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Enhancement of Decoherence by Chaotic-like Behavior

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We demonstrate by use of a simple one-dimensional model of a square barrier imbedded in an infinite potential well that decoherence is enhanced by chaotic-like behavior. We, moreover, show that the transition $\hbar \to 0$ *is singular. Finally it is argued that the time scale on which decoherence occurs depends on the degree of complexity of the underlying quantum mechanical system, i.e., more complex systems decohere relatively faster than less complex ones.*

Chaos is expected, at least in some cases, to lead to irreversibility. This in spite of the fact that chaos is deterministic although noncomputable. The noncomputability is due to the exponential divergence of neighboring trajectories which necessitates the computation of the trajectory of every single particle in an ensemble in order to know the dynamics of the ensemble, a task which in practice is impossible.

In order to find a quantum mechanical paraphrase to the definition of classical chaos as being the exponential divergence in time of neighboring points in phase space, we recognize that the fast spreading of the wave function can be considered as an indication for chaotic like behavior.[1] This is not a rigorous definition but rests on the fact that the wave function may be thought of as representing an ensemble of points in phase space and that its spread with time is a measure for the divergence of those points.

One evidence for chaos is the exponential decay of the autocorrelation function of the form

$$\langle f(t), f(0) \rangle$$

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We shall in the following consider the time development of a quasi-stationary state by studying barrier penetration. The study of resonance transmission provides us with an example of the exponential decay, although only to an approximation.

The quantum mechanical description for the decay of quasi-stationary states is based on the evolution under the action of the full Hamiltonian of an eigenstate $|\Phi_0\rangle$ of the unperturbed Hamiltonian. The probability amplitude for the persistence of the unperturbed state is given by

$$A(t) = \langle \Phi_0 | e^{-iHt} | \Phi_0 \rangle \tag{1}$$

and the probability $P(t)$ that the system has not decayed by time t is the square of the survival amplitude

$$P(t) = |\langle \Phi_0 | e^{-iHt} | \Phi_0 \rangle|^2 \tag{2}$$

Wigner and Weisskopf[2] showed that this probability is approximately exponential for t not to large and not too small. It is clear that it is not exponential for very small t, since for hermitic H, when $H|\Phi_0\rangle$ is defined

$$\left.\frac{d}{dt} P(t)\right|_{t=0} = 0 \tag{3}$$

There is also a large time deviation from the exponential mode—called the long time tail—due to the boundness from below of the energy spectrum.[3]

Let H be a Hilbert space embedded in a larger Hilbert space

$$H = H_U \oplus H_B \tag{4}$$

H_U being the Hilbert space of undecayed states and H_D the space of the decay products. Let further $Z(t)$ be a family of operators in H. The semigroup law

$$Z(t_1)\, Z(t_2) = Z(t_1 + t_2) \tag{5}$$

which ensures exact exponential decay, can only be valid—as proved independently by Horwitz *et al.*[4] and Williams[5]—if the generator of $U(t) = \exp(-iHt)$ has the whole real line as its spectrum.

For an unstable quantum system for which the unperturbed Hamiltonian has discrete states embedded in a continuous spectrum on $(-\infty, \infty)$ the time dependence of the decay is a sum of exponential contributions plus a background contribution which may be arbitrarily small for any positive t. For small times it is the expectation value of $(H - \langle H \rangle)^2$ in the initial state $|\Phi_0\rangle$ which leads to a measure for the deviation from the pure exponential.[6] It should noticed that the boundness of the Hamiltonian on $|\Phi_0\rangle$ also is crucial for the short time deviation.

The existence of an exact semigroup law for the contracted evolution $Z(t)$ implies that there is no regeneration.[4] This establishes the connection between exact exponential decay and irreversibility.

It can be shown that the initial quadratic behavior of the decay is possibly followed by an oscillating period of enhanced and hindered decay before the state enters the exponential mode.[7]

A broad state goes more smoothly to the exponential mode, while in the case of a narrow state there is a relative big interval between the time where the quadratic behavior terminates and the time where the exponential sets in.

We conjecture that the onset of the exponential era corresponds to the onset of chaotic-like behavior in the system and that the initial non-exponential era may be understood as the preparation time for the manifestation of this behavior. Such a conjecture implies that a broad state should exhibit a faster approach to and possibly also a higher degree of complex behavior. This behavior is verified in the observations of the time evolution of a broad and a narrow state below.

We shall also demonstrate that the time scale on which decoherence takes place depends on the degree of complexity of the underlying quantum mechanical system, i.e., more complex systems decohere faster than less complex ones. This is in accordance with the expectation that decoherence effects suppress the possibility of interference and hence reduce the off-diagonal elements of the reduced density matrix.[8] The Schrödinger evolution cannot transform a pure state into a mixture and hence not make the off-diagonal elements vanish completely. Although a precise definition of quantum chaos has not been established, we study here phenomena which appear to be closely related to this concept. Quantum chaos, if it exists even in some approximate sense, on the other hand is expected to lead at least, very closely, to irreversibility and hence also to manifest itself by implying that the nondiagonal elements vanish (or become small), so it appears that quantum chaos facilitates decoherence. This is demonstrated by a rapid entropy production; see also Ref. 9 and 10.

We have recently utilized a simple one-dimensional model of a square barrier embedded in an infinite potential well in order to demonstrate that tunneling leads to very complex behavior of the wave function.[11] Many parallels to the well-known characteristics of classical chaos emerge, for example, an exponential decreasing correlation function of the peak-to-peak time series, a phase space plot of the expectation values $\langle x\rangle(t)$ and $\langle p\rangle(t)$ revealing a phenomenon similar to period doubling and attractor-like behavior of a chaotic double-well system such as a driven

Duffing oscillator. Another indication of chaotic-like behavior are the level statistics slightly above the barrier which show signs of Wigner statistics. Time-dependent adiabatic modulation of the barrier, which is tantamount to coupling the system to the environment, enhances the complex, chaotic-like behavior, which is obvious from the increased correspondence to Wigner statistics when an ensemble of slightly different barrier heights is considered.[11]

But most striking is the behavior of the spatial entropy function

$$S(t) = \int |\Psi(x, t)|^2 \ln |\Psi(x, t)|^2 \, dx \tag{6}$$

This entropy shows a rapid rise at early times to a nonperiodic fluctuating function around a smooth almost constant asymptotic value, illustrating the early burst of chaotic-like behavior.

In order to obtain an exponential decay of the initial wave packet and hence both chaotic-like and irreversible behavior, the energy spectrum must be continuous. For a finite bounded quantum system of the form

$$\Psi(x, t) = \sum_N a_N u_n(x) \, e^{-iE_n t/h} \tag{7}$$

where E_n and $U_n(x)$ are the eigenvalues and the eigenfunctions, the spectrum is discrete. The wave function is almost periodic, i.e, Ψ makes near returns to every achieved value. This excludes true decay of correlations.[12] However, those near periods are relevant only for long time behavior of the system. On laboratory time scales the system mimics a chaotic time evolution. Moreover, one can, by enlarging the extension of the double-well system, delay the onset of the reversible behavior ad infinitum.

By displacing the barrier in the double-well system to the right or to the left, certain positions are passed where the system becomes almost degenerate. These positions occur at almost commensurate intervals. It is exactly for those positions that one may find significant tunneling accompanied by chaotic-like behavior.[13] We have recently shown that, in cases of high-degeneracy, tunneling from the left to the right has exponential decay for times not too small and not too large, while at other positions, where almost degenerate conditions are somewhat weaker, the transition curve is nonexponential and develops strong oscillations.[13]

Taking into account that chaos should facilitate tunneling and vice versa, together with the fact that tunneling is enhanced by a high degree of almost degeneracy, we conclude that systems with a higher degree of complex, chaotic-like, behavior have a sharper increase of the entropy, i.e., higher entropy production, and a faster approach to equilibrium than systems with lower complexity.

Ballentine *et al.* have shown that Ehrenfest's theorem is neither necessary nor sufficient in order to identify the classical regime and argued that the classical limit of a quantum state is not a single classical trajectory but rather an ensemble of classical trajectories However, Ehrenfest's theorem breaks down much sooner for a chaotic ensemble than for a regular one.[14]

One can use the present model to illustrate some crucial differences between classical and quantum dynamics. Let us start with a wave packet initially located on the left-hand side of the square barrier placed in the

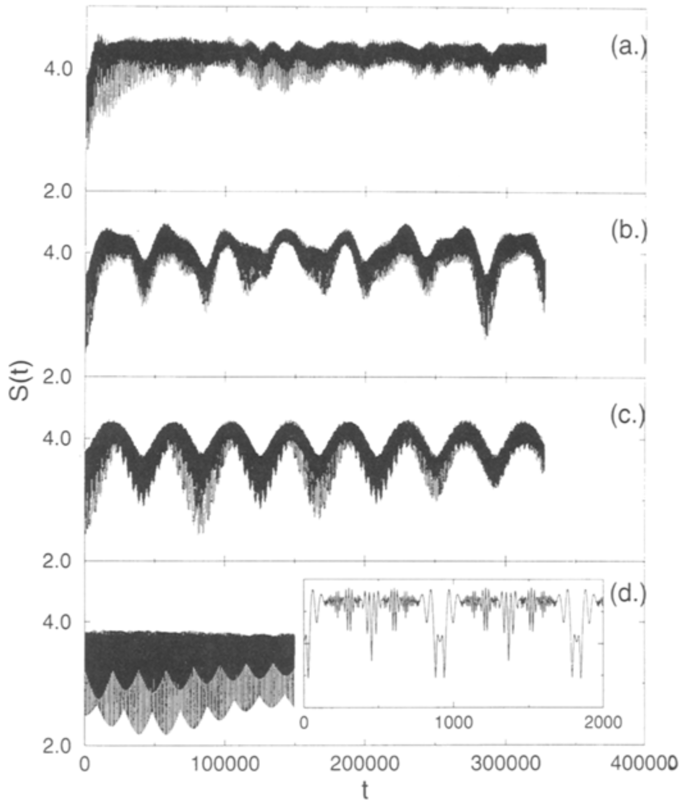

Fig. 1. The entropy function as a function of time for different locations in the well ($X \in [-55, 55]$). (a) The center of the barrier in the middle of the well $X = 0$; (b) the center of the barrier in $X = -10, 89$; (c) the center of the barrier in $X = -3, 63$; (d) the center of the barrier in $X = -1$. (The inset shows an enlargement of typical periodic oscillations).

infinite potential well. The parameters are as follows: the barrier height $V=5$, the half-width of the well $l=55$, the width of the Gaussian wave packet of the form $|\Psi|^2 \cong c\exp[-(x-x_0)^2/2\sigma^2]$, $\sigma=5$, the barrier width $2a=2$, and the mass $m=1/2$. The Gaussian wave packet is constructed from the first 30 energy levels, and the standard deviation from an exact (normalized) Gaussian is $8.81 \cdot 10^{-5}$. The computation preserves the norm to high accuracy, 0.999997. The problem is solved analytically as a function of time. There is no additional error for large t, since t enters in periodic exponential form.

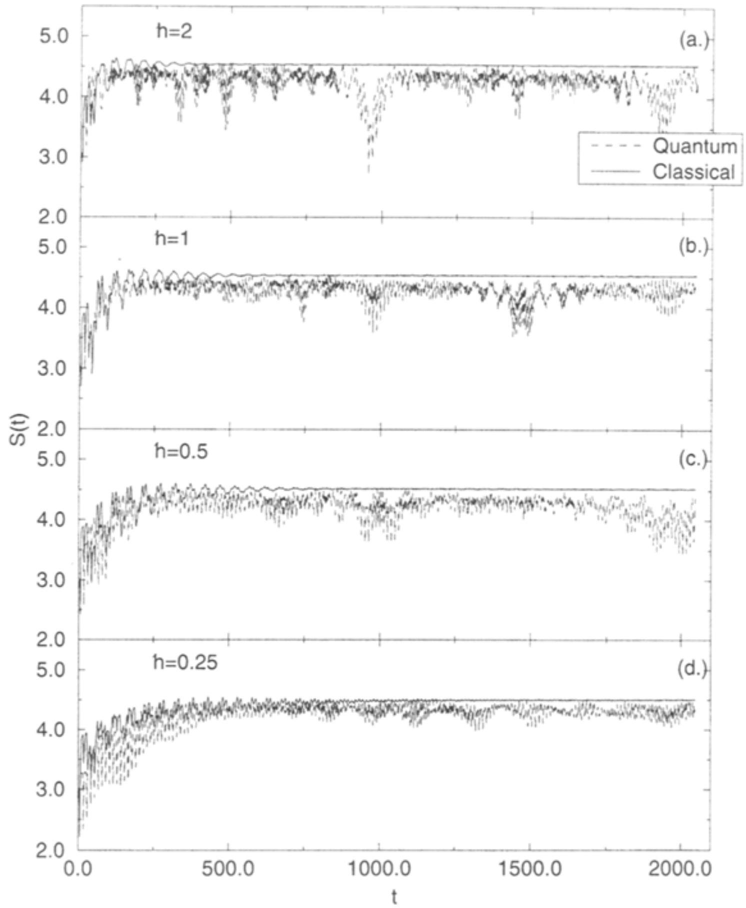

Fig. 2. The entropy function as a function of time for the classical case (solid line) an the quantum case (dotted), for different values of $\hbar$. The barrier is placed in the middle of the potential well and the mean energy of the incoming particle $\langle E\rangle$ is equal to the barrier height V, $V=5$. (a) $\hbar=2$; (b) $\hbar=1$; (c) $\hbar=0.5$; (d) $\hbar=0.25$.

The motion and time evolution for the spatial entropy function of this quantum state is compared to those of a classical ensemble whose initial position and momentum distributions are equal to those of the quantum state, the initial state being the product of the position and momentum distributions (this method is introduced and worked out in Ref. 14).

In Figs. 2a–2d we have compared the time evolution of the spatial entropy functions $S(t)$ for the quantum case and the classical case for decreasing values of $\hbar$ for an energy of the incoming particle high relative

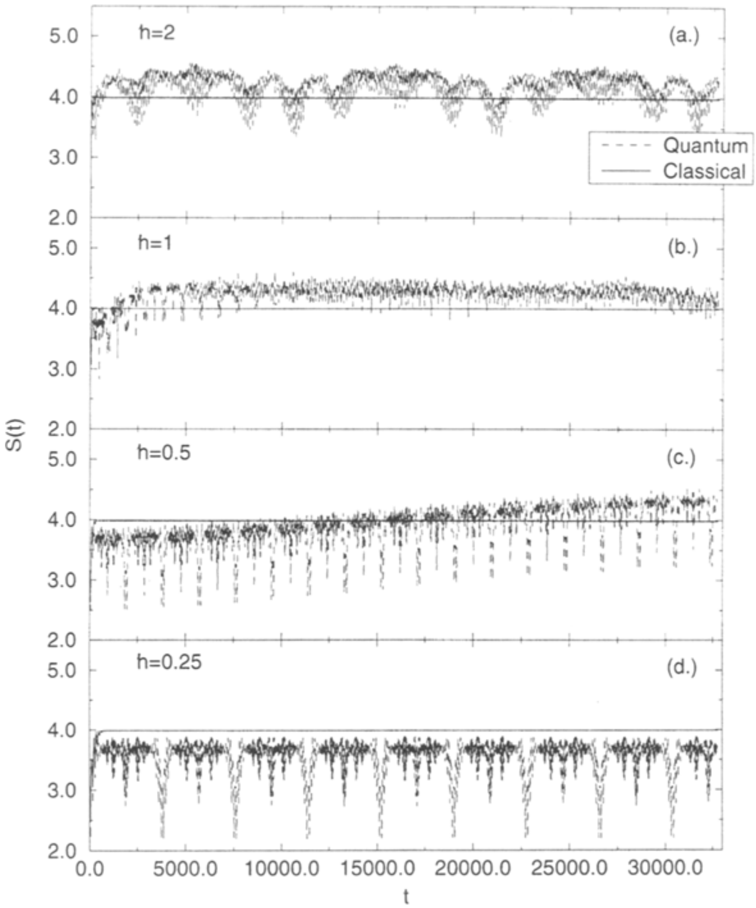

Fig. 3. The entropy function as a function of time for the classical case (solid line) an the quantum case (dotted), for different values of $\hbar$. The barrier is placed in the middle of the potential well and the mean energy of the incoming particle $\langle E\rangle = 0.05$ is low relative to the barrier height V, $V = 5$. (a) $\hbar = 2$; (b) $\hbar = 1$; (c) $\hbar = 0.5$; (d) $\hbar = 0.25$.

to the barrier height, $\langle E \rangle = 5$. In Figs. 3a–3d the same comparison is performed for an energy of the particle low relative to the barrier height, $\langle E \rangle = 0.005$.

In neither of the two cases—high energy relative to the barrier height (respectively low energy relative to the barrier height)—does a decreasing value of $\hbar$ lead to a true recovering of the classical time evolution, although it is noticed that significant tunneling ($\langle E \rangle \approx V$) leads to increased similarity between the classical case and the quantum case.

We are therefore led to the conclusions, that

(1) $\hbar \to 0$ is a singular transition

(2) tunneling and thus chaotic-like behavior weakens the difference between classical and quantum mechanics

We remark that in order to investigate the possible existence of a connection between chaos and decoherence in a rigorous way one needs to couple the system to the environment. This may be obtained by a harmonic modulation of the barrier height.

One can show that for an adiabatic harmonic perturbation of the barrier one can obtain enhanced as well as diminished tunneling. It appears, however, that enhancement is much prevailing (for a detailed discussion of this problem see Refs. 15–17, and it can be concluded that such a perturbation tends to increase the complex behavior.

The conclusion that tunneling and hence chaotic-like behavior tends to erase the differences between classical and quantum behavior becomes even more apparent by comparing the classical and the quantum case in connection with the position of the barrier inside the potential well. In Ref. 12 it is shown that placing the barrier in the center of the well results in a strong complex behavior of the wave function, whereas placing the barrier off center leads to a less complex, almost regular behavior. Indeed, by comparing for fixed $\hbar$ the two positions of the barrier—centered in Figs. 2a–2d and off center in Figs. 4a–4d, one observes that a similarity in the time development between the classical case and the quantum analog is reached much more quickly when the barrier is placed in the center, i.e, when there is a higher degree of almost degeneracy and hence more chaotic-like behavior. We therefore conjecture that the time scale on which decoherence takes place depends on the degree of complexity of the underlying quantum mechanical system, i.e., more complex systems decohere relatively faster than less complex ones.

In Fig. 5 we compare the time development of the spatial entropy function for a (relatively) narrow state and a (relatively) broad state. It is seen that the "uniform" behavior of the entropy function is an excellent

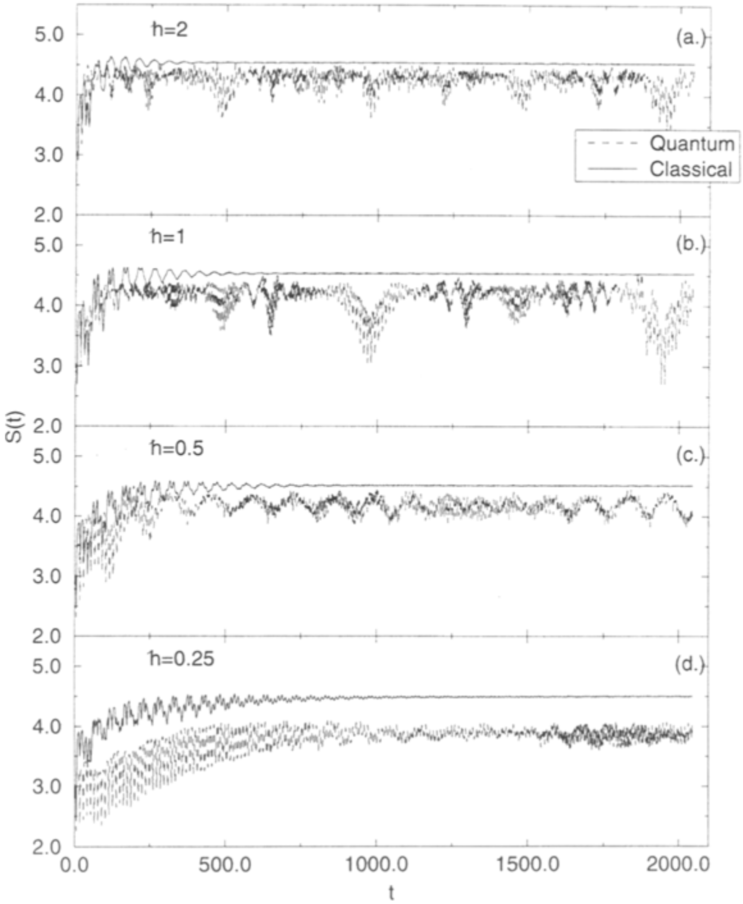

Fig. 4. The entropy function as a function of time for the classical case (solid line) an the quantum case (dotted), for different values of $\hbar$. The barrier is placed off center, $X=-1$, and the mean energy of the incoming particle $\langle E \rangle = 0.05$ is equal to the barrier height V, $V=5$. (a) $\hbar=2$; (b) $\hbar=1$; (c) $\hbar=0.5$; (d) $\hbar=0.25$.

indicator for the onset of exponential decay. The broad state has, as mentioned above, a faster approach to exponential—"irreversible"—behavior. This is accompanied by a strong increase of $S(t)$, i.e., a fast entropy production and hence a faster approach to equilibrium. A narrow state has a slower approach to the exponential with alternating intervals of enhanced and hindered decay. This is matched by the oscillating nature of $S(t)$ for the narrow state. This type of behavior is connected with a lower degree of complexity, which becomes eminent by comparison with Figs. 1a–1d.

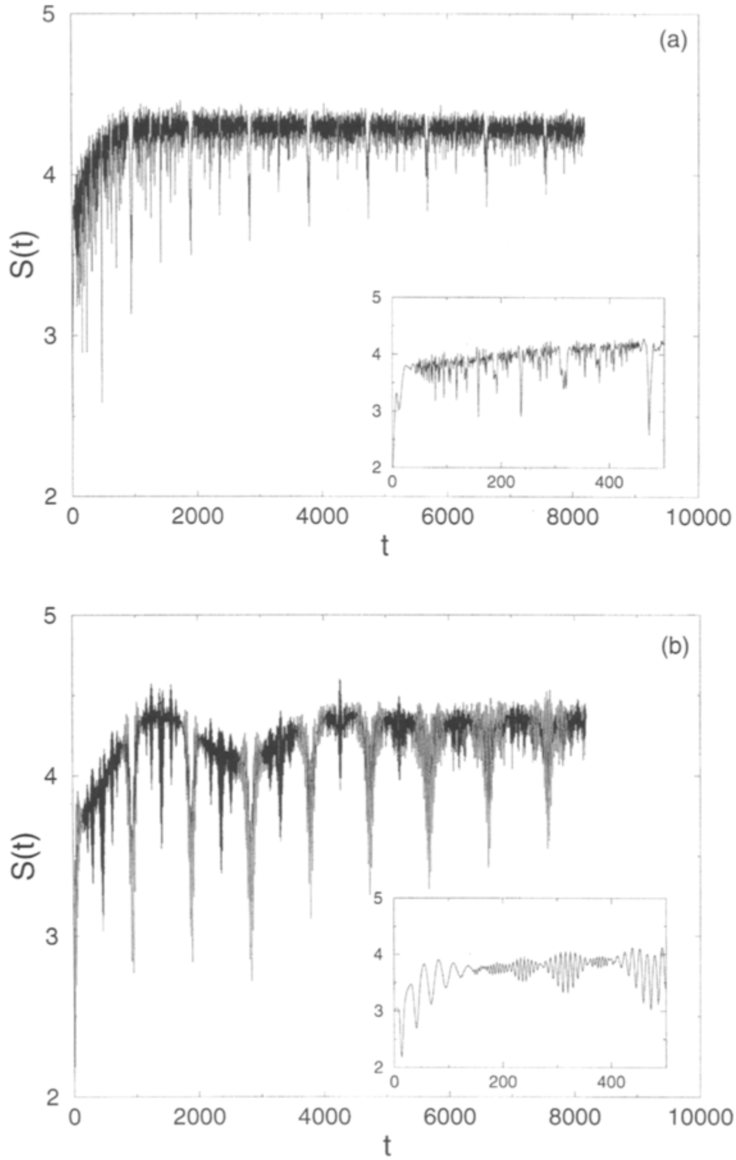

Fig. 5. The entropy function as a function of time for a relatively broad state and a relatively narrow state. (a) The wave packet constructed form the first 64 energy levels; (b) the wave packet constructed from the first 16 energy levels.

We have established the connection between tunneling and chaotic-like behavior in Ref. 1. We have, moreover, shown that the behavior of the spatial entropy function is a good measure for the degree of complexity. It is well known that barrier penetration provides a possibility to study the approach to exponential decay.[6, 15, 18] It is striking that for short times before the exponential region is reached, oscillations occur in accordance with our description of the short-time behavior above. We have, finally established the link between a fast approach to exponential decay—and hence approach to irreversibility—of a broad state and of the fast entropy production. A broad state decays faster (enters the exponential mode sooner), has a sharper entropy increase, and shows more chaotic-like behavior than a narrower one. Taking into account that fast entropy production is an indicator of chaos we provide evidence for our conjecture that chaos and irreversibility are linked together. It also appears that decoherence is enhanced by an increased complexity of the system.

Although it does not appear that chaos is a sufficient condition for irreversibility it seems that it may be a necessary condition. It is, moreover, made plausible that the irreversible transition from a coherent quantum description to the observed decoherent classicality is mediated by the agency of quantum chaos.

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It appears from the reference list that this study is based on the theory for decaying systems which to a large extent has been developed by Prof. L. P. Horwitz. Knowing that it is only one of many fields where Larry has contributed profoundly, we are indebted to him for having introduced us to this branch of physics as well as having stimulated us and participated with us in the study of decaying systems, chaos, and irreversibility. The inspiration and insight we have gained from our cooperation with Larry has been as invaluable as his friendship.

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