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Turbulent fluxes of entropy and internal energy in temperature stratified flows

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We derive equations for the mean entropy and the mean internal energy in low-Mach-number temperature stratified turbulence (i.e. for turbulent convection or stably stratified turbulence), and show that turbulent flux of entropy is given by $\boldsymbol{F}_s = \overline{\rho}\, \overline{\boldsymbol{u} s}$, where $\overline{\rho}$ is the mean fluid density, s is fluctuation of entropy and overbars denote averaging over an ensemble of turbulent velocity fields, $\boldsymbol{u}$. We demonstrate that the turbulent flux of entropy is different from the turbulent convective flux, $\boldsymbol{F}_c = \overline{T}\, \overline{\rho}\, \overline{\boldsymbol{u} s}$, of the fluid internal energy, where $\overline{T}$ is the mean fluid temperature. This turbulent convective flux is well-known in the astrophysical and geophysical literature, and it cannot be used as a turbulent flux in the equation for the mean entropy. This result is exact for low-Mach-number temperature stratified turbulence and is independent of the model used. We also derive equations for the velocity–entropy correlation, $\overline{\boldsymbol{u} s}$, in the limits of small and large Péclet numbers, using the quasi-linear approach and the spectral τ approximation, respectively. This study is important in view of different applications to astrophysical and geophysical temperature stratified turbulence.

1. Introduction

Temperature stratified turbulence (e.g. turbulent convection or stably stratified turbulence) plays a crucial role in astrophysics (Shakura, Sunyaev & Zilitinkevich 1978; Peebles 1980; Zeldovich, Ruzmaikin & Sokolov 1983; Ruzmaikin, Sokolov & Shukurov 1988; Zeldovich, Ruzmaikin & Sokoloff 1990; Clarke & Carswell 2007) and geophysics (Monin & Yaglom 1975; Zilitinkevich 1991; Zilitinkevich *et al.* 2008, 2013; Canuto 2009). The large-scale properties of temperature stratified turbulence are determined in the framework of the mean-field approach in which all quantities are decomposed into the mean and fluctuating parts, where the fluctuating parts have zero mean values and overbars denote averaging over an ensemble of turbulent velocity fields.

In the astrophysical and geophysical literature on low-Mach-number temperature stratified turbulence two different formulae for the turbulent flux of entropy are used. The first formula coincides with the turbulent convective flux of internal energy, $\boldsymbol{F}_c = \overline{T}\, \overline{\rho}\, \overline{\boldsymbol{u} s}$, so that the equation for the mean entropy is (Kitchatinov & Mazur 2000;

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Brun, Miesch & Toomre 2004; Miesch *et al.* 2008; Jones & Kuzanyan 2009; Jones, Kuzanyan & Mitchell 2009; Käpylä, Mantere & Brandenburg 2012)

$$\overline{\rho}\left(\frac{\partial \overline{S}}{\partial t}+(\overline{\boldsymbol{U}}\cdot\nabla)\overline{S}\right)+\frac{1}{\overline{T}}\nabla\cdot(\overline{T}\,\overline{\rho}\,\overline{\boldsymbol{u}s})=\frac{1}{\overline{T}}[\nabla\cdot(K\nabla\overline{T})+\overline{J}], \tag{1.1}$$

where $\overline{\rho}$, $\overline{T}$, $\overline{S}$ and $\overline{\boldsymbol{U}}$ are the mean fluid density, temperature, specific entropy and mean velocity, respectively; $\boldsymbol{u}$ and s are the fluctuations of fluid velocity and entropy, respectively; $\overline{J}$ is the mean source and/or sink of the entropy (that also includes the viscous heating) and K is the coefficient of molecular heat conductivity. The last term on the left-hand side of (1.1) corresponds to the turbulent flux of entropy.

The other form of the turbulent flux of entropy is $\boldsymbol{F}_s=\overline{\rho}\,\overline{\boldsymbol{u}s}$, and the equation for the mean entropy is (Braginsky & Roberts 1995; Glatzmaier & Roberts 1996*a*,*b*)

$$\overline{\rho}\left(\frac{\partial \overline{S}}{\partial t}+(\overline{\boldsymbol{U}}\cdot\nabla)\overline{S}\right)+\nabla\cdot(\overline{\rho}\,\overline{\boldsymbol{u}s})=\frac{1}{\overline{T}}[\nabla\cdot(K\nabla\overline{T})+\overline{J}]. \tag{1.2}$$

Equations (1.1) and (1.2) are essentially different. In particular, the last term on the left-hand sides of (1.1) and (1.2) are different.

The goal of the present paper is to derive equations for the mean entropy and the mean internal energy which yield formulae for the turbulent flux of entropy and the turbulent flux of internal energy, and to clarify which equation for the mean entropy ((1.1) or (1.2)) used in temperature stratified turbulence, is correct. When the fluid temperature profile is not uniform, the above question is crucial.

2. Turbulent convective flux of mean internal energy and turbulent flux of mean entropy

In this section we will derive equations for the mean entropy and the mean internal energy. We consider low-Mach-number temperature stratified fluid flows.

2.1. *Governing equations*

The budget equation for the instantaneous internal energy density $E=c_vT$ is (Landau & Lifshitz 1959)

$$\frac{\partial(\rho E)}{\partial t}+\nabla\cdot(\rho\boldsymbol{U}W-\boldsymbol{U}P-K\nabla T)=Q, \tag{2.1}$$

where $\boldsymbol{U}$ is the instantaneous velocity determined by the Navier–Stokes equation for fluid motion, ρ, T and P are the instantaneous density, temperature and pressure, respectively, which satisfy the equation of state for a perfect gas, K is the coefficient of molecular heat conductivity, $W=c_pT=c_vT+P/\rho=E+P/\rho$ is the instantaneous enthalpy, where c_v and c_p are the specific heats at constant volume and pressure, and $Q=-P\nabla\cdot\boldsymbol{U}+\hat{\sigma}_{ij}(\boldsymbol{U})\nabla_jU_i$, where $\hat{\sigma}(\boldsymbol{U})=2\nu\rho\hat{\mathcal{S}}(\boldsymbol{U})$, $\hat{\mathcal{S}}(\boldsymbol{U})=\mathcal{S}_{ij}=(U_{i,j}+U_{j,i})/2-(\delta_{ij}\nabla\cdot\boldsymbol{U})/3$, ν is the kinematic viscosity, and δ_{ij} is the Kronecker tensor. The instantaneous density ρ is determined by the continuity equation:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\,\boldsymbol{U})=0. \tag{2.2}$$

The budget equation for the instantaneous kinetic energy density $\rho \boldsymbol{U}^2/2$ is (Landau & Lifshitz 1959)

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho \boldsymbol{U}^2\right) + \boldsymbol{\nabla}\cdot\left[\boldsymbol{U}\left(\frac{1}{2}\rho \boldsymbol{U}^2 + P\right) - \boldsymbol{U}\hat{\sigma}(\boldsymbol{U})\right] = -Q. \tag{2.3}$$

The sum of (2.1) and (2.3) yields the conservation law for the instantaneous total (kinetic plus internal) energy densities $(\rho \boldsymbol{U}^2/2 + \rho E)$ (Landau & Lifshitz 1959):

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho \boldsymbol{U}^2 + \rho E\right) + \boldsymbol{\nabla}\cdot\left[\boldsymbol{U}\left(\frac{1}{2}\rho \boldsymbol{U}^2 + \rho W\right) - \boldsymbol{U}\hat{\sigma}(\boldsymbol{U}) - K\boldsymbol{\nabla}T\right] = 0. \tag{2.4}$$

2.2. *Turbulent convective flux and equation for mean internal energy*

Averaging (2.1) over the ensemble we obtain the budget equation for the mean internal energy density $\overline{E} = c_v\overline{T}$:

$$\frac{\partial(\overline{\rho}\,\overline{E})}{\partial t} + \boldsymbol{\nabla}\cdot(\overline{\rho}\,\overline{\boldsymbol{U}}\,\overline{E} + \overline{\rho}\,\overline{\boldsymbol{u}w} - \overline{\boldsymbol{u}p} - K\boldsymbol{\nabla}\overline{T}) = \overline{Q}, \tag{2.5}$$

where $\boldsymbol{u}$ is the velocity fluctuations, $w = \overline{T}\,s + p/\overline{\rho}$ are the enthalpy fluctuations, s and p are the entropy and pressure fluctuations, respectively, and $\overline{Q} = -\overline{P}\boldsymbol{\nabla}\cdot\overline{\boldsymbol{U}} - \overline{p\boldsymbol{\nabla}\cdot\boldsymbol{u}} + \hat{\sigma}_{ij}(\overline{\boldsymbol{U}})\nabla_j\overline{U}_i + \overline{\hat{\sigma}_{ij}(\boldsymbol{u})\nabla_j u_i}$. In the derivation of (2.5) we used the identity $\overline{W} = \overline{E} + \overline{P}/\overline{\rho}$, and since we consider a low-Mach-number turbulent flow, we took into account that $|\overline{\boldsymbol{u}\rho}|/|\overline{\rho}| \ll |\overline{\boldsymbol{u}s}|/|\overline{S}|$ (Chassaing *et al.* 2002). The turbulent flux of enthalpy is

$$\overline{\boldsymbol{u}w} = \overline{T}\,\overline{\boldsymbol{u}s} + \frac{\overline{\boldsymbol{u}p}}{\overline{\rho}}. \tag{2.6}$$

Since $\overline{W} = \overline{E} + \overline{P}/\overline{\rho}$ we obtain that $\overline{\rho}\,\overline{\boldsymbol{U}}\,\overline{E} = \overline{\boldsymbol{U}}\,(\overline{\rho}\overline{W} - \overline{P})$. Substituting the latter equation and (2.6) into (2.5), we obtain

$$\frac{\partial\overline{\rho}\,\overline{E}}{\partial t} + \boldsymbol{\nabla}\cdot(\overline{\rho}\,\overline{\boldsymbol{U}}\,\overline{E} + \overline{T}\,\overline{\rho}\,\overline{\boldsymbol{u}s} - K\boldsymbol{\nabla}\overline{T}) = \overline{Q}. \tag{2.7}$$

The mean fluid density $\overline{\rho}$ is determined by the continuity equation:

$$\frac{\partial\overline{\rho}}{\partial t} + \boldsymbol{\nabla}\cdot(\overline{\rho}\,\overline{\boldsymbol{U}}) = 0. \tag{2.8}$$

Equations (2.7) and (2.8) yield the following equation for the evolution of the mean internal energy:

$$\overline{\rho}\left(\frac{\partial\overline{E}}{\partial t} + (\overline{\boldsymbol{U}}\cdot\boldsymbol{\nabla})\overline{E}\right) + \boldsymbol{\nabla}\cdot(\overline{T}\,\overline{\rho}\,\overline{\boldsymbol{u}s} - K\boldsymbol{\nabla}\overline{T}) = \overline{Q}, \tag{2.9}$$

where the turbulent convective flux of the mean internal energy is

$$\boldsymbol{F}_c = \overline{T}\,\overline{\rho}\,\overline{\boldsymbol{u}s}. \tag{2.10}$$

2.3. *Equation for the sum of mean and turbulent kinetic energies and conservation law for total mean energy*

Averaging (2.3) for the instantaneous kinetic energy density $\rho U^2/2$ we obtain an equation for the sum of the mean and turbulent kinetic energies $\overline{\rho}\,\overline{\boldsymbol{U}^2}/2+\overline{\rho}\,\overline{\boldsymbol{u}^2}/2$:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{\rho}\,\overline{\boldsymbol{U}^2}+\frac{1}{2}\overline{\rho}\,\overline{\boldsymbol{u}^2}\right)+\boldsymbol{\nabla}\cdot\left[\overline{\boldsymbol{U}}\left(\frac{1}{2}\overline{\rho}\overline{\boldsymbol{U}}^2+\overline{P}\right)-\overline{\boldsymbol{U}}\hat{\sigma}(\overline{\boldsymbol{U}})\right.$$
$$\left.+\,\overline{\boldsymbol{u}\left(\frac{1}{2}\rho\boldsymbol{u}^2+p\right)}-\overline{\boldsymbol{u}\hat{\sigma}(\boldsymbol{u})}\right]=-\overline{Q}. \tag{2.11}$$

The sum of (2.7) and (2.11) yields the conservation law for the total mean energy $E_{tot}=\overline{\rho}\overline{\boldsymbol{U}}^2/2+\overline{\rho}\,\overline{\boldsymbol{u}^2}/2+\overline{\rho}\,\overline{E}$:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{\rho}\,\overline{\boldsymbol{U}}^2+\frac{1}{2}\overline{\rho}\,\overline{\boldsymbol{u}^2}+\overline{\rho}\,\overline{E}\right)+\boldsymbol{\nabla}\cdot\left[\overline{\boldsymbol{U}}\left(\frac{1}{2}\overline{\rho}\overline{\boldsymbol{U}}^2+\overline{P}+\overline{\rho}\,\overline{E}\right)-\overline{\boldsymbol{U}}\hat{\sigma}(\overline{\boldsymbol{U}})\right.$$
$$\left.-\,\overline{\boldsymbol{u}\hat{\sigma}(\boldsymbol{u})}+\overline{\boldsymbol{u}\left(\frac{1}{2}\rho\boldsymbol{u}^2+p\right)}+\overline{T}\,\overline{\rho}\,\overline{\boldsymbol{u}s}-K\boldsymbol{\nabla}\overline{T}\right]=0. \tag{2.12}$$

This equation contains the turbulent convective flux $\boldsymbol{F}_c=\overline{T}\,\overline{\rho}\,\overline{\boldsymbol{u}s}$. The conservation law (2.12) for the total mean energy E_{tot} can be rewritten in terms of the mean, $\overline{\boldsymbol{U}}\,\overline{W}$, and turbulent, $\overline{\boldsymbol{u}w}$, fluxes of enthalpy (see (2.6)):

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{\rho}\overline{\boldsymbol{U}}^2+\frac{1}{2}\overline{\rho}\,\overline{\boldsymbol{u}^2}+\overline{\rho}\,\overline{E}\right)+\boldsymbol{\nabla}\cdot\left[\overline{\boldsymbol{U}}\left(\frac{1}{2}\overline{\rho}\,\overline{\boldsymbol{U}}^2+\overline{\rho}\,\overline{W}\right)-\overline{\boldsymbol{U}}\hat{\sigma}(\overline{\boldsymbol{U}})\right.$$
$$\left.-\,\overline{\boldsymbol{u}\hat{\sigma}(\boldsymbol{u})}+\overline{\boldsymbol{u}\left(\frac{1}{2}\rho\boldsymbol{u}^2\right)}+\overline{\rho}\,\overline{\boldsymbol{u}w}-K\boldsymbol{\nabla}\overline{T}\right]=0. \tag{2.13}$$

2.4. *Equation for mean entropy*

The evolution equation for the instantaneous entropy $S=c_v\,\ln(P\rho^{-\gamma})$ is (Landau & Lifshitz 1959)

$$\left(\frac{\partial}{\partial t}+\boldsymbol{U}\cdot\boldsymbol{\nabla}\right)S=\frac{1}{\rho T}[\boldsymbol{\nabla}\cdot(K\boldsymbol{\nabla}T)+J], \tag{2.14}$$

where $\gamma=c_p/c_v$ is the ratio of specific heats and J is a source and/or sink of entropy (that also includes the viscous heating). Multiplying the equation for the entropy (2.14) by the fluid density ρ, and the continuity equation (2.2) by the fluid entropy S, and add them, we obtain the following equation:

$$\frac{\partial(\rho\,S)}{\partial t}+\boldsymbol{\nabla}\cdot(\rho\,\boldsymbol{U}S)=\frac{1}{T}[\boldsymbol{\nabla}\cdot(K\boldsymbol{\nabla}T)+J]. \tag{2.15}$$

The second term on the left-hand sides of (2.14) and (2.15), which contributes to the turbulent diffusion of the mean entropy, does not contain the temperature field. This is a reason why the turbulent flux of the mean entropy for stratified turbulence cannot contain the mean temperature.

Averaging (2.15) over the ensemble we obtain the equation for the mean entropy, $\overline{S}$:

$$\frac{\partial(\overline{\rho}\,\overline{S})}{\partial t} + \nabla\cdot(\overline{\rho}\,\overline{\boldsymbol{U}}\,\overline{S} + \overline{\rho}\,\overline{\boldsymbol{u}s}) = \frac{1}{\overline{T}}[\nabla\cdot(K\nabla\overline{T}) + \overline{J}]. \tag{2.16}$$

In the derivation of (2.16) we have taken into account that, for a low-Mach-number turbulent flow, $|\overline{\rho\, s}| \ll |\overline{\rho}|\,|\overline{S}|$, $|\overline{\rho\, s}|/|\overline{\rho}| \ll |\overline{\boldsymbol{u}s}|/u_{rms}$ and $|\overline{\boldsymbol{u}\rho}|/|\overline{\rho}| \ll |\overline{\boldsymbol{u}s}|/|\overline{S}|$ (Chassaing *et al.* 2002). To get the simplest form of the molecular diffusion term and the source term on the right-hand side of (2.16) we assumed that: (*a*) the temperature fluctuations, θ, are much smaller than the mean fluid temperature, $\overline{T}$, i.e. $|\theta| \ll \overline{T}$; (*b*) in the framework of the mean-field theory there is a separation of scales, $\ell_0 \ll L_T$, where L_T is the characteristic scale of the mean temperature variation and ℓ_0 is the integral scale of turbulence (the random velocity field); (*c*) the coefficient of molecular heat conductivity K is independent of the temperature fluctuations and (*d*) fluctuations of the source or sink of the entropy, J', are independent of the temperature fluctuations. Equations (2.8) and (2.16) yield the following equation for the evolution of the mean entropy:

$$\overline{\rho}\left(\frac{\partial\overline{S}}{\partial t} + (\overline{\boldsymbol{U}}\cdot\nabla)\overline{S}\right) + \nabla\cdot(\overline{\rho}\,\overline{\boldsymbol{u}s}) = \frac{1}{\overline{T}}[\nabla\cdot(K\nabla\overline{T}) + \overline{J}], \tag{2.17}$$

which coincides with (1.2), and the turbulent flux of the mean entropy for stratified turbulence with non-uniform profiles of the mean fluid temperature and density is

$$\boldsymbol{F}_s = \overline{\rho}\,\overline{\boldsymbol{u}s}. \tag{2.18}$$

Other forms of the turbulent flux of entropy (see (1.1)) used in the astrophysical and geophysical literature are incorrect. This is an exact statement for low-Mach-number temperature stratified turbulence and is independent of the model.

3. The velocity–entropy correlation

Now let us determine the velocity–entropy correlation, $\overline{\boldsymbol{u}s}$. For simplicity we consider turbulent flows with a zero mean velocity, $\overline{\boldsymbol{U}} = 0$. Subtracting (2.16) from (2.15) we obtain the equation for the entropy fluctuations:

$$\frac{\partial s}{\partial t} + \mathcal{N} - \chi\nabla^2 s = I, \tag{3.1}$$

where $\mathcal{N} = (\overline{\rho})^{-1}\nabla\cdot[\overline{\rho}(\boldsymbol{u}s - \overline{\boldsymbol{u}s})]$ is the nonlinear term, $\chi = K/c_p\overline{\rho}$ is the molecular diffusion coefficient of entropy and $I = -(\overline{\rho})^{-1}\nabla\cdot(\overline{\rho}\overline{S}\,\boldsymbol{u}) = -(\boldsymbol{u}\cdot\nabla)\overline{S}$ is the source term. In the derivation of (3.1) we took into account the anelastic approximation, $[\nabla\cdot(\overline{\rho}\,\boldsymbol{u}) = 0]$, we assumed that fluctuations of the source or sink of the entropy, J', are very small, and we also assumed that the molecular diffusion term can be simplified as $(\overline{\rho}\overline{T})^{-1}\nabla\cdot[(K/c_p)\nabla(\overline{T}s)] \sim \chi\nabla^2 s$. In the latter estimate we assumed that: (i) the temperature fluctuations are much smaller than the mean fluid temperature; (ii) $\ell_0 \ll L_T$; (iii) the coefficient of molecular heat conductivity is independent of the coordinates; (iv) for low Mach numbers the entropy fluctuations are given by

$$s = c_p\left(\frac{\theta}{\overline{T}} + \frac{(1-\gamma)p}{c_s^2\overline{\rho}}\right) \approx c_p\frac{\theta}{\overline{T}}, \tag{3.2}$$

where $c_s = (\gamma \overline{P}/\overline{\rho})^{1/2}$ is the sound speed. Equation (3.2) follows from the equation of state for a perfect gas: $P = (c_p - c_v)\rho T$ and the definition of entropy. Let us derive the equation for the velocity–entropy correlation $\overline{\boldsymbol{u}s}$ in two limiting cases for small and large Péclet number, where $Pe = u_0 \ell_0/\chi$ is the Péclet number and u_0 is the characteristic turbulent velocity in the integral scale of turbulence, ℓ_0.

3.1. *Small Péclet numbers*

In order to study entropy fluctuations for small Péclet numbers we use a quasi-linear approach (Moffatt 1978; Krause & Raedler 1980), that for a given velocity field is valid only for small Péclet numbers ($Pe \ll 1$). In the framework of this approximation we neglect the nonlinear term and keep the molecular diffusion term in (3.1). We rewrite (3.1) in Fourier space and solve this equation. The solution is

$$s(\omega, \boldsymbol{k}) = G_\chi(\omega, \boldsymbol{k}) I(\omega, \boldsymbol{k}), \tag{3.3}$$

where $G_\chi(\omega, \boldsymbol{k}) = (\chi k^2 + \mathrm{i}\omega)^{-1}$, ω is the frequency and $\boldsymbol{k}$ is the wave vector. We apply a standard two-scale approach, whereby the non-instantaneous two-point second-order correlation function is written as follows:

$$\begin{aligned} \overline{u_i(t_1, \boldsymbol{x})\, s(t_2, \boldsymbol{y})} &= \int \overline{u_i(\omega_1, \boldsymbol{k}_1) s(\omega_2, \boldsymbol{k}_2)} \\ &\quad \times \exp[\mathrm{i}(\boldsymbol{k}_1 \boldsymbol{\cdot} \boldsymbol{x} + \boldsymbol{k}_2 \boldsymbol{\cdot} \boldsymbol{y}) + \mathrm{i}(\omega_1 t_1 + \omega_2 t_2)]\, \mathrm{d}\omega_1\, \mathrm{d}\omega_2\, \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2 \\ &= \int F_i(\omega, \boldsymbol{k}) \exp[\mathrm{i}\boldsymbol{k} \boldsymbol{\cdot} \boldsymbol{r} + \mathrm{i}\omega\, \tilde{\tau}]\, \mathrm{d}\omega\, \mathrm{d}\boldsymbol{k}, \end{aligned} \tag{3.4}$$

where we use large scale variables: $\boldsymbol{R} = (\boldsymbol{x} + \boldsymbol{y})/2$, $\boldsymbol{K} = \boldsymbol{k}_1 + \boldsymbol{k}_2$, $t = (t_1 + t_2)/2$, $\varOmega = \omega_1 + \omega_2$, and small scale variables: $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$, $\boldsymbol{k} = (\boldsymbol{k}_1 - \boldsymbol{k}_2)/2$, $\tilde{\tau} = t_1 - t_2$, $\omega = (\omega_1 - \omega_2)/2$, and

$$F_i(\omega, \boldsymbol{k}) = \int \overline{u_i(\omega_1, \boldsymbol{k}_1)\, s(\omega_2, \boldsymbol{k}_2)} \exp[\mathrm{i}\varOmega t + \mathrm{i}\boldsymbol{K} \boldsymbol{\cdot} \boldsymbol{R}]\, \mathrm{d}\varOmega\, \mathrm{d}\boldsymbol{K}. \tag{3.5}$$

Here $\omega_1 = \omega + \varOmega/2$, $\omega_2 = -\omega + \varOmega/2$, $\boldsymbol{k}_1 = \boldsymbol{k} + \boldsymbol{K}/2$, and $\boldsymbol{k}_2 = -\boldsymbol{k} + \boldsymbol{K}/2$ (see e.g. Roberts & Soward 1975). We assume here that there is a separation of scales, i.e. the maximum scale of random motions ℓ_0 is much smaller than the characteristic scales of inhomogeneities of the mean entropy and fluid density. Equations (3.3)–(3.5) yield the velocity–entropy correlation $\overline{\boldsymbol{u}s}$:

$$\overline{u_i s} = \int \overline{u_i(\omega, \boldsymbol{k}) I(-\omega, -\boldsymbol{k})} G_\chi^*\, \mathrm{d}\omega\, \mathrm{d}\boldsymbol{k} = -(\nabla_j \overline{S}) \int \overline{u_i(\omega, \boldsymbol{k}) u_j(-\omega, -\boldsymbol{k})} G_\chi^*\, \mathrm{d}\omega\, \mathrm{d}\boldsymbol{k}. \tag{3.6}$$

We use the simple model for the second moments, $\overline{u_i(\omega, \boldsymbol{k}) u_j(-\omega, -\boldsymbol{k})}$, of a random velocity field in Fourier space for inhomogeneous, isotropic and incompressible flow:

$$\overline{u_i(\omega, \boldsymbol{k}) u_j(-\omega, -\boldsymbol{k})} = \frac{\tilde{E}(k)\varPhi(\omega)}{8\pi k^2} \left[\delta_{ij} - \frac{k_i k_j}{k^2} + \frac{\mathrm{i}}{2k^2}(k_i \nabla_j - k_j \nabla_i) \right] \overline{\boldsymbol{u}^2}. \tag{3.7}$$

This model is obtained using symmetry arguments (see e.g. Batchelor 1971). Here δ_{ij} is the Kronecker tensor, the energy spectrum function is $\tilde{E}(k) = C_E k_0^{-1}\, (q-1)\, (k/k_0)^{-q}$ for the range of wavenumbers $k_0 < k < k_d$, the wavenumber $k_0 = 1/\ell_0$, the length ℓ_0 is the maximum scale of random motions, the exponent $1 < q < 3$, and the coefficient

$C_E = [1 - (k_0/k_d)^{q-1}]^{-1}$. We use the frequency function $\Phi(\omega)$ in the form of the Lorentz profile: $\Phi(\omega) = [\pi\tau_c\,(\omega^2 + \tau_c^{-2})]^{-1}$, where τ_c is the correlation time of a random velocity field. This model for the frequency function corresponds to the following non-instantaneous correlation function: $\overline{u_i(t)u_j(t+\tau)} \propto \exp(-\tau/\tau_c)$.

We use (3.6) and (3.7), and after integration in ω-space and in $\boldsymbol{k}$-space in (3.6) we obtain the formula for the velocity–entropy correlation $\overline{u_i\, s}$:

$$\overline{u_i\, s} = -\chi_T \nabla_i \overline{S}, \tag{3.8}$$

$$\chi_T = C_\chi u_0 \ell_0\, Pe, \quad C_\chi = \frac{q-1}{3(q+1)}\left[\frac{1-(k_0/k_d)^{q+1}}{1-(k_0/k_d)^{q-1}}\right], \tag{3.9a,b}$$

where χ_T is the coefficient of turbulent diffusion of the mean entropy and $u_0 = \sqrt{\overline{\boldsymbol{u}^2}}$ is the characteristic velocity in the maximum scale of random motions. Here we used that $I_0 \equiv \int \Phi(\omega) G_\chi(\omega, \boldsymbol{k})\, \mathrm{d}\omega = \tau_c/(1 + \tau_c \eta k^2)$, and for small Péclet numbers $I_0 \approx (\eta k^2)^{-1}$. The coefficient $C_\chi = 1/3$ for a narrow range of the random velocity field in the wavenumbers, $k_d - k_0 \ll k_d$, and $C_\chi = (q-1)/3(q+1)$ for a wide range in the wavenumbers, $k_d \gg k_0$. Contributions (which are proportional to $\boldsymbol{\nabla}$ in (3.7)) to the velocity–entropy correlation $\overline{u_i\, s}$, after the integration over the angles in $\boldsymbol{k}$-space, vanish. However, the coefficient of turbulent diffusion χ_T depends on the coordinates, due to the inhomogeneous turbulence. Equations (3.8) and (3.9*a,b*) are in agreement with those obtained by means of dimensional arguments (Batchelor, Howells & Townsend 1959) and by the Lagrangian-history direct-interaction approximation (Kraichnan 1968).

3.2. *Large Péclet numbers*

In this subsection we derive a formula for the velocity–entropy correlation $\overline{\boldsymbol{u}s}$ using the spectral τ approach that is valid for large Péclet and Reynolds numbers ($Pe \gg 1$). Using (3.1) written in Fourier space we derive an equation for the instantaneous two-point second-order correlation functions $F_i(t, \boldsymbol{k}) = \overline{u_i(t, \boldsymbol{k})\, s(t, -\boldsymbol{k})}$:

$$\frac{\mathrm{d}F_i}{\mathrm{d}t} = \overline{u_i(t, \boldsymbol{k})\, I(t, -\boldsymbol{k})} + \hat{\mathcal{M}} F_i^{(III)}(\boldsymbol{k}), \tag{3.10}$$

where $\hat{\mathcal{M}} F_i^{(III)}(\boldsymbol{k}) = -[\overline{u_i \mathcal{N}} + \overline{(\partial u_i/\partial t)\, s} - \chi\, \overline{u_i\, \nabla^2 s}]_{\boldsymbol{k}}$ are the third-order moment terms appearing due to the nonlinear terms which also include the molecular diffusion term.

The equation for the second moment includes the first-order spatial differential operators applied to the third-order moments. A problem arises regarding how to close the system, i.e. how to express the third-order terms $\hat{\mathcal{M}} F^{(III)}$ through the lower moments $F^{(II)}$ (Orszag 1970; Monin & Yaglom 1975; McComb 1990). We use the spectral τ approximation which postulates that the deviations of the third-moment terms, $\hat{\mathcal{M}} F^{(III)}(\boldsymbol{k})$, from the contributions to these terms by the background turbulence, $\hat{\mathcal{M}} F^{(III,0)}(\boldsymbol{k})$, can be expressed through similar deviations of the second moments, $F^{(II)}(\boldsymbol{k}) - F^{(II,0)}(\boldsymbol{k})$ (Orszag 1970; Pouquet, Frisch & Leorat 1976):

$$\hat{\mathcal{M}} F^{(III)}(\boldsymbol{k}) - \hat{\mathcal{M}} F^{(III,0)}(\boldsymbol{k}) = -\frac{1}{\tau_r(k)}[F^{(II)}(\boldsymbol{k}) - F^{(II,0)}(\boldsymbol{k})], \tag{3.11}$$

where $\tau_r(k)$ is the scale-dependent relaxation time, which can be identified with the correlation time $\tau(k)$ of the turbulent velocity field for large Reynolds and

Péclet numbers. The functions with the superscript (0) correspond to the background turbulence with a zero gradient of the mean entropy. Validation of the τ approximation for different situations has been performed in numerous numerical simulations and analytical studies (see e.g. the review by Brandenburg & Subramanian 2005; and also discussions by Rogachevskii & Kleeorin 2007, Rogachevskii *et al.* 2011).

Note that the contributions of the terms with the superscript (0) vanish because when the gradient of the mean entropy is zero, the turbulent heat flux and the entropy fluctuations vanish. Consequently, (3.11) for $\hat{\mathcal{M}}F_i^{(III)}(\boldsymbol{k})$ is reduced to $\hat{\mathcal{M}}F_i^{(III)}(\boldsymbol{k}) = -F_i(\boldsymbol{k})/\tau(k)$. We also assume that the characteristic time of variation of the second moment $F_i(\boldsymbol{k})$ is substantially larger than the correlation time $\tau(k)$ for all turbulence scales. Therefore, in a steady state (3.10) yields the following formula for the velocity–entropy correlation:

$$F_i = \int \tau(k)\langle u_i(t,\boldsymbol{k})I(t,-\boldsymbol{k})\rangle\,\mathrm{d}\boldsymbol{k} = -(\nabla_j\overline{S})\int \tau(k)\langle u_i(\boldsymbol{k})u_j(-\boldsymbol{k})\rangle\,\mathrm{d}\boldsymbol{k}. \tag{3.12}$$

We use the following simple model for the second moments, $\overline{u_i(\boldsymbol{k})u_j(-\boldsymbol{k})}$, of a turbulent velocity field in Fourier space for inhomogeneous, isotropic and incompressible flow for large Reynolds numbers:

$$\overline{u_i(\boldsymbol{k})u_j(-\boldsymbol{k})} = \frac{\tilde{E}(k)}{8\pi k^2}\left[\delta_{ij} - \frac{k_ik_j}{k^2} + \frac{\mathrm{i}}{2k^2}(k_i\nabla_j - k_j\nabla_i)\right]\overline{\boldsymbol{u}^2}. \tag{3.13}$$

This model is obtained using symmetry arguments (see e.g. Batchelor 1971). After integration in $\boldsymbol{k}$-space of (3.12) we arrive at an equation for the velocity–entropy correlation, $\langle u_is\rangle$:

$$\langle u_is\rangle = -\chi_T\nabla_i\overline{S}, \quad \chi_T = u_0\ell_0/3, \tag{3.14}$$

where $u_0 = \sqrt{\overline{\boldsymbol{u}^2}}$ is the characteristic turbulent velocity. In the derivation of (3.14) we used the following expression for the turbulent correlation time: $\tau(k) = 2\tau_0(k/k_0)^{1-q}$, where $\tau_0 = \ell_0/u_0$ is the characteristic turbulent time. Contributions (which are proportional to $\boldsymbol{\nabla}$ in (3.13)) to the velocity–entropy correlation $\overline{u_is}$, after integration over the angles in $\boldsymbol{k}$-space, vanish. However, the coefficient of turbulent diffusion χ_T depends on the coordinates, due to the inhomogeneous turbulence. Therefore, the formulae for the velocity–entropy correlation, $\langle u_is\rangle$, are similar for small and large Péclet numbers, while the coefficients of turbulent diffusion of the mean entropy are different in these two limiting cases. Equation (3.14) is in agreement with that derived by means of the path integral approach (Elperin, Kleeorin & Rogachevskii 1995), by dimensional arguments and by the renormalization procedure used for large Péclet numbers (Elperin, Kleeorin & Rogachevskii 1996).

4. Conclusions

In the present study we have demonstrated that for a low-Mach-number compressible fluid flow the turbulent flux of entropy, $\boldsymbol{F}_s = \overline{\rho}\,\overline{\boldsymbol{u}s} = -\overline{\rho}\chi_T\boldsymbol{\nabla}\overline{S}$, is different from the turbulent convective flux of the mean internal energy, $\boldsymbol{F}_c = \overline{T}\,\overline{\rho}\,\overline{\boldsymbol{u}s} = -\overline{T}\,\overline{\rho}\chi_T\boldsymbol{\nabla}\overline{S}$. As follows from the analysis performed in § 3, the coefficient of turbulent diffusion of entropy χ_T depends on the Péclet number. For small Péclet numbers, applying the quasi-linear approach for an isotropic and inhomogeneous background random velocity field we obtain the following coefficient of turbulent diffusion of the mean entropy: $\chi_T = C_\chi Pe u_0 \ell_0$, where the constant C_χ depends on the energy spectrum of the random velocity field. For large Péclet and Reynolds numbers, applying the spectral τ approximation we get the following coefficient of turbulent diffusion of the mean entropy: $\chi_T = u_0 \ell_0/3$.

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