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Letter to the Editor

Helicity balance and steady-state strength of the dynamo generated galactic magnetic field

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Abstract. We demonstrate that the inclusion of the helicity flux in the magnetic helicity balance in the nonlinear stage of galactic dynamo action results in a radical change in the magnetic field dynamics. The equilibrium value of the large-scale magnetic field is then approximately the equipartition level. This is in contrast to the situation without the flux of helicity, when the magnetic helicity is conserved locally, which leads to substantially subequipartition values for the equilibrium large-scale magnetic field.

Key words: galaxies: magnetic fields – magnetic fields – turbulence

1. Introduction

The large-scale magnetic fields of galaxies are thought to be generated by a galactic dynamo due to the simultaneous action of the helicity of interstellar turbulence and differential rotation (see, e.g., Ruzmaikin et al. 1988). The kinematic stage of the galactic dynamo, i.e. the evolution of a weak magnetic field with negligible influence on the turbulent flows, seems to be clear, while the nonlinear stage of dynamo evolution is a topic of intensive discussions (for reviews, see Beck et al. 1996, Kulsrud 1999). The most contentious issue is the question of the equilibrium magnetic field strength at which dynamo action saturates.

A naive viewpoint is that the saturation level for the *large-scale* magnetic field is given by the equipartition between kinetic energy and the energy of the large-scale magnetic field $\mathbf{B}$ (see, e.g., Zeldovich et al. 1983). The motivation is that the equations describing large-scale dynamo action contain the mean, but not the total, magnetic field. This naive outlook leads to models of dynamo generated magnetic fields which are in basic agreement with the available observational information.

Vainshtein and Cattaneo (1992) formulated a more sophisticated argument, suggesting that the equilibrium magnetic field should be determined by a balance between the kinetic energy and the energy of the *total* magnetic field. The simplest models of dynamo generation then result in the estimate $b/B \sim Rm^{1/2}$, where b is the small-scale magnetic field, and the magnetic Reynolds number $Rm \approx 10^8$ for the interstellar turbulence (or even much larger if a microscopic diffusivity instead of ambipolar diffusion is used; cf. Brandenburg & Zweibel, 1995). Thus the ideas of Vainshtein and Cattaneo lead to the conclusion that a dynamo generated large-scale galactic magnetic field must be negligible in comparison with that observed, and so the generation of the observed field must be connected with another mechanism. However, no other general and realistic mechanism for galactic magnetic field generation is currently available.

The arguments of Vainshtein and Cattaneo do not seem inevitable. For example, a dynamo generated magnetic field can itself produce helicity, so the nonlinear effects can even amplify rather than suppress field generation at the initial stages of nonlinear evolution (Parker 1992, Moss et al. 1999); other suggestions are discussed by, e.g., Beck et al. (1996), Kulsrud (1999), Field et al. (1999) and Blackman & Field (1999). In particular, Blackman & Field (2000) argue that the Rm-dependent quenching seen in the simulations of Cattaneo & Hughes (1996) is a consequence of helicity conservation when using closed or periodic boundaries, while simulations with open boundaries by Brandenburg & Donner (1997) (see also Brandenburg 2000) do not show this effect.

The aim of this letter is to demonstrate that with open boundaries the scenario of Vainshtein and Cattaneo results in basically the same estimate for the equilibrium magnetic field strength as is given by the naive viewpoint.

The essence of our arguments can be presented as follows. According to Vainshtein and Cattaneo, the suppression of dynamo action by the small-scale magnetic field that is generated together with the large-scale is connected with the magnetic helicity of the small-scale magnetic field. Because the total magnetic helicity is an inviscid invariant of motion, the magnetic helicity of the small-scale magnetic field can be connected with the magnetic helicity of the large-scale magnetic field. The governing equation for magnetic helicity has been proposed

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by Kleeorin and Ruzmaikin (1982; see the discussion by Zeldovich et al., 1983), investigated by Kleeorin et al. (1995) for stellar dynamos, and self-consistently derived by Kleeorin and Rogachevskii (1999). During nonlinear stages of the dynamo, the α-effect is thought to be determined by the hydrodynamic and magnetic helicities, so a closed system of equations can be obtained for the evolution of the magnetic field and the α-coefficient (see below, Sect. 2). This governing system (with helicity locally conserved) leads to magnetic field behaviour which is consistent with the prediction of Vainshtein and Cattaneo (we are grateful to M. Reshetnyak, who provided us with the relevant numerical results, which will be published elsewhere).

We stress that Eq. (4) takes into account the local helicity balance at a given point inside the galactic disc $|z| < h, r < R$, where r, φ, z are cylindrical coordinates. However, the kinematic galactic dynamo is impossible without a turbulent flux of magnetic field through the surface $|z| = h$ (see, e.g., Zeldovich et al. 1983, Ch. 11). It is more than natural to believe that this flux can transport magnetic helicity to the outside of the disc. The methods of Kleeorin and Rogachevskii (1999) allow us to introduce the corresponding term into the governing equations for the galactic dynamo. We demonstrate by numerical simulations, and to some extent analytically, that this term leads to a drastic change in the magnetic field evolution. Now the steady-state large-scale magnetic field strength is approximately in equipartition with the kinetic energy of the interstellar turbulence.

2. Equations for magnetic helicity

Following Kleeorin and Ruzmaikin (1982), we parameterize the back-reaction of dynamo generated magnetic field in terms of a differential equation for the α-coefficient, using arguments from the magnetic helicity conservation law. It is necessary to introduce the large-scale vector potential $\mathbf{A}$, small-scale vector potential $\mathbf{a}$, and the corresponding representations for the magnetic fields, $\mathbf{B}$ and $\mathbf{b}$. We then write the total magnetic field as $\mathbf{H} = \mathbf{B} + \mathbf{b}$, and the total vector potential as $\boldsymbol{\mathcal{A}} = \mathbf{A} + \mathbf{a}$, thus decomposing the fields into mean and fluctuating parts. The equation for the vector potential $\boldsymbol{\mathcal{A}}$ follows from the induction equation for the total magnetic field $\mathbf{H}$

$$\partial \boldsymbol{\mathcal{A}}/\partial t = \mathbf{v} \times \mathbf{H} - \eta\, \mathbf{curl}\, \mathbf{H} + \boldsymbol{\nabla} \varphi \;, \quad (1)$$

where $\mathbf{v} = \mathbf{V} + \mathbf{u}$, and $\mathbf{V} = \langle \mathbf{v} \rangle$ is the mean fluid velocity field, η is the magnetic diffusion due to the electrical conductivity of the fluid, φ is an arbitrary scalar function. Now we multiply the induction equation for the total magnetic field $\mathbf{H}$ by $\mathbf{a}$ and Eq. (1) by $\mathbf{b}$, add them and average over the ensemble of turbulent fields. This yields an equation for the magnetic helicity $\chi^h = \langle \mathbf{a} \cdot \mathbf{b} \rangle$ in the form

$$\partial \chi^h/\partial t + \boldsymbol{\nabla} \cdot \mathbf{F} = -2\langle \mathbf{u} \times \mathbf{b} \rangle \cdot \mathbf{B} - 2\eta \langle \mathbf{b} \cdot \mathbf{curl}\, \mathbf{b} \rangle \;, \quad (2)$$

where $\mathbf{F} = (2/3)\mathbf{V}\chi^h + \langle \mathbf{a} \times (\mathbf{u} \times \mathbf{B}) \rangle - \eta \langle \mathbf{a} \times \mathbf{curl}\, \mathbf{b} \rangle + \langle \mathbf{a} \times (\mathbf{u} \times \mathbf{b}) \rangle - \langle \mathbf{b} \varphi \rangle$ is the flux of magnetic helicity. The electromotive force for isotropic and homogeneous turbulence is

$$\langle \mathbf{u} \times \mathbf{b} \rangle = \alpha \mathbf{B} - \eta_T \mathbf{curl}\, \mathbf{B}, \quad (3)$$

where η_T is the turbulent magnetic diffusivity, and it is assumed that α is the total alpha-effect which at the nonlinear stage includes both the original hydrodynamical, and the magnetic, contributions. Note that the magnetic part of the α effect is proportional to the magnetic helicity, i.e. $\alpha^h = \chi^h/(18\pi\eta_T\rho)$ (see, e.g., Kleeorin and Rogachevskii, 1999), where ρ is the density. The simplest form of the magnetic helicity flux for an isotropic turbulence is given by $\mathbf{F} = \mathbf{V}\chi^h$, where $\mathbf{V}$ is the mean fluid velocity, e.g. that of the differential rotation (see Kleeorin and Ruzmaikin, 1982; Kleeorin and Rogachevskii, 1999). Thus, the equation for the magnetic part of the α effect in dimensionless form is given by

$$\frac{\partial \alpha^h}{\partial t} + \frac{\alpha^h}{T} + \boldsymbol{\nabla} \cdot (\mathbf{V}\chi^h) = 4(h/l)^2 (R_\alpha^{-1} \mathbf{B} \cdot \mathbf{curl}\, \mathbf{B} - \alpha B^2), \quad (4)$$

(see Kleeorin and Ruzmaikin, 1982), where $l \approx 100\, pc$ is the scale of turbulent motions. We adopt here the standard dimensionless form of the galactic dynamo equation from Ruzmaikin et al. 1988; in particular, the length is measured in units of the disc thickness h, the time is measured in units of h^2/η_T and B is measured in units of the equipartition energy $B_{\rm eq} = \sqrt{4\pi\rho}\, u$. Here u is the characteristic turbulent velocity in the scale l, $\eta_T = lu/3, \quad T = (1/3)(l/h)^2 Rm$ and $R_\alpha = l\alpha_*/\eta_T$, where α^h and α are measured in units of α_* (the maximum value of the hydrodynamic part of the α effect). For an axisymmetric dynamo $\boldsymbol{\nabla} \cdot (\mathbf{V}\chi^h) = 0$.

When $\partial \alpha^h/\partial t = 0$ and $R_\alpha^{-1} \mathbf{B} \cdot \mathbf{curl}\, \mathbf{B} \ll \alpha B^2$, Eq. (4) yields $\alpha = \alpha^v/[1 + (4/3) Rm B^2]$ (see, e.g., Vainshtein and Cattaneo, 1992). However, the latter equation is not valid for galaxies because $\partial \alpha^h/\partial t \gg \alpha^h/T$. In addition, the condition $R_\alpha^{-1} \mathbf{B} \cdot \mathbf{curl}\, \mathbf{B} \ll \alpha B^2$ seems not to be valid for galaxies.

Eq. (4) has been later reproduced, e.g. by Gruzinov and Diamond (1995). However, although this equation has never been included into detailed galactic dynamo calculations, nevertheless its qualitative properties are more or less clear. Provided that dissipative losses are taken into account, Eq. (4) leads to the same type of behaviour as that obtained by the *ad hoc* prescription of the result of Vainshtein and Cattaneo (1992), i.e. the steady state strength of magnetic field is about $B_{\rm eq} Rm^{-1/2}$ (see, e.g. Field, 1999). The real advantage of Eq. (4) is the fact that it is derived from first principles rather than prescribed *ad hoc*. If the dissipative losses in Eq. (4) are neglected, the magnetic field decays for $t \to \infty$. We stress that Eq. (4) contains a large factor $4(h/l)^2 \sim 100$ typically.

Kleeorin and Rogachevskii (1999) extended the calculations to include a flux of magnetic helicity. Based on Eq. (13) of that paper, the approximate relation

$$\frac{\partial \alpha^h}{\partial t} \;=\; 4\left(\frac{h}{l}\right)^2 [(\mathbf{B} \cdot \mathbf{curl}\, \mathbf{B} R_\alpha^{-1} - \alpha(B) B^2) + \frac{\partial}{\partial z}(\alpha^v(z)\phi(B) B^2 h f_1(z))] \quad (5)$$

can be formulated. In Eq. (5), $f_1(z)$ describes the inhomogeneity of the turbulent diffusivity, and we define $f(z) = \alpha^v(z) f_1(z)$. The profile $f(z)$ depends on details of the galactic structure. Also, $\alpha(B)$ is the total α effect and $\alpha = \alpha^v \phi(B) + \alpha^h \phi_1(B)$, where $B = |\mathbf{B}|$. Here α^v is the hydrodynamic part of the α effect, with $\alpha^v \phi(B)$ its modification due to nonlinear effects. Correspondingly, α^h is the magnetic part of the α effect, and $\alpha^h \phi_1(B)$ is the modification caused by nonlinear effects (see Rogachevskii and Kleeorin, 2000). $\phi_1(B) = (3/8B^2)(1 - \arctan(\sqrt{8}B)/\sqrt{8}B)$ and the function $\phi(B)$ is defined below. The magnetic part of the α effect is proportional to the magnetic helicity, i.e., $\alpha^h = \chi^h/(18\pi\eta_T\rho)$ (see, e.g., Kleeorin and Rogachevskii, 1999). For galaxies the term α^h/T is very small and can be dropped. The gauge conditions $\boldsymbol{\nabla}\cdot\mathbf{A} = \boldsymbol{\nabla}\cdot\mathbf{a} = \mathbf{0}$ have been used; our results can be shown to be gauge invariant (see Berger and Ruzmaikin, 2000).

The last term in Eq. (5) is related to the turbulent flux of magnetic helicity. This turbulent flux is proportional to the hydrodynamic part of the α effect and the turbulent diffusivity (see Kleeorin and Rogachevskii, 1999). The turbulent flux of magnetic helicity serves as an additional nonlinear source in the equation for the magnetic part of the α effect and it causes a drastic change in the dynamics of the large-scale magnetic field.

For simplicity we replace the flux divergence in the right hand side of Eq. (5) by a decay term, i.e. we replace $\frac{\partial}{\partial z}$ by $1/h$ (in principle, there is no problem in treating this point more carefully).

3. The equilibrium magnetic field configuration

We now present some asymptotic expansions for galactic dynamo models with the nonlinearity (5). First of all, we recognize that, because of the large parameter $4(h/l)^2$ in the right hand side of Eq. (5), we can take

$$\alpha(B) = f(z)\phi(B) + R_\alpha^{-1}B^{-2}\mathbf{B}\cdot\mathbf{curl}\,\mathbf{B}, \qquad (6)$$

where

$$\phi(B) \;=\; \frac{3}{14B^2}\Bigg(1 - \frac{\arctan(\sqrt{8}B)}{\sqrt{8}B} + \; 2B^2[1 - 16B^2 + 128B^4\ln(1 + (8B^2)^{-1})]\Bigg).$$

Thus $\phi(B) = 1/(4B^2)$ for $B \gg 1/\sqrt{8}$ and $\phi = 1 - (48/5)B^2$ for $B \ll 1/\sqrt{8}$. The function ϕ is derived by Rogachevskii and Kleeorin (2000). Note that in a more simplified model of turbulence the function $\phi(B) = \phi_1(B) = (3/8B^2)(1 - \arctan(\sqrt{8}B)/\sqrt{8}B)$ (see Field et al. 1999). We stress that the qualitative behaviour of the model does not depend on these uncertainties in estimates for the scaling functions ϕ and ϕ_1.

Now we insert the α-coefficient given by Eq. (6) into local disc dynamo problem to obtain the following equations:

$$\frac{\partial b_r}{\partial t} \;=\; -(\alpha(B)B_\phi)' + b_r'', \qquad (7)$$

$$\frac{\partial B_\phi}{\partial t} \;=\; Db_r + B_\phi'' \qquad (8)$$

(here $B_r = R_\alpha b_r$). We can then obtain the steady-state solution of Eqs. (7) and (8). Recognizing that in cylindrical coordinates

$$\mathbf{B}\cdot\mathbf{curl}\,\mathbf{B} = R_\alpha(B_\phi b_r' - b_r B_\phi'), \qquad (9)$$

we obtain for fields of quadrupole symmetry (cf. Kvasz et al., 1992)

$$B_\phi''' + D\alpha(B)B_\phi = 0 \qquad (10)$$

in a steady state. The corresponding equation in kinematic theory reads

$$B_\phi''' + D\alpha_0 B_\phi = 0. \qquad (11)$$

Substituting (6) into (10) we obtain,

$$B_\phi''' B^2 + DB_\phi[f(z)\phi(B)B^2 + R_\alpha^{-1}\mathbf{B}\cdot\mathbf{curl}\,\mathbf{B}] = 0. \qquad (12)$$

Using Eq. (9) we rewrite Eq. (12) in the form

$$B_\phi'''(B^2 - B_\phi^2) + B_\phi[B_\phi'' B_\phi' + Df(z)\phi(B)B^2] = 0. \qquad (13)$$

For the $\alpha\Omega$ dynamo $B \approx B_\phi$. This assumption is justified if $|D| >> R_\alpha$, i.e. $|R_\omega| >> 1$. Eq. (13) then becomes

$$B''B' + Df(z)\phi(B)B^2 = 0, \qquad (14)$$

Note that Eq. (14) differs from Eq. (11), arising from kinematic theory. For the specific choice of helicity profile $f(z) = \sin\pi z$ and negative dynamo number D, there is an explicit steady solution, if we assume $B^2 \approx B_\phi^2$ (remember that also $B \gg 1/\sqrt{8}$, i.e. super-equipartition), of the form

$$B_\phi \;=\; \frac{2\sqrt{|D|}}{\pi^{3/2}}B_{\rm eq}\cos\,\frac{\pi z}{2}, \qquad (15)$$

$$B_r \;=\; -\frac{\sqrt{\pi R_\alpha}}{2\sqrt{|R_\omega|}}B_{\rm eq}\cos\,\frac{\pi z}{2}, \qquad (16)$$

where we have restored the dimensional factor $B_{\rm eq}$. (Note that $\mathbf{B}\cdot\mathbf{curl}\,\mathbf{B} = 0$ for this approximate solution.) This solution is remarkably close to the results from the naive *Ansatz* $\alpha = \alpha_0(1-(B/B_{\rm eq})^2)$ or $\alpha = \alpha_0/(1+(B/B_{\rm eq})^2)$, or the model of Moss et al. (1999). For example, the pitch angle of the magnetic field lines is $p = -\arctan(\pi^2/4|R_\omega|) \approx 14^\circ$ for $|D| = 10$ and $R_\alpha = 1$.

4. Numerical results

We verified numerically that the initially weak magnetic field approaches the equilibrium configuration (15) with accuracy 1% for $|D| > 1000$, and an accuracy of 50% for $|D| > 10$. As is anticipated in the previous section, the equilibrium magnetic field near to the generation threshold value is more complicated. The threshold value for the nonlinear solution of Eqs. (7) and (8) is $D \approx -3.14$, while the linear threshold value is $D \approx -8$. This is because the nonlinear solution arranges itself so that the term $\mathbf{B}\cdot\mathbf{curl}\,\mathbf{B}/B_\phi^2$ in α (see Eq. (6)) is of order 1. Thus, for the nonlinear solution with $D = -8$, the maximal value of α is about 1.25, whereas for $D = -5$, the maximal value is about 1.76. For $|D| \lesssim 10$ we obtain numerically

$$B_\phi(0) \approx 0.23|D - D_{\rm cr}|^{0.52}, \qquad (17)$$

where $D_{\rm cr}$ is the nonlinear threshold value. As $|D - D_{\rm cr}|$ increases towards 10, the slope increases slightly, but Eq. (17)

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remains a reasonable estimate. (Note that accurately estimating the exponent in Eq. (17), and subsequently, is a quite delicate matter even in this one-dimensional problem, and that the quoted figures may be uncertain in the last digit.)

This result is robust under variations of the helicity profile. For $f(z) = z$ we get in the nonlinear case $D_{\rm cr} = -7.49$, while the linear threshold value is $D_{\rm cr} = -12.5$ and $B_\phi(0) \approx 0.15|D - D_{\rm cr}|^{0.50}$ near $D = D_{\rm cr}$, i.e. again a square root dependence to within the errors of our procedure. Further, with $f(z) = z/|z|$ in $|z| > 0.2$, and a smooth interpolation to zero in $|z| \le 0.2$, we find $D_{\rm cr} \approx -2.41$ and $B_\phi(0) \approx 0.25|D - D_{\rm cr}|^{0.50}$, again closely the same dependence. (In this case the linear threshold value is $D_{\rm cr} = -6.53$.)

5. Discussion

We have demonstrated that the nonlinear evolution of the helicity following from Eq. (4) gives a basically different type of galactic magnetic field evolution to that following from Eq. (5). Eq. (4), being based on local helicity conservation, results in magnetic field decay, after a stage of kinematic growth. If the molecular diffusivity of the magnetic field is taken into account, this decay is followed by a stabilization at a very low magnetic field strength, corresponding to the estimate of Vainshtein and Cattaneo (1992). The scenario of magnetic field and helicity dynamics can then be described as follows. Large-scale dynamo action produces large-scale magnetic helicity. Due to the local conservation of helicity, suppression of field generation results. An equilibrium is possible if molecular diffusivity is present, so the equilibrium magnetic field strength is very low.

Eq. (5) allows for the transport of helicity, so the local value of the helicity changes during magnetic field evolution. The scenario of magnetic field and helicity dynamics can be presented as follows. As usual, magnetic helicity of the large-scale magnetic field is produced, however the total magnetic helicity is not now conserved locally, but the magnetic helicity of the small-scale magnetic field is redistributed by a helicity flux. The equilibrium state is given by a balance between helicity production and transport. The helicity conservation law now expresses the conservation of an integral of the helicity over the galactic disc. However this conservation law is trivial, because the integral vanishes identically as helicity is an odd function with respect to z. Now the equilibrium strength of the large-scale magnetic field is of order that of the equipartition field: this is our main result.

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