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ON THE CONSISTENCY CONDITIONS FOR FORCE SYSTEMS

REUVEN SEGEV and GAL DE BOTTON

The Pearlstone Center for Aeronautical Engineering Studies, Department of Mechanical Engineering, Ben-Gurion University of the Negev, Beer-Sheva, Israel

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Abstract—In analogy with the classical Cauchy conditions, this work presents conditions so that a force system, the assignment of a force to each subbody of a given body, can be represented by a stress. The setting in which the theory is formulated is more general than that of classical continuum mechanics as stresses can be as irregular as measures, equilibrium is not assumed, it applies to continuum mechanics of order higher than one and it may be extended to the case where the body and space are modelled by general differentiable manifolds. The consistency conditions presented are that of additivity of the force system on pairs of disjoint subbodies, continuity and boundedness.

1. INTRODUCTION

The present paper is concerned with the conditions so that a force system on a body can be represented by a stress. Here, by a force system, we mean an assignment of a force to each subbody of a given body. In the classical formulation of continuum mechanics, it is assumed that force systems on bodies satisfy the following conditions: (i) The force f_P on any subbody P is given in terms of a body force field b_P and a continuous surface force field t_P in the form $f_P = \int_P b_P \, dv + \int_{\partial P} t_P \, da$. (ii) The body force on a subbody is independent of the subbody and the surface force on a subbody depends on the subbody only through its unit normal n to the boundary. It is also assumed that the surface force depends on the normal continuously. (iii) The total force on each subbody vanishes. On the basis of these conditions it is proven that there is a unique tensor field σ over the body, called the stress, so that the forces on the various bodies are given by $b = -\operatorname{div} \sigma$ and $t_P = \sigma(n)$. Thus, we may say that the stress σ *represents* the force system $\{f_P\}$ with these two equations and that the aforementioned conditions are *consistency* conditions for the existence of a stress representing the given force system.

Modern formulations of stress theory replace some of these assumptions by other, more general assumptions. In [1], which summarizes the works of Noll, Gurtin, Williams and others (see references therein), the relevant structure is as follows. Forces are defined as interactions for pairs of subbodies so that $f(P, C)$ represents the force, a vector in some inner product space, that the subbody C exerts on the subbody P. The consistency conditions in this formulation are shown to be the following: (i) $f(P, C_1 \cup C_2) = f(P, C_1) + f(P, C_2)$ for separate C_1, C_2, and $f(P_1 \cup P_2, C) = f(P_1, C) + f(P_2, C)$ for separate P_1, P_2. (ii) The force $f(P, C)$ for separate P and C is bounded by both the volume of P and the area of the common boundary of P and C. (iii) The total force on each subbody vanishes. (iv) Proving the existence of a surface force field on the basis of the previous assumptions it is assumed that it is continuous.

In previous works (see [2, 3]), an alternative framework for the theory of forces in continuum mechanics was suggested. A force on a body was defined as a continuous linear functional on a tangent space to an appropriate configuration Banach manifold. Assuming that bodies are compact submanifolds with boundary of space, and using the axiom of impenetrability of continuum mechanics as a guideline, it was shown that it is natural to take the collection of differentiable embeddings of the body in space equipped with the C^r topology as the configuration manifold. Restricting ourselves for simplicity to the case where the space is modelled by $\mathscr{R}^3$ and a body B is a 3-dimensional submanifold of $\mathscr{R}^3$, it follows that the various tangent spaces can be identified with $C^r(B, \mathscr{R}^3)$ and that forces can

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be identified with elements of the dual space $C^r(B, \mathcal{R}^3)^*$. It is shown that such forces can be represented in the form

$$f(u) = \sum_{p,\alpha_1,\alpha_2,\alpha_3} \int_B \frac{\partial^{\alpha_1+\alpha_2+\alpha_3} u_p}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} \, \mathrm{d}\sigma_{p\alpha_1\alpha_2\alpha_3},$$

where $p = 1, 2, 3$, $\alpha_1, \alpha_2, \alpha_3$ are non-negative integers with $\alpha_1 + \alpha_2 + \alpha_3 \leq r$, and $\sigma_{p\alpha_1\alpha_2\alpha_3}$ are Borel measures to which we will refer as *stress measures*. From the construction it follows that the collection of stress measures that represent the force f is not unique. The representation presented in the previous equations corresponds to high-order stresses, those appearing in rth order continuum mechanics. For the simplest case $r = 1$ the equation can be rewritten as

$$f(u) = \sum_p \int_B u_p \, \mathrm{d}\sigma_p + \sum_{p,q} \int_B \frac{\partial u_p}{\partial x_q} \, \mathrm{d}\sigma_{pq},$$

where here, σ_{pq} are measures corresponding to the components of the usual stress tensor and σ_p, which are not present in usual formulations, appear in the equation because no equilibrium hypothesis was made (i.e. if one assumes that the total force on each subbody vanishes it follows that $\sigma_p = 0$). If these stress measures are given by differentiable densities in terms of the volume measure in $\mathcal{R}^3$, it can be easily shown that forces can be represented by body forces and surface forces. Since, unlike forces, stress measures can be restricted, a given collection of stresses $\sigma_{p\alpha_1\alpha_2\alpha_3}$ induces a force system on B in which the force f_P on a subbody P is represented by

$$f_P(u) = \sum_{p,\alpha_1,\alpha_2,\alpha_3} \int_P \frac{\partial^{\alpha_1+\alpha_2+\alpha_3} u_p}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} \, \mathrm{d}\sigma_{p\alpha_1\alpha_2\alpha_3}.$$

In addition to the fact that it does not require the assumption of equilibrium and permits stresses that are irregular as Borel measures, this formulation of the theory of stresses in continuum mechanics has the advantage that it applies in the general geometrical setting of differentiable manifolds.

While the suggested formulation models the representation of a single force on a body by nonunique stress measures, the aforementioned papers do not contain the appropriate consistency conditions on a force system that guarantee that there is a collection of stress measures that induce it in the form presented in the last equation. In this paper, it is shown that the following conditions on the force system $\{f_P \in C^r(P, \mathcal{R}^3)^*;\ P \text{ is a subbody of } B\}$, which are clearly necessary, imply that there is a unique collection of Borel stress measures $\sigma_{p\alpha_1\alpha_2\alpha_3}$, $p = 1, 2, 3$, $\alpha_1 + \alpha_2 + \alpha_3 \leq r$, that represent it as in the last equation. (We use the convention that the empty set is a subbody and the force on it is zero.)

(i) *Additivity.* If P_1 and P_2 are disjoint subbodies of B then for any $u \in C^r(B, \mathcal{R}^3)$,

$$f_{P_1 \cup P_2}(u|_{P_1 \cup P_2}) = f_{P_1}(u|_{P_1}) + f_{P_2}(u|_{P_2}).$$

(ii) *Continuity.* We recall that the set A is the *limit* of the sequence of sets P_i if and only if for every element x in A there is an integer i_0 so that $x \in P_i$ for every $i \geq i_0$, and every point that is contained in an infinite number of sets in the sequence is contained in A. It is required that if $A \in \Phi$, where Φ is the minimal field containing the open subsets of the body, and P_i is a sequence of subbodies whose limit is A, then, for any $u \in C^r(B, \mathcal{R}^3)$, the sequence $f_{P_i}(u|_{P_i})$ converges and its limit is independent of the particular sequence of subbodies P_i.

(iii) *Boundedness.* There is a finite bound K such that for any subbody P and any $u \in C^r(B, \mathcal{R}^3)$,

$$|f_P(u|_P)| < K \, \| u|_P \|.$$

We note that as the traditional consistency conditions hold only for continuum mechanics of order one, and while it is only very recently that Noll and Virga were able to present consistency conditions for second order continuum mechanics [4], the consistency conditions presented in this paper hold for continuum mechanics of any order.

The reason why the representation of a force system by a stress is unique while the representation of a force by stresses is not unique may be described roughly as follows. In the expression for the representation of a given force by stress measures we know the values of the integrals only for "compatible" collections of continuous functions $w_{p\alpha_1\alpha_2\alpha_3}$, $p = 1, 2, 3$, $\alpha_1 + \alpha_2 + \alpha_3 \leq r$, i.e. collections for which there are differentiable functions u_p such that

$$w_{p\alpha_1\alpha_2\alpha_3} = \frac{\partial^{\alpha_1+\alpha_2+\alpha_3} u_p}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}.$$

Hence, the measures cannot be determined uniquely. However, we can approximate any collection $w_{p\alpha_1\alpha_2\alpha_3}$ by a family of compatible collections such that the members of the family are defined on a family of disjoint subbodies. If we know the force on each subbody we can approximate the integrals for "non-compatible" collections and determine the measures uniquely.

Section 2 discusses the approximation by subbodies of sets that belong to the minimal field of subsets containing the open subsets. For each set A in the minimal field, a sequence of subbodies whose limit is A is constructed and some useful properties of this sequence are presented. Section 3 uses the construction of Section 2 in order to give sufficient conditions (which are also necessary conditions) so that a real valued set function defined on the collection of subbodies can be extended to a Borel measure. In addition to its use as a tool for the proof of the sufficiency of the consistency conditions, this result is of some interest as it specifies the conditions under which quantities such as electric charge can be extended to measures if they are given for the various subbodies. Section 4 presents the consistency conditions and proves that they are sufficient (again, these conditions are also necessary conditions).

2. THE APPROXIMATIONS OF SETS BY BODIES

As was mentioned a *body* is defined as a three-dimensional compact submanifold with boundary of $\mathscr{R}^3$. We now add formally the empty set to the collection of bodies. Given a body B, a subset D of B which is a body is a *subbody* of B. Clearly, the complement of a subbody in the body is not a subbody and the same holds in general for the unions and intersections of subbodies. In this section it will be shown that although our collection of subbodies is small, any set in the smallest field of subsets containing the open subsets of B can be approximated by a sequence of subbodies of B. This approximation has some additional properties that will be used in the following sections.

Proposition 2.1. For any compact subset C of $\mathscr{R}^3$ and a bounded open subset U of $\mathscr{R}^3$ containing C, there is a body B which is contained in U whose interior contains C.

Proof. By a standard theorem of differential topology (see [5, p. 12]) it is possible to construct a smooth real function $h: \mathscr{R}^3 \to \mathscr{R}$ with the following properties: (i) its value at any point in C is 1; (ii) its value at any point in the complement of U is 0; and (iii) its value at any other point is in the interval $[0, 1]$. By Sard's theorem (see [6, p. 204]) any neighborhood of the value 1/2 contains a point x such that h is not singular at any point in the inverse image of x. From the implicit function theorem it follows that $h^{-1}(\{x\})$ is a smooth submanifold of $\mathscr{R}^3$ that is clearly compact and which is the boundary of $h^{-1}([x, 1])$. It follows that $h^{-1}([x, 1])$ is a body. ∎

Proposition 2.2. Let D be a body, U a bounded open subset of $\mathscr{R}^3$ and C a closed set contained in U. There exists a body B contained in U whose interior contains C such that $B \cup D$ and $B \cap D$ are bodies.

Proof. Let us construct a body B_2 contained in U whose interior contains C as in the previous proposition. In the case where $U \cap D$ is empty, or $D \supset U$ it follows from the construction that $B_2 \cup D$ and $B_2 \cap D$ are bodies. However, in general, when the boundaries of B_2 and D interest, $B_2 \cup D$ and $B_2 \cap D$ will not be bodies. We will modify B_2 so that the modified body will have the required properties.

We recall that if X and Y are submanifolds of the differentiable manifold Z, any C^r neighborhood, $r \geq 1$, of the inclusion of X in Z contains an embedding $g: X \to Z$ which is transversal to the inclusion of Y in Z (see [7, p. 78]). It follows that we may modify B_2 slightly to obtain a diffeomorphic body B_1 so that the properties guaranteed by the previous proposition still hold, and in addition, ∂B_1 is transversal to ∂D. Hence, the intersection $\partial B_1 \cap \partial D$ is a one-dimensional submanifold of both ∂B_1 and ∂D.

Next we modify B_1 to obtain a diffeomorphic body B so that the intersection between the boundary of B and the boundary of D contains an open subset. In other words, the two boundaries will overlap on an open set so that $B \cup D$ and $B \cap D$ are bodies. This modification will take place in a small neighborhood of the intersection of the transversal submanifolds ∂B_1 and ∂D.

In constructing the neighborhood in which the modification will take place we use the following theorems of differential topology (see [7, pp. 109–115]).

(i) If Y is an orientable submanifold of the differentiable manifold Z of codimension one, then, there is an embedding $g: Y \times (-1, 1) \to Z$, called a *tubular neighborhood* of Y in Z having the following properties: (a) $g|_{Y \times \{0\}}$ is the inclusion of Y in Z; and (b) the image of g is an open neighborhood of Y in Z. We will also use the term tubular neighborhood for the image of the mapping g, and we will identify a point in the image with the "coordinates" given to it by g.

(ii) If Y is a neat submanifold of Z, then, there is a tubular neighborhood of Y in Z.

(iii) If Y is a neat submanifold of Z, then, every tubular neighborhood of ∂Y in ∂Z is the intersection of ∂Z with a tubular neighborhood of Y in Z.

We first construct tubular neighborhoods V_1 and V_2 of $\partial B_1 \cap \partial D$ in ∂B_1 and in ∂D, respectively, with the convention that the parameters in $(-1, 1)$ are positive in D and the complement of B_1, respectively. Next, considering the neat submanifold $\partial B_1 \cap D$ in D, we can use theorem (iii) above to show that there is a tubular neighborhood V_3 of $\partial B_1 \cap D$ in D such that $V_2 = V_3 \cap \partial D$, where we set the values of the parameter $s \in (-1, 1)$ to be positive outside B_1. Let

$$V = \{p = (x, s); p \in V_3, x \in V_1, s \in [0, 1)\}.$$

By definition, every point in V has three coordinates (x, t, s) such that $x \in \partial B_1 \cap \partial D$, $t, s \in [0, 1)$, where $t = 0$ for points on ∂D and $s = 0$ for points on ∂B_1. Since there is a finite positive distance between ∂B_1 and the complement of U, there is a number $\xi > 0$ such that a point in V is in U if the coordinate s is less than ξ.

Consider the following functions:

$$\alpha: \mathscr{R} \to \mathscr{R} \qquad \alpha(r) = 0 \quad \text{for } r \leq 0$$
$$\alpha(r) = \exp(-1/r) \quad \text{for } r > 0,$$
$$\beta: \mathscr{R} \to \mathscr{R} \qquad \beta(r) = \alpha(r - \tfrac{1}{4}\xi)\alpha(\tfrac{3}{4}\xi - r)$$
$$\theta: \mathscr{R} \to [0, \pi/2] \qquad \theta(t) = \frac{\pi}{2} \frac{\int_t^{3/4\xi} \beta(r)\,dr}{\int_{1/4\xi}^{3/4\xi} \beta(r)\,dr}.$$

The function θ is smooth and has the following properties:

(i) $\theta(t) = \pi/2 \quad \text{if } t \leq \xi/4$

(ii) $0 \leq \theta(t) \leq \pi/2 \quad \text{if } \xi/4 \leq t \leq 3\xi/4$

(iii) $\theta(t) = 0 \quad \text{if } t \geq 3\xi/4$.

We will now modify the portion of ∂B_1 that is a subset of V (which is clearly in D) as follows. Let $g: V_1 \to V$ be defined by

$$g(x, t) = (x, t\cos\theta(t), t\sin\theta(t)) \quad \text{if } 0 \leq t < \xi,$$
$$g(x, t) = (x, t, 0) \quad \text{otherwise}.$$

Clearly, g is an embedding and has the following properties:

(i) $\mathrm{Im}(g) \cap \partial D = \{p = g(x, t); 0 \leq t \leq \xi/4\}$ contains an open set of ∂D,
(ii) $\mathrm{Im}(g) \cap \partial B_1 = \{p = g(x, t); 3\xi/4 \leq t \leq 1\}$ contains an open set of ∂B_1,
(iii) $\mathrm{Im}(g)$ is a subset of U and is disjoint from C.

In a similar way we will define a function h that will modify a neighborhood of $\partial B_1 \cap \partial D$ in $\partial B_1 \cap (D^0)^c$ whose image will be contained in a neighborhood of $\partial B_1 \cap \partial D$ in $B_1 \cap (D^0)^c$. (The superscript "c" denotes the complement of the corresponding set.) This mapping will have the analogous properties to those of g, i.e. it is an embedding, its image is contained in U–C, $\mathrm{Im}(h) \cap \partial D$ contains an open set of ∂D and $\mathrm{Im}(h) \cap \partial B_1$ contains an open set of ∂B_1.

We can define now the mapping ψ: $\partial B_1 \to U$–C by $\psi(y) = g(y)$ if $y \in \mathrm{domain}(g)$, $\psi(y) = h(y)$ if $y \in \mathrm{domain}(h)$ and $\psi(y) = y$ otherwise. Clearly, ψ is an embedding and its image overlaps with ∂D on an open set. By the Jordan–Brower Separation theorem (see [6, p. 89]), $\mathrm{Im}(\psi)$ is a boundary of a body B and it follows from the construction that $B \cap D$ and $B \cup D$ are bodies as their boundaries are smooth. ∎

We can now state the analog of proposition 2. for the case where both C and U are subsets of the body B.

Proposition 2.3. If U is an open subset of the body D and C is a closed subset of D contained in U, there is a subbody P of D contained in U whose interior contains C.

Proof. It is clear that C is closed in $\mathscr{R}^3$ and that it is contained in an open subset V of $\mathscr{R}^3$ such that $V \cap D = U$. By proposition 2.2 we can construct a body B contained in V whose interior contains C such that $B \cap D$ is a body, hence, a subbody of D which clearly has the required properties. ∎

Henceforth, unless otherwise stated, we will refer by open and closed sets to open and closed subsets of the body under consideration.

Proposition 2.4. Let Φ be the smallest field of subsets containing the open subsets of a body B. For each A in Φ there is a sequence P_i of subbodies of B such that $P_i \to A$.

Proof. It follows from a standard representation theorem for the members of the smallest field containing a given collection of sets (see [8, p. 7]) that a subset A of B belongs to Φ if and only if $A = \bigcup_{k=1}^{n} U_k \cap C_k$, where U_k and C_k are open and closed sets, respectively, with $(U_j \cap C_j) \cap (U_k \cap C_k) = \varnothing$ for $j \neq k$. Let A be a member of Φ and assume that such a representation of A is given.

For each k let C_{ki} be defined by

$$C_{ki} = \mathrm{comp}\left(\bigcup_{x \in U_k^c} \mathbf{B}\left(\frac{1}{i}, x\right)\right) \cap C_k,$$

where $\mathbf{B}(d, x)$ denotes the open ball of radius d centered at x. Clearly, for a given i the sets C_{ki} are disjoint. In case all $C_{ki} = \varnothing$ we will set $P_i = \varnothing$ and henceforth we may assume that at least for one k, $C_{ki} \neq \varnothing$. In the case where there is only one $C_{ki} \neq \varnothing$, we will set $\delta_i = 1/2i$. Otherwise, since for any given i the sets C_{ki} are disjoint and compact there is a number $\delta_i > 0$ smaller than half the distance between any two of them and $1/2i$. For each $C_{ki} \neq \varnothing$, let

$$U_{ki} = \bigcup_{x \in C_{ki}} \mathbf{B}(\delta_i, x).$$

Clearly, any U_{ki} is a subset of U_k and for a fixed i the sets U_{ki} are disjoint. Using proposition 2.3 it is possible to construct for a fixed i and each k such that $C_{ki} \neq \varnothing$, a subbody P_{ki} contained in U_{ki} whose interior contains C_{ki}. In case $C_{ki} = \varnothing$ we just set $P_{ki} = \varnothing$. Define

$$P_i = \bigcup_{k=1}^{n} P_{ki}, \quad E_i = \bigcup_{k=1}^{n} C_{ki}.$$

Since the various P_{ki} are disjoint, P_i is a body.

Lemma 2.4. The sequence P_i satisfies $\lim\inf_i P_i \supset A$.

Proof of lemma 2.4.1. We first note that for each k the limit of the increasing sequence C_{ki} is $C_k \cap U_k$. Clearly, each C_{ki} is a subset of $C_k \cap U_k$ hence their union is also a subset of $C_k \cap U_k$. On the other hand, assuming that $x \in C_k \cap U_k$, there is a minimal distance $d > 0$ between x and U_k^c. Choosing an i such that $1/i < d$, we have

$$x \in \operatorname{comp}\left(\bigcup_{x \in U_k^c} \mathbf{B}\left(\frac{1}{i}, x\right)\right) \cap C_k = C_{ki}$$

and it follows that

$$\bigcup_i C_{ki} \supset C_k \cap U_k.$$

We conclude that

$$\bigcup_i E_i = \bigcup_i \bigcup_{k=1}^{n} C_{ki} = \bigcup_{k=1}^{n} C_k \cap U_k = A.$$

Since, $P_i \supset E_i$ for each i we have $\lim\inf_i P_i \supset \lim\inf_i E_i = A$.

Lemma 2.4.2. The sequence P_i satisfies $A \supset \lim\sup_i P_i$.

Proof of lemma 2.4.2. Let us construct the sequence

$$V_{ki} = \bigcup_{x \in C_k} \mathbf{B}\left(\frac{1}{i}, x\right)$$

which is decreasing to C_k. By our construction P_{ki} is a subset of V_{ki} for each k and i, hence, $C_k = \lim\sup_i V_{ki} \supset \lim\sup_i P_{ki}$. On the other hand, $U_k \supset P_{ki}$ for each i and it follows that $U_k \cap C_k \supset \lim\sup_i P_{ki}$.

Assume that $x \notin A$, then, for each k, $x \notin U_k \cap C_k$ and $x \notin \lim\sup_i P_{ki}$. In other words, $x \in P_{ki}$ for a finite number of elements, say, N_k elements of the sequence only. Thus, $x \in \bigcup_{k=1}^{n} P_{ki} = P_i$ for a finite number of elements and hence, $x \notin \lim\sup_i P_i$.

From lemma 2.4.1 and lemma 2.4.2, we have $\lim\inf_i P_i \supset \lim\sup_i P_i$, and by the definitions of the limit of the sets it follows that $\lim_i P_i = A$. ■

Corollary 2.5. Let $A = \bigcup_{k=1}^{n} U_k \cap C_k$. Then for each k the sequence of subbodies P_{ki} constructed in the previous proposition converges to $U_k \cap C_k$ and it has the following property: for each $\varepsilon > 0$ there is an N independent of k such that for each $i > N$ and each $x \in P_{ki}$ there is a point in $U_k \cap C_k \cap P_{ki}$ whose distance from x is less than ε.

Proof. This corollary follows immediately from the construction of the previous proposition.

Proposition 2.6. If $A_1, A_2, \ldots, A_n$ are disjoint members of Φ, then, there are n sequences of subbodies $P_{1i}, P_{2i}, \ldots, P_{ni}$ converging to $A_1, A_2, \ldots, A_n$, respectively, such that for any i, $P_{1i}, P_{2i}, \ldots, P_{ni}$ are mutually disjoint and the sequence of subbodies $P_i = P_{1i} \cup P_{2i} \cup \ldots \cup P_{ni}$ converges to $A_1 \cup A_2 \cup \ldots \cup A_n$.

Proof. It is sufficient to prove the proposition for the case $n = 2$. Let A_1 and A_2 be represented in the form $A_1 = \bigcup_{j=1}^{J} U_{1j} \cap C_{1j}$, $A_2 = \bigcup_{k=1}^{K} U_{2k} \cap C_{2k}$ as in proposition 2.4. We can represent $A_1 \cup A_2$ in the form

$$A_1 \cup A_2 = \left[\bigcup_{j=1}^{J} U_{1j} \cap C_{1j}\right] \cup \left[\bigcup_{k=1}^{K} U_{2k} \cap C_{2k}\right]$$

so that we can apply to it proposition 2.4 and corollary 2.5. It follows that for each j and k there are sequences P_{1ji} and P_{2ki} of subbodies converging to $U_{1j} \cap C_{1j}$ and $U_{2k} \cap C_{2k}$, respectively, such that for any fixed i the various subbodies are disjoint. Hence, the sequences $P_{1i} = \bigcup_{j=1}^{J} P_{1ji}$, $P_{2i} = \bigcup_{k=1}^{K} P_{2ki}$ have the required properties. ■

Proposition 2.7. Let $A = \bigcup_{k=1}^{K} W_k \cap E_k$, $A' = \bigcup_{j=1}^{J} V_j \cap F_j$ be members of Φ represented as in proposition 2.4 (i.e. W_k, V_j are open and E_k, F_j are closed) such that $A' \supset A$. Then, it is possible to represent A in the form $A = \bigcup_{i=1}^{I} U_i \cap C_i$ such that the following conditions hold:

(i) The subsets U_i are open, the subsets C_i are closed and the various $U_i \cap C_i$ are mutually disjoint.

(ii) For each i there is a unique $j(i)$ such that $V_{j(i)} \supset U_i$ and $F_{j(i)} \supset C_i$ so that $V_{j(i)} \cap F_{j(i)} \supset U_i \cap C_i$.

Proof. Define the closed sets $C_{kj} = E_k \cap F_j$ and the open sets $U_{kj} = W_k \cap V_j$. We note that $A = \bigcup_{k,j} U_{kj} \cap C_{kj}$ and that the various $U_{kj} \cap C_{kj}$ are mutually disjoint. In addition, by the construction, for each pair j, k there is a unique $j(k)$ such that $V_{j(k)} \supset U_{kj}$ and $F_{j(k)} \supset C_{kj}$ so that $V_{j(k)} \cap F_{j(k)} \supset U_{kj} \cap C_{kj}$. Finally, we obtain the required representation $A = \bigcup_{i=1}^{I} U_i \cap C_i$ by re-enumerating the subsets U_{kj} and C_{kj}. ■

Proposition 2.8. Let A_k be a decreasing sequence of members of Φ. Then, there are sequences P_{ki} of subbodies such that $P_{ki} \to A_k$, and for a fixed i, $P_{ki} \supset P_{(k+1)i}$.

Proof. We first construct the sequence P_{1i} converging to A_1 as in proposition 2.4. Inductively, we construct the sequence P_{ki} on the basis of the sequence $P_{(k-1)i}$. Hence, we assume that the sequence $P_{(k-1)i} \to A_{k-1}$ has the following properties as in proposition 2.4:

(a) $A_{k-1} = \bigcup_{n=1}^{N} U_{(k-1)n} \cap C_{(k-1)n}$.

(b) $P_{(k-1)i} = \bigcup_{n=1}^{N} P_{(k-1)ni}$.

(c) For each subbody $P_{(k-1)ni}$ we have interior $(P_{(k-1)ni}) \supset C_{(k-1)ni}$ where

$$C_{(k-1)ni} = \text{comp}\left\{ \bigcup_x \mathbf{B}\left(\frac{1}{i}, x\right); x \in \text{comp}(U_{(k-1)n}) \right\} \cap C_{(k-1)n}.$$

(d) For each subbody $P_{(k-1)ni}$ we have $U_{(k-1)ni} \supset P_{(k-1)ni}$ where

$$U_{(k-1)ni} = \bigcup_x \{ \mathbf{B}(\delta_{(k-1)i}, x); x \in C_{(k-1)ni} \}.$$

Using the fact that $A_{k-1} \supset A_k$ and proposition 2.7 we can represent A_k in the form $A_k = \bigcup_{i=1}^{J} U_{kj} \cap C_{kj}$ such that the following hold:

(i) The subsets U_{kj} are open, the subsets C_{kj} are closed and the various $U_{kj} \cap C_{kj}$ are mutually disjoint.

(ii) For each j there is a unique $n(j)$ such that $U_{(k-1)n(j)} \supset U_{kj}$ and $C_{(k-1)n(j)} \supset C_{kj}$ so that $U_{(k-1)n(j)} \cap C_{(k-1)n(j)} \supset U_{kj} \cap C_{kj}$.

We now construct the subbody P_{ki} such that $P_{(k-1)i} \supset P_{ki}$. For each $U_{kj} \cap C_{kj}$ let

$$C_{kji} = \text{comp}\left\{ \bigcup_x \mathbf{B}\left(\frac{1}{i}, x\right); x \in \text{comp}(U_{kj}) \right\} \cap C_{kj}.$$

Lemma 2.8.1. For each j there is one $n(j)$ such that interior $(P_{(k-1)n(j)i}) \supset C_{kji}$.

Proof of lemma 2.8.1. By (ii) above $U_{(k-1)n(j)} \supset U_{kj}$, it follows that

$$\text{comp}\left\{ \bigcup_x \mathbf{B}\left(\frac{1}{i}, x\right); x \in \text{comp}(U_{(k-1)n}) \right\} \supset \text{comp}\left\{ \bigcup_x \mathbf{B}\left(\frac{1}{i}, x\right); x \in \text{comp}(U_{kj}) \right\},$$

hence, $C_{(k-1)n(j)i} \supset C_{kji}$. By (c) above interior $(P_{(k-1)n(j)i}) \supset C_{(k-1)n(j)i}$ which completes the proof of our Lemma.

In case all $C_{kji} = \varnothing$ we will set $P_{ki} = \varnothing$ and henceforth we may assume that at least for one j, $C_{kji} \neq \varnothing$. In the case where there is only one $C_{kji} \neq \varnothing$, we will set $\delta_{ki} = 1/2i$. Otherwise, since for any given i the sets C_{kji} are disjoint and compact there is a number $\delta_{ki} > 0$ smaller than half the distance between any two of them and $1/2i$. For each $C_{kji} \neq \varnothing$ let

$$U_{kji} = \text{interior}(P_{(k-1)n(j)i}) \cap \bigcup_{x \in C_{kji}} \mathbf{B}(\delta_{ki}, x).$$

Clearly, any U_{kji} is a subset of U_{kj} and for a fixed i the sets U_{kji} are disjoint. Using proposition 2.3 it is possible to construct for a fixed i and each j such that $C_{kji} \neq \varnothing$, a subbody P_{kji} contained in U_{kji} whose interior contains C_{kji}. In case $C_{kji} = \varnothing$ we just set $P_{kji} = \varnothing$. Define

$$P_{ki} = \bigcup_{j=1}^{J} P_{kji}, \qquad E_{ki} = \bigcup_{j=1}^{J} C_{kji}.$$

Since the various subbodies P_{kji} are disjoint, their union P_{ki} is a body and in addition, as interior $(P_{(k-1)n(j)i}) \supset U_{kji} \supset P_{kji}$ it is clear that $P_{(k-1)i} \supset P_{ki}$. The proof that $P_{ki} \to A_k$ follows from lemma 2.4.1 and lemma 2.4.2. ■

3. THE EXTENSION OF A SET FUNCTION DEFINED ON SUBBODIES TO A MEASURE

In the next section we will need a condition such that a set function defined over the collection of subbodies of a given body can be extended to a unique measure. Such a condition is of interest in general because it can be applied to various physical instances such as mass, electric charge, etc.

Just as we formally added the empty set as a subbody, we now set formally the value of the set function for the empty set to be zero.

Proposition 3.1. Let μ be a bounded real valued set function defined on the collection of subbodies of a body B that satisfies the following conditions:

(i) If P_1 and P_2 are disjoint subbodies, then, $\mu(P_1 \cup P_2) = \mu(P_1) + \mu(P_2)$.

(ii) If $A \in \Phi$ and $P_i \to A$, then, the sequence $\mu(P_i)$ converges and its limit is independent of the particular sequence P_i.

Then, there is a unique Borel measure ν on B such that $\nu(P) = \mu(P)$ for all subbodies P of B.

We will refer to the first and second conditions as additivity and continuity conditions, respectively.

Proof. Define the real valued set function ν on Φ by $\nu(A) = \lim_i \mu(P_i)$, where P_i is a sequence of subbodies whose limit is A. By proposition 2.4 ν is well defined, and in addition, it follows from the continuity assumption that for any subbody P, $\nu(P) = \mu(P)$. To show that ν is additive, consider the disjoint sets A_1 and A_2 in Φ. We now construct two sequences P_{1i} and P_{2i} converging to A_1 and A_2, respectively, satisfying the properties guaranteed by proposition 2.6, so in particular, $P_{1i} \cap P_{2i} = \varnothing$ for a fixed i. Since $A_1 \cup A_2 \doteq \lim_i(P_{1i} \cup P_{2i})$, we have

$$\nu(A_1 \cup A_2) = \lim_i \mu(P_{1i} \cup P_{2i}) = \nu(A_1) + \nu(A_2).$$

So far, we obtained a finitely additive set function on a field. We next show that ν is countably additive. We recall (see [9, p. 10]) that in order to prove that an additive set function ν on a field is countably additive it is sufficient to show that ν is continuous from above at the empty set, i.e. if the decreasing sequence A_i converges to the empty set, then $\lim_i \nu(A_i) = 0$.

Lemma 3. The set function ν is continuous from above at the empty set.

Proof of lemma 3.1.1. Let A_k be a sequence of sets in Φ decreasing to the empty set and for each k let P_{ki} be the sequence converging to A_k as in proposition 2.8, i.e. for a fixed i, $P_{ki} \supset P_{(k+1)i}$. We now extract from the double sequence P_{ki} a sequence $P_{k,i(k)}$, $k = 1, 2, \ldots$, as follows. For each k let N_k be the integer such that for any $i \geq N_k$, $|v(A_k) - v(P_{ki})| < 1/k$. We set $i(1) = N_1$ and we construct the sequence inductively by choosing $i(k+1) = \max\{N_{k+1}, i(k)+1\}$.

We first show that for each integer n, $A_n \supset \limsup_k P_{k,i(k)}$. Let $x \in \limsup_k P_{k,i(k)}$, then, for each $k \geq n$ there are some $j, j > k$ such that $x \in P_{j,i(j)}$. From the construction of the double sequence it follows that $P_{n,i(j)} \supset P_{j,i(j)}$. Hence, $x \in P_{n,i(j)}$ for some $j > k$ for each k so that $x \in \limsup_j P_{n,i(j)}$. Since $P_{n,i(j)}$ is a subsequence of P_{ni} it follows that $x \in \limsup_i P_{ni} = A_n$. We can conclude that $\limsup_k P_{k,i(k)} \subset \bigcap_n A_n = \varnothing$ since $A_n \supset \limsup_k P_{k,i(k)}$ for any n.

By the continuity assumption on μ we have $\lim_k \mu(P_{k,i(k)}) = 0$. In addition, our construction implies that the sequence $v(A_k) - \mu(P_{k,i(k)})$ converges to zero so that we conclude that $\lim_k v(A_k) = 0$.

Now, the proof of the proposition follows from the fact (see [8, p. 50]) that a bounded countably additive set function on a field, such as v in our case, can be represented as the difference of two positive bounded countably additive set functions each of which can be extended uniquely (see [9, p. 13]) to a positive measure on the smallest σ-field containing the original field. In our case this σ-field is the collection of the Borel sets and the difference between the two positive measures gives us the required Borel measure. ■

4. THE REPRESENTATION OF A FORCE SYSTEM BY A STRESS

Definition 4.1. A force system $\{f_P; P \text{ is a subbody of } B\}$ is *consistent* if the following conditions hold.

(i) If P_1 and P_2 are disjoint subbodies of B, then, for any $u \in C^r(B, \mathscr{R}^3)$,

$$f_{P_1 \cup P_2}(u|_{P_1 \cup P_2}) = f_{P_1}(u|_{P_1}) + f_{P_2}(u|_{P_2}).$$

(ii) If $A \in \Phi$ and $P_i \to A$, then, for any $u \in C^r(B, \mathscr{R}^3)$ the sequence $f_{P_i}(u|_{P_i})$ converges and its limit is independent of the particular sequence of subbodies P_i.

(iii) There is a finite bound $K > 0$ such that for any subbody P and any $u \in C^r(B, \mathscr{R}^3)$, $|f_P(u|_P)| < K\|u|_P\|$.

We will refer to these conditions as the *additivity, continuity* and *boundedness* conditions, respectively.

In the sequel we will use the following notation of multi-indices. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, where α_p is a positive integer. We will write x^α for $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}$, $|\alpha|$ for $\alpha_1 + \alpha_2 + \alpha_3$, $\alpha!$ for $\alpha_1!\alpha_2!\alpha_3!$, and given the real function w, we use $w_{,\alpha}$ for

$$\frac{\partial^{|\alpha|} w}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}.$$

We say that a function w on P is a *piecewise* r*th order polynomial* over a subbody P if the restriction of w to any connected component is a polynomial of order r, i.e. if the restriction is of the form $\sum_{|\alpha| \leq r} a_\alpha x^\alpha$, $a_\alpha \in \mathscr{R}$.

In proving that a consistent force system can be represented by a unique stress, we will need the following proposition.

Proposition 4.2. For each $u \in C^r(B, \mathscr{R}^3)$ and $\varepsilon_0 > 0$ there is a sequence of subbodies $P_i \to B$, a sequence of piecewise rth order polynomials $s_i \in C^r(P_i, \mathscr{R}^3)$ and an integer i_0 such that if $i > i_0$, then, $\|u|_{P_i} - s_i\|_{P_i} < \varepsilon_0$.

Proof. Let $u \in C^r(B, \mathscr{R}^3)$ and $\varepsilon_0 > 0$ be given. Set $\|u\|^* = \|u\| + 1$, $a = \min_{p,\alpha}\{\inf_{x \in B}\{u_{p,\alpha}\}\}$ and $b = \max_{p,\alpha}\{\sup_{x \in B}\{u_{p,\alpha}\}\} + 1$, where $p = 1, 2, 3$, $|\alpha| \leq r$. We choose the positive number ε and the integer N so that $\varepsilon = \dfrac{b-a}{N} \leq \dfrac{\varepsilon_0}{m}$ where m is a constant depending only on r that we will specify later. We define the index $\lambda = \{\lambda_{p\alpha};\ p = 1, 2, 3, |\alpha| \le r, \lambda_{p\alpha} = 0, \ldots, N-1\}$. Clearly, for each component $\lambda_{p\alpha}$ of λ we have $a \le a + \lambda_{p\alpha}\varepsilon < b$.

We also set

$$L = \min_\alpha\{a, \inf_{x\in B}\{\|u\|^* x^\alpha\}\},$$
$$H' = \max_\alpha\{b, \sup_{x\in B}\{\|u\|^* x^\alpha\}\},$$
$$H = L + N_1\varepsilon,$$

where N_1 is the smallest integer such that $N_1 > (H' - L)/\varepsilon$, and finally

$$\beta = \{\beta_1, \beta_2, \beta_3\}, \qquad \beta_p = 0, \ldots, N_1 - 1.$$

With these definitions we have $L \le L + \beta_p\varepsilon < H$.

Divide the body B into the subsets $A_{\lambda\beta}$ defined by

$$A_{\lambda\beta} = \bigcap_{\substack{p=1,2,3\\ |\alpha|\le r}} \{(u_{p,\alpha})^{-1}[a + \lambda_{p\alpha}\varepsilon, a + (\lambda_{p\alpha} + 1)\varepsilon)\} \cap$$
$$\{x; L + \beta_p\varepsilon \le \|u\|^* x_p^{\alpha_p} < L + (\beta_p + 1)\varepsilon\}.$$

By definition, if $x \in A_{\lambda\beta}$ then for each p and α we have

$$a + \lambda_{p\alpha}\varepsilon \le u_{p,\alpha}(x) < a + (\lambda_{p\alpha} + 1)\varepsilon,$$
$$L + \beta_p\varepsilon \le \|u\|^* x_p^{\alpha_p} < L + (\beta_p + 1)\varepsilon.$$

Clearly, the various $A_{\lambda\beta}$ are disjoint and their union is B. Moreover, each $A_{\lambda\beta}$ belongs to Φ and can be represented as the intersection of an open set and a closed set.

Using the constructions of Section 2, for each pair of indices λ and β there is a sequence of subbodies $P_{\lambda\beta i} \to A_{\lambda\beta}$ so that for a fixed i the various $P_{\lambda\beta i}$ are disjoint and their union is a subbody that we denote by P_i. We choose a point $y_{\lambda\beta}$ in each of the sets $A_{\lambda\beta}$ and define $s_{\lambda\beta} \in C^r(B, \mathscr{R}^3)$ by its pth component as

$$(s_{\lambda\beta})_p(x) = \sum_{|\alpha|\le r} \frac{1}{\alpha!}(a + \lambda_{p\alpha}\varepsilon)(x - y_{\lambda\beta})^\alpha.$$

Set $s_{\lambda\beta i} \in C^r(P_{\lambda\beta i}, \mathscr{R}^3)$ by $s_{\lambda\beta i} = s_{\lambda\beta}|_{P_{\lambda\beta i}}$ and the piecewise rth order polynomial $s_i \in C^r(P_i, \mathscr{R}^3)$ by $s_i = \sum_{\lambda,\beta} s_{\lambda\beta}\chi_{P_{\lambda\beta i}}$, where χ_A denotes the characteristic function of the subset A.

We recall that

$$\|u|_{P_{\lambda\beta i}} - S_{\lambda\beta i}\|_{P_{\lambda\beta i}} = \sup\{|u_{p,\alpha}(x) - (s_{\lambda\beta i})_{p,\alpha}(x)|; x \in P_{\lambda\beta i}, |\alpha| \le r, p = 1, 2, 3\}.$$

By adding and subtracting $u_{p,\alpha}(z_{\lambda\beta}(x))$, $a + \lambda_{p\alpha}\varepsilon$ and by using the triangle inequality we obtain for any choice of a point $z_{\lambda\beta}(x) \in A_{\lambda\beta}$,

$$\begin{aligned}\|u|_{P_{\lambda\beta i}} - s_{\lambda\beta i}\|_{P_{\lambda\beta i}} \le{} & \sup_{x,\alpha,p}\{|u_{p,\alpha}(x) - u_{p,\alpha}(z_{\lambda\beta}(x))|\} \\ & + \sup_{x,\alpha,p}\{|u_{p,\alpha}(z_{\lambda\beta}(x)) - (a + \lambda_{p\alpha}\varepsilon)|\} \\ & + \sup_{x,\alpha,p}\{|(a + \lambda_{p\alpha}\varepsilon) - (s_{\lambda\beta i})_{p,\alpha}(x)|\}.\end{aligned}$$

We now examine the terms on the right hand side of this inequality.

(i) The functions $u_{p,\alpha}|_{P_i}$ are uniformly continuous for each p and α. Hence, there is a $\delta > 0$ such that $|x - y| < \delta$ implies that $|u_{p,\alpha}|_{P_i}(x) - u_{p,\alpha}|_{P_i}(y)| < \varepsilon$, for each p and α. We choose an integer i_1 such that $i_1 > 1/\delta$ and it follows from corollary 2.5 that for each $x \in P_{\lambda\beta i}$, $i > i_1$, there is a $z_{\lambda\beta}(x) \in A_{\lambda\beta} \cap P_{\lambda\beta i}$ with $|x - z_{\lambda\beta}(x)| < \delta$. Clearly, for $i > i_1$,

$$\sup\{|u_{p,\alpha}(x) - u_{p,\alpha}(z_{\lambda\beta}(x))|; x \in P_{\lambda\beta i}, |\alpha| \le r, p = 1, 2, 3\} < \varepsilon,$$

and from that corollary i_1 is independent of the values of λ and β.

(ii) Since $z_{\lambda\beta}(x) \in A_{\lambda\beta}$ we have by our construction

$$\sup\{|u_{p,\alpha}(z_{\lambda\beta}(x)) - (a + \lambda_{p\alpha}\varepsilon)|; x \in P_{\lambda\beta i}, |\alpha| \leq r, p = \ ,2,3\} < \varepsilon.$$

Clearly, the last inequality holds for each λ and β.

(iii) Consider the term

$$|(a + \lambda_{p\alpha}\varepsilon) - (s_{\lambda\beta i})_{p,\alpha}(x)| = \left|(a + \lambda_{p\alpha}\varepsilon) - \sum_{\substack{|\gamma| \leq r \\ \gamma \geq \alpha}} \frac{(a + \lambda_{p\gamma}\varepsilon)}{(\gamma - \alpha)!}(x - y_{\lambda\beta})^{\gamma - \alpha}\right|$$

$$= \left|\sum_{\substack{|\gamma| \leq r \\ \gamma > \alpha}} \frac{(a + \lambda_{p\gamma}\varepsilon)}{(\gamma - \alpha)!}(x - y_{\lambda\beta})^{\gamma - \alpha}\right|$$

where we used

$$x^{\gamma}_{,\alpha} = \frac{\gamma!}{(\gamma - \alpha)!} x^{\gamma - \alpha}$$

and where $\alpha \leq \gamma$ means $\alpha_p \leq \gamma_p$ for $p = 1, 2, 3$. By the definition of $\|u\|^*$ we have

$$|(a + \lambda_{p\alpha}\varepsilon) - (s_{\lambda\beta i})_{p,\alpha}(x)| \leq \|u\|^* \left|\sum_{\substack{|\gamma| \leq r \\ \gamma > \alpha}} (x - y_{\lambda\beta})^{\gamma - \alpha}\right|$$

$$\leq \|u\|^* \left|\sum_{\substack{|\gamma| \leq r \\ \gamma > \alpha}} (x - z(x) + z(x) - y_{\lambda\beta})^{\gamma - \alpha}\right|$$

where $z(x) \in A_{\lambda\beta}$ will be specified later. The last sum contains a finite number of binomials raised to powers smaller or equal to r in the variables x–z and z–$y_{\lambda\beta}$. Let us denote the number of all such powers by $m_1(r)$. It follows that

$$\left|\sum_{\substack{|\gamma| \leq r \\ \gamma > \alpha}} (x - z(x) + z(x) - y_{\lambda\beta})^{\gamma - \alpha}\right| \leq m_1(r) \max_{\substack{|\gamma| \leq r \\ \gamma > \alpha}} |x - z(x) + z(x) - y_{\lambda\beta}|^{\gamma - \alpha}.$$

Now, if $m_2(r)$ is the maximal number of terms in the binomial expansion for powers smaller or equal to r and $m_3(r)$ is the largest binomial coefficient for such expansions we have

$$|(x - z(x)) + (z(x) - y_{\lambda\beta})|^{\gamma - \alpha}$$
$$\leq m_2(r) m_3(r) \sup\{|x - z(x)|^{\eta}; 0 \leq |\eta| \leq r\} \sup\{|z(x) - y_{\lambda\beta}|^{\psi}; 0 \leq |\psi| \leq r\}.$$

We can choose by corollary 2.5 an integer i_2 independent of λ and β such that for each $x \in P_{\lambda\beta i}$, $i > i_2$ there is a $z(x) \in A_{\lambda\beta} \cap P_{\lambda\beta i}$ with $|x - z(x)| \leq \varepsilon/\|u\|^*$. In addition, by our construction, for any ψ, $|\psi| \leq r$, $|z(x) - y_{\lambda\beta}|^{\psi} \leq \varepsilon/\|u\|^*$. It follows that

$$\sup_{x,\alpha,p} \{|(a + \lambda_{p\alpha}\varepsilon) - (s_{\lambda\beta i})_{p,\alpha}(x)|\} \leq 2m_1(r)m_2(r)m_3(r)\varepsilon.$$

Finally, the proof of the proposition is obtained by choosing $m = 2m_1(r)m_2(r)m_3(r) + 2$ and choosing an i_0 which is greater than i_1 and i_2. ■

Corollary 4.3. Let $\{f_P\}$ be a consistent force system on B, then, for each $u \in C^r(B, \mathscr{R}^3)$ and $\varepsilon_0 > 0$, there is a sequence of subbodies $P_i \to B$, a sequence of piecewise rth order polynomials $s_i \in C^r(P_i, \mathscr{R}^3)$ and an integer i_0 such that if $i > i_0$, then, $|f_B(u) - f_{P_i}(s_i)| < \varepsilon_0$.

Proof. Let $u \in C^r(B, \mathscr{R}^3)$ and $\varepsilon_0 > 0$ be given. By proposition 4.2 there is a sequence of subbodies $P_i \to B$, a sequence of piecewise rth order polynomials $s_i \in C^r(P_i, \mathscr{R}^3)$ and an integer i_1 such that if $i > i_1$ then, $\|u|_{P_i} - s_i\|_{P_i} < \frac{\varepsilon_0}{2K}$, where K is the bound in definition 4.1 (iii). Using the linearity of the forces and the boundedness property we have for each $i > i_1$

$$|f_{P_i}(u|_{P_i}) - f_{P_i}(s_i)| = |f_{P_i}(u|_{P_i} - s_i)| < K\|u|_{P_i} - s_i\|_{P_i} < \frac{\varepsilon_0}{2}.$$

By the continuity of the force system there is an integer i_2 such that if $i > i_2$ then, $|f_B(u) - f_{P_i}(u|_{P_i})| < \varepsilon_0/2$. Now, if i_0 is greater than i_1 and i_2 we have for each $i > i_0$

$$|f_B(u) - f_{P_i}(s_i)| \le |f_B(u) - f_{P_i}(u|_{P_i})| + |f_{P_i}(u|_{P_i}) - f_{P_i}(s_i)| < \varepsilon_0.$$

Remark. We note that the sequences P_i and s_i depend only on u and B and do not depend on the force system.

Proposition 4.4. Let $\{f_P\}$ and $\{g_P\}$ be two consistent force systems on B. Assume that $g_P(s) = f_P(s)$ for each subbody P and each piecewise rth order polynomial s on P, then, $g_P = f_P$ for each subbody P.

Proof. Assume that there is a subbody P and a $u \in C^r(B, \mathscr{R}^3)$ such that $g_P(u|_P) \neq f_P(u|_P)$ and let $\varepsilon_0 = |g_P(u|_P) - f_P(u|_P)|$. Since $g_P(s) = f_P(s)$ for each subbody P and each piecewise rth order polynomial s on P we have

$$\begin{aligned}|g_P(u|_P) - f_P(u|_P)| &\le |g_P(u|_P) - f_{P_i}(s_i)| + |f_{P_i}(s_i) - f_P(u|_P)| \\ &= |g_P(u|_P) - g_{P_i}(s_i)| + |f_{P_i}(s_i) - f_P(u|_P)|.\end{aligned}$$

Since the sequence s_i is independent of the force system, by corollary 4.3 we may choose an i_g so that $|g_P(u|_P) - g_{P_i}(s_i)| < \varepsilon_0/2$ for $i > i_g$ and an i_f so that $|f_{P_i}(s_i) - f_P(u|_P)| < \varepsilon_0/2$ for $i > i_f$. A choice of an i_0 greater than both i_f and i_g will result in a contradiction. ■

Proposition 4.5. A force system $\{f_P\}$ on B is consistent if and only if there is a unique collection of bounded Borel measures $\{\sigma_{p\alpha}\}$, $p = 1, 2, 3$, $|\alpha| \le r$ that represents the force system in the form

$$f_P(u) = \sum_{\substack{|\alpha| \le r \\ p=1,2,3}} \int_P u_{p,\alpha}\, d\sigma_{p\alpha} \qquad u \in C^r(P, \mathscr{R}^3).$$

Proof. It is clear that a force system which is represented by a collection of bounded stress measures as in the previous representation equation is consistent. We now show that consistency is indeed a sufficient condition for the representation. Define the functions $u_{(q\alpha)} \in C^r(B, \mathscr{R}^3)$ by their pth components as $u_{(q\alpha)p}(x) = \delta_{qp} x^\alpha$, $p, q = 1, 2, 3$, $|\alpha| < r$, where δ_{qp} denotes the Kronecker δ. For $|\alpha| = 0$ we define the set functions $\mu_{q\alpha}$ on the collection of subbodies of B by $\mu_{q\alpha}(P) = f_P(u_{(q\alpha)}|_P)$. Clearly, the set functions $\mu_{q\alpha}$ satisfy the conditions of proposition 3.1 and it follows that they can be extended to Borel measures $\sigma_{q\alpha}$, $|\alpha| = 0$, on B. Next we define inductively

$$\mu_{q\alpha}(P) = \frac{1}{\alpha!} f_P(u_{(q\alpha)}|_P) - \sum_{\gamma<\alpha} \frac{1}{(\alpha-\gamma)!} \int_P x^{\alpha-\gamma}\, d\sigma_{q\gamma}$$

for $0 < |\alpha| \le r$. Since the conditions of proposition 3.1 hold for both terms in the previous equation, they hold for the set functions $\mu_{q\alpha}$, $0 < |\alpha| \le r$ as well, and hence, we can extend them to the Borel measures $\sigma_{q\alpha}$.

Construct the force system $\{g_P\}$ on B by

$$g_P(u) = \sum_{\substack{|\alpha| \le r \\ p=1,2,3}} \int_P u_{p,\alpha}\, d\sigma_{p\alpha}, \qquad u \in C^r(P, \mathscr{R}^3).$$

It is obvious that $\{g_P\}$ is a consistent force system. We have

$$\begin{aligned}g_P(u_{(q\alpha)}|_P) &= \sum_{\substack{|\gamma| \le r \\ p=1,2,3}} \int_P (u_{(q\alpha)})_{p,\gamma}\, d\sigma_{p\gamma} \\ &\sum_{\substack{\gamma \le \alpha \\ p=1,2,3}} \int_P \frac{\alpha!}{(\alpha-\gamma)!} \delta_{pq} x^{\alpha-\gamma}\, d\sigma_{p\gamma} \\ &= \alpha!\sigma_{q\alpha}(P) + \sum_{\gamma<\alpha} \int_P \frac{\alpha!}{(\alpha-\gamma)!} x^{\alpha-\gamma}\, d\sigma_{q\gamma}.\end{aligned}$$

By using $\sigma_{q\alpha}(P) = \mu_{q\alpha}(P)$ and the definition of $\mu_{q\alpha}$, in the last line we obtain $g_P(u_{(q\alpha)}|_P) = f_P(u_{(q\alpha)}|_P)$. By the linearity of forces it follows that the force systems $\{f_P\}$ and $\{g_P\}$ have equal values for any rth order polynomial, by the additivity of the force systems they have equal values for any piecewise rth order polynomial and it follows from proposition 4.4 that they are equal.

It remains to show that if $\{\sigma_{1p\alpha}\}$ and $\{\sigma_{2p\alpha}\}$, $p = 1, 2, 3$, $|\alpha| \leq r$ represent the force system $\{f_P\}$, then $\sigma_{1p\alpha} = \sigma_{2p\alpha}$. Let A be an arbitrary set in Φ and let P_i be a sequence of subbodies converging to A. We have for $|\alpha| = 0, f_{P_i}(u_{(p\alpha)}) = \sigma_{1p\alpha}(P_i) = \sigma_{2p\alpha}(P_i)$ for each i and p. Using the continuity of the measures $\sigma_{1p\alpha}$ and $\sigma_{2p\alpha}$ we conclude that $\sigma_{1p\alpha}(A) = \sigma_{2p\alpha}(A)$ for $|\alpha| = 0$ and every $A \in \Phi$. We recall that if two measures are equal on a field they are equal on the minimal σ-field containing it, hence, $\sigma_{1p\alpha} = \sigma_{2p\alpha}$ for $|\alpha| = 0$. To use an induction process, assume that $\sigma_{1p\alpha} = \sigma_{2p\alpha}$ for all $|\alpha| < r_0$. Again, for an arbitrary $A \in \Phi$, $P_i \to A$ and γ with $|\gamma| = r_0$, we have by using the definition of $u_{(p\alpha)}$,

$$f_{P_i}(u_{(q\gamma)}) = \sum_{\substack{|\alpha| \leq r_0 \\ p=1,2,3}} \int_{P_i} (u_{(q\gamma)})_{p,\alpha} \, \mathrm{d}\sigma_{1p\alpha} = \sum_{\substack{|\alpha| \leq r_0 \\ p=1,2,3}} \int_{P_i} (u_{(q\gamma)})_{p,\alpha} \, \mathrm{d}\sigma_{2p\alpha}.$$

Hence,

$$\sum_{\substack{|\alpha| < r_0 \\ p=1,2,3}} \int_{P_i} (u_{(q\gamma)})_{p,\alpha} \, \mathrm{d}\sigma_{1p\alpha} + \sum_{p=1,2,3} \int_{P_i} (u_{(q\gamma)})_{p,\gamma} \, \mathrm{d}\sigma_{1p\gamma}$$

$$= \sum_{\substack{|\alpha| < r_0 \\ p=1,2,3}} \int_{P_i} (u_{(q\gamma)})_{p,\alpha} \, \mathrm{d}\sigma_{2p\alpha} + \sum_{p=1,2,3} \int_{P_i} (u_{(q\gamma)})_{p,\gamma} \, \mathrm{d}\sigma_{2p\gamma}$$

and from the induction hypothesis it follows that $\sigma_{1p\gamma}(P_i) = \sigma_{2p\gamma}(P_i)$ for each i and p. Again, the properties of measures imply that $\sigma_{1p\gamma} = \sigma_{2p\gamma}$. ■

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