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High-rank nonlinear sequentially laminated composites and their possible tendency towards isotropic behavior

G. deBotton*, I. Hariton

The Pearlstone Center for Aeronautical Studies, Department of Mechanical Engineering, Ben-Gurion University, P.O. Box 653, Beer-Sheva 84105, Israel

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Abstract

This work is concerned with the determination of the effective behavior of sequentially laminated composites with nonlinear behavior of the constituting phases. An exact expression for the effective stress energy potential of two-dimensional and incompressible composites is introduced. This allows to determine the stress energy potential of a rank-N sequentially laminated composite with arbitrary volume fractions and lamination directions of the core laminates in terms of an N-dimensional optimization problem.

Stress energy potentials for sequentially laminated composites with pure power-law behavior of the phases are determined. It is demonstrated that as the rank of the lamination becomes large the behaviors of certain families of sequentially laminated composite tend to be isotropic. Particulate composites with both, stiffer and softer inclusions are considered. The behaviors of these almost isotropic composites are, respectively, softer and stiffer than the corresponding second-order estimates recently introduced by Ponte Castañeda (1996).

Keywords: Inhomogeneous material; Constitutive behavior; Elastic–plastic behavior; Anisotropic material; Layered material

1. Introduction

The evaluation of the constitutive properties of composite materials is a classical problem in physics of solid state. The main idea is to define these properties in terms of the effective, or overall, energy of the heterogeneous system resulting from

* Corresponding author. Tel.: +972-8-647-7105; fax: +972-8-647-2813.
E-mail address: debotton@bgu.ac.il (G. deBotton).

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special classes of uniform boundary conditions. The effective energy function involves dependence on the properties of the constituents phases as well as on microstructural parameters such as the volume fractions of the constituents. Required conditions for a coherent definition of the effective energy function were stated by Hill (1963) and Hashin (1964). Principally, it is required that the size of a typical heterogeneity is small compared to the size of the specimen under consideration. For a summary of recent methods available for characterizing the overall properties of composites with *nonlinear* constitutive behavior of the phases we refer the reader to Ponte Castañeda and Suquet (1998).

Different methods have been proposed to predict the effective properties of heterogeneous systems. These can be classified into three main categories. The first deals with the development of approximate models that capture the essential features of the microstructure. Effective medium theories such as the self-consistent scheme (Kröner, 1958) and the generalized self-consistent scheme (Christensen and Lo, 1979) are examples of such approximate models. Budiansky and Wu (1962) and Hill (1965) extended these concepts to the class of nonlinear composites, and Hutchinson (1976) made use of these extensions for estimating the effective properties of polycrystals. In the context of the transport problem, Stroud and Hui (1988) and Zeng et al. (1988) obtained expressions for the effective behavior of weakly nonlinear composites accurate to the first order in the nonlinear term. Ponte Castañeda (1991) introduced a method for extending such estimates obtained for classes of linear composites to corresponding classes of nonlinear composites with identical microstructure. More advanced estimates for the behavior of nonlinear composites, which are accurate to the second order in the contrast, were recently introduced by Ponte Castañeda (1996).

The second category deals with the determination of the range of possible behaviors for a given composite class. This approach characterizes the effective behavior of classes of heterogeneous systems by specifying bounds on the effective properties. A well-known example of this approach are the Hashin and Shtrikman (1962) bounds on the effective properties of statistically isotropic composite materials. Willis (1977) derived these bounds differently, extending them to classes of anisotropic composites with anisotropic constituents. Milton and Kohn (1988) extended the Hashin–Shtrikman inequalities to arbitrary dimension via the relations between the eigenvalues of the effective tensors and the anisotropic comparison media. Talbot and Willis (1985) proposed an extension of the Hashin–Shtrikman variational principles to nonlinear composites. Talbot and Willis (1992) applied this method to calculate bounds on the effective properties of nonlinear composites. Ponte Castañeda (1992a) developed a variational method aimed at studying the effective properties of nonlinear composites from which a strict lower bound and an estimate for the upper bound can be generated. Applications of this method to the class of transversely isotropic fiber-reinforced composites have been carried out by deBotton and Ponte Castañeda (1993). Interestingly, by application of a different method, Suquet (1993) arrived at bounds and estimates for the class of power-law hardening materials which coincide with those obtained by Ponte Castañeda (1992a). Whereas the above-mentioned methods for nonlinear composites result in a single side bound for the effective energy and an estimate for the other bound, Talbot and Willis (1997) developed a method for determining the "other" bound, and

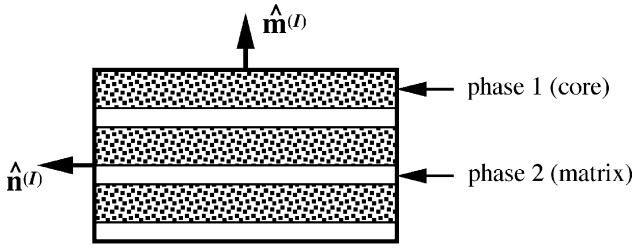


Fig. 1. A rank-1 laminated composite.

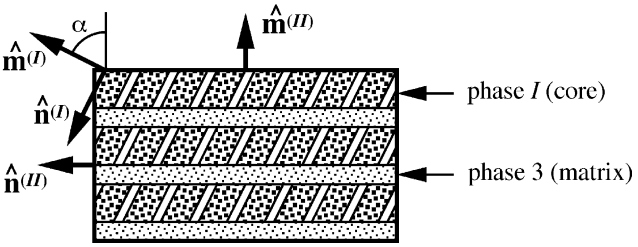


Fig. 2. A rank-2 sequentially laminated composite.

used it to obtain a corresponding bound for the class of isotropic power-law hardening composites.

The third category is based on identifying specific microstructures for which the effective properties can be obtained exactly. An example of this approach is the composite-sphere assemblage of Hashin and Shtrikman (1962). Suquet (1987) developed a systematic method for determining the exact effective properties of composites with periodic structure. An important class of composites for which the effective properties can be determined exactly is the class of sequentially laminated composites. A simple laminated composite, denoted as a rank-1 laminate, is constructed by layering two materials in an alternate order (Fig. 1). A rank-2 laminate is constructed by layering a rank-1 composite as a core phase with yet another constituent phase as illustrated in Fig. 2. A rank-N composite is constructed by following this procedure N times. It was realized by Bruggeman (1935) that when the thickness of the core laminate used to create the next rank lamina is sufficiently small, the effective properties of the resulting composite can be determined exactly since the local fields are piecewise constant. Schulgasser (1976) made use of this technique to demonstrate the optimality of the Voigt bound for classes of statistically isotropic polycrystals. Lurie and Cherkaev (1984), Milton (1986) and Francfort and Murat (1986) demonstrated that certain composites belonging to this class can be constructed so that their effective properties attain the Hashin–Shtrikman bounds. The construction of these extremal structures demonstrated in fact the optimality of the Hashin–Shtrikman bounds on the effective shear modulus. A somewhat different family of composites, also attaining these bounds, was constructed by Norris (1985). Kohn and Lipton (1988) introduced an optimal bound for anisotropic three-dimensional composites which is sharper than the classical lower

bound. Extensions of these ideas to the class of anisotropic composites were carried out by Milton (1986), who explored the general relations between the functional form of the bounds and the corresponding laminated microstructures attaining them. deBotton and Ponte Castañeda (1992) provided an expression for the exact energy function of a rank-1 nonlinear laminate, and demonstrated that it attains the nonlinear classical Reuss and Voigt bounds. It was further shown that these expressions can be derived from the variational principle of Ponte Castañeda (1991) or, alternatively, directly from the classical principles of minimum energy. Ponte Castañeda (1992b) demonstrated that, unlike the case with linear composites, isotropic rank-2 nonlinear sequentially laminated composites cannot be constructed.

The present work deals with the class of nonlinear sequentially laminated composites in the limit of a large lamination sequence. It is demonstrated that as the rank of the lamination increases the behaviors of certain sequentially laminated composites tend to be isotropic. The paper is structured as follows. In Section 2 the definition of the effective properties is reviewed and their characterization in terms of the classical variational principles is given. In Section 3 a method for determining the effective energy function of sequentially laminated composites with arbitrary microstructure is introduced. Numerical results for certain families of composites with pure power-hardening behavior of the phases are provided in Section 4, and a comparison with different available bounds and estimates is carried out. Some concluding remarks are finally given in Section 5.

2. Effective properties and the classical variational principle

Consider an n-phase composite occupying a unit volume Ω, with boundary $\partial\Omega$. Each homogeneous phase in the composite occupies a volume $\Omega^{(r)}$, $r=1,2,\ldots,n$. The stress strain relations are characterized via strain energy density functions $W^{(r)}$ such that in each phase

$$\sigma_{ij} = \frac{\partial W^{(r)}}{\partial \varepsilon_{ij}}(\boldsymbol{\varepsilon}). \tag{1}$$

The strain energy density of the composite, in terms of the strain energy densities of the phases, is

$$W(\boldsymbol{\varepsilon}, \mathbf{x}) = \sum_{r=1}^{n} \chi^{(r)}(\mathbf{x}) W^{(r)}(\boldsymbol{\varepsilon}), \tag{2}$$

where the characteristic function of the rth phase is such that $\chi^{(r)}=1$ if $\mathbf{x}$ is in phase r and $\chi^{(r)}=0$ otherwise. The volume fraction of the rth phase is

$$c^{(r)} = \int_{\Omega} \chi^{(r)}(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Following Hill (1963), the constitutive law defining the relation between the mean stress

$$\bar{\boldsymbol{\sigma}} = \int_{\Omega} \chi^{(r)}(\mathbf{x}) \boldsymbol{\sigma}^{(r)}(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{3}$$

and the mean strain $\bar{\boldsymbol{\varepsilon}}$, which is defined in a similar way in terms of $\boldsymbol{\varepsilon}(\mathbf{x})$, may be expressed in the form

$$\bar{\sigma}_{ij} = \frac{\partial \tilde{W}}{\partial \bar{\varepsilon}_{ij}}(\bar{\boldsymbol{\varepsilon}}). \tag{4}$$

The *effective* strain energy function $\tilde{W}$ is determined via the principle of minimum energy, namely

$$\tilde{W}(\bar{\boldsymbol{\varepsilon}}) = \inf_{\boldsymbol{\varepsilon} \in K(\bar{\boldsymbol{\varepsilon}})} \left\{ \int_{\Omega} W(\boldsymbol{\varepsilon}, \mathbf{x}) \, \mathrm{d}\mathbf{x} \right\}, \tag{5}$$

where

$$K(\bar{\boldsymbol{\varepsilon}}) = \left\{ \boldsymbol{\varepsilon} | \varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \text{ in } \Omega, \quad u_i = \bar{\varepsilon}_{ij} x_j \text{ on } \partial\Omega \right\},$$

is the set of admissible strains and $\mathbf{u}(\mathbf{x})$ is a displacement field which is continuous across the interfaces between the phases.

A dual formulation can be given by means of the principle of the minimum complementary energy in terms of the stress density potential

$$U(\boldsymbol{\sigma}, \mathbf{x}) = \sum_{r=1}^{n} \chi^{(r)}(\mathbf{x}) U^{(r)}(\boldsymbol{\sigma}), \tag{6}$$

where $U^{(r)}$ are the stress energy potentials of the phases and in each phase

$$\varepsilon_{ij} = \frac{\partial U^{(r)}}{\partial \sigma_{ij}}(\boldsymbol{\sigma}).$$

The effective stress energy function is

$$\tilde{U}(\bar{\boldsymbol{\sigma}}) = \inf_{\boldsymbol{\sigma} \in S(\bar{\boldsymbol{\sigma}})} \left\{ \int_{\Omega} U(\boldsymbol{\sigma}, \mathbf{x}) \, \mathrm{d}\mathbf{x} \right\}, \tag{7}$$

where

$$S(\bar{\boldsymbol{\sigma}}) = \{ \boldsymbol{\sigma} | \sigma_{ij,j} = 0 \text{ in } \Omega, \quad \sigma_{ij} n_j = \bar{\sigma}_{ij} n_j \text{ on } \partial\Omega \},$$

is the set of admissible stresses that satisfy the traction continuity conditions across the interfaces between the phases. The effective stress–strain constitutive relation is then

$$\bar{\varepsilon}_{ij} = \frac{\partial \tilde{U}}{\partial \bar{\sigma}_{ij}}(\bar{\boldsymbol{\sigma}}). \tag{8}$$

For the sake of conciseness, the rest of the paper is written using the constitutive formulation based on the complementary energy function.

3. Sequentially laminated composites

As mentioned in Section 1 the construction of a laminated material consists of layering alternate slices of the two constituent phases. The direction normal to the layers plane is defined as the laminate direction ($\hat{\mathbf{m}}^{(1)}$ in Fig. 1). The next rank laminate

is obtained by layering this laminate as the *core* phase with a third phase, possibly in a different layering direction (e.g. $\hat{\mathbf{m}}^{(\mathrm{II})}$ in Fig. 2). Clearly, the material from which the third phase is made out of may be identical to one of the constituent materials composing the core laminate itself.

If we begin the procedure with a rank-1 material made out of two isotropic phases and repeat the layering process N times while layering the homogeneous phase 2 with the nonhomogeneous core laminate from the previous rank, we end up with a *rank-N sequentially laminated composite*. An important observation concerning this procedure is that the length scale of the core laminates is assumed to be small compared with the length scale of the embedding ones. This requirement allows for the effective properties to be determined precisely since at each stage the core laminate can be treated as a homogeneous material. Further, it is recalled that when this sequentially laminated composites are subjected to uniform boundary conditions, the stress and the strain fields are piecewise-constant with a different constant in each phase.

The complementary energy density function of a rank-1 sequentially laminated composite is determined via Eq. (6). In the case of isotropic and incompressible constituent phases the stress energy potentials $U^{(r)}(\boldsymbol{\sigma})$ depend only on the two invariants of the stress deviators $\mathbf{S}=\boldsymbol{\sigma}-\frac{1}{3}(\boldsymbol{\sigma}\cdot\mathbf{I})$ within each phase. Without loss of generality, under plane strain conditions the stress energy potentials can be expressed in terms of a two-dimensional "reduced" stress tensor (see Appendix)

$$\boldsymbol{\Sigma}=(\mathbf{P}_d\cdot\boldsymbol{\sigma})\mathbf{P}_d+(\mathbf{P}_m\cdot\boldsymbol{\sigma})\mathbf{P}_m, \tag{9}$$

in the form

$$U^{(r)}(\boldsymbol{\sigma})=\psi^{(r)}(\boldsymbol{\Sigma}),\quad r=1,2, \tag{10}$$

where

$$\begin{aligned}\mathbf{P}_d&=\tfrac{1}{\sqrt{2}}(\hat{\mathbf{m}}\otimes\hat{\mathbf{m}}-\hat{\mathbf{n}}\otimes\hat{\mathbf{n}}),\\ \mathbf{P}_m&=\tfrac{1}{\sqrt{2}}(\hat{\mathbf{m}}\otimes\hat{\mathbf{n}}+\hat{\mathbf{n}}\otimes\hat{\mathbf{m}})\end{aligned} \tag{11}$$

and where $\hat{\mathbf{n}}$ and $\hat{\mathbf{m}}$ are two orthogonal unit vectors in the deforming plane under consideration. It is further noted that $\boldsymbol{\Sigma}\cdot\boldsymbol{\Sigma}=\mathbf{S}\cdot\mathbf{S}$.

Consider a rank-1 lamina subjected to a uniform traction field which is compatible with a constant stress matrix $\bar{\boldsymbol{\Sigma}}$. The composite is made out of a core and a matrix phases which are characterized, respectively, by the energy density functions $\psi^{(1)}$ and $\psi^{(2)}$. The volume fraction of the core phase is $c^{(\mathrm{I})}$ and that of the matrix is $(1-c^{(\mathrm{I})})$. The superscript I refers to quantities associated with the rank-1 laminate. The normal to the layers is denoted $\hat{\mathbf{m}}^{(\mathrm{I})}$, and the corresponding projections defined in (11) are $\mathbf{P}_d^{(\mathrm{I})}$ and $\mathbf{P}_m^{(\mathrm{I})}$. The traction continuity condition across the interfaces $(\boldsymbol{\Sigma}^{(1)}-\boldsymbol{\Sigma}^{(2)})\cdot\hat{\mathbf{m}}^{(\mathrm{I})}=\mathbf{0}$, implies that

$$\mathbf{P}_m^{(\mathrm{I})}\cdot\boldsymbol{\Sigma}^{(1)}=\mathbf{P}_m^{(\mathrm{I})}\cdot\boldsymbol{\Sigma}^{(2)}=\mathbf{P}_m^{(\mathrm{I})}\cdot\bar{\boldsymbol{\Sigma}}. \tag{12}$$

From Eq. (3) we have (recall that the stresses are constant within each phase)

$$\mathbf{P}_d^{(\mathrm{I})} \cdot (c^{(\mathrm{I})}\boldsymbol{\Sigma}^{(1)} + (1-c^{(\mathrm{I})})\boldsymbol{\Sigma}^{(2)}) = \mathbf{P}_d^{(\mathrm{I})} \cdot \bar{\boldsymbol{\Sigma}}.$$

Following deBotton and Ponte Castañeda (1992), we define

$$\mathbf{P}_d^{(\mathrm{I})} \cdot (\boldsymbol{\Sigma}^{(1)} - \boldsymbol{\Sigma}^{(2)}) = \omega^{(\mathrm{I})}\mathbf{P}_d^{(\mathrm{I})} \cdot \bar{\boldsymbol{\Sigma}},$$

and hence, in terms of $\omega^{(\mathrm{I})}$,

$$\begin{aligned}
\mathbf{P}_d^{(\mathrm{I})} \cdot \boldsymbol{\Sigma}^{(1)} &= (1 + (1-c^{(\mathrm{I})})\omega^{(\mathrm{I})})\mathbf{P}_d^{(\mathrm{I})} \cdot \bar{\boldsymbol{\Sigma}}, \\
\mathbf{P}_d^{(\mathrm{I})} \cdot \boldsymbol{\Sigma}^{(2)} &= (1 - c^{(\mathrm{I})}\omega^{(\mathrm{I})})\mathbf{P}_d^{(\mathrm{I})} \cdot \bar{\boldsymbol{\Sigma}}.
\end{aligned} \tag{13}$$

Finally, from the principle of minimum complementary energy (7) together with Eqs. (9), (12) and (13), the effective stress potential of the rank-1 laminate may be expressed in the form:

$$\begin{aligned}
\tilde{\psi}^{(\mathrm{I})}(\bar{\boldsymbol{\Sigma}}) = \inf_{\omega^{(\mathrm{I})}} \{ & c^{(\mathrm{I})}\psi^{(1)}[(1 + (1-c^{(\mathrm{I})})\omega^{(\mathrm{I})})(\mathbf{P}_d^{(\mathrm{I})} \cdot \bar{\boldsymbol{\Sigma}})\mathbf{P}_d^{(\mathrm{I})} + (\mathbf{P}_m^{(\mathrm{I})} \cdot \bar{\boldsymbol{\Sigma}})\mathbf{P}_m^{(\mathrm{I})}] \\
& + (1-c^{(\mathrm{I})})\psi^{(2)}[(1 - c^{(\mathrm{I})}\omega^{(\mathrm{I})})(\mathbf{P}_d^{(\mathrm{I})} \cdot \bar{\boldsymbol{\Sigma}})\mathbf{P}_d^{(\mathrm{I})} + (\mathbf{P}_m^{(\mathrm{I})} \cdot \bar{\boldsymbol{\Sigma}})\mathbf{P}_m^{(\mathrm{I})}]\}.
\end{aligned} \tag{14}$$

A rank-2 sequentially laminated composite is constructed by layering layers of the rank-1 laminate as the core phase in volume fraction $c^{(\mathrm{II})}$ with layers of the matrix phase in volume fraction $(1-c^{(\mathrm{II})})$. The normal to the layers direction is $\hat{\mathbf{m}}^{(\mathrm{II})}$ and the corresponding projections are $\mathbf{P}_d^{(\mathrm{II})}$ and $\mathbf{P}_m^{(\mathrm{II})}$. The resulting structure corresponds to a particulate composite with distinct inclusions of the first homogeneous material and a continuous matrix phase of the second homogeneous material. The volume fraction of the inclusions' phase is $c^{(1)} = c^{(\mathrm{II})}c^{(\mathrm{I})}$. When a uniform traction field compatible with a constant stress matrix $\bar{\boldsymbol{\Sigma}}$ is applied on the boundaries of the rank-2 composite, the stress fields $\boldsymbol{\Sigma}^{(\mathrm{I})}$ in the core and $\boldsymbol{\Sigma}^{(2)}$ in the matrix phases are uniform. It is emphasized that the stress field in the layers of the matrix phase in the core rank-1 laminate is also uniform but, in general, with a value different from $\boldsymbol{\Sigma}^{(2)}$. As in Eq. (12), from the traction continuity condition across the interfaces

$$\mathbf{P}_m^{(\mathrm{II})} \cdot \boldsymbol{\Sigma}^{(\mathrm{I})} = \mathbf{P}_m^{(\mathrm{II})} \cdot \boldsymbol{\Sigma}^{(2)} = \mathbf{P}_m^{(\mathrm{II})} \cdot \bar{\boldsymbol{\Sigma}}. \tag{15}$$

In a manner similar to the one followed for the rank-1 composite, we express the difference between the deviatoric projections of $\boldsymbol{\Sigma}^{(\mathrm{I})}$ and $\boldsymbol{\Sigma}^{(2)}$ in terms of $\omega^{(\mathrm{II})}$. Consequently, we have that

$$\begin{aligned}
\mathbf{P}_d^{(\mathrm{II})} \cdot \boldsymbol{\Sigma}^{(\mathrm{I})} &= (1 + (1-c^{(\mathrm{II})})\omega^{(\mathrm{II})})\mathbf{P}_d^{(\mathrm{II})} \cdot \bar{\boldsymbol{\Sigma}}, \\
\mathbf{P}_d^{(\mathrm{II})} \cdot \boldsymbol{\Sigma}^{(2)} &= (1 - c^{(\mathrm{II})}\omega^{(\mathrm{II})})\mathbf{P}_d^{(\mathrm{II})} \cdot \bar{\boldsymbol{\Sigma}}.
\end{aligned} \tag{16}$$

The stresses in the core and the matrix phases can be expressed in terms of the corresponding two pairs of projections determined in Eqs. (15) and (16). From the

principle of minimum complementary energy, the stress energy potential of the rank-2 sequentially laminated composite is then

$$\tilde{\psi}^{(\mathrm{II})}(\bar{\boldsymbol{\Sigma}}) = \inf_{\omega^{(\mathrm{II})}} \{c^{(\mathrm{II})}\tilde{\psi}^{(\mathrm{I})}[(1+(1-c^{(\mathrm{II})})\omega^{(\mathrm{II})})(\mathbf{P}_d^{(\mathrm{II})}\cdot\bar{\boldsymbol{\Sigma}})\mathbf{P}_d^{(\mathrm{II})} + (\mathbf{P}_m^{(\mathrm{II})}\cdot\bar{\boldsymbol{\Sigma}})\mathbf{P}_m^{(\mathrm{II})}] + (1-c^{(\mathrm{II})})\psi^{(2)}[(1-c^{(\mathrm{II})}\omega^{(\mathrm{II})})(\mathbf{P}_d^{(\mathrm{II})}\cdot\bar{\boldsymbol{\Sigma}})\mathbf{P}_d^{(\mathrm{II})} + (\mathbf{P}_m^{(\mathrm{II})}\cdot\bar{\boldsymbol{\Sigma}})\mathbf{P}_m^{(\mathrm{II})}]\}, \quad (17)$$

where $\tilde{\psi}^{(\mathrm{I})}$ is determined from Eq. (14).

By following the same steps followed for the ranks-1 and 2 composites higher order sequentially laminated composites can be constructed. At each rank, the corresponding stress energy potential is determined with an additional optimization variable that corresponds to the difference between the deviatoric projections of the stresses in the core and the matrix phases. Thus, in terms of the stress energy potential $\tilde{\psi}^{(N-1)}$ of a core rank-$(N-1)$ sequentially laminated composite, the potential of a rank-N composite subjected to a uniform traction field compatible with a uniform stress $\bar{\boldsymbol{\Sigma}}$ is

$$\tilde{\psi}^{(N)}(\bar{\boldsymbol{\Sigma}}) = \inf_{\omega^{(N)}} \{c^{(N)}\tilde{\psi}^{(N-1)}[(1+(1-c^{(N)})\omega^{(N)})(\mathbf{P}_d^{(N)}\cdot\bar{\boldsymbol{\Sigma}})\mathbf{P}_d^{(N)} + (\mathbf{P}_m^{(N)}\cdot\bar{\boldsymbol{\Sigma}})\mathbf{P}_m^{(N)}] + (1-c^{(N)})\psi^{(2)}[(1-c^{(N)}\omega^{(N)})(\mathbf{P}_d^{(N)}\cdot\bar{\boldsymbol{\Sigma}})\mathbf{P}_d^{(N)} + (\mathbf{P}_m^{(N)}\cdot\bar{\boldsymbol{\Sigma}})\mathbf{P}_m^{(N)}]\}. \quad (18)$$

In the above expression the projections $\mathbf{P}_d^{(N)}$ and $\mathbf{P}_m^{(N)}$ are derived via Eq. (11) with $\hat{\mathbf{m}}^{(N)}$ a unit vector normal to the layers of the rank-N composite. The volume fraction of the inclusions phase is

$$c^{(1)} = \prod_{J=1}^{N} c^{(J)}. \quad (19)$$

The expression obtained in Eq. (18) enables one to determine the exact stress energy potential for any incompressible sequentially laminated composite under plane strain conditions. The number of optimization variables is equal to the rank of the composite. It is noted that, due to the convexity of the optimization functionals, the optimization can be performed as a single N-dimensional optimization problem instead of N iterated one dimensional optimizations.

4. Applications to pure power-law composites

It is known that two-phase linear sequentially laminated composites of rank greater than, or equal to, the dimension of the underlying space can be constructed to admit overall isotropic behavior. This is accomplished by appropriate choice of the relative volume fractions and the layering directions of the core laminates at each step (Tartar, 1985). Ponte Castañeda (1992b) determined the effective potentials of ranks-2 and 3 nonlinear sequentially laminated composites and demonstrated that, unlike their linear

counterparts, these materials cannot be constructed to behave isotropically. In this section we make use of the general iterative procedure developed in the previous section to determine the effective behavior of high-rank sequentially laminated composites with power-law behavior of the constituent phases. It is demonstrated, albeit numerically, that as the rank of the sequentially laminated composites becomes large the overall behavior of these materials tends to be isotropic.

Consider a sequentially laminated composite made out of two isotropic phases with stress energy functions

$$\psi^{(r)}(\mathbf{\Sigma}) = \frac{\gamma_0 \tau_0^{(r)}}{n+1} \left(\frac{\sqrt{(1/2)\mathbf{\Sigma}\cdot\mathbf{\Sigma}}}{\tau_0^{(r)}} \right)^{n+1}, \quad r = 1, 2, \tag{20}$$

where n is the hardening exponent, γ_0 is a reference shear strain, and $\tau_0^{(r)}$ are the reference shear stresses of the two phases. In the limit $n = 1$ the phases are linear with shear moduli $\tau_0^{(r)}/\gamma_0$, and in the limit of rigid-perfectly plastic behavior (i.e., $n \to \infty$), $\tau_0^{(r)}$ are the yield shear stresses.

In the following we take $\tau_0^{(1)} > \tau_0^{(2)}$ and construct two complementary families of sequentially laminated composites. The first, with stiff inclusions and a softer matrix, is constructed by layering at each rank layers of the core laminate with layers of the material with reference stress $\tau_0^{(2)}$. The second family, with a stiffer matrix phase, is constructed by using the material with the reference stress $\tau_0^{(1)}$ as the matrix phase. There is clearly an infinite number of ways to realize these materials by choosing different volume fractions and lamination directions of the core laminates at each stage. Here, we proceed in the simplest way of choosing an identical volume fraction for the core laminates at each rank, and a fixed lamination angle of each successive rank with respect to the lamination direction of the core laminate. From Eq. (19), in a rank-N composite the fixed volume fractions of the core laminates are

$$c^{(J)} = \sqrt[N]{c^{(i)}}, \quad J = 1, 2, \ldots, N,$$

where the $c^{(i)}$ is the volume fraction of the stiffer and the softer phases in the two families of composites. The lamination direction of each rank, relative to the corresponding direction in the core lamina, is

$$\alpha = \pi \left(\frac{\eta}{N} \right), \tag{21}$$

where the number of rounds η is the number of times the absolute lamination direction, measured relative to the initial rank-1 laminate, has been repeated during the lamination sequence.

Uniform boundary conditions, which are compatible with a constant stress of the form

$$\bar{\mathbf{\Sigma}} = \sqrt{2}(\cos 2\theta \mathbf{P}_d + \sin 2\theta \mathbf{P}_m)$$

are applied, and the effective stress energy potentials of the sequentially laminated composites are determined for different values of θ by application of Eq. (18). As mentioned before, the optimization problem is solved as an N-dimensional optimization

Table 1
Mean values of $\tilde{\tau}^{(N)}/\tau_0^{(2)}$ and their maximal variations with θ for $n=3$

Rank		4	8	16	16	32	64	64
α		$\pi/4$	$\pi/8$	$\pi/4$	$\pi/8$	$\pi/8$	$\pi/8$	$\pi/16$
η		1	1	4	2	4	8	4
Family 2	$\tilde{\tau}^{(N)}/\tau_0^{(2)}$	1.192	1.194	1.192	1.194	1.194	1.194	1.194
	Δ%	2.80	0.57	2.65	0.47	0.41	0.40	0.08
Family 1	$\tilde{\tau}^{(N)}/\tau_0^{(2)}$	1.147	1.145	1.145	1.143	1.143	1.143	1.143
	Δ%	5.00	2.13	3.85	1.08	0.50	0.28	0.50

problem. Since the equations are homogeneous of degree $n+1$, the effective potential of the pure power-law hardening sequentially laminated composite can be expressed in the form

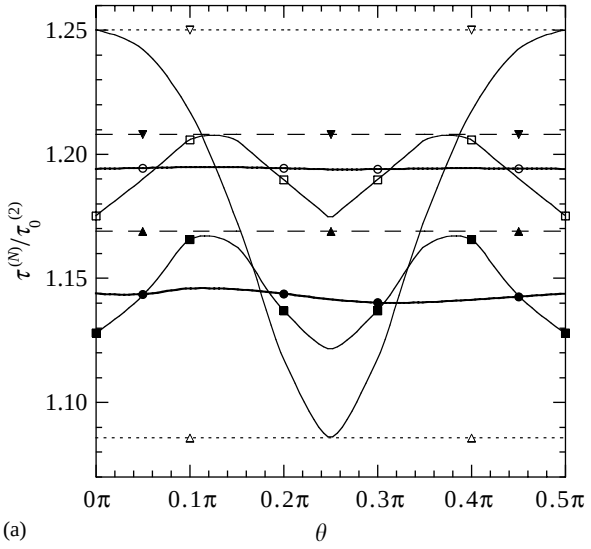
$$\tilde{\psi}^{(N)}(\bar{\boldsymbol{\Sigma}}) = \frac{\gamma_0 \tilde{\tau}^{(N)}(\theta)}{n+1}\left(\frac{\sqrt{(1/2)\bar{\boldsymbol{\Sigma}}\cdot\bar{\boldsymbol{\Sigma}}}}{\tilde{\tau}^{(N)}(\theta)}\right)^{n+1},$$

where $\tilde{\tau}^{(N)}$ is an effective reference shear stress which is a function of θ.

The variations of $\tilde{\tau}^{(N)}$ as functions of θ for sequentially laminated composites of ranks 16 and 64, normalized by $\tau_0^{(2)}$, are shown in Fig. 3. The hardening exponent is $n=3$, the contrast between the properties of the constituting phases is $\tau_0^{(1)}/\tau_0^{(2)}=2$, and the volume fraction of the stiffer phase is $c^{(1)}=0.25$. Fig. 3a corresponds to the case of a fixed number of rounds $\eta=4$ where the relative lamination angle α is determined according to Eq. (21). Results for a rank-1 composite are shown for comparison. Fig. 3b depicts the variation of $\tilde{\tau}^{(N)}$ when the relative lamination angle is kept fixed $\alpha=\pi/8$ while this time, the number of rounds (η) is obtained from relation (21). Both families of composites, with the harder and the softer inclusions, are shown in the figures. The curves for the rank 16 and 64 composites are the continuous ones which are marked by squares and circles, respectively. The curves for the first family, with the harder inclusions, are marked with dark markers and those for the second family are marked with clear markers. It is noted that the composites with the stiffer inclusions are essentially the softer ones and hence the curves corresponding to this family are in the lower sections of the plots. The continuous, unmarked curves correspond to the rank-1 laminate and these are of course identical for the two families.

It is evident that as the rank of the sequentially laminated composites increases, the variations of $\tilde{\tau}^{(N)}$ with θ become smaller, suggesting that the overall behavior tends towards an isotropic one. In fact, regardless of the precise lamination procedure (i.e., figure a or b), and for both families, the variations between the maximal and the minimal values of $\tilde{\tau}^{(64)}$ are smaller than 0.5%. The mean values of $\tilde{\tau}^{(N)}$ as functions of θ and the corresponding maximal variations are summarized in Table 1 for different sequentially laminated composites.

It is interesting to note that the variations in $\tilde{\tau}^{(N)}$ depend on the lamination process. When the inclusions are stiffer, it seems that for the same rank, smaller variations are obtained by increasing the number of rounds while the relative lamination angle



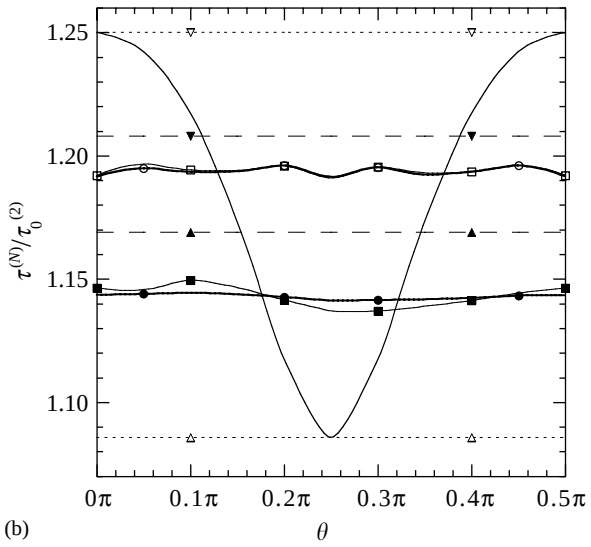


Fig. 3. $\tilde{\tau}^{(N)}$ as functions of θ for rank 16 (squares) and 64 (circles) sequentially laminated composites with hardening exponent $n = 3$. Figs. 3a and b correspond to different lamination sequences. The dark and clear marks are for the composites from the first and the second families, respectively. Also shown are $\tilde{\tau}^{(1)}$ (unmarked curves), the Reuss and Voigt bounds (clear triangles) and the Hashin–Shtrikman bounds and estimates of Ponte Castañeda (1992a) (dark triangles).

α is relatively large. On the other hand, for the second family, the variations in the effective reference stress are smaller when both, the lamination angles and the number of rounds, are relatively small.

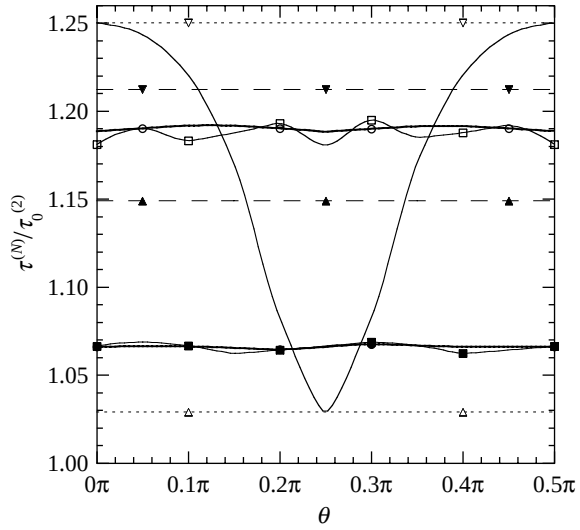


Fig. 4. $\tilde{\tau}^{(N)}$ as functions of θ for rank 32 (squares) and 64 or 96 (circles) sequentially laminated composites with hardening exponent $n=10$. The dark and clear marks are for the composites from the first and the second families, respectively. Also shown are $\tilde{\tau}^{(1)}$ (unmarked curve), the classical Reuss and Voigt bounds (clear triangles) and the Hashin–Shtrikman bounds and estimates of Ponte Castañeda (1992a) (dark triangles).

For comparison, also shown in Fig. 3 are the classical Voigt and Reuss bounds which are marked by short dashed curves. As expected, at $\theta = 0$ and $\pi/4$ these bounds are attained by the rank-1 laminate. The Hashin–Shtrikman lower bound and the estimate for the upper bound of Ponte Castañeda (1992a) are shown by the top and bottom long dashed curves, respectively (see also Suquet, 1993; deBotton and Ponte Castañeda, 1993). It is noted that the curves for the high-rank sequentially laminated composites do not violate the bound. However, the curves corresponding to the softer family of composites lie well below the estimate for the upper bound. This result is in agreement with the higher order estimates determined recently by Ponte Castañeda (1996).

The effective reference stresses, as functions of θ, for sequentially laminated composites with a higher hardening exponent $n = 10$, are shown in Fig. 4 for the two complementary families. The contrast between the properties of the two phases is $\tau_0^{(1)}/\tau_0^{(2)} = 2$, and the volume fraction of the stiffer phase is $c^{(1)} = 0.25$. It is anticipated that as the hardening exponent becomes large the variations in $\tilde{\tau}^{(N)}(\theta)$ increase. For the first family of composites variations smaller than 0.5%, as those obtained for $n=3$, were achieved only with rank-96 sequentially laminated composites. The continuous curves correspond to the results determined for rank-32, 64 for the second family and 96 for the first family of composites. The curves for the ranks 32, 64 or 96 composites are marked, respectively, by squares and circles. The dark markers correspond to the family of composites with the stiffer inclusions and the clear markers to the second family with the softer inclusions. The unmarked curve is for the common rank-1 laminate.

Table 2
Mean values of $\tilde{\tau}^{(N)}/\tau_0^{(2)}$ and their maximal variations with θ for $n = 10$

Rank		8	16	32	64	64	96	96
α		$\pi/8$	$\pi/8$	$\pi/8$	$\pi/8$	$\pi/16$	$\pi/8$	$\pi/16$
η		1	2	4	8	4	12	6
Family 2	$\tilde{\tau}^{(N)}/\tau_0^{(2)}$	1.189	1.189	1.187	1.188	1.190		
	Δ%	1.30	1.26	1.20	1.20	0.30		
Family 1	$\tilde{\tau}^{(N)}/\tau_0^{(2)}$	1.070	1.067	1.066		1.066	1.065	1.066
	Δ%	3.33	2.00	0.60		0.60	0.86	0.40

As before, the variations of $\tilde{\tau}^{(N)}(\theta)$ for two different composites with the same rank depend on the lamination sequence (as in Fig. 3a and b). Large η and α result in smaller variations for the first family and vice versa. The results depicted in Fig. 4 correspond to those for which the variations in θ are the smallest. The average values of $\tilde{\tau}^{(N)}$, their maximal variation with θ and the lamination sequence are summarized in Table 2.

Also shown in Fig. 4 are the short dashed upper and lower curves that correspond to the classical Voigt and Reuss bounds. The upper and lower long dashed curves correspond, respectively, to the Hashin–Shtrikman lower bound and the estimate for the upper bound of Ponte Castañeda (1992a). The curves for the high-rank sequentially laminated composites do not violate any of the bounds. The curves for the first family of composites, with the stiffer inclusions, are closer to the Reuss upper bound than they are to the estimate of Ponte Castañeda (1992a) for the upper bound.

In Fig. 5 the average values of $\tilde{\tau}^{(N)}(\theta)$, normalized by $\tau_0^{(2)}$, as functions of the hardening exponent n are shown for the two families of sequentially laminated composites. The curves are plotted versus the reciprocal of the hardening exponent. The volume fraction of the stiffer phase, the inclusions in the first family and the matrix in the second, is $c^{(1)} = 0.25$. The continuous lower (clear circles) and upper (dark circles) curves correspond to the results obtained for the first and the second families, respectively. The ranks of the composites for which the results are shown in the figure were chosen such that the variations in the effective reference stresses (as functions of θ) are smaller than 0.5%. There was no need to consider sequentially laminated composites with rank higher than 96. Fig. 5a corresponds to the case where the ratio between the reference stresses of the two phases is $\tau_0^{(1)}/\tau_0^{(2)} = 2$ and 5b to $\tau_0^{(1)}/\tau_0^{(2)} = 10$. Also shown in the figure are the classical Voigt and Reuss bounds which are represented by short dashed curves marked by clear upside down and right triangles, respectively. (The curve for the Voigt bound in Fig. 5b, for which $\tilde{\tau}^{(V)}/\tau_0^{(2)} = 3.25$ for all n, is not shown.) The short dashed curves marked by dark upside down and right triangles correspond to the lower bound and the estimate for the upper bound of Ponte Castañeda (1992a), respectively. Additionally, the second-order estimates of Ponte Castañeda (1996) for the class of two-dimensional isotropic composites are depict by the long dashed curves which are marked with diamonds.

First, it is emphasized that the curves for the sequentially laminated composites do not violate any of the three bounds shown in the figures. Further, in the linear limit

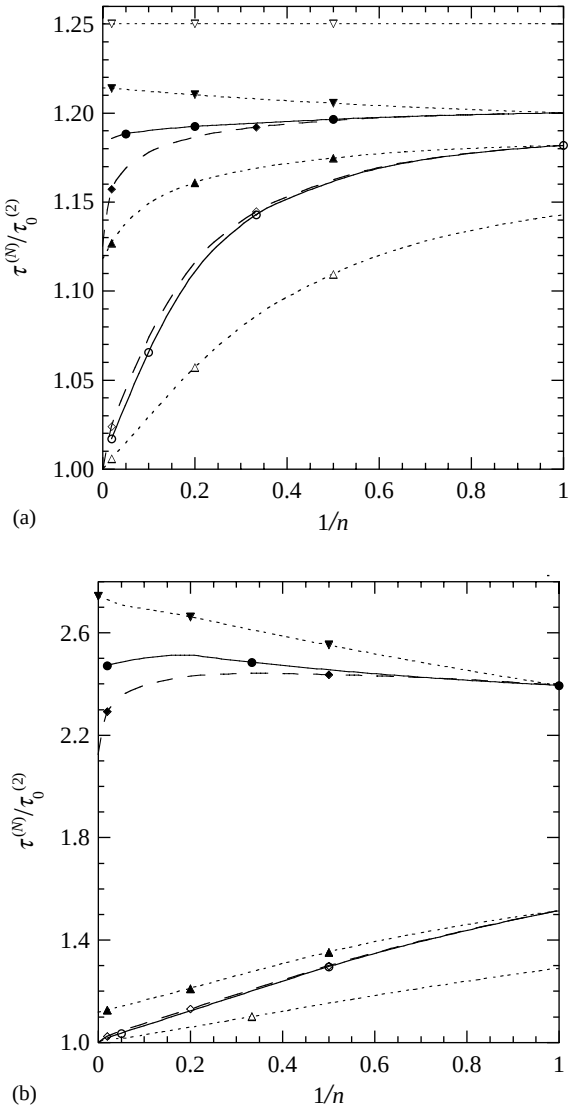


Fig. 5. $\tilde{\tau}^{(N)}$ as functions of $1/n$ for high-rank sequentially laminated composites. Figs. 5a and b correspond to contrasts $\tau_0^{(1)}/\tau_0^{(2)}=2$ and 10, respectively. The continuous curves with clear (dark) circles are for composites from the first (second) family. Also shown are the classical bounds (dashed curves, clear triangles), the Hashin–Shtrikman bounds and estimates of Ponte Castañeda (1992a) (dashed curves, dark triangles), and the second-order estimates of Ponte Castañeda (1996) (long dashed curves, diamonds).

$n=1$ the effective reference stresses of the sequentially laminated composites agree with the corresponding Hashin and Shtrikman (1962) bounds together with the appropriate estimates and bounds of Ponte Castañeda (1992a, 1996). As the hardening exponent increases the effective reference stress of the composites from the first family, with the

stiffer inclusions (clear circles) tends towards the classical Reuss bound while diverging from the estimate for the upper Hashin–Shtrikman bound. The trend of the curves is remarkably close to that for the upper second-order estimates of Ponte Castañeda (1996). Yet the curves corresponding to the sequentially laminated composites lie between the curves for second-order estimate and the classical Reuss bound. We recall that an upper bound for statistically isotropic composites with a power-law hardening behavior was recently determined by Talbot and Willis (1997). Direct comparison of the results obtained for the two-dimensional sequentially laminated composites with this bound is not available at present. Nonetheless, the comparison between the second-order estimates of Ponte Castañeda (1996) and this bound (see Fig. 2 of Ponte Castañeda and Willis, 1999) suggest that the results for the sequentially laminated composites do not violate this rigorous upper bound. The curves for the second family of composites (dark circles) also diverge from the lower bound of Ponte Castañeda (1992a) with the increase of the hardening exponent. In this case, however, at high values of the hardening exponent ($n > 10$) the curves for $\tilde{\tau}^{(N)}$ diverge from the corresponding second-order estimates too, and lie halfway between these estimates and the lower Hashin–Shtrikman bounds.

5. Concluding remarks

In this work a simple expression for the effective stress energy potential of two-dimensional nonlinear incompressible sequentially laminated composites has been introduced. Originating from the principle of minimum complementary energy, the effective stress energy potential of the core laminate at each stage is expressed in terms of the continuous and the discontinuous projections of the stress tensor. This form of presentation allows to determine the exact stress energy potential of sequentially laminated composites with arbitrary volume fractions and lamination directions of the core laminates. The expression for the effective stress energy potential is obtained in terms of an N-dimensional optimization problem.

Stress energy potentials for sequentially laminated composites with pure power-law behavior of the phases were determined. For the sake of simplicity, these composites were constructed with fixed volume fractions and fixed relative lamination directions of the core laminates at each stage. Two complementary families of composites were considered, one corresponds to particulate composites with soft matrix and stiffer inclusions and the second to the opposite case. It was found that the stress energy potentials of high-rank sequentially laminated composites constructed this way tends to be isotropic in the sense that their variations as functions of the direction of the applied stress become negligible. The variations of the stress energy potentials of rank-96 composites with an order of magnitude contrast between the behaviors of the phases and power-hardening exponent larger than 10 were found to be smaller than 0.5%. It was further found that sequentially laminated composites with stiffer inclusions and soft matrix behave in a manner which is slightly more ductile than the one predicted by the second-order estimate of Ponte Castañeda (1996). The composites from the second family, with the stiffer matrix phase, exhibit a behavior which is noticeably stiffer than

the corresponding second-order estimate of Ponte Castañeda (1996). The fact that the behaviors of these almost isotropic composites from the two complementary families are softer and stiffer than the corresponding estimates of Ponte Castañeda (1996) is in agreement with the conclusion of Ponte Castañeda and Willis (1999) that these estimates are not rigorous bounds.

Acknowledgements

The authors wish to thank Prof. J.R. Willis for his insight concerning the extension of this work to incorporate materials whose behavior depends on the third invariant of the stress.

Appendix

We consider an isotropic incompressible layer whose behavior is governed by a smooth *strictly convex* stress energy potential $U(\boldsymbol{\sigma}) = \phi(J_2, J_3)$, where $J_2 = \mathbf{S}\cdot\mathbf{S}$ and $J_3 = |\mathbf{S}|$ are the second and third invariants of the stress deviator $\mathbf{S}$. We assume that ϕ is an even function of J_3. The layer is subjected to plane strain conditions and, for convenience, we choose a coordinate system such that the x_3-axis is normal to the layer's plane and the axes x_1 and x_2 are along the unit vectors $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$, respectively. Under plane strain conditions the fields developing in the layer must be independent of x_3 and the strain components along the x_3 axis must vanish, that is

$$\varepsilon_{i3} = \frac{\partial\phi(J_2,J_3)}{\partial J_2}\frac{\partial J_2}{\partial\sigma_{i3}} + \frac{\partial\phi(J_2,J_3)}{\partial J_3}\frac{\partial J_3}{\partial\sigma_{i3}} = 0, \quad i = 1,2,3.$$

Explicitly, these conditions may be expressed in the form

$$\varepsilon_{13} = 4\frac{\partial\phi}{\partial J_2}\sigma_{13} + 2\frac{\partial\phi}{\partial J_3}\left(\sigma_{23}\sigma_{12} - (\sigma_{22} - \frac{1}{3}\sigma_{mm})\sigma_{13}\right) = 0,$$
$$\varepsilon_{23} = 4\frac{\partial\phi}{\partial J_2}\sigma_{23} + 2\frac{\partial\phi}{\partial J_3}\left(\sigma_{13}\sigma_{12} - (\sigma_{11} - \frac{1}{3}\sigma_{mm})\sigma_{23}\right) = 0$$

and

$$\varepsilon_{33} = 2\frac{\partial\phi}{\partial J_2}\left(\sigma_{33} - \frac{1}{3}\sigma_{mm}\right) + \frac{1}{3}\frac{\partial\phi}{\partial J_3}\left(\left(\sigma_{11} - \frac{1}{3}\sigma_{mm}\right)(\sigma_{22} - \sigma_{33})\right.$$
$$\left. + \left(\sigma_{22} - \frac{1}{3}\sigma_{mm}\right)(\sigma_{11} - \sigma_{33}) - 2\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2\right) = 0.$$

The contour of the layer, that is the portion of its boundary whose normal is perpendicular to the x_3-axis, is subjected to a *uniform* inplane traction field which is compatible with a constant symmetric tensor $T_{\alpha\beta}$, $(\alpha,\beta) = (1,2)$. The uniform field

$\sigma_{11}^{(0)} = T_{11},\ \sigma_{22}^{(0)} = T_{22},\ \sigma_{12}^{(0)} = T_{12},\ \sigma_{13}^{(0)} = \sigma_{23}^{(0)} = 0$ and $\sigma_{33}^{(0)} = \frac{1}{2}(T_{11} + T_{22})$ together with the accompanying constant strain field

$$\varepsilon_{ij}^{(0)} = \frac{\partial U}{\partial \sigma_{ij}}(\boldsymbol{\sigma}^{(0)}),$$

clearly satisfy the governing equilibrium and compatibility equations. With this choice for the stresses $J_3^{(0)} = 0$ and, due to the assumption that ϕ is an even function of J_3, the partial derivative $\partial\phi/\partial J_3$ vanishes. Additionally, since $\sigma_{33}^{(0)} = \frac{1}{3}\sigma_{mm}^{(0)}$, the above conditions for vanishing strains along the x_3 direction are all satisfied. Thus, we conclude that the field $\sigma_{ij}^{(0)}$ is a possible solution for the imposed boundary value problem.

Assume that there exists a second field $\sigma_{ij}^{(1)}(x_1, x_2)$ which also provides a solution for the imposed problem. On the layer's contour this field must satisfy the uniform boundary conditions $\sigma_{11}^{(1)} = T_{11},\ \sigma_{22}^{(1)} = T_{22},\ \sigma_{12}^{(1)} = T_{12}$. In terms of $\sigma'_{ij} = \sigma_{ij}^{(1)} - \sigma_{ij}^{(0)}$, a Taylor series expansion about $\sigma_{ij}^{(0)}$ of the total energy stored in the layer is

$$\begin{aligned}\int_V U(\boldsymbol{\sigma}^{(1)})\,\mathrm{d}\mathbf{x} &= \int_V U(\boldsymbol{\sigma}^{(0)} + \boldsymbol{\sigma}')\,\mathrm{d}\mathbf{x}\\ &= VU(\boldsymbol{\sigma}^{(0)}) + \int_V \frac{\partial U}{\partial \sigma_{ij}}(\boldsymbol{\sigma}^{(0)})\sigma'_{ij}\,\mathrm{d}\mathbf{x}\\ &\quad + \frac{1}{2}\int_V \frac{\partial^2 U}{\partial \sigma_{ij}\partial \sigma_{kl}}(\boldsymbol{\sigma}^{(0)} + \lambda\boldsymbol{\sigma}')\sigma'_{ij}\sigma'_{kl}\mathrm{d}\mathbf{x},\end{aligned}$$

for some $0 \leqslant \lambda \leqslant 1$. The first-order term of the Taylor expansion may be expressed in the form

$$\int_V \frac{\partial U}{\partial \sigma_{ij}}(\boldsymbol{\sigma}^{(0)})\sigma'_{ij}\,\mathrm{d}\mathbf{x} = \int_V \varepsilon_{ij}^{(0)}\sigma'_{ij}\,\mathrm{d}\mathbf{x} = \int_V (u_i^{(0)}\sigma'_{ij})_{,j}\ \mathrm{d}\mathbf{x} - \int_V u_i^{(0)}\sigma'_{ij,j}\,\mathrm{d}\mathbf{x}.$$

The last term vanishes since $\sigma'_{ij,j} = 0$. Recalling that all the fields are independent of x_3, and by making use of Green's theorem we have

$$\begin{aligned}\int_V (u_i^{(0)}\sigma'_{ij})_{,j}\ \mathrm{d}\mathbf{x} &= \int_V (u_i^{(0)}\sigma'_{i1})_{,1}\ \mathrm{d}\mathbf{x} + \int_V (u_i^{(0)}\sigma'_{i2})_{,2}\ \mathrm{d}\mathbf{x}\\ &= \int_{\partial V} u_i^{(0)}\sigma'_{i1}n_1\,\mathrm{d}\mathbf{x} + \int_{\partial V} u_i^{(0)}\sigma'_{i2}n_2\,\mathrm{d}\mathbf{x}.\end{aligned}$$

We note that this last term also vanishes since everywhere $u_3^{(0)} = 0$, on the faces of the layer $n_1 = n_2 = 0$, and on the contour $\sigma'_{11} = \sigma'_{22} = \sigma'_{12} = 0$. Finally, due to the assumed convexity of the stress potential the second-order term of the Taylor series expansion is positive unless $\boldsymbol{\sigma}' = 0$ (Van Tiel, 1984), and from the principle of minimum complementary energy (7) it follows that $\boldsymbol{\sigma}^{(0)}$ is indeed the solution for the boundary value problem.

The components of the associated stress deviator are $S_{11}^{(0)} = -S_{22}^{(0)} = \frac{1}{2}(\sigma_{11}^{(0)} - \sigma_{22}^{(0)})$ and $S_{12}^{(0)} = \sigma_{12}^{(0)}$. It can be easily verified that within the above described coordinate

system $S_{ij}^{(0)} = \Sigma_{ij}$, where the reduced stress tensor $\mathbf{\Sigma}$ is defined in Eq. (9) in terms of the projections $\mathbf{P}_d$ and $\mathbf{P}_m$. Additionally, since $\mathbf{P}_d \cdot \mathbf{P}_d = 1$, $\mathbf{P}_m \cdot \mathbf{P}_m = 1$ and $\mathbf{P}_d \cdot \mathbf{P}_m = 0$, we note that $\mathbf{\Sigma} \cdot \mathbf{\Sigma} = (\mathbf{P}_d \cdot \boldsymbol{\sigma})^2 + (\mathbf{P}_m \cdot \boldsymbol{\sigma})^2$.

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