$$\frac{1}{\sqrt{\rho_0}} \frac{\partial \boldsymbol{v}(\boldsymbol{x}, t)}{\partial t} = -\nabla\left(\frac{p'}{\rho_0}\right) + \frac{1}{\sqrt{\rho_0}}[2\boldsymbol{v} \times \boldsymbol{\varOmega} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{U} - G^U \boldsymbol{v} - \boldsymbol{g} s] + \boldsymbol{v}^N, \tag{A 1}$$

$$\frac{\partial s(\boldsymbol{x}, t)}{\partial t} = -\frac{\varOmega_b^2}{g}(\boldsymbol{v}\cdot\boldsymbol{e}) - G^U s + s^N, \tag{A 2}$$

where $G^U = (1/2)\mathrm{div}\boldsymbol{U} + \boldsymbol{U}\cdot\nabla$, $\boldsymbol{v}^N$ and s^N are the nonlinear terms which include the molecular viscous and dissipative terms. The fluid velocity fluctuations $\boldsymbol{v}$ satisfy the equation $\nabla\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{\lambda}/2$.

Let us derive equations for the second-order moments. For this purpose we rewrite the momentum equation and the entropy equation in Fourier space. In particular,

$$\frac{\mathrm{d}v_i(\boldsymbol{k})}{\mathrm{d}t} = [D^{\Omega}_{im}(\boldsymbol{k}) + \tilde{J}^U_{im}(\boldsymbol{k})]v_m(\boldsymbol{k}) + g e_m P_{im}(\boldsymbol{k}) s(\boldsymbol{k}) + v_i^N(\boldsymbol{k}), \tag{A 3}$$

$$\frac{\mathrm{d}s(\boldsymbol{k})}{\mathrm{d}t} = -G^U(\boldsymbol{k})s(\boldsymbol{k}) + s^N, \tag{A 4}$$

where

$$\tilde{J}^U_{ij}(\boldsymbol{k}) = 2k_{in}\nabla_j U_n - \nabla_j U_i - G^U(\boldsymbol{k})\delta_{ij}, \quad G^U(\boldsymbol{k}) = \tfrac{1}{2}\,\mathrm{div}\,\boldsymbol{U} + \mathrm{i}(\boldsymbol{U}\boldsymbol{\cdot}\boldsymbol{k}), \tag{A 5a,b}$$

$D^{\Omega}_{ij}(\boldsymbol{k}) = 2\varepsilon_{ijm}\Omega_n k_{mn}$, δ_{ij} is the Kronecker tensor, $k_{ij} = k_i k_j/k^2$ and ε_{ijk} is the Levi-Civita tensor. To derive (A 3) we multiply the momentum equation written in $\boldsymbol{k}$-space by $P_{ij}(\boldsymbol{k}) = \delta_{ij} - k_{ij}$ to exclude the pressure term. We also use the following identities:

$$\sqrt{\rho_0}\big[\nabla\times[\nabla\times(\boldsymbol{u}'\times\boldsymbol{\Omega})]\big] = \big(\boldsymbol{\Omega}\times\nabla^{(\lambda)}\big)(\boldsymbol{\lambda}\boldsymbol{\cdot}\boldsymbol{v}) + \big(\boldsymbol{\Omega}\boldsymbol{\cdot}\nabla^{(\lambda)}\big)\big(\nabla^{(\lambda)}\times\boldsymbol{v}\big), \tag{A 6}$$

$$\sqrt{\rho_0}\big[\nabla\times[\nabla\times(\boldsymbol{g}s')]\big] = -g_j\big[\delta_{ij}\big(\nabla^{(\lambda)}\big)^2 - \nabla_i^{(\lambda)}\nabla_j^{(\lambda)}\big]s, \tag{A 7}$$

$$\sqrt{\rho_0}[\nabla\times[\nabla\times\boldsymbol{u}]]_{\boldsymbol{k}} = -[\Lambda^2\delta_{ij} - \Lambda_i\Lambda_j]v_j(\boldsymbol{k}), \tag{A 8}$$

where $\nabla^{(\lambda)} = \nabla + \boldsymbol{\lambda}/2$, $\boldsymbol{\lambda} = -(\nabla\rho_0)/\rho_0$, $\boldsymbol{\Lambda} = \mathrm{i}\boldsymbol{k} + \boldsymbol{\lambda}/2$. Using (A 3) and (A 4) we derive equations for the second moments which are given by (2.5)–(2.7).

Appendix B. Solutions for the second moments

Equations (2.5)–(2.7) in a steady state and after applying the spectral τ approximation (2.13), read

$$f_{ij}(\boldsymbol{k}) = L^{-1}_{ijmn}\big[f^{(0)}_{mn} + \tau\tilde{M}^F_{mn} + \tau(I^U_{mnpq} + L^{\nabla}_{mnpq} + L^{\lambda}_{mnpq} + L^{\nabla^2}_{mnpq} + L^{\lambda^2}_{mnpq})f_{pq}\big], \tag{B 1}$$

$$F_i(\boldsymbol{k}) = D^{-1}_{im}\big[F^{(0)}_m(\boldsymbol{k}) + \tau(J^U_{mn} + D^{\nabla}_{mn} + D^{\lambda}_{mn} + D^{\nabla^2}_{mn} + D^{\lambda^2}_{mn})F_n\big], \tag{B 2}$$

and $\Theta(\boldsymbol{k}) = [1 - \tau(\boldsymbol{U}\boldsymbol{\cdot}\nabla)]\Theta^{(0)}(\boldsymbol{k})$, where

$$\begin{aligned}\tilde{M}^F_{ij} \;=\; & g e_m\big\{\big[P_{im}(\boldsymbol{k}) + k^{\nabla}_{im} + k^{\lambda}_{im} - k^{\nabla^2}_{im} + k^{\lambda^2}_{im}\big]\tilde{F}_j(\boldsymbol{k}) \\ & + \big[P_{jm}(\boldsymbol{k}) - k^{\nabla}_{jm} - k^{\lambda}_{jm} - k^{\nabla^2}_{jm} + k^{\lambda^2}_{jm}\big]\tilde{F}_i(-\boldsymbol{k})\big\},\end{aligned} \tag{B 3}$$

and $\tilde{F}_i = F_i - F_i^{\Omega=0}$ and we have neglected terms $\sim O(\nabla^3, \lambda^3)$. Here the operator $D^{-1}_{ij} = \chi(\psi)(\delta_{ij} + \psi\varepsilon_{ijm}\hat{k}_m + \psi^2 k_{ij})$ is the inverse of $\delta_{ij} - \tau\tilde{D}_{ij}$ and the operator $L^{-1}_{ijmn}(\boldsymbol{\Omega})$ is the inverse of $\delta_{im}\delta_{jn} - \tau\tilde{L}_{ijmn}$ (Kleeorin & Rogachevskii 2003; Elperin *et al.* 2005), where

$$\begin{aligned}L^{-1}_{ijmn}(\boldsymbol{\Omega}) \;=\; & \tfrac{1}{2}\big[B_1\delta_{im}\delta_{jn} + B_2 k_{ijmn} + B_3(\varepsilon_{imp}\delta_{jn} + \varepsilon_{jnp}\delta_{im})\hat{k}_p + B_4\big(\delta_{im}k_{jn} + \delta_{jn}k_{im}\big) \\ & + B_5\varepsilon_{ipm}\varepsilon_{jqn}k_{pq} + B_6(\varepsilon_{imp}k_{jpn} + \varepsilon_{jnp}k_{ipm})\big],\end{aligned} \tag{B 4}$$

and $\hat{k}_i = k_i/k$, $\chi(\psi) = 1/(1+\psi^2)$, $\psi = 2\tau(k)(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{\Omega})/k$, $B_1 = 1 + \chi(2\psi)$, $B_2 = B_1 + 2 - 4\chi(\psi)$, $B_3 = 2\psi\chi(2\psi)$, $B_4 = 2\chi(\psi) - B_1$, $B_5 = 2 - B_1$ and $B_6 = 2\psi[\chi(\psi) - \chi(2\psi)]$.

To obtain solutions for the second moments, we extract in tensors D_{ij}^{Ω} and L_{ijmn}^{Ω} the parts which depend on the large-scale spatial derivatives and on the density stratification effects:

$$D_{ij}^{\Omega} = \tilde{D}_{ij} + D_{ij}^{\nabla} + D_{ij}^{\nabla^2} + D_{ij}^{\lambda} + D_{ij}^{\lambda^2} + O(\nabla^3), \tag{B 5}$$

$$L_{ijmn}^{\Omega} = \tilde{L}_{ijmn} + L_{ijmn}^{\nabla} + L_{ijmn}^{\nabla^2} + L_{ijmn}^{\lambda} + L_{ijmn}^{\lambda^2} + O(\nabla^3), \tag{B 6}$$

where

$$\tilde{L}_{ijmn} = 2\Omega_q(\varepsilon_{imp}\delta_{jn} + \varepsilon_{jnp}\delta_{im})k_{pq}, \quad L_{ijmn}^{\nabla} = -2\Omega_q(\varepsilon_{imp}\delta_{jn} - \varepsilon_{jnp}\delta_{im})k_{pq}^{\nabla}, \tag{B 7a,b}$$

$$L_{ijmn}^{\lambda} = -2\Omega_q\left[(\varepsilon_{imp}\delta_{jn} - \varepsilon_{jnp}\delta_{im})k_{pq}^{\lambda} + \frac{\mathrm{i}}{k^2}(\varepsilon_{ilq}\delta_{jn}\lambda_m - \varepsilon_{jlq}\delta_{im}\lambda_n)k_l\right], \tag{B 8}$$

$$L_{ijmn}^{\nabla^2} = 2\Omega_q(\varepsilon_{imp}\delta_{jn} + \varepsilon_{jnp}\delta_{im})k_{pq}^{\nabla^2}, \quad L_{ijmn}^{\lambda^2} = 2\Omega_q(\varepsilon_{imp}\delta_{jn} + \varepsilon_{jnp}\delta_{im})k_{pq}^{\lambda^2}, \tag{B 9a,b}$$

and $\tilde{D}_{ij} = 2\varepsilon_{ijp}\Omega_q k_{pq}$, $D_{ij}^{\nabla} = 2\varepsilon_{ijp}\Omega_q k_{pq}^{\nabla}$, $D_{ij}^{\lambda} = 2\varepsilon_{ijp}\Omega_q k_{pq}^{\lambda}$, $D_{ij}^{\nabla^2} = 2\varepsilon_{ijp}\Omega_q k_{pq}^{\nabla^2}$, and $D_{ij}^{\lambda^2} = 2\varepsilon_{ijp}\Omega_q k_{pq}^{\lambda^2}$. Here

$$k_{ij}^{\nabla} = \frac{\mathrm{i}}{2k^2}[k_i\nabla_j + k_j\nabla_i - 2k_{ij}(\boldsymbol{k}\boldsymbol{\cdot}\nabla)], \quad k_{ij}^{\lambda} = \frac{\mathrm{i}}{2k^2}[k_i\lambda_j + k_j\lambda_i - 2k_{ij}(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{\lambda})], \tag{B 10a,b}$$

$$k_{ij}^{\nabla^2} = \frac{1}{4k^2}\left[k_{ij}\nabla^2 + 2(k_{ip}\nabla_j + k_{jp}\nabla_i)\nabla_p - 4k_{ijpq}\nabla_p\nabla_q - \nabla_i\nabla_j\right], \tag{B 11}$$

$$k_{ij}^{\lambda^2} = \frac{1}{4k^2}\left[\lambda_i\tilde{\nabla}_j + \lambda_j\tilde{\nabla}_i - 2\lambda_m(k_{im}\tilde{\nabla}_j + k_{jm}\tilde{\nabla}_i + k_{ij}\tilde{\nabla}_m) + \lambda_i\lambda_j - k_{ij}\lambda^2 + 4k_{ijpq}\lambda_p\lambda_q\right], \tag{B 12a,b}$$

and $\tilde{\nabla}_i = \nabla_i - 4k_{il}\nabla_l$.

After integration in $\boldsymbol{k}$ space we obtain contributions to the Reynolds stress caused by turbulence anisotropy due to the rapid rotation:

$$\begin{aligned} f_{ij}^{(u)} &= \frac{\lambda}{20}\{[(\hat{\boldsymbol{\omega}}\times\boldsymbol{e})_i e_j + (\hat{\boldsymbol{\omega}}\times\boldsymbol{e})_j e_i](\lambda - \nabla_z) + (\hat{\boldsymbol{\omega}}\boldsymbol{\cdot}\boldsymbol{e})\,[(\hat{\boldsymbol{\omega}}\times\boldsymbol{e})_i\hat{\omega}_j \\ &\quad + (\hat{\boldsymbol{\omega}}\times\boldsymbol{e})_j\hat{\omega}_i](\lambda + \nabla_z)\}\frac{\varepsilon_u}{1+\varepsilon_u}\rho_0\langle\boldsymbol{u}'^2\rangle^{(0)}\Omega\tau_\Omega\ell_0^2. \end{aligned} \tag{B 13}$$

To derive (B 13), we use the following integrals:

$$\int k_{ij}^{\perp}\,\mathrm{d}\varphi = \pi\delta_{ij}^{(2)}, \quad \int k_{ijmn}^{\perp}\,\mathrm{d}\varphi = \frac{\pi}{4}\Delta_{ijmn}^{(2)}, \tag{B 14a,b}$$

where $\delta_{ij}^{(2)} \equiv P_{ij}(\Omega) = \delta_{ij} - \Omega_i\Omega_j/\Omega^2$ and $\Delta_{ijmn}^{(2)} = \delta_{ij}^{(2)}\delta_{mn}^{(2)} + \delta_{im}^{(2)}\delta_{jn}^{(2)} + \delta_{in}^{(2)}\delta_{jm}^{(2)}$. The tensors $k_{ij}^{\perp} = \tilde{k}_i\tilde{k}_j/\tilde{k}^2$ and $k_{ijmn}^{\perp} = \tilde{k}_i\tilde{k}_j\tilde{k}_m\tilde{k}_n/\tilde{k}^4$ are based on the vector k_i in the plane perpendicular to the angular velocity Ω_i.

The contributions to the Reynolds stress caused by the turbulent heat flux are given by (3.6) and (3.7), where the functions $\Phi_1(\omega)$ and $\Phi_2(\omega)$ are given by

$$\Phi_1(\omega) = 2\Psi_1(\omega) + \Psi_2(\omega/2), \quad \Phi_2(\omega) = 2\Psi_2(\omega) + \Psi_2(\omega/2), \tag{B 15a,b}$$

$$\Psi_1(\omega) = -\frac{6}{\omega^4}\left[\frac{\arctan\omega}{\omega}(1+\omega^2) - \frac{8\omega^2}{3} - 1 + 2\omega Y(\omega)\right], \tag{B 16}$$

$$\Psi_2(\omega) = \frac{6}{\omega^4}\left[5\frac{\arctan\omega}{\omega}(1+\omega^2) + \frac{8\omega^2}{3} - 5 - 6\omega Y(\omega)\right], \tag{B 17}$$

$\omega = 8\Omega\tau_\Omega$ and $Y(\omega) = \int_0^\omega [\arctan y/y]\,\mathrm{d}y$. When the turbulent correlation time is independent of the rotation rate, equations (3.6) and (3.7) coincide with those obtained by Kleeorin & Rogachevskii (2006).

To determine the profile of the differential rotation, we use the rotation rate dependence of the turbulent viscosity $\nu_T(\omega) = \nu_T^* \Phi_\nu(\omega)$, where $\nu_T^* = \tau_0\langle \boldsymbol{u}'^2\rangle^{(0)}/6$ and the function $\Phi_\nu(\omega)$ given by

$$\begin{aligned}\Phi_\nu(\omega) \ = \ & \frac{1}{8(1+\varepsilon_u)}\left\{(q+3)\varepsilon_u + 2\left[A_1^{(1)}(\omega) - A_1^{(1)}(0) + (q+2)C_1^{(1)}(0) + C_1^{(1)}(\omega)\right]\right. \\ & \left. + A_2^{(1)}(\omega) + C_3^{(1)}(\omega)\right\}.\end{aligned} \tag{B 18}$$

Here

$$A_1^{(1)}(\omega) = 12\left[\frac{\arctan(\omega)}{\omega}\left(1-\frac{1}{\omega^2}\right) + \frac{1}{\omega^2}[1-\ln(1+\omega^2)]\right], \tag{B 19}$$

$$A_2^{(1)}(\omega) = -12\left[\frac{\arctan(\omega)}{\omega}\left(1-\frac{3}{\omega^2}\right) + \frac{1}{\omega^2}[3-2\ln(1+\omega^2)]\right], \tag{B 20}$$

$$C_1^{(1)}(\omega) = \frac{\arctan(\omega)}{\omega}\left(3-\frac{6}{\omega^2}-\frac{1}{\omega^4}\right) + \frac{1}{\omega^2}\left(\frac{17}{3}+\frac{1}{\omega^2}-4\ln(1+\omega^2)\right), \tag{B 21}$$

where $C_3^{(1)}(\omega) = A_1^{(1)}(\omega) - 5C_1^{(1)}(\omega)$, and we use equations derived by Elperin *et al.* (2005), which are adapted for the spherical geometry.

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