G. deBotton

K. Schulgasser
Mem. ASME.

The Pearlstone Center for Aeronautical Engineering Studies, Department of Mechanical Engineering, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel

Bifurcation of Orthotropic Solids

We consider a large deformation plane-strain problem involving a compressible orthotropic solid subjected to uniaxial compressive loading along one of the principle directions which is aligned with the boundary of a half-space. An exact solution for the displacement field is obtained and a condition for the smallest compressive load corresponding to the onset of a surface instability is determined. It is shown that when the compression occurs along the stiffest direction this condition is expressible in terms of a cubic polynomial, and that the corresponding critical load is lower than the well-known estimate which determines the critical load to be equal to the inplane shear modulus.

Introduction

Solids of orthotropic symmetry have become common in various engineering applications, and accordingly, the problem of predicting their stability limited compressive strength is of practical importance. Among the first works dealing with this subject we may cite the work of Biot (1965) and earlier references to his work therein; he developed a theory for dealing with prestressed solids and applied his approach to the study of the behavior of nonlinear rubber-like materials. Hill and Hutchinson (1975) carried out a general discussion concerning the bifurcation phenomena and obtained estimates for the critical tensile stresses of orthotropic solids.

With the growing usage of fiber-reinforced and other composites, the concept of bifurcation and loss of stability at the micro level was introduced. Rosen (1965) obtained, by making use of beam theory, the well-known estimate that the compressive strength is equal to the effective in-plane shear modulus of the composite. This result has been rederived by different methods in other works dealing with fiber and laminated composites. Among these works we mention the article by Ben-chin and Pipkin (1972) dealing with a class of inextensible fiber-reinforced incompressible matrices, the article by Steif (1987) demonstrating that beam theory is applicable only when the fiber diameter is very small compared with the variation of the deformation, and the article by Christensen (1994) providing an asymptotic solution for linear elastic composites in the limit of large ratios between the longitudinal Young's modulus and the other elastic moduli. Methods which take into account imperfections of reinforced composites, such as initial waviness or partial debonding of the fibers are also available, but this is outside the scope of the present study. (See, for example, the work by Budiansky (1983) and the comprehensive study by Kyriakides et al. (1995) for the former, and Steif (1988) for the later.)

An extensive and rigorous study concerning the microscopic and macroscopic instability mechanisms of nonlinear heterogeneous solids was carried out by Geymonat et al. (1993), generalizing previous work of Triantafyllidis and Maker (1985). These investigators obtained, within the general framework of finite deformations, rigorous results for the complete boundary value problem of elasticity in the long and short wave limits. Among other topics, they also demonstrated that under appropriate convexity assumptions homogenization and linearization procedures commute. Finally, a procedure for estimating the compressive strength of linearly elastic fiber-reinforced composites can be deduced from the work of Shield et al. (1994), who obtained rigorous results for the critical compressive strain of a thin layer bonded to an infinite matrix.

The present study deals with an orthotropic, linearly elastic macroscopically homogeneous solid subjected to uniaxial compression perpendicular to one of the symmetry planes. Beginning from the equilibrium equations we determine the general expression for the displacement field. Next, from the boundary conditions we find an exact condition for the lowest compressive stress which is compatible with a displacement field other than the homogeneous one. We stress that while Geymonat et al. (1993) solved an analogous problem for a class of materials more general than the one we consider here, the advantage of the present approach is centered on the simplicity of the resulting expressions. Thus, we obtain a simple *analytical* solution for the boundary value problem which leads to predictions for the critical compressive strength in a straightforward manner. The solution may also be used as a *benchmark* for more advanced, numerically based models for the prediction of compressive failures.

We specialize the general result to the practically important situation involving compression along the stiffest direction, and compare the resulting predictions for the critical loading with the well-known, beam theory based estimate of Rosen (1965). As was noticed by Christensen (1994), in the limit of highly anisotropic materials the exact result tends towards the beam theory estimate. However, we demonstrate that for more realistic materials the sharpness of this estimate strongly depends on some of the other elastic moduli characterizing the orthotropic solid. We also demonstrate the ability of the results obtained in this work to provide predictions for the critical compressive load of slightly anisotropic solids, circumstances where the beam theory estimate would apparently be of no practical usefulness.

Governing Equations for the Plane-Strain Problem

We consider a half-space, $x_2 \leq 0$, of an orthotropic solid whose principle axes are aligned with the Cartesian axes. Initially, the solid is in a state of homogeneous strain corresponding to compressive uniform stress p parallel to the free boundary along the x_1-direction. In terms of the second Piola-Kirchhoff stress, which measures the tractions per unit undeformed area, the equilibrium equations are (see, for example, Malvern (1969))

$$\nabla \cdot [(\boldsymbol{\sigma} + \boldsymbol{\Sigma}) \cdot \nabla(\mathbf{x} + \mathbf{u})] = \mathbf{0}, \tag{1}$$

where we have neglected body forces. In the above relation, $\mathbf{x}$ and $\mathbf{u}$ are the position and displacement vectors, respectively, and $\boldsymbol{\sigma}$ is a small stress increment beyond the initially applied stress $\boldsymbol{\Sigma}$. We assume that the product terms $\boldsymbol{\sigma} \cdot \nabla \mathbf{u}$ and their

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derivatives are small and ignore them in the subsequent analysis. Consequently, for plane-strain conditions, where $\Sigma_{11} = -p$ is the only nonvanishing component of the initial stress, the remaining two equilibrium equations may be expressed in the form

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} - p\frac{\partial^2 u_1}{\partial x_1^2} = 0,$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} - p\frac{\partial^2 u_2}{\partial x_1^2} = 0. \qquad (2)$$

We note that, alternatively, we could obtain these equations by following the variational procedure outlined by Biot (1965) (chapter 2, Section 5), with the only difference being that instead of using his expressions for the strains in terms of the displacements, we should have used the more common definition of the Lagrangian strain tensor

$$\mathbf{e} = \tfrac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T + \nabla \mathbf{u}\cdot(\nabla \mathbf{u})^T], \qquad (3)$$

which is the appropriate conjugate variable to the second Piola-Kirchhoff stress in the undeformed configuration.

Following Biot (1965) (Chapter 2, Section 3), we assume a linear relationship between the stress increment and the *infinitesimal* strain increment. Accordingly, for an orthotropic solid in plane-strain condition we may write

$$\sigma_{11} = C_{11}\frac{\partial u_1}{\partial x_1} + C_{12}\frac{\partial u_2}{\partial x_2},$$

$$\sigma_{22} = C_{12}\frac{\partial u_1}{\partial x_1} + C_{22}\frac{\partial u_2}{\partial x_2},$$

$$\sigma_{12} = C_{66}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \qquad (4)$$

where C_{11}, C_{12}, C_{22}, and C_{66} are the instantaneous elastic moduli of the solid in the vicinity of the prestressed configuration. Substituting these relations into the equilibrium Eqs. (2) we obtain the following expressions for the equilibrium equations in terms of the displacements, namely,

$$(C_{11} - p)\frac{\partial^2 u_1}{\partial x_1^2} + C_{66}\frac{\partial^2 u_1}{\partial x_2^2} + (C_{12} + C_{66})\frac{\partial^2 u_2}{\partial x_1 \partial x_2} = 0,$$

$$(C_{66} - p)\frac{\partial^2 u_2}{\partial x_1^2} + C_{22}\frac{\partial^2 u_2}{\partial x_2^2} + (C_{12} + C_{66})\frac{\partial^2 u_1}{\partial x_1 \partial x_2} = 0. \qquad (5)$$

We recall that similar relations were previously obtained by Shield et al. (1994) for solids of isotropic symmetry and by Christensen (1994) for the same class of orthotropic solids considered above.

Generalizing the solution proposed by Shield et al. (1994) for the isotropic case, we find that the following choice for the displacement field components, namely

$$u_1 = [AH_a e^{ax_2} + Be^{bx_2}]\sin(x_1),$$

$$u_2 = [Ae^{ax_2} + BH_b e^{bx_2}]\cos(x_1), \qquad (6)$$

satisfies the equilibrium Eqs. (5), where

$$H_a = -\frac{a[C_{22}(C_{11} - E) + a^2 C_{22}C_{66} + C_{66}(2C_{12} + p)]}{(C_{12} + C_{66})(C_{66} - p)},$$

$$H_b = -\frac{b[C_{22}(E - p) - b^2 C_{22}C_{66} - C_{66}(2C_{12} + C_{66})]}{(C_{12} + C_{66})(C_{66} - p)}, \qquad (7)$$

a and b are the positive roots of the bi-quadratic

$$z^4 - 2H_2 z^2 + H_0 = 0, \qquad (8)$$

where

$$H_2 = \frac{C_{22}(E - p) - C_{66}(2C_{12} + p)}{2C_{22}C_{66}},$$

$$H_0 = \frac{(C_{11} - p)(C_{66} - p)}{C_{22}C_{66}}, \qquad (9)$$

and A and B are arbitrary constants. In the above relations

$$E = \frac{C_{11}C_{22} - C_{12}^2}{C_{22}} \qquad (10)$$

is a Young's modulus such that, for a linear solid in *plane strain*, $\epsilon = p/E$ is the uniform compressive strain associated with the initial loading. It can be verified that the above displacements u_1 and u_2 decay as x_2 tend to negative infinity, and reduce to the appropriate expressions obtained by Shield et al. (1994) in the isotropic limit. We note that due to the lack of a characteristic length scale for this problem, the wavelength of the displacement field is undetermined. Accordingly, the displacement field components in Eqs. (6) may be expressed in terms of dimensionless axes $\xi_i = x_i/\lambda$, $i = 1, 2$, where λ is an arbitrary wave length.

The expressions for the stress components are determined by making use of relation (6) in the stress-displacement relations (4). These must satisfy the boundary conditions $\sigma_{12} = 0$ and $\sigma_{22} = 0$ at the free boundary $x_2 = 0$, providing a set of two homogeneous linear equations for the constants A and B. To obtain a nontrivial solution for these equations the determinant of the coefficients must vanish, and it can be shown that the roots of the determinant must satisfy the relation

$$(C_{11} - p)(C_{66} - p)(H_2^2 - H_0)(Y(p))^2 = 0, \qquad (11)$$

where we note that the third term is a quadratic polynomial in p, and

$$Y(p) = C_{22}(C_{66} - p)(E - p)^2 - C_{66}(C_{11} - p)p^2 \qquad (12)$$

is a cubic polynomial in p. Equation (11) is a condition for the onset of a surface instability, stating that a nontrivial solution to the problem can exists only when $p \geq p_{cr}$, where p_{cr} is the smallest real positive root.

Applications to Orthotropic Solids

We apply the results of the previous section to determine the critical compressive loading for which a solid of orthotropic symmetry becomes unstable. From a practical point of view, the interesting configuration is the one that involves compression along the stiff direction, e.g., along the fibers or the layers of fiber-reinforced or laminated composites, respectively. In this section, we restrict ourselves to this common loading situation.

For convenience, we express the elastic properties of the solid in terms of the engineering constants, E_i the Young's moduli along the x_i axes, ν_{ij} the Poisson ratios measuring the lateral contraction along the x_j axis due to uniaxial tension along the x_i-axis, and G_{12} the shear modulus in the (x_1, x_2)-plane. In terms of these engineering constants, the four independent components of the *plane-strain* compliance matrix S are

$$S_{11} \equiv \frac{1}{E} = \frac{1}{E_1}\left(1 - \frac{E_3}{E_1}\nu_{13}^2\right),$$

$$S_{12} = -\frac{1}{E_1}\left(\nu_{12} + \frac{E_3}{E_2}\nu_{13}\nu_{23}\right),$$

$$S_{22} = \frac{1}{E_2}\left(1 - \frac{E_3}{E_2}\nu_{23}^2\right), \qquad (13)$$

and

$$S_{66} = \frac{1}{G_{12}},$$

and where the moduli C_{11}, C_{12}, C_{22}, and C_{66} are the appropriate components of $\mathbf{S}^{-1}$.

The predictions for the critical loading p_{cr} can be easily obtained from the solutions of the four polynomials composing relation (11). However, due to our assumption that the compression occurs along the stiff direction, we have that $G_{12} = C_{66} < C_{11}$ and we can rule out the root $p = C_{11}$. Furthermore, it can be shown that if

$$\frac{E_1}{E_2} > \frac{E_l - G_{12}}{E_2 - G_{12}}, \quad (14)$$

where $E_l = 2G_{12}(1 + \nu_{12})$, the roots of the quadratic term in (11) are complex, and since this inequality is satisfied for many orthotropic solids with $E_1 > E_2$, we may exclude the quadratic term as well. Thus, practically, the only possible roots of interest are the smallest root of the cubic polynomial $Y(p)$ and the root $p = G_{12}$. However, we also note that if $G_{12} < C_{11}$ the first and the second terms of Y are positive as long as $p < G_{12}$. This suggests that at least one real root of Y must lie in the interval $p < G_{12}$, and since $Y(G_{12}) < 0$ and $Y(0) > 0$ this root must also be positive. We conclude then, that p_{cr} is the smallest root of the cubic polynomial $Y(p)$, and it is always lower than the well-known estimate $p_R = G_{12}$ which was first proposed by Rosen (1965).

Estimates for the critical loading normalized by the in-plane shear modulus plotted versus the ratio of the longitudinal to transverse Young's moduli are shown in Fig. 1, for the choice of the elastic constants $E_3 = E_2$, $\nu_{12} = \nu_{13} = 0.3$ and $\nu_{23} = 0.4$. Results for two different values of the ratio between the in-plane shear modulus and the transverse Young's modulus (0.5 and 0.8) are shown. The continuous curves correspond to the exact results which were obtained from the solution of $Y = 0$ and the long dashed curve to the estimate $p_R = G_{12}$. We note that in this figure, and the subsequent ones, a circle at the point where a curve stops marks that the limit of the validity of inequality (14) has been reached. We observe that, as expected, in the limit $E_1 \gg E_2$ the exact results approach the beam theory based estimate p_R. A less obvious observation is that when the ratio G_{12}/E_2 is small enough (which is the case for many reinforced composites), Rosen's estimate provides an excellent approximation even for relatively low values of E_1/E_2. This is not the case for larger values of the ratio between the shear modulus and the transverse Young modulus. Thus, for moderate values of E_1 and G_{12}/E_2 approaching unity, the relative error between the exact and the approximate predictions for the critical loading is approximately ten percent.

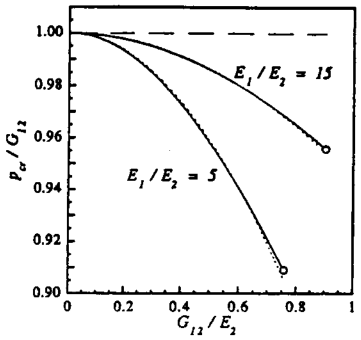

Fig. 2 Estimates for the critical compressive load normalized by the in-plane shear modulus versus the ratio between the in-plane shear and transverse Young's moduli, for two different values of the ratio between the longitudinal and the transverse Young's moduli and $E_3 = E_2$, $\nu_{12} = \nu_{13} = 0.3$, and $\nu_{23} = 0.4$. The continuous curves correspond to the exact estimate, the long dashed curve to the beam theory based estimate, and the short dashed curves to the first-order approximation (15). The circles at the points where the curves stop mark the limit of the validity of inequality (14).

The short dashed curves correspond to the first-order approximation for the smallest root of Y

$$p_e = G_{12}\left[1 - \left(1 - \frac{E_3}{E_2}\nu_{23}^2\right)\left(\frac{G_{12}}{E_2}\right)\left(\frac{G_{12}}{E}\right)\right], \quad (15)$$

in the limit when E_1 is larger than the other elastic moduli. Clearly, p_e provides an excellent estimate for p_{cr} throughout the entire range of possible ratios between the longitudinal and transverse Young's moduli. We note that the zero-order term in (15) is precisely the estimate p_R from beam theory, as well as the asymptotic result obtained by Christensen (1994) from relation (5) by an order of magnitude analysis. However, we stress that in spite of the fact that the solution proposed by Christensen (1994) for (5) delivers the zero-order approximation for the critical load, the associated displacement field does not decay as x_2 tends to negative infinity (as would happened for any *finite* E_1). In accordance with our earlier observations concerning Fig. 1, we note that the ratio G_{12}/E_2 appears already in the first order term and dominates the sharpness of the beam theory approximation.

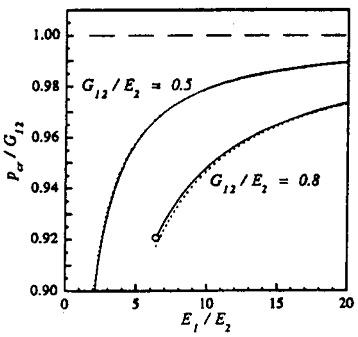

Fig. 1 Estimates for the critical compressive load normalized by the in-plane shear modulus versus the ratio between the longitudinal and transverse Young's moduli, for two different values of the ratio between the in-plane shear and the transverse Young's moduli and $E_3 = E_2$, $\nu_{12} = \nu_{13} = 0.3$, and $\nu_{23} = 0.4$. The continuous curves correspond to the exact estimate, the long dashed curve to the beam theory based estimate, and the short dashed curves to the first-order approximation (15). The circle at the point where the curve stops marks the limit of the validity of inequality (14).

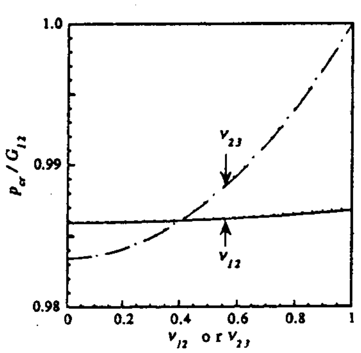

Fig. 3 Estimates for the critical compressive load normalized by the in-plane shear modulus versus the inplane and out-of-plane Poisson's ratios, where $E_1 = 15E_2$, $G_{12} = E_2/2$ and $E_3 = E_2$. The continuous curve corresponds to the in-plane Poisson's ratio with $\nu_{13} = \nu_{12}$ and $\nu_{23} = 0.4$, and the long dashed curve to the out-of-plane ratio ν_{23} with $\nu_{12} = \nu_{13} = 0.3$. The short dashed curves are the corresponding first-order approximations (15).

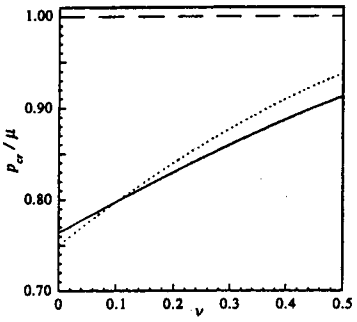

Fig. 4 Estimates for the critical compressive load of a slightly anisotropic solid normalized by the shear modulus versus the Poisson's ratio. The continuous curve corresponds to the exact estimate, the long dashed curve to the beam theory based estimate, and the short dashed curve to the first-order approximation (15).

Figure 2 shows values for the critical compressive load as functions of the ratio between the inplane shear and the transverse Young's moduli for the same choice of the elastic constants as in Fig. 1. The continuous curves correspond to the exact result and the long and short dashed curves to the zero and first-order approximations, respectively. Results for two different ratios of the longitudinal Young's modulus to the transverse Young's modulus are presented ((5) and (15)). We observe that, regardless of the value of E_2, in the limit of small in-plane shear modulus the critical compressive load is precisely G_{12}. We also observe the proximity of the first-order approximation p_e to the exact result, implying that this simple estimate captures the principal features of the solution.

The dependence of the critical load on the *in-plane* and *out-of-plane* Poisson's ratios (ν_{12} and ν_{23}, respectively) is shown in Fig. 3 for the choice of the elastic constants $E_1 = 15E_2$, $G_{12} = E_2/2$ and $E_3 = E_2$. The continuous curve depicts the dependence on the in-plane Poisson's ratio with $\nu_{13} = \nu_{12}$ and $\nu_{23} = 0.4$, and the long dashed curve the dependence on the out-of-plane ratio ν_{23} with $\nu_{12} = \nu_{13} = 0.3$. We note that the inplane Poisson's ratio has only negligible influence on the critical load, while the influence of the out-of-plane ratio is much more noticeable. Once again, we observe that the first-order approximation p_e is in near perfect agreement with the exact solution (note that the dependence on ν_{13} appears implicitly through the dependence on E).

An interesting limit is, of course, the isotropic limit. In this case relation (12) reduces to the form

$$Y(p) = (1-\nu)p^3 - 8(1-\nu)\mu p^2 + 8(2-\nu)\mu^2 p - 8\mu^3, \quad (16)$$

where μ and ν are the shear modulus and the Poisson's ratio of the isotropic solid, respectively. We emphasize that for isotropic solids equality holds in (14), and thus, the solution provided by (16) should be viewed as a limiting case for slightly anisotropic solids for which $E_1 \rightarrow E_2 = 2\mu(1+\nu)$. We note that the strains associated with p_{cr} in this limit are large (more than 20 percent) and it is unlikely that solids would undergo such strains while still in the linear regime. Results for the compressive strength normalized by the shear modulus of isotropic solids are shown in Fig. 4 as functions of the Poisson's ratio. We observe that, in a manner similar to that noted for orthotropic solids, for fixed μ the compressive strength is lower for solids with smaller Poisson's ratios. We also observe that the relative error between p_R (long dashed curve) and the exact result is more than ten percent, while the next order approximation p_e (short dashed curve) provides reasonable estimates even in this limit.

Concluding Remarks

A rigorous solution for a finite deformation plane-strain problem involving uniaxial compression along one of the principle axes of a linearly elastic orthotropic solid was determined. Based on this solution, an explicit condition for the smallest compressive stress which is compatible with a nonhomogeneous state of deformation was found. Special attention was given to the practically common situation of compression along the stiffest principle direction, and it was shown that in this case the stability condition can be expressed in terms of a cubic polynomial. It was further shown that the well-known beam theory estimate, which is commonly cited in applications for reinforced materials, overestimates the critical compressive strength.

Numerical predictions of the critical compressive loading of certain orthotropic solids were computed, demonstrating the domain of applicability of the beam theory estimate and highlighting the relative contribution of the various elastic coefficients. In particular, the high sensitivity of the critical loading to the ratio between the in-plane shear modulus and the transverse Young's modulus was detected. An explicit, higher order approximation for the compressive strength was determined, exhibiting excellent agreement with the exact result. Finally, it is stressed that the simple solution that was presented in this work can be incorporated into more sophisticated models that take into account initial imperfections of reinforced solids.

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