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Rank-infinity laminated composites attaining the Hashin–Shtrikman bounds

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Abstract

The attainability of the Hashin–Shtrikman bounds by a class of sequentially laminated composites with prescribed and generally different volume fractions of the core laminates at each rank is well-known. It is demonstrated that in the limit of a rank-infinity laminate these bounds are attained by choosing identical volume fractions for the core laminates. © 2002 Elsevier Science B.V. All rights reserved.

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Lurie and Cherkaev [1], Milton [2] and Francfort and Murat [3] demonstrated that certain classes of sequentially laminated composites attain the Hashin–Shtrikman bounds. The number of times the laminating sequence is repeated is denoted as the *rank* of the laminate. A rank-1 laminate is constructed by layering two materials in an alternate order (Fig. 1). A rank-2 laminate is constructed by layering a core rank-1 laminate with a third phase (Fig. 2). Higher rank laminates are constructed through a sequence of similar layering procedures.

As was noted by Bruggeman [4], the overall properties of these sequentially laminated composites can be determined exactly since, when subjected to uniform boundary conditions, the fields within the composite are piecewise constant. This statement holds only when the thickness of the layers in the core laminate is sufficiently small so that the lamina can be regarded as a homogeneous phase in the subsequent layering stage [2]. In this Letter the lamination sequence is followed to determine the effective *elastic* properties of a two-dimensional, incompressible rank-infinity laminated composite.

Consider a two-dimensional laminated composite constructed by layering a pair of incompressible isotropic constituents in volume fractions $c^{(1)}$ and $c^{(2)} = 1 - c^{(1)}$. In the following, superscripts with Arabic numerals or lowercase letters indicate properties associated with the constituting homogeneous phases whereas superscripts with Roman numerals or uppercase letters indicate the rank of the lamination. The unit vectors normal and along the layers are $\hat{\mathbf{n}}$ and $\hat{\mathbf{m}}$,

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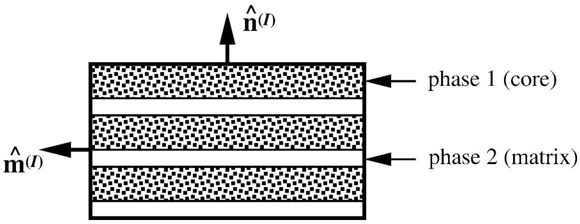

Fig. 1. A rank-1 laminated composite.

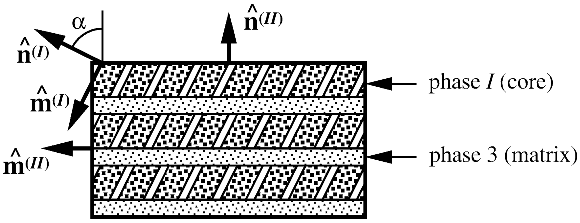

Fig. 2. A rank-2 sequentially laminated composite.

respectively. Within each phase the stress strain relation may be expressed in the form

$$\begin{pmatrix} \gamma_d^{(r)} \\ \gamma_n^{(r)} \end{pmatrix} = \frac{1}{2\mu^{(r)}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau_d^{(r)} \\ \tau_n^{(r)} \end{pmatrix}, \quad r = 1, 2, \quad (1)$$

where

$$\tau_d = \frac{1}{2}\boldsymbol{\sigma} \cdot \big(\hat{\mathbf{n}} \otimes \hat{\mathbf{n}} - \hat{\mathbf{m}} \otimes \hat{\mathbf{m}}\big),$$
$$\tau_n = \frac{1}{2}\boldsymbol{\sigma} \cdot \big(\hat{\mathbf{n}} \otimes \hat{\mathbf{m}} + \hat{\mathbf{m}} \otimes \hat{\mathbf{n}}\big)$$

are the deviatoric and inplane shear stress components, respectively. Their shear strain counterparts γ_d and γ_n are defined analogously in terms of the strain tensor $\boldsymbol{\epsilon}$. In (1), $\mu^{(r)}$ are the shear moduli of the phases, and the symbol $\otimes$ denotes the direct or the outer product between two vectors.

The continuity conditions across the interfaces are

$$\big(\boldsymbol{\epsilon}^{(1)} - \boldsymbol{\epsilon}^{(2)}\big) \cdot \big(\hat{\mathbf{m}} \otimes \hat{\mathbf{m}}\big) = 0,$$
$$\big(\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}\big) \cdot \hat{\mathbf{n}} = 0. \quad (2)$$

The mean stress and the strain tensors are, respectively, $\bar{\boldsymbol{\sigma}} = c^{(1)}\boldsymbol{\sigma}^{(1)} + c^{(2)}\boldsymbol{\sigma}^{(2)}$ and $\bar{\boldsymbol{\epsilon}} = c^{(1)}\boldsymbol{\epsilon}^{(1)} + c^{(2)}\boldsymbol{\epsilon}^{(2)}$. The continuity of the inplane shear stress component τ_n implies that

$$\bar{\gamma}_n = \left(\frac{c^{(1)}}{2\mu^{(1)}} + \frac{c^{(2)}}{2\mu^{(2)}}\right)\bar{\tau}_n. \quad (3)$$

From the incompressibility condition $\boldsymbol{\epsilon} \cdot (\hat{\mathbf{n}} \otimes \hat{\mathbf{n}} + \hat{\mathbf{m}} \otimes \hat{\mathbf{m}}) = 0$, and the first of (2),

$$\tau_d^{(r)} = -2\mu^{(r)}\boldsymbol{\epsilon}^{(r)} \cdot \big(\hat{\mathbf{m}} \otimes \hat{\mathbf{m}}\big)$$
$$= -2\mu^{(r)}\bar{\boldsymbol{\epsilon}} \cdot \big(\hat{\mathbf{m}} \otimes \hat{\mathbf{m}}\big), \quad r = 1, 2. \quad (4)$$

A weighted sum over the two equations of (4) together with the incompressibility equation, leads to the relation $\bar{\tau}_d = 2(c^{(1)}\mu^{(1)} + c^{(2)}\mu^{(2)})\bar{\gamma}_d$. In conjunction with (3), the relation between the mean stress and strain may be expressed in the form

$$\begin{pmatrix} \bar{\gamma}_d \\ \bar{\gamma}_n \end{pmatrix} = \begin{pmatrix} \frac{1}{2\mu_d^{(\mathrm{I})}} & 0 \\ 0 & \frac{1}{2\mu_n^{(\mathrm{I})}} \end{pmatrix} \begin{pmatrix} \bar{\tau}_d \\ \bar{\tau}_n \end{pmatrix}, \quad (5)$$

where the effective shear moduli of the laminate are

$$\mu_d^{(\mathrm{I})} = c^{(1)}\mu^{(1)} + c^{(2)}\mu^{(2)},$$
$$\mu_n^{(\mathrm{I})} = \left(\frac{c^{(1)}}{\mu^{(1)}} + \frac{c^{(2)}}{\mu^{(2)}}\right)^{-1},$$

and where the superscript I identifies properties associated with the rank-1 laminate.

Consider next a rank-2 laminate constructed by layering layers of the above described rank-1 laminate in volume fraction $c^{(\mathrm{II})}$ with an isotropic (incompressible) homogeneous phase with a shear modulus $\mu^{(3)}$. The superscript II refers to properties associated with the rank-2 laminate. The lamination direction is such that the angle between the normals to the layers of the rank-2 and the core laminates is $\pi/4$ (α in Fig. 2). In terms of $\hat{\mathbf{n}}$ and $\hat{\mathbf{m}}$, the unit vectors normal and along the layers of the rank-2 laminate, the stress strain relation of the core rank-1 laminate is

$$\begin{pmatrix} \bar{\gamma}_d^{(\mathrm{I})} \\ \bar{\gamma}_n^{(\mathrm{I})} \end{pmatrix} = \begin{pmatrix} \frac{1}{2\mu_n^{(\mathrm{I})}} & 0 \\ 0 & \frac{1}{2\mu_d^{(\mathrm{I})}} \end{pmatrix} \begin{pmatrix} \bar{\tau}_d^{(\mathrm{I})} \\ \bar{\tau}_n^{(\mathrm{I})} \end{pmatrix}. \quad (6)$$

The stress strain relation for the third isotropic phase is in the form of Eq. (1), and the continuity conditions are identical to (2). As before, the mean stress and strain are $\bar{\boldsymbol{\sigma}} = c^{(\mathrm{II})}\bar{\boldsymbol{\sigma}}^{(\mathrm{I})} + (1 - c^{(\mathrm{II})})\boldsymbol{\sigma}^{(3)}$ and $\bar{\boldsymbol{\epsilon}} = c^{(\mathrm{II})}\bar{\boldsymbol{\epsilon}}^{(\mathrm{I})} + (1 - c^{(\mathrm{II})})\boldsymbol{\epsilon}^{(3)}$. Following precisely the same steps followed in going from (1) to (5), the relation between the mean stress and strain of the rank-2 lamina may be expressed in the form

$$\begin{pmatrix} \bar{\gamma}_d \\ \bar{\gamma}_n \end{pmatrix} = \begin{pmatrix} \frac{1}{2\mu_d^{(\mathrm{II})}} & 0 \\ 0 & \frac{1}{2\mu_n^{(\mathrm{II})}} \end{pmatrix} \begin{pmatrix} \bar{\tau}_d \\ \bar{\tau}_n \end{pmatrix}, \quad (7)$$

where

$$\mu_d^{(\mathrm{II})} = c^{(\mathrm{II})}\mu_n^{(\mathrm{I})} + \left(1 - c^{(\mathrm{II})}\right)\mu^{(3)},$$
$$\mu_n^{(\mathrm{II})} = \left(\frac{c^{(\mathrm{II})}}{\mu_d^{(\mathrm{I})}} + \frac{(1 - c^{(\mathrm{II})})}{\mu^{(3)}}\right)^{-1}.$$

A rank-3 laminate can now be constructed by laminating the rank-2 laminate as the core phase with an isotropic phase in volume fractions $c^{(\mathrm{III})}$ and $(1 - c^{(\mathrm{III})})$, respectively. Once again, the lamination direction is rotated by $\pi/4$ relative to the lamination direction of the rank-2 composite. The effective shear moduli $\mu_d^{(\mathrm{III})}$ and $\mu_n^{(\mathrm{III})}$ will be functions of $\mu_n^{(\mathrm{II})}$ and $\mu_d^{(\mathrm{II})}$, respectively.

Consider a sequentially laminated composite constructed by layering, at each stage, a core rank-$(N-1)$ laminate with one of the constituent phases, say phase 2. At each successive rank the lamination direction is rotated by $\pi/4$. Following the steps followed in going from (6) to (7), the effective shear moduli of a rank-$2N$ composite are

$$\mu_d^{(2N)} = c^{(2N)}\mu_n^{(2N-1)} + \left(1 - c^{(2N)}\right)\mu^{(2)},$$
$$\mu_n^{(2N)} = \left(\frac{c^{(2N)}}{\mu_d^{(2N-1)}} + \frac{(1 - c^{(2N)})}{\mu^{(2)}}\right)^{-1},$$

and where, for consistency, the rank-0 laminate is defined as the constituting phase 1. In terms of the properties of the rank-$(2N-2)$ laminate,

$$\mu_d^{(2N)} = c^{(2N)}\left(\frac{c^{(2N-1)}}{\mu_d^{(2N-2)}} + \frac{1 - c^{(2N-1)}}{\mu^{(2)}}\right)^{-1} + \left(1 - c^{(2N)}\right)\mu^{(2)},$$
$$\mu_n^{(2N)} = \left(\frac{c^{(2N)}}{c^{(2N-1)}\mu_n^{(2N-2)} + (1 - c^{(2N-1)})\mu^{(2)}} + \frac{1 - c^{(2N)}}{\mu^{(2)}}\right)^{-1}. \tag{8}$$

The first of (8) can be simplified by substituting

$$\hat{\mu}_d^{(J)} = \left(\frac{\mu^{(2)}}{\mu_d^{(J)}} - 1\right)^{-1}, \quad J = 2, 4, \ldots, 2N, \tag{9}$$

to obtain

$$\hat{\mu}_d^{(2N)} = \frac{\hat{\mu}_d^{(2N-2)} + c^{(2N-1)} - c^{(2N)}c^{(2N-1)}}{c^{(2N)}c^{(2N-1)}}. \tag{10}$$

From (10), the expression for $\hat{\mu}_d^{(2N)}$ becomes

$$\hat{\mu}_d^{(2N)} = \frac{\hat{\mu}_d^{(0)}}{c^{(1)}} - \frac{1}{c^{(1)}}\sum_{J=1}^{2N}\left[(-1)^J \prod_{K=1}^{J} c^{(K)}\right],$$

where it is noted that

$$c^{(1)} = \prod_{J=1}^{2N} c^{(J)}.$$

If the volume fractions of the core laminates within the preceding ranks are all equal, that is $c^{(J)} = \sqrt[2N]{c^{(1)}}$, $J = 1, 2, \ldots, 2N$, the expression for $\hat{\mu}_d^{(2N)}$ further reduces to

$$\hat{\mu}_d^{(2N)} = \frac{\hat{\mu}_d^{(0)}}{c^{(1)}} - \frac{1}{c^{(1)}}\sum_{J=1}^{2N}\left(-\sqrt[2N]{c^{(1)}}\right)^J.$$

The second term, which is the sum of a geometric series, can be readily determined. Thus,

$$\hat{\mu}_d^{(2N)} = \frac{\hat{\mu}_d^{(0)}}{c^{(1)}} + \frac{1}{c^{(1)}}\frac{\sqrt[2N]{c^{(1)}}\,(1 - c^{(1)})}{1 + \sqrt[2N]{c^{(1)}}}.$$

In the limit as N becomes large, $\sqrt[2N]{c^{(1)}} \to 1$ and

$$\hat{\mu}_d^{(\infty)} = \frac{\hat{\mu}_d^{(0)}}{c^{(1)}} + \frac{1 - c^{(1)}}{2c^{(1)}}.$$

From definition (9), together with the fact that $\mu_d^{(0)} = \mu^{(1)}$, it can be shown that

$$\mu_d^{(\infty)} = \mu^{(2)}\frac{(1 + c^{(1)})\mu^{(1)} + (1 - c^{(1)})\mu^{(2)}}{(1 - c^{(1)})\mu^{(1)} + (1 + c^{(1)})\mu^{(2)}}. \tag{11}$$

It is noted that (11) is precisely the expression for the Hashin–Shtrikman bound for statistically isotropic two-dimensional composites (e.g., [5]).

In a similar manner, the second of (8) can be modified by defining

$$\hat{\mu}_n^{(J)} = \left(\frac{\mu_n^{(J)}}{\mu^{(2)}} - 1\right)^{-1}, \quad J = 2, 4, \ldots, 2N, \tag{12}$$

to obtain an expression for $\hat{\mu}_n^{(2N)}$ in terms of $\hat{\mu}_n^{(2N-2)}$ which is identical to the one obtained in (10) for $\hat{\mu}_d^{(2N)}$. Therefore, in the limit of large N and equal volume fractions of the core laminates in the subsequent ones,

$$\hat{\mu}_n^{(\infty)} = \frac{\hat{\mu}_n^{(0)}}{c^{(1)}} + \frac{1 - c^{(1)}}{2c^{(1)}}.$$

From definition (12), it follows that in the limit $N \to \infty$ the expression for $\mu_n^{(\infty)}$ is identical to the one obtained for $\mu_d^{(\infty)}$ in (11). Thus, the above constructed rank-infinity laminated composite assumes isotropic elastic symmetry and attains the corresponding Hashin–Shtrikman bounds.

In conclusion, we note that analogous procedure can be repeated for the corresponding transport problems of conductivity, thermal conductivity and the dielectric and magnetic behavior of heterogeneous media. This is due to the similarity between the above described set of equations characterizing the continuity conditions and the constitutive relations in plane-strain incompressible elasticity and the corresponding set of equations for the two-dimensional transport problem. The only difference being the relative lamination angle at each lamination stage which is $\pi/2$ for the transport problem instead of $\pi/4$ for the problem considered herein.

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