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THE EFFECTIVE YIELD STRENGTH OF FIBER-REINFORCED COMPOSITES

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Abstract—This work is concerned with the determination of the effective properties of transversely isotropic fiber composites made up of two rigid–perfectly plastic phases in prescribed volume fractions. The phases are assumed to satisfy incompressible, isotropic yield criteria of the Mises type. To study the behavior of these composites we make use of variational principles, recently developed by Ponte Castañeda (1991, *J. Mech. Phys. Solids* **39**, 45–71), that provide a method for generating estimates for the effective properties of nonlinear composites from corresponding estimates for the effective properties of linear composites. We demonstrate that this method allows us to obtain simple expressions for the effective yield functions of rigid-perfectly plastic composites. Explicit results, corresponding to the Hashin–Shtrikman bounds, the self consistent and the generalized self consistent estimates, and the composite cylinder assemblage model are obtained for the class of rigid-perfectly plastic fiber composites. These estimates exhibit the existence of two distinct yielding modes, in agreement with corresponding experimental results.

1. INTRODUCTION

Over the years, different approaches have been employed to study the effective behavior of rigid–perfectly plastic composites. Among the first results available in the literature, we may cite the work of Drucker (1959) who made use of plasticity limit theorems to establish bounds for the effective behavior of perfectly plastic, nonhomogeneous materials. Shu and Rosen (1967) and Majumdar and McLaughlin (1975) determined upper and lower bounds for the effective yield strength domain of fiber composites by application of these plasticity limit theorems. A well-known estimate for the overall strength domain for rigid–perfectly plastic matrices weakened by cylindrical voids was introduced by Gurson in (1977). Ponte Castañeda and deBotton (1992) determined bounds and estimates of the Hashin–Shtrikman (1962) type for statistically isotropic and fiber-reinforced composites by making use of a variational method proposed by Ponte Castañeda in (1991). By application of a different method, Suquet (1993) obtained various bounds and estimates for the effective tensile yield strength of statistically isotropic composites. Olson (1994), by application of yet a different method, also obtained bounds of the Hashin–Shtrikman type for the effective yield stress of statistically isotropic composites. In other cases, numerical schemes were used to determine estimates for the effective strength domain of rigid–perfectly plastic composites with specific microstructures [see, for example, Bao *et al.* (1991)].

In this work, we obtain estimates for the effective yield strength domain of two-phase rigid–perfectly plastic fiber composites with statistically isotropic distribution of the fibers in the transverse plane. To this end, we utilize a variational procedure developed by Ponte Castañeda (1991, 1992) that enables the generation of bounds and estimates for the effective properties of nonlinear composites in terms of optimization problems involving corresponding bounds and estimates for the effective moduli of appropriate families of linear comparison composites. Applications of the procedure to obtain various bounds and estimates for the effective properties of nonlinear, statistically isotropic and incompressible fiber composites were carried out by Ponte Castañeda in these two works. DeBotton and Ponte Castañeda (1993) made use of this procedure to determine bounds and estimates for the effective behavior of nonlinear, fiber-reinforced composites. The procedure was also applied to the class of rigid–perfectly plastic composites by Ponte Castañeda and deBotton

(1992), and here, we make use of these latter results to develop new, simple estimates for the effective yield functions of two-phase composites.

Five types of estimates for the effective yield functions of fiber-reinforced composites will be considered. The first two estimates are the Hashin–Shtrikman upper bound and lower estimate which were determined previously by Ponte Castañeda and deBotton (1992). These two estimates are obtained from the linear Hashin–Shtrikman upper and lower bounds introduced by Hill (1964) and Hashin (1965). The third estimate corresponds to the generalized self consistent estimate first proposed for linear elastic fiber composites by Hermans (1967) and later modified by Christensen and Lo (1979). The significance of this estimate is centered on the fact that in some approximate way the interactions among the fibers are taken into account. In the context of linear elasticity, this method was found to provide satisfactory approximations for the effective moduli of linear elastic composites. The fourth estimate is obtained from the composite cylinder assemblage model of Hashin and Rosen (1964). This model corresponds to a space filling assemblage of composite cylinders in various diameters, each of which is made up of a fiber embedded in a concentric matrix shell. The fifth estimate is the self consistent estimate which is obtained from the corresponding linear estimates of Hill (1965). In spite of the fact that these self consistent estimates always lie between the Hashin–Shtrikman bounds, there are some limitations on their range of applicability, in particular, when the contrast between the properties of the phases is large [see related discussion in Hashin (1983)].

Explicit calculations are carried out for representative classes of fiber-reinforced composites comprised of two isotropic phases of the Mises type with different tensile yield strengths. The resulting estimates for the effective yield functions of the composites are presented in terms of yield surfaces in the space of the three incompressible transversely isotropic invariants of the stress tensor. These estimates are compared with corresponding results obtained by application of other methods and with available experimental results.

2. THE EFFECTIVE YIELD STRENGTH OF TWO-PHASE COMPOSITES

In this section we consider the class of two-phase rigid–perfectly plastic composites with periodic microstructure. The effective properties of composites belonging to this class can be determined by considering a unit cell of the microstructure Y. For simplicity, we will restrict ourselves to the class of composites made up of incompressible, isotropic phases of the Mises type with yield strengths $k^{(1)}$ and $k^{(2)}$, in prescribed volume fractions $c^{(1)}$ and $c^{(2)} = 1-c^{(1)}$, respectively. In terms of a position vector $\mathbf{y}$ defined over the unit cell, the *strength domain* of the composite P is given by the set

$$P(\mathbf{y}) = \{\boldsymbol{\sigma}(\mathbf{y}) | \boldsymbol{\sigma}_e \leqslant k(\mathbf{y}), \text{ with } k(\mathbf{y}) = k^{(r)} \text{ if } \mathbf{y} \in Y^{(r)}, \ r = 1, 2\}, \tag{1}$$

where $\boldsymbol{\sigma}_e = \sqrt{\frac{3}{2}\boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}'}$ is the equivalent stress, $\boldsymbol{\sigma}'$ is the deviatoric part of $\boldsymbol{\sigma}$, and $Y^{(1)}$ and $Y^{(2)}$ are the regions of the unit cell occupied by phases 1 and 2, respectively. The *effective strength domain* of the composite may be described by the set (Suquet, 1983)

$$\tilde{\mathrm{P}} = \{\Sigma \,|\, \exists\, \boldsymbol{\sigma}(\mathbf{y}),\ \operatorname{div}_y \boldsymbol{\sigma} = 0,\ \boldsymbol{\sigma}(\mathbf{y}) \in P(\mathbf{y}),\ \boldsymbol{\sigma}\mathbf{n} \text{ is antiperiodic}, \int_Y \boldsymbol{\sigma}(\mathbf{y})\, \mathrm{d}y = \boldsymbol{\Sigma}\}, \tag{2}$$

where $\boldsymbol{\Sigma} = \int_Y \boldsymbol{\sigma}(\mathbf{y})\, \mathrm{d}y$ is the average stress tensor and $\mathbf{n}$ is the unit outward normal to the boundary of Y.

Alternatively, the strength domain of the composite may be represented by means of the *support function* of P,

$$\pi(\mathbf{y}, \mathbf{d}) = \sup_{\sigma \in P(\mathbf{y})} \{\boldsymbol{\sigma} \cdot \mathbf{d}\}, \tag{3}$$

which corresponds physically to the plastic dissipation function associated with the rate of deformation tensor $\mathbf{d}(\mathbf{y})$. In particular, for composites with incompressible, isotropic phases of the Mises type,

$$\pi(\mathbf{y}, \mathbf{d}) = k(\mathbf{y})d_e + \delta_0(d_m), \tag{4}$$

where $k(\mathbf{y})$ is given in eqn (1), $d_e = \sqrt{\frac{2}{3}\mathbf{d}' \cdot \mathbf{d}'}$, $\mathbf{d}'$ is the deviatoric part of $\mathbf{d}$, d_m is the hydrostatic component of $\mathbf{d}$, and $\delta_0 = 0$ if $d_m = 0$, or $\delta_0 = \infty$ otherwise. In terms of the support function we have that

$$P(\mathbf{y}) = \text{dom}\,\{\pi^*(\mathbf{y}, \boldsymbol{\sigma})\}, \tag{5}$$

where $\pi^*(\mathbf{y}, \boldsymbol{\sigma}) = \sup_{\mathbf{d}} \{\boldsymbol{\sigma} \cdot \mathbf{d} - \pi(\mathbf{y}, \mathbf{d})\}$ is the Legendre–Fenchel polar of π, and

$$\text{dom}\,\{\pi^*(\mathbf{y}, \boldsymbol{\sigma})\} \equiv \{\boldsymbol{\sigma} \,|\, \pi^*(\mathbf{y}, \boldsymbol{\sigma}) < \infty\}, \tag{6}$$

(Van Tiel, 1984, §§ 6.1 and 5.11). The *effective support function* $\tilde{\pi}$ is defined via the relation

$$\tilde{\pi}(\mathbf{D}) = \inf_{\mathbf{d} \in K} \int_Y \pi(\mathbf{y}, \mathbf{d})\, \mathrm{d}y, \tag{7}$$

where $\mathbf{D} = \int_Y \mathbf{d}(\mathbf{y})\, \mathrm{d}y$ is the average strain-rate tensor, and the infimum is taken over the set of kinematically admissible strain rates,

$$K = \{\mathbf{d} \,|\, \exists\, \mathbf{v}, \mathbf{d} = \tfrac{1}{2}[\nabla\mathbf{v} + (\nabla\mathbf{v})^T],\ \mathbf{v} = \mathbf{D}\mathbf{y} + \mathbf{v}', \text{ and } \mathbf{v}' \text{ is periodic}\}. \tag{8}$$

The corresponding effective strength domain $\tilde{P}$ may be determined, via relations analogous to the local relations (5) and (6), from the Legendre–Fenchel polar of $\tilde{\pi}$.

We note that, in some cases, applications of eqns (2) or (7) can be made to obtain bounds or estimates for the effective strength domain. For instance, Majumdar and McLaughlin (1975) and de Buhan *et al.* (1991) obtained bounds for the effective strength domain for certain classes of fiber composites under plane stress conditions and general loading conditions, respectively. However, in most cases, the task of obtaining expressions for the effective yield strength domains from these expressions is very complicated. For this reason, in the following section, we make use of an alternative variational procedure that enables us to obtain relatively simple expressions for bounds and estimates for the effective yield strength domain.

3. ESTIMATES FOR THE EFFECTIVE YIELD STRENGTH OF TWO-PHASE COMPOSITES

The developments followed in this section are based on a general variational procedure, originally developed by Ponte Castañeda (1991, 1992), for estimating the effective properties of composite materials with arbitrary nonlinear behavior of the constituent phases. This variational procedure was applied to the class of rigid–perfectly plastic composites by Ponte Castañeda and deBotton (1992), and here, we advance and obtain additional results for this particular class of composites.

The procedure is based on the following representation for the support function π (Ponte Castañeda and deBotton, 1992), namely,

$$\pi(\mathbf{y}, \mathbf{d}) = \inf_{\mu \geqslant 0} \{w(\mathbf{y}, \mathbf{d}) + v(\mathbf{y}, \mu)\} \tag{9}$$

where

$$w(\mathbf{y}, \mathbf{d}) = \tfrac{3}{2}\mu(\mathbf{y})d_e^2 + \delta_0(d_m), \tag{10}$$

is the local dissipation function of an incompressible, isotropic, linear comparison composite with viscosity coefficient μ, and

$$v(\mathbf{y}, \mu) = \sup_{\mathbf{d}} \{\pi(\mathbf{y}, \mathbf{d}) - w(\mathbf{y}, \mathbf{d})\}. \tag{11}$$

Equation (9) can be substituted into eqn (7) to obtain an alternative representation for the effective support function in the form (Ponte Castañeda, 1992),

$$\tilde{\pi}(\mathbf{D}) = \inf_{\mu(\mathbf{y}) \geq 0} \left\{ \tilde{w}(\mathbf{D}) + \int_Y v(\mathbf{y}, \mu)\, \mathrm{d}y \right\}, \tag{12}$$

where

$$\tilde{w}(\mathbf{D}) = \inf_{\mathbf{d} \in K} \int_Y w(\mathbf{y}, \mathbf{d})\, \mathrm{d}y, \tag{13}$$

is the effective dissipation function of a linear, heterogeneous comparison material governed by the local dissipation function w of eqn (10).

We stress that, in practice, it is difficult to obtain a solution for the variational procedure introduced in eqn (12) since the viscosity coefficient μ is an arbitrary function of the position vector. Nevertheless, a simple expression for an upper bound for $\tilde{\pi}$ can be obtained by restricting the function μ to the class of piecewise constant functions with a different constant in each phase (Ponte Castañeda, 1992). This upper bound may be expressed in the form

$$\tilde{\pi}(\mathbf{D}) \leq \tilde{\Pi}(\mathbf{D}) = \inf_{\mu^{(1)}, \mu^{(2)} \geq 0} \left\{ \tilde{w}(\mathbf{D}) + \sum_{r=1}^{2} c^{(r)} v^{(r)}(\mu^{(r)}) \right\}, \tag{14}$$

where $\tilde{w}$ is given in eqn (13), and

$$v^{(r)}(\mu^{(r)}) = \frac{1}{6\mu^{(r)}} (k^{(r)})^2. \tag{15}$$

In eqn (14), besides the advantage we gained from the relaxation of eqn (12) into a two-dimensional minimization problem, we also have that $\tilde{w}$ corresponds to the effective dissipation function of a linear comparison composite with two *homogeneous* phases distributed similarly to the two phases in the rigid–perfectly plastic composite. The two incompressible, isotropic phases in this comparison composite are characterized by the viscosity coefficients $\mu^{(1)}$ and $\mu^{(2)}$ and are prescribed in volume fractions $c^{(1)}$ and $c^{(2)}$, respectively. Bounds and estimates for $\tilde{w}$ can be found in the literature by analogy of the governing equations for linear viscous materials and linear elastic materials. However, we should note that because of the inequality introduced in eqn (14), upper bounds for $\tilde{w}$ will be translated into rigorous upper bounds for $\tilde{\pi}$, but lower bounds for $\tilde{w}$ will provide only estimates for the corresponding lower bounds for $\tilde{\pi}$.

From the variational procedure in eqn (14), first introduced by Ponte Castañeda and deBotton (1992), we proceed to obtain a simple estimate for the effective strength domain of the composite. This is obtained via the following relation between $\tilde{\Pi}$ and its Legendre–Fenchel polar $\tilde{\Pi}^*$ (Ponte Castañeda, 1992), namely,

$$\tilde{\Pi}^*(\mathbf{\Sigma}) = \sup_{\mu^{(1)},\mu^{(2)} \geqslant 0} \left\{ \tilde{w}^*(\mathbf{\Sigma}) - \sum_{r=1}^{2} \frac{c^{(r)}}{6\mu^{(r)}} (k^{(r)})^2 \right\}, \tag{16}$$

where $\tilde{w}^*$ is the Legendre–Fenchel polar of $\tilde{w}$, and where we made use of eqn (15) for the functions $v^{(r)}$. We note that $\tilde{\Pi}^*$ is a lower bound for $\tilde{\pi}^*$.

In the context of linear elasticity, it can be shown that $\tilde{w}^*$ is equal to the effective complementary energy function (in its rate form) of the linear comparison composite, and thus, it may be expressed in the form

$$\tilde{w}^*(\mathbf{\Sigma}) = \tfrac{1}{2}\mathbf{\Sigma} \cdot (\tilde{\mathbf{M}}\Sigma), \tag{17}$$

where $\tilde{\mathbf{M}} = \tilde{\mathbf{M}}(\mu^{(1)}, \mu^{(2)})$ is the *effective compliance tensor*. We further note that in many of the estimates for $\tilde{\mathbf{M}}$, and particularly in those that will be used in this work, by appropriate change of variables we may write

$$\tilde{\mathbf{M}}(\mu^{(1)}, \mu^{(2)}) = \frac{1}{\mu^{(1)}} \tilde{\mathbf{m}}(\rho), \tag{18}$$

where $\tilde{\mathbf{m}}$ is a dimensionless, fourth order tensor and $\rho = \mu^{(1)}/\mu^{(2)}$. By making use of eqns (17) and (18) in eqn (16) we have that

$$\tilde{\Pi}^*(\mathbf{\Sigma}) = \sup_{\mu^{(1)} \geqslant 0} \left\{ \frac{1}{\mu^{(1)}} \tilde{\Phi}(\mathbf{\Sigma}) \right\}, \tag{19}$$

where

$$\tilde{\Phi}(\mathbf{\Sigma}) = \sup_{\rho \geqslant 0} \{ \tfrac{1}{2}\mathbf{\Sigma} \cdot (\tilde{\mathbf{m}}(\rho)\mathbf{\Sigma}) - \tfrac{1}{6}c^{(2)}\rho(k^{(2)})^2 \} - \tfrac{1}{6}c^{(1)}(k^{(1)})^2. \tag{20}$$

From eqn (19) it is clear that $\tilde{\Pi}^* < \infty$ only if $\tilde{\Phi}(\mathbf{\Sigma})$ is non-positive, and hence, on account of eqns (5) and (6), we obtain the following estimate for $\tilde{P}$, namely,

$$\hat{P} = \{ \mathbf{\Sigma} | \tilde{\Phi}(\mathbf{\Sigma}) \leqslant 0 \}. \tag{21}$$

We note that since $\tilde{\Pi}^*$ is a lower bound for $\tilde{\pi}^*$, the effective strength domain of the composite is contained in the set $\hat{P}$ (i.e. $\tilde{P} \subset \hat{P}$).

Next, we recall that the boundary of $\tilde{P}$ in the $\mathbf{\Sigma}$-space defines an *effective yield function* for the composite in the form $\tilde{\phi}(\mathbf{\Sigma}) = 0$. Accordingly, the boundary of $\hat{P}$ provides an estimate for the effective yield function, and from the convexity of eqn (20), it follows that this boundary is characterized by the relation

$$\tilde{\Phi}(\mathbf{\Sigma}) \equiv 0. \tag{22}$$

Equation (22) provides a procedure for estimating the effective yield functions of rigid-perfectly plastic composites in terms of a one-dimensional minimization problem over corresponding estimates for the effective compliance tensor of linear comparison composites (with identical distribution of the phases). We stress that since $\tilde{\Pi}^* \leqslant \tilde{\pi}^*$, estimates for $\tilde{\Phi}$ from upper bounds for $\tilde{\mathbf{M}}$ will bound regions that are strictly larger than the region bounded by $\tilde{\phi}$, and thus, we may regard them as upper bounds for the effective yield function. On the other hand, estimates for $\tilde{\Phi}$ from lower bounds for $\tilde{\mathbf{M}}$ will bound regions that are not guaranteed to be contained in the region bounded by $\tilde{\phi}$. (In this sense, they are not rigorous lower bounds.)

We notice that Suquet (1993) proposed a different method for estimating the properties of rigid–perfectly plastic composites, and demonstrated that his method provides a one-dimensional minimization procedure for estimating the effective yield stress of statistically isotropic composites. The results obtained via his procedure may be recovered by application of eqn (22) to the particular class of statistically isotropic composites.

Finally, before we specialize the above results to the class of fiber-reinforced composites, we note that if phase 2 is rigid (i.e. formally $k^{(2)} = \infty$), eqn (20) for $\tilde{\Phi}$ reduces to the explicit form

$$\tilde{\Phi}(\mathbf{\Sigma}) = \tfrac{1}{2}\mathbf{\Sigma}\cdot(\tilde{\mathbf{m}}(0)\mathbf{\Sigma}) - \tfrac{1}{6}c^{(1)}(k^{(1)})^2, \tag{23}$$

which follows from the fact that in this case the optimal value for the minimization variable is $\rho = 0$.

4. APPLICATIONS TO FIBER COMPOSITES

We make use of the variational procedure in eqn (22) to obtain bounds and estimates for the effective strength domains of fiber-reinforced composites with statistically isotropic distribution of the fibers in the transverse plane. To this end, we require an expression for the effective complementary energy function $\tilde{w}^*$ of incompressible, linear elastic fiber composites with similar microstructure. Such an expression can be written in the form

$$\tilde{w}^*(\mathbf{\Sigma}) = \frac{1}{6\tilde{\mu}_d}\Sigma_d^2 + \frac{1}{6\tilde{\mu}_n}\Sigma_n^2 + \frac{1}{6\tilde{\mu}_p}\Sigma_p^2, \tag{24}$$

where $\tilde{\mu}_d$, $\tilde{\mu}_n$ and $\tilde{\mu}_p$ are the three shear moduli that suffice to characterize the behavior of such transversely isotropic materials (Lipton, 1992),

$$\begin{aligned} \Sigma_d &= \tfrac{3}{2}[\tfrac{1}{3}tr(\mathbf{\Sigma}) - \mathbf{n}\cdot(\mathbf{\Sigma}\mathbf{n})], & &\{\tfrac{1}{2}(\Sigma_{22}+\Sigma_{33}-2\Sigma_{11})\} \\ \Sigma_n &= \sqrt{3[(\mathbf{\Sigma}\mathbf{n})\cdot(\mathbf{\Sigma}\mathbf{n}) - [\mathbf{n}\cdot(\mathbf{\Sigma}\mathbf{n})]^2]}, & &\{\sqrt{3(\Sigma_{12}^2+\Sigma_{13}^2)}\} \\ \Sigma_p &= \sqrt{\Sigma_e^2 - \Sigma_d^2 - \Sigma_n^2}, & &\{\sqrt{3[\Sigma_{23}^2+\tfrac{1}{4}(\Sigma_{22}-\Sigma_{33})^2]}\} \end{aligned} \tag{25}$$

are the three (incompressible) transversely isotropic invariants of order two or less of $\mathbf{\Sigma}$, $\mathbf{n}$ is a unit normal aligned with the fibers and Σ_e is the equivalent stress. (Given in brackets are the expressions for these invariants for the choice of $\mathbf{n}$ aligned with the 1 direction.) Physically, Σ_d corresponds to axisymmetric loading (aligned with $\mathbf{n}$), Σ_n to shear along the fibers, and Σ_p to shear transverse to the fibers.

By making use of eqn (24) in eqn (20), we obtain the following estimate for the effective yield function for the class of rigid–perfectly plastic fiber composites, namely,

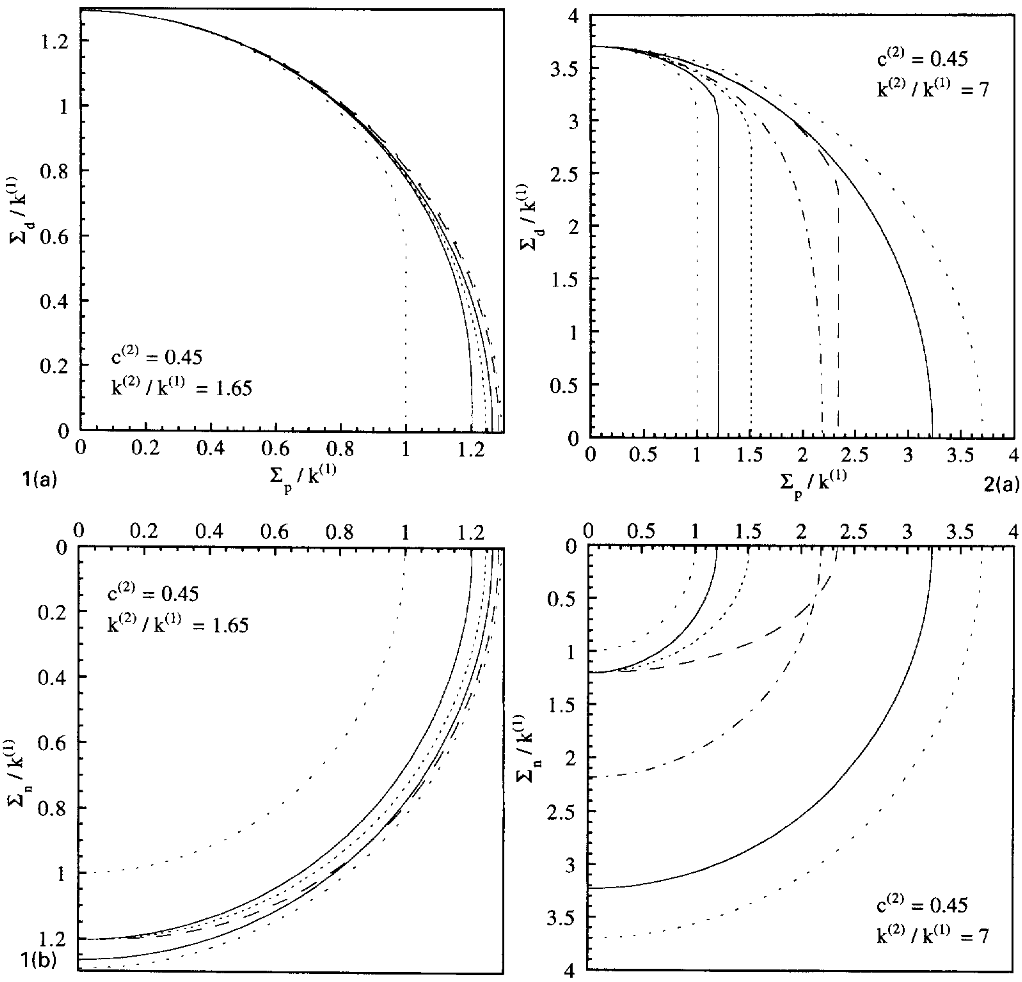
$$\tilde{\Phi}(\mathbf{\Sigma}) = \sup_{\rho\geq 0}\left\{\frac{\Sigma_d^2}{(\tilde{\mu}_d/\mu^{(1)})} + \frac{\Sigma_n^2}{(\tilde{\mu}_n/\mu^{(1)})} + \frac{\Sigma_p^2}{(\tilde{\mu}_p/\mu^{(1)})} - c^{(2)}\rho(k^{(2)})^2\right\} - c^{(1)}(k^{(1)})^2, \tag{26}$$

where, we recall that $\rho = \mu^{(1)}/\mu^{(2)}$. We note that $\tilde{\Phi}$ depends only on the three transversely isotropic invariants of $\mathbf{\Sigma}$, and thus, to obtain a complete description of the six-dimensional yield surface in the stress space it is sufficient to know its projection on the three-dimensional $(\Sigma_d, \Sigma_n, \Sigma_p)$-space.

As mentioned in the introduction, we will consider five estimates for the effective yield strength of fiber composites corresponding to the Hashin–Shtrikman (HS) upper and lower bounds, the generalized self consistent (GSC) estimate, the composite cylinder assemblage (CCA) model, and the self consistent (SCS) estimate. Estimates for the three independent shear moduli $\tilde{\mu}_d$, $\tilde{\mu}_n$ and $\tilde{\mu}_p$ from the five models are given in the appendix, and upon substitution of these estimates into eqn (26), the corresponding estimates for the effective yield functions $\tilde{\Phi}^{(HS+)}$, $\tilde{\Phi}^{(HS-)}$, $\tilde{\Phi}^{(GSC)}$, $\tilde{\Phi}^{(CCA)}$ and $\tilde{\Phi}^{(SCS)}$, are obtained.

In general, it is not possible to derive explicit expressions for the five estimates for the effective yield function, but, on the other hand, numerical results can be calculated readily. Representative results for two composites with volume fraction of the fibers $c^{(2)} = 0.45$, and different values for the ratio $k^{(2)}/k^{(1)}$ are shown in Figs 1 and 2. Figures 1(a) and 1(b) show, respectively, the intersections with the (Σ_d, Σ_p) and the (Σ_n, Σ_p) planes of the estimates for the effective yield surface of a fiber composite with $k^{(2)}/k^{(1)} = 1.65$. Figures 2(a) and 2(b) show, respectively, the same intersections of the estimates for the effective yield surface of a fiber composite with $k^{(2)}/k^{(1)} = 7$. In all figures, the outer and inner continuous curves correspond to the upper HS bounds and the estimates for the lower HS bounds, respectively; the short dash curves to the GSC estimates; the long dash curves to the CCA results; and the dashed-dotted curves to the SCS results.

The intersections of these estimates with the (Σ_d, Σ_n)-plane may be deduced from Figs 1(a) and 2(a) as follows. First, we recall that the expressions for the effective longitudinal and transverse shear moduli from the HS upper bound are similar, and hence, the intersections of $\tilde{\Phi}^{(HS+)}$ with the (Σ_d, Σ_n)-plane and the (Σ_d, Σ_p)-plane are identical. We also note that the same is true for the estimates $\tilde{\Phi}^{(HS-)}$ and $\tilde{\Phi}^{(SCS)}$. Next, we note that the expressions for $\tilde{\mu}_n$



Figs 1 and 2. Intersections with the (Σ_p, Σ_d)-plane [1(a) and 2(a)] and (Σ_p, Σ_n)-plane [1(b) and 2(b)] of the estimates for the effective yield surfaces of rigid-perfectly plastic fiber composites in volume fraction of the fibers $c^{(2)} = 0.45$, and $k^{(2)}/k^{(1)} = 1.65$ and $k^{(2)}/k^{(1)} = 7$, respectively. The continuous inner and outer curves correspond to the estimates from the HS lower and upper bounds, respectively; the short dash curves to the GSC estimates; the long dash curves to the CCA model; the dashed-dotted curves to the SCS results; and the outer and inner dotted curves to the trivial isotropic upper bounds from the principle of minimum work and the lower bounds from eqn (27), respectively.

from the GSC and the CCA estimates are similar to the expression for $\tilde{\mu}_n$ from the HS lower bound, and consequently, the intersections of $\tilde{\Phi}^{(\mathrm{GSC})}$, $\tilde{\Phi}^{(\mathrm{CCA})}$ and $\tilde{\Phi}^{(\mathrm{HS}-)}$ with the (Σ_d, Σ_n)-plane are identical. Thus, it suffices to consider only the intersections of the five estimates with the (Σ_d, Σ_p)-plane and the (Σ_n, Σ_p)-plane, recalling that the intersections of $\tilde{\Phi}^{(\mathrm{GSC})}$ and $\tilde{\Phi}^{(\mathrm{CCA})}$ with the (Σ_d, Σ_n)-plane are similar to the intersection of $\tilde{\Phi}^{(\mathrm{HS}-)}$ with the (Σ_d, Σ_p)-plane.

For comparison, also shown in Figs 1 and 2 are the trivial isotropic upper bounds obtained from the classical principle of minimum work (analogous to the Voigt bound in linear elasticity). The innermost dotted curves correspond to lower bounds for the effective yield strength domains obtained by de Buhan *et al.* (1991) by assuming, in eqn (2), a piecewise constant trial stress field satisfying the optimization constraints. In terms of the three transversely isotropic invariants, the associated lower bound for the effective yield function $\tilde{\Phi}^{(\mathrm{LB})}$ may be expressed in the form,

$$\tilde{\Phi}^{(\mathrm{LB})}(\boldsymbol{\Sigma}) = \begin{cases} \Sigma_d - c^{(1)}\sqrt{(k^{(1)})^2 - (\Sigma_n^2 + \Sigma_p^2)} - c^{(2)}\sqrt{(k^{(2)})^2 - (\Sigma_n^2 + \Sigma_p^2)}, & \Sigma_n^2 + \Sigma_p^2 < (k^{(1)})^2, \\ \Sigma_n^2 + \Sigma_p^2 - (k^{(1)})^2, & \Sigma_n^2 + \Sigma_p^2 = (k^{(1)})^2. \end{cases} \tag{27}$$

We note that under plane stress loading conditions, where the loading plane is aligned with the direction of the fibers, the above expression is completely equivalent to the lower bound introduced by Majumdar and McLaughlin (1975). This is anticipated since the trial stress field assumed by de Buhan *et al.* (1991) reduces to the corresponding stress field determined by Majumdar and McLaughlin (1975) for the plane stress case. Interestingly, eqn (27) is reminiscent of the exact expression for the effective yield function for laminated composites obtained by Ponte Castañeda and deBotton (1992).

We observe that according to the five estimates the principal yield stress along the Σ_d axis (when $\Sigma_n = \Sigma_p = 0$) coincide with the trivial upper bound, and may be expressed in the form

$$\tilde{\Sigma}_d = c^{(1)}k^{(1)} + c^{(2)}k^{(2)}. \tag{28}$$

This results from the fact that the expressions for $\tilde{\mu}_d$, obtained from the corresponding five estimates for the class of linear fiber composites, agree with the classical Voigt bound. Additionally, as mentioned earlier, the GSC and CCA estimates for the principal yield stress along the Σ_n axis (when $\Sigma_d = \Sigma_p = 0$) agree with the corresponding estimate from the HS lower bound.

In Figs 1(a) and 1(b), that correspond to a composite with small ratio of the yield strength of fibers to that of the matrix, the estimates for the effective yield surface are almost isotropic and the principle stresses along the three axes are very close to the classical upper bound (dotted curves). This suggests that a quadratic interpolation between the three principle yield stresses, in the form of Hill's (1948) extension of the Mises yield criteria to slightly anisotropic materials, may be used to approximate the effective yield functions of fiber composites in this limit. Interestingly, we note that the CCA estimate (long dash curves) violates the HS upper bound (outer continuous curves) when the transverse shear Σ_p is the dominating loading mode. The curves from the SCS estimate are not shown in these two figures since they are disposed almost on top of the curves for the HS upper bound.

The differences between the five estimates for the effective yield function become more noticeable when the ratio $k^{(2)}/k^{(1)}$ increases [e.g. Figs 2(a), (b)]. In particular, we note that while the intersection of $\tilde{\Phi}^{(\mathrm{HS}+)}$ with the (Σ_d, Σ_p)-plane resembles the classical upper bound, the intersections of $\tilde{\Phi}^{(\mathrm{HS}-)}$, $\tilde{\Phi}^{(\mathrm{GSC})}$ and $\tilde{\Phi}^{(\mathrm{CCA})}$ with the (Σ_d, Σ_p)-plane are characterized by two distinct sections, a flat section parallel to the Σ_d axis and a curved part associated with large axisymmetric loads. The flat section corresponds to an overall yielding of the composite due to matrix yielding in shear loads along or transverse to the fibers, and the other section to simultaneous yielding of the matrix and the fibers under axisymmetric loads. In spite of the fact that the yielding mechanism is only one of the sources of failure in composites, it is interesting to note that the prediction of two different yielding modes is reminiscent of the predictions of Hashin (1980) who argued, on physical grounds, that the failure surfaces of fiber composites are composed of two primary failure modes, a matrix mode and a fiber mode. We also note that the prediction of a matrix yielding mode for fiber composites with plastically deforming matrices is in good agreement with experimental results of Dvorak *et al.* (1988) for boron-reinforced aluminum composites.

The lack of a flat section in the intersection of $\tilde{\Phi}^{(HS+)}$ with the (Σ_d, Σ_p)-plane can be motivated via the following alternative interpretation for the HS upper bound. First, we recall that Lipton (1992) demonstrated that for the class of linear, incompressible transversely isotropic fiber composites the Hashin–Shtrikman bounds are optimal, that is, there are specific microstructures that attain the bounds. In particular, the upper bound is attained by a fiber composite in which the stiffer phase plays the role of the matrix and the softer phase that of the fibers. Therefore, we may regard the Hashin–Shtrikman upper bound for $\tilde{w}^*$ as an estimate for the effective energy function of a linear fiber composite with stiff matrix weakened by softer fibers. This alternative point of view can be extended to the class of rigid-perfectly plastic fiber composites since the expression for $\tilde{\Phi}^{(HS+)}$ was obtained from the linear bound. Consequently, we may regard $\tilde{\Phi}^{(HS-)}$ as an estimate for the effective yield function of a fiber composite made up of a matrix with yield strength $k^{(2)}$ and fibers with yield strength $k^{(1)}$, such that $k^{(2)} > k^{(1)}$. Clearly, it is unlikely that such a composite will exhibit a matrix yielding mode in which only the stiff matrix phase yields.

An analogous, but opposite, interpretation can be given to the HS lower estimate by following the same steps followed for the HS upper bound. Thus, we may regard $\tilde{\Phi}^{(HS-)}$ as an estimate for the effective yield function of a fiber composite made up of a matrix with yield strength $k^{(1)}$ and fibers with yield strength $k^{(2)}$, such that $k^{(2)} > k^{(1)}$. This interpretation puts the HS lower estimate on an equal footing with the GSC and the CCA estimates in the sense that these three estimates correspond to the class of fiber composites with soft matrices reinforced by stiffer fibers.

We observe that the yield surface associated with the SCS estimate bounds a volume of the stress space which is larger than the corresponding volumes bounded by the HS$-$, the GSC or the CCA estimates. (In particular, despite the fact that the CCA curve in Fig. 2(a) bounds a region larger than the region bounded by the SCS curve, from Fig. 2(b) we can deduce that the SCS estimate bounds a volume larger than the volume bounded by the CCA estimate.) In contrast with the other four estimates, in the SCS model the phases can not be identified with a matrix phase and a fiber phase [a property inherited from the fact that the linear SCS estimate is invariant to phase properties interchange, Hashin (1983)]. Nevertheless, as will be demonstrated in the sequel, if the contrast between the yield stresses of the two phases is large and the volume fraction of the stiffer phase is small, the SCS estimate predicts an overall yielding of the composite due to yielding of the soft phase in shear transverse or along the fibers.

Figure 3 shows various estimates for $\tilde{\Sigma}_p$, the principal yield stress along the Σ_p axis, as functions of the volume fraction of the fibers $c^{(2)}$ for a fixed ratio $k^{(2)}/k^{(1)} = 5$. As outlined before, the corresponding estimates for the principal yield stresses $\tilde{\Sigma}_d$ and $\tilde{\Sigma}_n$ can be easily extracted from this plot. It is interesting to note that while the curves for the classical upper bound, the HS upper bound and the SCS estimate tend towards the fibers yield strength in the limit as $c^{(2)}$ approaches 1, this is not the case with the other three estimates that correspond to the class of fiber composites with a soft matrix phase. We note that similar observations were made by Ponte Castañeda and deBotton (1992) and Suquet (1993) in connection with analogous estimates for the effective yield stress of statistically isotropic composites.

The dependence of the estimates for the principal yield stress $\tilde{\Sigma}_p$ on the ratio $k^{(2)}/k^{(1)}$ for a fixed volume fraction of the fibers $c^{(2)} = 0.3$ is shown in Fig. 4. Once again, this plot is reminiscent of corresponding plots obtained by Ponte Castañeda and deBotton (1992) and Suquet (1993) for the effective yield stress of statistically isotropic composites. The horizontal asymptotes of the curves from the GSC, the CCA, the SCS and the HS$-$

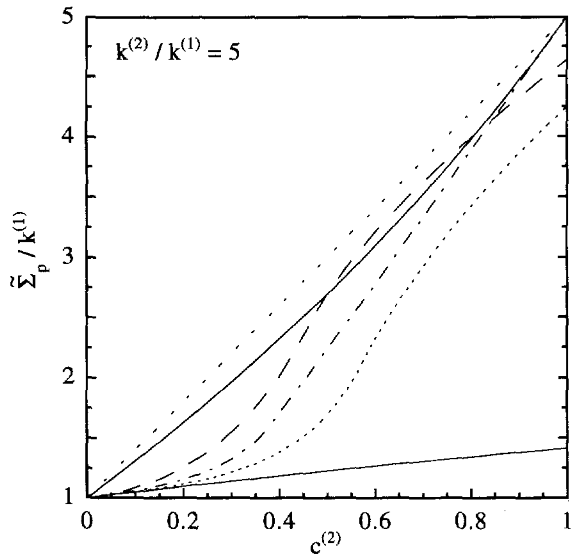


Fig. 3. Estimates for the principal yield stress $\tilde{\Sigma}_p$ as functions of the volume fraction of the fibers $c^{(2)}$ for a fixed ratio $k^{(2)}/k^{(1)} = 5$. The continuous upper and lower curves correspond to the estimates obtained from the HS upper and lower bounds, the short-dash curve to the GSC estimate, the long-dash curve to the CCA estimate, the dashed-dotted curve to the SCS estimate, and the dotted curve to the classical upper bound.

estimates indicate that for large enough values of $k^{(2)}/k^{(1)}$ the principal stresses $\tilde{\Sigma}_p$ (and $\tilde{\Sigma}_n$) become independent of $k^{(2)}$. This is of course in agreement with our previous observation concerning the existence of a matrix yielding mode for composites with soft matrices reinforced by considerably stiffer fibers.

In fact, the existence of a matrix yielding mode suggests that if $k^{(2)}/k^{(1)} > \beta$, for some $\beta > 1$, the intersection of the effective yield surface with the (Σ_n, Σ_p)-plane is independent of the fibers yield strength. The expressions for $\beta^{(HS-)}$, $\beta^{(GSC)}$, $\beta^{(CCA)}$ and $\beta^{(SCS)}$ for which the intersections of $\tilde{\Phi}^{(HS-)}$, $\tilde{\Phi}^{(GSC)}$, $\tilde{\Phi}^{(CCA)}$ and $\tilde{\Phi}^{(SCS)}$ with the (Σ_n, Σ_p)-plane become independent of $k^{(2)}$ are,

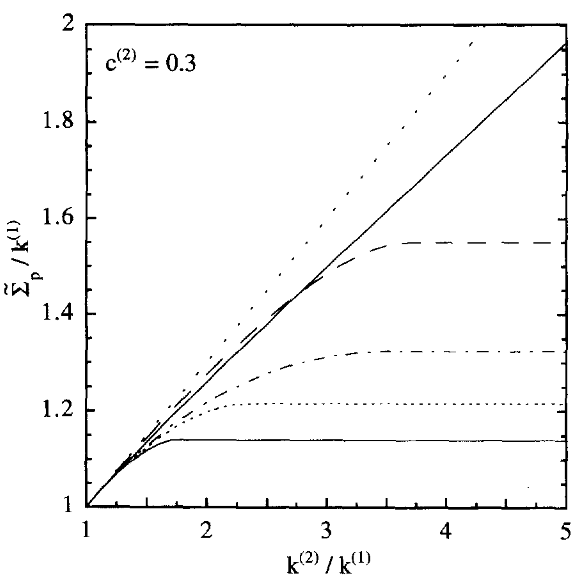


Fig. 4. Estimates for the principal yield stress $\tilde{\Sigma}_p$ as functions of the ratio $k^{(2)}/k^{(1)}$ for a fixed volume fraction of the fibers $c^{(2)} = 0.3$. The continuous upper and lower curves correspond to the estimates obtained from the HS upper and lower bounds, the short-dash curve to the GSC estimate, the long-dash curve to the CCA estimate, the dashed-dotted curve to the SCS estimate, and the dotted curve to the classical upper bound.

$$\beta^{(\mathrm{HS}-)} = \frac{2}{\sqrt{1+c}}, \tag{29}$$

$$\beta^{(\mathrm{GSC})} = \frac{2}{1-c}\sqrt{\frac{(1+c+c^2+c^3)((1-2c)\eta-2+7c-7c^2+c^3-c^4+2c^5)}{2c(1-2c)\eta-1-c+11c^2-13c^3+c^4+c^5+c^6+c^7}}, \tag{30}$$

where $\eta = \sqrt{1-12c^2+24c^3-14c^4+c^8}$,

$$\beta^{(\mathrm{CCA})} = \frac{2}{1-c}\sqrt{\frac{1+2c+3c^2+2c^3+4c^4}{1-c+5c^2+c^3}}, \tag{31}$$

and

$$\beta^{(\mathrm{SCS})} = \frac{2(1-c)}{1-2c}, \quad c < 0.5, \tag{32}$$

where, for simplicity, in eqns (29)–(32) $c = c^{(2)}$ is the volume fraction of the fibers. The above four estimates for β versus the volume fraction of the fibers are plotted in Fig. 5. We note that in the limit $c^{(2)} = 0$ all four estimates agree. However, while according to the GSC, the CCA and the SCS predictions β increases as the fiber volume fraction increases, according to the HS lower estimate the value of β decreases.

We also note that for fiber composites with volume fraction of the fibers $c^{(2)} < 0.5$ and $k^{(2)}/k^{(1)} > 5$, three estimates predict that the intersection of the effective yield surface with the (Σ_n, Σ_p)-plane is independent of $k^{(2)}$. Since this is the case for most fiber-reinforced composites, it follows that usually the intersection of the effective yield surface with the (Σ_n, Σ_p)-plane is identical to the corresponding intersection of the effective yield surface of a fiber composite with rigid fibers (and identical distribution of the fibers). This class of fiber composites is dealt with in the following section.

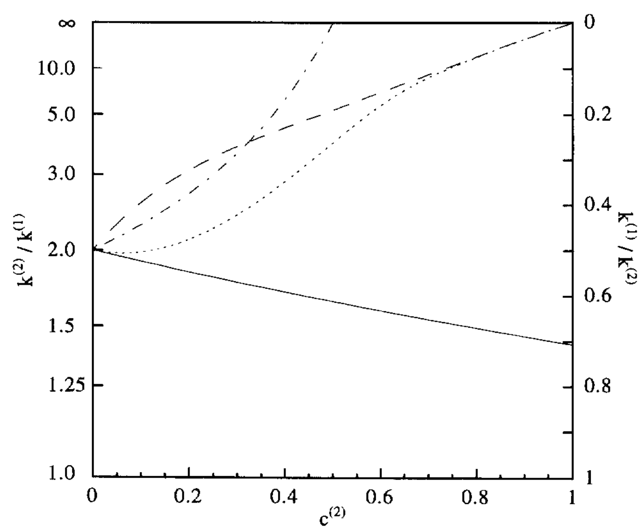


Fig. 5. Estimates for $\beta = k^{(2)}/k^{(1)}$ for which the intersection of the yield surface with the (Σ_n, Σ_p)-plane becomes independent of $k^{(2)}$ plotted as functions of the fibers volume fraction $c^{(2)}$. The continuous curve corresponds to the estimate for the HS lower bound, the short-dash curve to the GSC estimate, the long-dash curve to the CCA estimate, and the dashed-dotted curve to the SCS estimate.

5. APPLICATIONS TO FIBER COMPOSITES WITH RIGID FIBERS

We consider the class of fiber composites made up of rigid–perfectly plastic matrices reinforced by rigid fibers. The yield surface bounding the strength domain of these composites may be described in the $(\Sigma_d, \Sigma_n, \Sigma_p)$-space as an infinitely long cylinder whose generatrix is aligned with the Σ_d-axis, and thus, it is completely characterized by the cross-section of the cylinder on the (Σ_n, Σ_p)-plane.

As demonstrated in section 3, explicit estimates for the effective yield function can be easily generated by letting $\rho = 0$ in eqn (26). By doing so, we obtain the following expressions for the yield surface according to the four estimates, namely,

$$\tilde{\Phi}^{(\mathrm{HS}-)} = \frac{1}{1+c}\left[\left(\frac{\Sigma_n}{k^{(1)}}\right)^2 + \left(\frac{\Sigma_p}{k^{(1)}}\right)^2\right] - 1, \tag{33}$$

$$\tilde{\Phi}^{(\mathrm{GSC})} = \frac{1}{1+c}\left(\frac{\Sigma_n}{k^{(1)}}\right)^2 + \frac{(1-c)^3}{\eta - 2c + 6c^2 - 4c^3}\left(\frac{\Sigma_p}{k^{(1)}}\right)^2 - 1, \tag{34}$$

where $\eta = \sqrt{1 - 12c^2 + 24c^3 - 14c^4 + c^8}$,

$$\tilde{\Phi}^{(\mathrm{CCA})} = \frac{1}{1+c}\left(\frac{\Sigma_n}{k^{(1)}}\right)^2 + \frac{(1-c)^2}{1 - c + 5c^2 + c^3}\left(\frac{\Sigma_p}{k^{(1)}}\right)^2 - 1, \tag{35}$$

and

$$\tilde{\Phi}^{(\mathrm{SCS})} = \frac{1-2c}{1-c}\left[\left(\frac{\Sigma_n}{k^{(1)}}\right)^2 + \left(\frac{\Sigma_p}{k^{(1)}}\right)^2\right] - 1, \quad c < 0.5. \tag{36}$$

In eqns (33)–(36), $c = c^{(2)}$ is the volume fraction of the fibers phase. The HS upper bound for the effective yield function, as well as the SCS estimate for volume fraction of the fibers $c^{(2)} \geqslant 0.5$, become boundless for this class of composites.

We note that for the class of fiber composites discussed in the previous section, when the ratio $k^{(2)}/k^{(1)}$ is greater than $\beta^{(\mathrm{HS}-)}$, $\beta^{(\mathrm{GSC})}$, $\beta^{(\mathrm{CCA})}$ or $\beta^{(\mathrm{SCS})}$, eqns (33), (34), (35) or (36) provide the expressions for the intersections of the corresponding estimates for the effective yield surface with the (Σ_n, Σ_p)-plane. Thus, for example, the intersections of the estimates $\tilde{\Phi}^{(\mathrm{HS}-)}$, $\tilde{\Phi}^{(\mathrm{GSC})}$ and $\tilde{\Phi}^{(\mathrm{CCA})}$ with the (Σ_n, Σ_p)-plane given in Fig. 2(b) can be obtained from eqns (33), (34) and (35) by letting $c^{(2)} = 0.45$.

To some extent, the yield strength domain of metal matrix composites reinforced with strong, brittle fibers may be approximated by the yield strength domain of composites made up of rigid–perfectly plastic matrices reinforced by rigid fibers. This approximation, which neglects the effects of compressibility and of the linear region in the stress–strain curve, make sense only when the ratio of the ultimate stress of the fibers to that of the matrix yield strength is very large. Clearly, the yielding of the composite due to axisymmetric loads can not be accounted for in this approximation.

Comparison of the estimates in eqns (33) through (36) with corresponding experimental off-axis uniaxial tensile results for aluminum matrices reinforced with boron fibers is shown in Fig. 6. We note that in terms of an off-axis uniaxial tensile load Σ, the expressions for the transversely isotropic invariants Σ_n and Σ_p are

$$\Sigma_n = \frac{\sqrt{3}}{2}\Sigma \sin(2\theta),$$

$$\Sigma_p = \frac{\sqrt{3}}{2}\Sigma \sin^2(\theta), \tag{37}$$

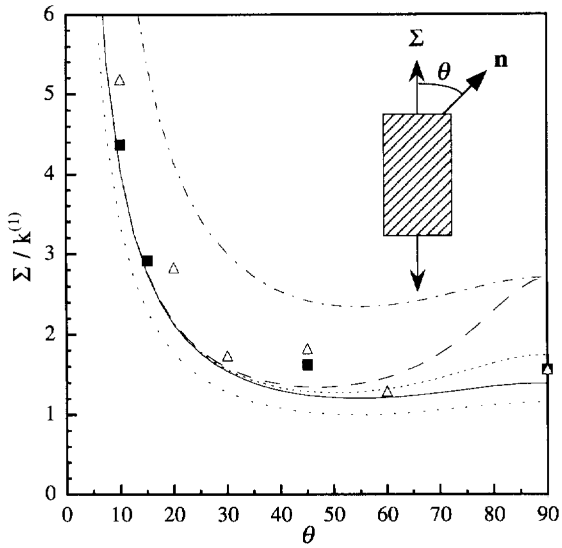


Fig. 6. Comparison between experimental estimates for the off-axis tensile yield strength of boron-reinforced aluminum composites and corresponding estimates for a rigid–perfectly plastic matrix reinforced with rigid fibers in volume fraction $c^{(2)} = 0.45$ plotted as functions of the off-axis loading angle θ. The continuous curve corresponds to the HS lower estimate; the short dash curve to the GSC estimate; the long dash curve to the CCA estimate; the dashed-dotted curve to the SCS estimate; the dotted curve to the lower bound from eqn (38); the dark squares to estimates extracted from experimental stress–strain curves of Pindera and Lin (1989); and the triangles to experimental estimates of Kenaga *et al.* (1987). A value of 48 MPa was assumed for the yield strength of the aluminum matrix.

where θ is the angle between the tension direction and the fiber direction (see Fig. 6). By making use of eqns (37) in eqns (33) through (36), explicit HS−, GSC, CCA and SCS estimates for the effective off-axis tensile yield strength of the composite are obtained.

In Fig. 6, the continuous curve shows the HS lower estimate for the effective off-axis tensile yield strength of a matrix with yield strength $k^{(1)}$ reinforced by rigid fibers in volume fraction $c^{(2)} = 0.45$ as a function of the off-axis loading angle θ. The short and long dash curves depict the corresponding GSC and CCA estimates, respectively, and the dashed-dotted curve corresponds to the SCS estimate. The dark squares correspond to estimates for the effective off-axis tensile yield strength of a boron-reinforced 6061 aluminum composite in a volume fraction of boron fibers $c^{(2)} = 0.45$. These estimates were extracted from experimental stress–strain curves given in Figs 3 and 4 of Pindera and Lin (1989). The estimate for the tensile yield strength of the 6061 aluminum matrix $k^{(1)} = 48$ MPa was extracted from Fig. 2 of the same reference. The triangles correspond to experimental estimates given in Table 1 of Kenaga *et al.* (1987) for the effective off-axis tensile yield strength of a boron-reinforced 6061 aluminum in a volume fraction of boron fibers $c^{(2)} = 0.475$. No estimate for the tensile yield strength of the aluminum matrix was provided in the later reference, and thus, the value $k^{(1)} = 48$ MPa was assumed.

For comparison, also shown in Fig. 6 (dotted curve) is the lower bound for the effective off-axis tensile yield strength of de Buhan *et al.* (1991). This can be easily determined from the following expression for $\tilde{\Phi}^{(LB)}$ for the class of fiber composites with rigid fibers, namely,

$$\tilde{\Phi}^{(LB)} = \left(\frac{\Sigma_n}{k^{(1)}}\right)^2 + \left(\frac{\Sigma_p}{k^{(1)}}\right)^2 - 1. \tag{38}$$

We note that the for low values of θ the HS−, the GSC and the CCA estimates are very close, and all three of them underestimate the corresponding experimental estimates. On the other hand, for large values of θ, the HS− and the GSC estimates demonstrate good agreement with the experimental data points. In particular, we note that these estimates may be used to obtain equitable approximations for the transverse yield strength ($\alpha = 90$) for fiber-reinforced metal matrices.

6. CONCLUDING REMARKS

We made use of the variational procedure of Ponte Castañeda (1992) to obtain simple estimates for the effective yield strength domains and the associated effective yield functions of two-phase rigid–perfectly plastic composites. The resulting expressions, in terms of one dimensional optimization problems, provide a procedure for generating bounds and estimates for the effective yield functions from corresponding bounds and estimates available for the class of linear elastic composites. Applications of the procedure were carried out for the class of fiber-reinforced composites.

Five explicit estimates, corresponding to the Hashin–Shtrikman upper and lower bounds, the generalized self consistent estimate, the composite cylinder assemblage and the self consistent estimate were obtained. The results, in terms of estimates for the effective yield surfaces in the stress-space, demonstrated the existence of two distinct failure modes: a matrix mode corresponding to overall yielding of the composite due to yielding of the matrix in shear loads along or transverse to the fibers, and a fiber mode in which the composite yields due to simultaneous yielding of the fibers and the matrix under axisymmetric loads. Explicit estimates were also obtained for the class of fiber composites with rigid fibers. The corresponding results from the Hashin–Shtrikman lower bound and the generalized self consistent model demonstrated favorable agreement with available off-axis uniaxial tensile tests for boron-reinforced aluminum composites.

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APPENDIX

Bounds and estimates for the effective shear moduli $\tilde{\mu}_d$, $\tilde{\mu}_n$ and $\tilde{\mu}_p$ of linear elastic fiber composites made up of two incompressible isotropic phases. The bounds and the estimates are given in terms of the ratio $\rho = \mu^{(1)}/\mu^{(2)}$, where $\mu^{(1)} \leqslant \mu^{(2)}$ are the shear moduli of phases 1 and 2 in volume fractions $c^{(1)}$ and $c^{(2)} = 1-c^{(1)}$, respectively. For the generalized self consistent estimate and the composite cylinder assemblage model, phase 2 corresponds to the fibers and phase 1 to the matrix.

The Hashin–Shtrikman lower bound for the longitudinal shear modulus may be expressed in the form,

$$\tilde{\mu}_n^{(\mathrm{HS}-)} = \mu^{(1)} \frac{c^{(1)}\rho + (1+c^{(2)})}{(1+c^{(2)})\rho + c^{(1)}}. \tag{A1}$$

The expression for the lower bound for the transverse shear modulus $\tilde{\mu}_p^{(\mathrm{HS}-)}$ is identical to the one for $\tilde{\mu}_n^{(\mathrm{HS}-)}$. The Hashin–Shtrikman upper bound for the longitudinal shear modulus is

$$\tilde{\mu}_n^{(\mathrm{HS}+)} = \mu^{(1)} \frac{1}{\rho}\left[\frac{(1+c^{(1)})\rho + c^{(2)}}{c^{(2)}\rho + (1+c^{(1)})}\right], \tag{A2}$$

and it is identical to the expression for $\tilde{\mu}_p^{(\mathrm{HS}+)}$.

The generalized self consistent estimate for the longitudinal shear modulus $\tilde{\mu}_n^{(\mathrm{GSC})}$ is similar to expression (A1) for $\tilde{\mu}_n^{(\mathrm{HS}-)}$. The corresponding expression for the transverse shear modulus is obtained from the quadratic equation

$$a(\rho)\left(\frac{\tilde{\mu}_p^{(\mathrm{GSC})}}{\mu^{(1)}}\right)^2 + 2b(\rho)\left(\frac{\tilde{\mu}_p^{(\mathrm{GSC})}}{\mu^{(1)}}\right) + c(\rho) = 0, \tag{A3}$$

where

$$\begin{aligned} a &= 2(1-(c^{(2)})^4)\rho + 4c^{(2)}\alpha\rho^2 + (c^{(1)})^4(1+\rho^2), \\ b &= 2c^{(2)}(\alpha-1)(1-\rho^2), \\ c &= -[2(1-(c^{(2)})^4)\rho + 4c^{(2)}\alpha + (c^{(1)})^4(1+\rho^2)], \end{aligned} \tag{A4}$$

and where $\alpha = 2 - 3c^{(2)} + 2(c^{(2)})^2$.

The estimate for the longitudinal shear modulus from the CCA model $\tilde{\mu}_n^{(\mathrm{CCA})}$ is also similar to expression (A1) for $\tilde{\mu}_n^{(\mathrm{HS}-)}$, and the corresponding expression for the upper bound for the transverse shear modulus is

$$\tilde{\mu}_p^{(\mathrm{CCA})} = \mu^{(1)} \frac{2(1-(c^{(2)})^4)\rho + 4c^{(2)}\alpha\rho^2 + (c^{(1)})^4(1+\rho^2)}{(1+\rho)^2 - (c^{(2)})^4(1-\rho)^2 - 2c^{(2)}(\alpha-1)(1-\rho^2)}, \tag{A5}$$

where α is given in (A4).

The SCS estimate for the longitudinal shear modulus is obtained from the positive root of the quadratic equation

$$\left(\frac{\tilde{\mu}_n^{(\mathrm{SCS})}}{\mu^{(1)}}\right)^2 + (c^{(1)} - c^{(2)})\left(\frac{1}{\rho} - 1\right)\left(\frac{\tilde{\mu}_n^{(\mathrm{SCS})}}{\mu^{(1)}}\right) - \frac{1}{\rho} = 0, \tag{A6}$$

and it is identical to the estimate $\tilde{\mu}_p^{(\mathrm{SCS})}$ for the transverse shear modulus.

Finally, the expression for the shear modulus $\tilde{\mu}_d$ is identical in all the models mentioned above, and may be expressed in the form

$$\tilde{\mu}_d = \mu^{(1)}\left(c^{(1)} + \frac{c^{(2)}}{\rho}\right). \tag{A7}$$