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Decision Aiding

A multi-period game theoretic model of venture capitalists and entrepreneurs

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Abstract

This study examines the relationship between a venture capitalist and an entrepreneur and follows it from its inception to the exit stage. The model we use is a multi-period game theoretic model with moral hazard where the contract is set in the first period. The contribution of the study lies in the insights it provides on optimal contracts and its characterization of an endogenous exit point. Specifically, the paper shows that the optimal incentive scheme should backload all incentive payments to the entrepreneur as much as possible. Consequently, a straight debt contract would be optimal in venture financing.

Keywords: Finance; Game theory; Venture capitalists; Entrepreneurs; Optimal contracts

1. Introduction

Venture capitalists (VCs) play a major role in North America and elsewhere in financing start-ups. Venture capital firms have backed in the past both high-tech firms and non-hi-tech firms. VCs have provided financing in the early stages to companies such as Apple, DEC, Federal Express, Genentech, Intel, Lotus, Microsoft, Staples, TCBY and Teledyne. In essence, most of the money raised from VCs is used to build the infrastructure required to grow the business in terms of manufacturing, marketing and sales.

Gompers and Lerner (1999) provide an excellent discussion on the VC industry and its history. Kaplan and Stromberg (2001) provide interesting insights into how the VC industry operates. The paper compares real world contracts in the VC industry with the theory of financial contracting.

Virtually all of the studies in the area view venture capital as a short-term source of financing. VCs aim to exit the firm once it reaches sufficient size and credibility so it can cash out on their investment. Financing

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from VCs comes usually in several rounds starting from seed financing and culminating with exit or cash out (please see, for example, Amit et al. (1997), Gompers and Lerner (1999) and Sahlman (1990) for discussions on the various stages of VC investing). Neher (1999) argues that financing the venture upfront is efficient but infeasible because of the entrepreneur's inability to commit to refraining from negotiating down the investor's claim once he has sunk his investment. Consequently, stage financing helps mitigate this problem because the early rounds of financing create collateral to support the later rounds.

Several studies examine the asymmetric information aspects of the venture capitalist versus other investors (notably, Chan (1983) Admati and Pfleiderer (1994), Bergemann and Hege (1998) and Neher (1999)). Cooper and Carleton (1979) examine the issue of optimal continuation decisions in a setting where the VC is the only source of funding for the project and there is no asymmetry of information between the parties. Chan (1983) examines ways to improve the efficiency of allocations through using financial intermediaries to overcome informational asymmetry between entrepreneurs and investors when investors have positive searching costs. Admati and Pfleiderer (1994) derive robust financial contracts in situations where VCs possess the same information as entrepreneurs and are better informed than other investors. [1]

Bergemann and Hege (1998) argue that both the expected share of the entrepreneur in the firm and the expected return of the VC decrease over time. Another result of their paper is that the VC should keep a "hard" (but not exclusively hard) claim in case of failure.

A recent paper by Elitzur and Gavious (2001) has studied optimal contracts between VCs and entrepreneurs when contracts are renegotiated each period.

Ravid and Spiegel (1997) examine the equilibrium financial structure for a start-up with unlimited operating discretion. Furthermore, the authors provide predictions about the "underpricing" of securities to outside investors.

The problem that we examine here deals mainly with the moral hazard aspect of the relationship that, in essence, starts after the adverse selection problem has been resolved.

This study examines the relationship between a VC and an entrepreneur in a multi-period game where the contract is set in the beginning of the game. As such, we follow the relationship between a VC and an entrepreneur from its inception to the exit stage. The model we use to model this relationship is a multi-period game theoretic model with moral hazard.

One of the important characteristics of VCs is their ability to reduce informational asymmetries. These information asymmetries stem from two sources: "hidden information" and "hidden action". Hidden information leads to adverse selection, i.e., the inability of an investor to distinguish among high and low quality entrepreneurs. "Hidden action" leads to a moral hazard problem, which is essentially the inability of an investor to judge the effort level and quality of the entrepreneur's decisions. To alleviate the moral hazard problem VCs prevalently use staged investments. This study focuses on the problem of moral hazard, i.e., the problem of the entrepreneur's hidden effort and the ways that VCs can cope with it, using staged investments. The effort in this study affects the probability of success of the firm.

Our study examines a game between the VC and the entrepreneur where the effort of the entrepreneur is unobservable by the VC. Further, this paper deals with a multi-period game between the parties when the contract is set in the beginning of the multi-period game rather than repeated contracts or a single period contract. This study captures the nested character of staged financing without starting from stylized facts on the staging except of its existence. The contribution of the study lies in the insights it provides on optimal contracts and its characterization of an endogenous exit point. Specifically, the paper shows that the optimal incentive scheme should backload all incentive payments to the entrepreneur as much as possible. Consequently, a straight debt contract would be optimal in venture financing.

[1] Admati and Pfleiderer (1994) define robust contracts as those which withstand an addition of another state of nature.

The paper is organized as follows. Section 2 presents the model and the setting that the study uses. Section 3 outlines the results obtained in this study. Section 4 provides a numerical example. Section 5 concludes.

2. The model

Our model involves a start-up company with two risk neutral players: a VC and an entrepreneur. The game has K stages where K is common knowledge. [2] It is assumed that the contract between the two players cannot be canceled for the duration of the game, i.e., until K, unless the firm fails. It is interesting to examine whether the VC exits and when would this exit occur. [3] In each stage, k, of the game the entrepreneur chooses effort level, $e_k \geqslant 0$. We assume that interest is zero. The assumption of zero interest is made because incorporating a positive interest rate would make the model more cumbersome and, as we found out, does not add any insights. The VC cannot observe or infer e_k but only whether the stage was a success or a failure. e_k, in turn, affects the probability of success in this stage, $\alpha_k(e_k)$. [4] The following assumptions are made with respect to α_k:

$$1 \geqslant \alpha_k(e_k) \geqslant 0, \;\; \alpha'_k(e_k) > 0, \;\; \alpha''_k(e_k) < 0, \quad k = 1, \ldots, K.$$

The above assumptions state that the probability of success, $\alpha_k(e_k)$,
(1) is between 0 and 1,
(2) increases in effort,
(3) has diminishing marginal returns to effort.
If the outcome of stage k is a success the value of the company increases by B_k and continues to the next stage, also the VC awards the entrepreneur $S_k \geqslant 0$ (this award is, of course, a decision variable of the VC). [5] B_k is assumed to be common knowledge. B_k is, in essence, the expected value increment of the VC's holdings in the firm, net of his investment in this period, $B_k = W_k - I_k$, where W_k represents the value increment that accrues if stage k is successfully completed, and I_k is the investment made by the VC in period k. Thus, B_k can take any sign. Note that I_k in our model is not a decision variable for the VC but is based on the financing needs of the project at each stage. From this characterization it follows that the final value of the company if all stages were completed is $\sum_{k=1}^{K} [W_k - I_k] = \sum_{k=1}^{K} B_k$. [6] Let V_k be the overall expected payoff of the VC from period k until K. Similarly, define U_k as the overall expected payoff of the entrepreneur from period k until K.

We assume that $B_k + V_{k+1} > 0$, for every k, i.e., that the VC won't invest in the company unless he believes that his total expected payoff from the investment is positive at each stage k. We also assume that $V_k \geqslant 0$ for every k, in other words that the VC's total expected payoff in each period k, from current and future periods, cannot be a loss. This assumption is, in essence, a limitation of our analysis as it assumes

[2] As studies on VCs indicate, staged investments are quite prevalent in the VC industry. Furthermore, both Sahlman (1990) and Neher (1999) argue that staged investment is the most potent control mechanism used by VCs, mostly to alleviate the adverse selection problem.

[3] From the discussion above it follows that K is not uniform among firms because each stage may take several rounds or may be skipped altogether.

[4] The inclusion of an endogenous probability of success, α_k, and, in, turn, probability of failure, $1 - \alpha_k$, is especially appropriate for hi-tech firms. Research and development in hi-tech firms usually involve high risks and high stakes. If research and development efforts are not successful the firm in most cases ceases to exist.

[5] From our assumptions it follows that the entrepreneur cannot be dismissed until the end of the stage and only if it is a failure.

[6] Based on this formulation it does not matter if immediate payoffs are earned right away or not. It is sufficient to have a final payoff, which is an increasing function of the number of successful stages.

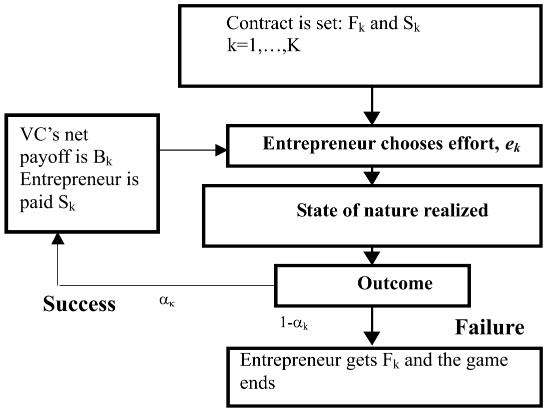

Fig. 1. The Time Line.

that the VC must be breaking even in every subgame. This assumption however is not an unreasonable one because suppose that there is some room for contest between the two parties about success or failure in each stage. Then if individual rationality does not hold in every stage, the VC would claim failure, simply in order to get out of the contract if his expected payoff is negative.

Another assumption is that until a certain point of the game B_k is negative, i.e., the investment by the VC exceeds his gross benefits, and beyond this point B_k is positive, in other words, the benefits to the VC exceed his investment. [7] This assumption makes sense because if B_k is not expected to become positive at a certain stage then no VC will enter such a relationship. If, on the other hand, the outcome is a failure the VC awards the entrepreneur $F_k \geqslant 0$, (F_k is also a decision variable for the VC) and the game terminates. If every stage of the game is a success the game then ends at stage K.

The entrepreneur incurs a cost of effort $C(e_k) \geqslant 0$ at each stage $k = 1, \ldots, K$. It is assumed that $C'(e_k) > 0$, i.e., that the cost of effort for the entrepreneur increases in effort, and that $C''(e_k) > 0$, i.e., that the marginal cost of effort is increasing. [8] We assume that both functions $\alpha_k(\cdot)$ for $k = 1, \ldots, K$, and $C(\cdot)$ are common knowledge. [9]

The payments to the entrepreneur in each stage k, S_k, F_k, are set in advance in the beginning of the game and are binding until the end of the game (stage K, or a failure at stage $k < K$). In this case the entrepreneur is informed about the payments before the game begins. We set, without loss of generality, all fixed payments to the entrepreneurs to zero. [10]

The sequence of events in the game is depicted in Fig. 1.

According to this time line, the actions of the VC, except investment, occur in the beginning of the game, and then in each period the actions of only the entrepreneur and nature occur (including B_k). [11]

[7] One could argue that it does not matter whether there is such a point of payoff reversal as long as the total expected payoff is positive.

[8] Alternatively, we could redefine the notations in the model as follows: $c_k = C(e_k)$, $k = 1, 2, \ldots, K$, $\tilde{\alpha}_k(c_k) = \alpha_k(C^{-1}(c_k))$, $k = 1, 2, \ldots, K$, where C^{-1} is the inverse function of C. This would streamline the results but, unfortunately, some of the model's results would seem to be less intuitive.

[9] Note that we assume that the functions of α and C are known but not their realized values. This is the case because they depend on the entrepreneur's effort, which cannot be observed by the VC.

[10] The purpose of such fixed payments is to satisfy reservation payoff for the entrepreneur. Setting these payments to zero simplifies our model without causing any effect on the insights derived.

[11] As opposed to this study, in most finance models, the entrepreneur has all the bargaining power (captures all surplus), not the VC.

Following our above definitions of V_k and U_k and the above discussion we obtain the following recursive equations:

$$U_k = \alpha_k(e_k)[S_k + U_{k+1}] + [1 - \alpha_k(e_k)]F_k - C(e_k)$$

and

$$V_k = \alpha_k(e_k)[B_k - S_k + V_{k+1}] - [1 - \alpha_k(e_k)]F_k, \quad k = 1, \ldots, K.$$

Define $U_{K+1} = 0$ and $V_{K+1} = 0$.

The contract in our model is set in the beginning of the first period ($k = 1$) and, thus, the relevant payoffs are V_1 for the VC and U_1 for the entrepreneur. Note that the payoffs for $k = 1$ contain the payoffs for $k = 2$, which in turn contain the payoffs for $k = 3$, and so forth. The VC sets the trajectory of payments first followed by the entrepreneur who then reacts to it by setting the trajectory of effort over time. In essence, the VC is a Stackelberg leader who sets up the trajectory of F_k and S_k over time, taking into account the self-interested behavior of the entrepreneur.

The VC's program is as follows:

$$\begin{aligned} &\operatorname*{Max}_{S_k, F_k} \quad V_1 = \alpha_1(e_1)[B_1 - S_1] - [1 - \alpha_1(e_1)]F_1 + \sum_{k=2}^{K} \left\{ \prod_{j=1}^{k-1} \alpha_j(e_j) \right\} \langle \alpha_k(e_k)[B_k - S_k] - [1 - \alpha_k(e_k)]F_k \rangle \\ &\text{s.t.} \quad B_k + V_{k+1} > 0, \ \ V_k \geqslant 0, \ \ k = 1, 2, \ldots, K, \ \ F_k, S_k \geqslant 0. \end{aligned} \tag{1}$$

The entrepreneur solves the following recursive program:

$$\operatorname*{Max}_{e_k} \ U_k = \alpha_k(e_k)[S_k + U_{k+1}] + [1 - \alpha_k(e_k)]F_k - C(e_k), \quad e_k \geqslant 0, \ \ k = 1, \ldots, K. \tag{2}$$

3. Results

This section outlines the findings of our model. We start by solving for the trajectory over time of the optimal effort of the entrepreneur. Given the self-interested choice of effort by the entrepreneur, we proceed to obtain the VC's trajectory over time of the compensation paid to the entrepreneur each period. Next, we examine the expected value for the VC of the payments made to the entrepreneur. The next result involves the payment to the entrepreneur for failure. We conclude this section by analyzing the optimal exit period and total payoffs of the VC over time.

The solution for the entrepreneur's program in (2) above shows that optimal effort of the entrepreneur in each period, e_k^*, $k = 1, \ldots, K$, is set as a function of the period payment for success, the period payment for failure, and the entrepreneur's next period payoff. [12] This is described in the following proposition:

Proposition 1. *The optimal effort of the entrepreneur in each period*, e_k^*, $k = 1, \ldots, K$, *has the following structure*:

$$e_k^* = e_k^*(S_k - F_k + U_{k+1}) \quad k = 1, \ldots, K. \tag{3}$$

All proofs are relegated to Appendix A.

[12] It should be noted that solving the programs above cannot be accomplished with dynamic programming because the optimal effort over time, e_k's, are a function of future payments, F_k and S_k, and, hence, the problem cannot be reduced to a simpler separable recursive equation.

Proposition 1 is intuitively appealing because it shows that the entrepreneur is going to exert effort depending on the rewards for either success or failure and the future prospects.

Subject to the choice of optimal effort by the entrepreneur, the VC chooses an optimal trajectory of compensation schemes. This scheme will be set according to the following proposition:

Proposition 2. *The optimal trajectory of compensation over time satisfies the following condition*:

$$\alpha'_k(e_k)[S_k - F_k + U_{k+1}] = C'(e_k), \quad k = 1, \ldots, K. \tag{4}$$

Proposition 2 implies that the optimal compensation awarded each period by the VC to the entrepreneur, anticipating effort selection by the entrepreneur, should set the marginal expected benefits from e_k in current and future periods, $\alpha'_k(e_k)[S_k - F_k + U_{k+1}]$ equal to the entrepreneur's marginal cost of effort, $C'(e_k)$.

When analyzing the dynamical aspects of the trajectory of optimal compensation schemes we find that the marginal utility for the entrepreneur from payments decreases over time. This is formalized in the following proposition:

Proposition 3. *For every* S_k, F_k, $k = 1, \ldots, K$, *under the optimal trajectory of effort*,

$$\left.\frac{\partial U_1}{\partial S_k}\right|_{e_j = e_j*(S_k - F_k + U_{k+1})} > \left.\frac{\partial U_1}{\partial S_{k+1}}\right|_{e_j = e_j*(S_k - F_k + U_{k+1})}, \quad k = 1, \ldots, K-1.$$

Further analysis of the dynamical aspects of the program reveals that for the VC the marginal utility from payments to the entrepreneur increases over time. This is formalized in the following proposition.

Proposition 4. *The optimal compensation scheme satisfies the following conditions*: *For every* $j > k > 1$ *such that*, $S_j^* > 0$, $S_k^* > 0$,

$$\frac{\partial V_2}{\partial S_j} > \frac{\partial V_2}{\partial S_k}. \tag{5}$$

Note that Propositions 3 and 4 hold even though we assumed that interest rate is zero. These propositions indicate that the marginal utility from future payments to the entrepreneur diminishes over time for the entrepreneur and increases over time for the VC. The intuition behind Propositions 3 and 4 is that because the probability of getting to any period depends on the past periods probabilities, the VC would like to pay the entrepreneur later on and the entrepreneur, in turn, would like to be paid as early as possible.

Analysis of the program shows that payment for failure has negative correlation with the effort of the entrepreneur and, in turn, has a negative effect on the VC's payoff. Thus, the VC pays the lowest possible payment for failure, zero. [13] This is formalized in the next proposition:

Proposition 5. *Not rewarding the entrepreneur for failure at each stage is an optimal strategy for the VC.*

Proposition 5 is consistent with, and complementary to, the work of Bergemann and Hege (1998) who argue that the optimal share contract should reward the entrepreneur only if he was successful.

Next we establish that there is an endogenous exit point for the VC, ξ and characterize it.

[13] More accurately, in case of failure, the entrepreneur would be paid the fixed payments, which we set to zero.

Theorem 1. *There is a point ξ, $1 \leqslant \xi \leqslant K$, such that*
1. $S_k^* = 0 \quad k < \xi$,
2. $S_k^* = B_k^* \quad k > \xi$,
3. $V_k^* = 0 \quad k > \xi$.

Theorem 1 states that up to a certain point, denoted here as ξ, the VC takes all profits (net of his investment in each stage which, in turn, is used to pay entrepreneurs as part of the R&D costs) and from this point and on the VC gets nothing. [14] In essence, Theorem 1 indicates that the optimal incentive scheme should backload all incentive payments to the entrepreneur as much as possible. The intuition for this result is that later success in this game stems from early success. Consequently, payments in latter stages of the game have incentive effects on all previous periods. It thus follows that backloading the compensation to the entrepreneur is the cheapest way to compensate him. Furthermore, this result indicates that a straight debt contract is optimal in this setting. [15] Suppose that the VC holds a debt claim, $D = \sum_{k=1}^{\xi} B_k$ that matures at K. If the project is liquidated before the VC gets everything up to this point including the proceeds of liquidation because of the Absolute Priority Rule. The entrepreneur is the only equity holder and, consequently, receives the entire residual claim from period ξ and on, $\sum_{k=1}^{K} B_k - D$. This result is also consistent with Innes (1990) and provides an alternative explanation why standard debt contracts arise in the presence of moral hazard. Innes (1990) utilizes strong assumptions on the distribution and the monotonicity of the pay-off schedule. Here, we derive the results without the use of these assumptions but rather in a particular structure of the project to be financed.

Next, we analyze the overall payoff over time from the relationship for the VC.

This discussion is formalized in the following theorem.

Theorem 2. *Let S_k^*, F_k^*, $k = 1, \dots, K$, be an optimal solution for the problem. Let ξ be a positive integer, such that $\xi < K - 1$, $e_\xi^* > 0$ and $S_\xi^* > 0$ and $S_{\xi-1}^* = 0$. Then*

$$\sum_{j=1}^{\xi-1} \left(\prod_{i=1}^{i=j} \alpha_i\left(e_i^*\right) \right) B_j \leqslant V_1 \leqslant \sum_{j=1}^{\xi} \left(\prod_{i=1}^{i=j} \alpha_i\left(e_i^*\right) \right) B_j. \tag{6}$$

Theorem 2 states that the overall payoff over time for the VC, V_1, is between the sum of expected net benefits from period 1 to ξ and the sum of expected net benefits from period 1 to $\xi - 1$. The intuition for this result stems from the previous theorem in which we showed that after the critical ξ the VC exits and, hence, the payoff for the VC takes account of the exit point. Put differently, VCs exit after they have reaped the maximum benefits that they could obtain from the company.

Note that Theorems 1 and 2 hold for any form of exit other than a write-off. A write-off exit is captured in our model by the possibility of failure in each stage. Other exits may include an IPO, a sale, or a company buy-back; all of these exits fit our model because of the generic nature of exits here.

This may shed some light on why companies that are backed by VCs usually do better than other companies in IPO situations (see Brav and Gompers, 1997). Theorem 2 implies that VCs who back firms have not exited yet and, thus, expect to gain some future benefits from the firm. Consequently, backing by VCs may serve as a signal on the promising future of the firm.

[14] Note that we set all fixed payments to the entrepreneur to zero.

[15] The authors wish to thank an anonymous reviewer for this intuition.

4. Numerical example

Assume $K = 3$ stages and that the probability of success in every stage as a function of effort is $\alpha_k(e_k) = \mathrm{MIN}(0.8e_k, 1)$. Although the probability is a non-smooth function of effort, we ignore this problem since it will not affect the results. In this example, as we show later on, the equilibrium effort satisfies $0.8e_k^* < 1$ and, thus, we can take the probabilities of success as $\alpha_k(e_k) = 0.8e_k$ for convenience of notations. Assume that the entrepreneur cost function at each stage is $C(e) = e^2$, thus his utility function is given by

$$u_k = 0.8e_k(S_k - F_k + U_{k+1}) + F_k - e_k^2, \quad k = 1, 2, 3,$$
$$u_4 = 0.$$

Differentiating the utility function at each stage with respect to effort and finding the optimal effort yield

$$e_k^* = 0.4(S_k - F_k + U_{k+1}), \quad k = 1, 2, 3.$$

As Proposition 1 shows, optimal effort is decreasing in compensation for failure. By Proposition 5 we have $F_k^* = 0$, $k = 1, 2, 3$. Thus, the optimal efforts by the entrepreneur are

$$e_1^* = 0.4S_1 + 0.064\left(S_2 + 0.16S_3^2\right)^2,$$
$$e_2^* = 0.4S_2 + 0.064S_3^2,$$
$$e_3^* = 0.4S_3.$$

The VCs utility function is given by

$$v_1 = 0.8e_1^*(B_1 - S_1) + 0.64e_1^*e_2^*(B_2 - S_2) + 0.512e_1^*e_2^*e_3^*(B_3 - S_3).$$

Assume incomes of $B_1 = 1$, $B_2 = 2$, $B_3 = 3$. Finding numerically the optimal compensations yields

$$S_1^* = 0, \quad S_2^* = 1.646, \quad S_3^* = 3.$$

Furthermore, the probabilities of successes are $\alpha_1(e_1^*) = 0.487$, $\alpha_2(e_2^*) = 0.987$, $\alpha_3(e_3^*) = 0.96$.

As Theorem 1 predicts, we have $\xi = 2$, where $S_1^* = 0$ and $S_3^* = B_3$ while $0 < S_2^* < B_2$. The expected utility of the VC is $v_1 = 0.658$ which satisfies the results of Theorem 2 namely,

$$\alpha_1(e_1^*)B_1 = 0.487 \leqslant v_1 \leqslant \alpha_1(e_1^*)B_1 + \alpha_2(e_2^*)B_2 = 0.836.$$

5. Summary

This paper examines in multi-period game the relationship between a VC and an entrepreneur. The model that we use is novel in that it uses a multi-period game with moral hazard where the contract is set in the beginning of the multi-period game. The multi-period aspects of the model let us derive the strategic behavior of the VCs and entrepreneurs over time. Further, our model is consistent with reality where the average duration of the relationship between venture capitalists is several years and the investment is made in stages. The contribution of the study lies in the insights it provides on optimal contracts and its characterization of an endogenous exit point. Specifically, the paper shows that the optimal incentive scheme should backload all incentive payments to the entrepreneur as much as possible. Consequently, a straight debt contract would be optimal in venture financing.

A possible extension to the paper could focus on the contracts among VCs who syndicate together. We ignored in this study altogether syndication by VCs and treated them essentially as one VC. Another extension to the paper could deal with the bargaining between the VC and entrepreneur, a facet that we ignored altogether in this study.

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Appendix A

Proof of Proposition 1. The first order condition for the entrepreneur, with respect to effort, is as follows (recall that the entrepreneur chooses the effort level e_k at every stage k):

$$\frac{\mathrm{d}U_k}{\mathrm{d}e_k} = \alpha_k'(e_k)[S_k + U_{k+1}] - \alpha_k'(e_k)F_k - C'(e_k) = 0, \quad k = 1, \ldots, K. \tag{A.1}$$

If Eq. (A.1) is binding, then we have

$$\alpha_k'(e_k)[S_k - F_k + U_{k+1}] = C'(e_k). \tag{A.2}$$

Thus, the optimal effort e_k^* is of the form

$$e_k^* = \begin{cases} e_k^*(S_k - F_k + U_{k+1}), & S_k - F_k + U_{k+1} \geqslant \frac{C'(0)}{\alpha_k'(0)}, \\ 0, & \text{otherwise.} \end{cases} \qquad \square \tag{A.3}$$

Proof of Proposition 2. See Eq. (A.2) above. □

To prove Proposition 3 we need the following lemma.

Lemma A.1. *Given the optimal efforts e_k^*'s, U_m satisfies the following condition for every m, k, $k = 1, \ldots, K$, $m \leqslant k$:*

$$\frac{\partial U_m}{\partial S_k} = \prod_{j=m}^{k} \alpha_j(e_j^*) \tag{A.4}$$

and

$$\frac{\partial U_m}{\partial F_k} = (1 - \alpha_k(e_k^*)) \prod_{j=m}^{k-1} \alpha_j(e_j^*), \tag{A.5}$$

where we define

$$\prod_{j=k}^{j=k-1} \alpha_j(e_j^*) \equiv 1.$$

Proof. Note that for every j and k such that $j < k$,

$$\frac{\partial e_j^*}{\partial S_k} = e_j^{*'}(S_j - F_j + U_{j+1}) \frac{\partial U_{j+1}}{\partial S_k}, \tag{A.6}$$

where

$$e_j^{*'}(S_j - F_j + U_{j+1}) = \left. \frac{\mathrm{d}e_j^*(x)}{\mathrm{d}x} \right|_{x = S_j - F_j + U_{j+1}}$$

(hereafter, we will write $e_j^{*'}$ instead of $e_j^{*'}(S_j - F_j + U_{j+1})$). Thus

$$\frac{\partial U_m}{\partial S_k} = \alpha_m'(e_m^*)e_m^{*'}\frac{\partial U_2}{\partial S_k}[S_k + U_2] + \alpha_m(e_m^*)\frac{\partial U_{m+1}}{\partial S_k} - \alpha_m'(e_m^*)e_m^{*'}\frac{\partial U_{m+1}}{\partial S_k}F_m - C'(e_m^*)e_m^{*'}\frac{\partial U_{m+1}}{\partial S_k}$$
$$= \frac{\partial U_{m+1}}{\partial S_k}e_m^{*'}[\alpha_m'(e_m^*)(S_m - F_m + U_{m+1}) - C'(e_m^*)] + \alpha_m(e_m^*)\frac{\partial U_2}{\partial S_k}.$$

If the parameters are such that

$$S_m - F_m + U_{m+1} < \frac{C'(0)}{\alpha_m'(0)},$$

then $e_m^* = 0$ and thus $e_m^{*'} = 0$. In this case we obtain the following condition:

$$\frac{\partial U_m}{\partial S_k} = \alpha_m(e_m^*)\frac{\partial U_{m+1}}{\partial S_k}. \tag{A.7}$$

Else, $\alpha_m'(e_m^*)(S_m - F_m + U_{m+1}) - C'(e_m^*) = 0$ is the first order condition for the entrepreneur and, thus, we have the same result as in Eq. (A.7). Repeating the same process for $j < k$ yields

$$\frac{\partial U_j}{\partial S_k} = \alpha_j(e_j^*)\frac{\partial U_{j+1}}{\partial S_k}. \tag{A.8}$$

Let us find $(\partial U_k)/(\partial S_k)$ for every k, $k = 1, \dots, K$.

$$\frac{\partial U_k}{\partial S_k} = \alpha_k'(e_k^*)e_k^{*'}[S_k - F_k + U_{k+1}] - C'(e_k^*)e_k^{*'} + \alpha_k(e_k^*) = \alpha_k(e_k^*).$$

Note that the last equation follows, again, from the first order condition of the entrepreneur. Applying the above equation yields the following:

$$\frac{\partial U_m}{\partial S_k} = \prod_{j=m}^{k}\alpha_j(e_j^*). \tag{A.9}$$

To prove the second part we should follow the same process where the only difference is that $\partial U_k/\partial F_k = 1 - \alpha_k(e_k^*)$. □

Proof of Proposition 3. From Lemma A.1

$$\frac{(\partial U_1)/(\partial S_k)}{(\partial U_1)/(\partial S_{k+1})} = \frac{\prod_{j=1}^{k}\alpha_j(e_j^*)}{\prod_{j=1}^{k+1}\alpha_j(e_j^*)} = \frac{1}{\alpha_{k+1}(e_{k+1}^*)} > 1. \quad □ \tag{A.10}$$

Proof of Proposition 4. If $S_k^* > 0$, then the VC's first order condition satisfies the following condition:

$$\left.\frac{\partial V_1}{\partial S_k}\right|_{S_i^*,F_i^*,i=1,\dots,K} = \alpha_1'(e_1^*)e_1^{*'}\frac{\partial U_2}{\partial S_k}[B_1 - S_1^* + F_1^* + V_2] + \alpha_1(e_1^*)\frac{\partial V_2}{\partial S_k} = 0. \tag{A.11}$$

Substituting $\partial U_2/\partial S_k = \prod_{j=2}^{k}\alpha_j(e_j^*)$ (from Lemma A.1) and rearranging yields

$$\left.\frac{\partial V_2}{\partial S_k}\right|_{S_i^*,F_i^*,i=1,\dots,K} = -\frac{1}{\alpha_1(e_1^*)}\alpha_1'(e_1^*)e_1^{*'}\left(\prod_{j=2}^{k}\alpha_j(e_j^*)\right)[B_1 - S_1^* + F_1^* + V_2] < 0. \tag{A.12}$$

(Note that the last inequality follows the condition that $B_1 - S_1^* + F_1^* + V_2 > 0$, otherwise the VC expected payoff V_1 is not positive.) Thus, if $S_k^* > 0$ and $S_j^* > 0$, $j > k$, then

$$\left.\frac{\partial V_2}{\partial S_j}\right|_{S_i^*,F_i^*,i=1,\dots,K} = \left(\prod_{i=k+1}^{j}\alpha_i(e_i^*)\right)\left.\frac{\partial V_2}{\partial S_k}\right|_{S_i^*,F_i^*,i=1,\dots,K} > \left.\frac{\partial V_2}{\partial S_k}\right|_{S_i^*,F_i^*,i=1,\dots,K}. \qquad \square \tag{A.13}$$

Proof of Proposition 5. To prove Proposition 5 we need the following lemmas.

Lemma A.2. *For every* k, $k = 1,\dots,K-1$

$$\frac{\partial V_1}{\partial S_{k+1}} = \frac{\partial V_1}{\partial S_k}\alpha_{k+1}(e_{k+1}^*) + \left(\prod_{j=1}^{k}\alpha_j(e_j^*)\right)\alpha_{k+1}'(e_{k+1}^*)e_{k+1}^{*\prime}[B_{k+1} - S_{k+1} + F_{k+1} + V_{k+2}]. \tag{A.14}$$

Proof. For $k > 1$,

$$\frac{\partial V_1}{\partial S_k} = \alpha_1'(e_1^*)e_1^{*\prime}\frac{\partial U_2}{\partial S_k}[B_1 - S_1 + F_1 + V_2] + \alpha_1(e_1^*)\frac{\partial V_2}{\partial S_k}. \tag{A.15}$$

For every m and k such that $k > m$,

$$\frac{\partial V_m}{\partial S_k} = \alpha_m'(e_m^*)e_m^{*\prime}\frac{\partial U_{m+1}}{\partial S_k}[B_m - S_m + F_m + V_{m+1}] + \alpha_m(e_m^*)\frac{\partial V_{m+1}}{\partial S_k} \tag{A.16}$$

and

$$\frac{\partial V_k}{\partial S_k} = \alpha_k'(e_k^*)e_k^{*\prime}[B_k - S_k + F_k + V_{k+1}] - \alpha_k(e_k^*). \tag{A.17}$$

Substituting Eqs. (A.16) and (A.17) in Eq. (A.15), applying Lemma A.1 and rearranging yields

$$\frac{\partial V_1}{\partial S_k} = \left(\prod_{j=1}^{k}\alpha_j(e_j^*)\right)\sum_{i=1}^{k}\frac{1}{\alpha_i(e_i^*)}\alpha_i'(e_i^*)e_i^{*\prime}[B_i - S_i + F_i + V_{i+1}] - \prod_{j=1}^{k}\alpha_j(e_j^*). \tag{A.18}$$

From Eq. (A.18) we have the lemma. $\square$

Lemma A.3. *For every* k, $k = 2,\dots,K$,

$$\frac{\partial V_1}{\partial F_k} = (1 - \alpha_k(e_k^*))\frac{\partial V_1}{\partial S_{k-1}} - \left(\prod_{j=1}^{k-1}\alpha_j(e_j^*)\right)\alpha_k'(e_k^*)e_k^{*\prime}[B_k - S_k + F_k + V_{k+1}] \tag{A.19}$$

and

$$\frac{\partial V_1}{\partial F_k} = \frac{\partial V_1}{\partial S_{k-1}} - \frac{\partial V_1}{\partial S_k}. \tag{A.20}$$

Proof. We first find the recursive equation for $\partial V_1/\partial F_k$:

$$\frac{\partial V_1}{\partial F_k} = \alpha_1'(e_1^*)e_1^{*\prime}\frac{\partial U_2}{\partial F_k}[B_1 - S_1 + F_1 + V_2] + \alpha_1(e_1^*)\frac{\partial V_2}{\partial F_k}. \tag{A.21}$$

For every m and k such that $m < k$,

$$\frac{\partial V_m}{\partial F_k} = \alpha_m'(e_m^*)e_m^{*\prime}\frac{\partial U_{m+1}}{\partial F_k}[B_m - S_m + F_m + V_{m+1}] + \alpha_m(e_m^*)\frac{\partial V_{m+1}}{\partial F_k} \tag{A.22}$$

and for every k,

$$\frac{\partial V_k}{\partial F_k} = -\alpha_k'(e_k^*)e_k^{*'}[B_k - S_k + F_k + V_{k+1}] - (1 - \alpha_k(e_k^*)). \tag{A.23}$$

Substituting (A.22), (A.23) and Lemma A.1 in (A.21) yields

$$\frac{\partial V_1}{\partial F_k} = (1 - \alpha_k(e_k^*))\left(\prod_{j=1}^{k-1}\alpha_j(e_j^*)\right)\sum_{j=1}^{k-1}\frac{1}{\alpha_j(e_j^*)}\alpha_j'(e_j^*)e_j^{*'}[B_j - S_j + F_j + V_{j+1}] - (1 - \alpha_k(e_k^*))\prod_{j=1}^{k-1}\alpha_j(e_j^*)$$
$$- \left(\prod_{j=1}^{k-1}\alpha_j(e_j^*)\right)\alpha_k'(e_k^*)e_k^{*'}[B_k - S_k + F_k + V_{k+1}].$$

Substituting $\partial V_1/\partial S_{k-1}$ and then $\partial V_1/\partial S_k$ from previous lemma yields the lemma. □

Lemma A.4. *For every k, $k = 1, \ldots, K$, if $S_k^* = 0$ and $e_k^* > 0$ then $S_m^* = 0$, $m = 1, \ldots, k$.*

Proof. From Lemma A.2

$$\left.\frac{\partial V_1}{\partial S_k}\right|_{S_j^*, F_j^*, j=1,\ldots,K} = \left.\frac{\partial V_1}{\partial S_{k-1}}\right|_{S_k^*, F_k^*, j=1,\ldots,K}\alpha_k(e_k^*) + \left(\prod_{j=1}^{k-1}\alpha_j(e_j^*)\right)\alpha_k'(e_k^*)e_k^{*'}[B_k - S_k^* + F_k^* + V_{k+1}].$$

Since $S_k^* = 0$ is the optimal solution it follows that

$$\left.\frac{\partial V_1}{\partial S_k}\right|_{S_j^*, F_j^*, j=1,\ldots,K} \leqslant 0.$$

Substituting $S_k^* = 0$ into the above equation and using the constraint $B_k + V_{k+1} > 0$ and the assumption $e_k^* > 0$ we have

$$\left(\prod_{j=1}^{k-1}\alpha_j(e_j^*)\right)\alpha_k'(e_k^*)e_k^{*'}[B_k - S_k^* + F_k^* + V_{k+1}] > 0.$$

Thus,

$$\left.\frac{\partial V_1}{\partial S_{k-1}}\right|_{S_k^*, F_k^*} < 0$$

and we find that $S_{k-1}^* = 0$ since it is a corner solution. □

Lemma A.5. *Let S_k^*, F_k^*, $k = 1, \ldots, K$, be an optimal solution for the problem. For every k, $k = 1, \ldots, K-1$, if $e_k^* > 0$ and $S_k^* > 0$, then*
1. $S_m^* > 0, \quad m = k+1, \ldots, K.$
2. $S_m^* = B_m + F_m^* + V_{m+1}, \quad m = k+1, \ldots, K.$

Proof. Since S_k^* is interior point, from the first order condition for optimality

$$\left.\frac{\partial V_1}{\partial S_k}\right|_{S_j^*, F_j^*, j=1,\ldots,K} = 0.$$

From Lemma A.2

$$\left.\frac{\partial V_1}{\partial S_{k+1}}\right|_{S_j^*,F_j^*,j=1,\dots,K} = \left(\prod_{j=1}^{k}\alpha_j(e_j^*)\right)\alpha'_{k+1}(e^*_{k+1})e^{*'}_{k+1}[B_{k+1} - S^*_{k+1} + F^*_{k+1} + V_{k+2}].$$

Assume that $S^*_{k+1} = 0$, and given the constraints $B_{k+1} + V_{k+2} > 0$, $F_{k+1} \geqslant 0$, we find that

$$\left.\frac{\partial V_1}{\partial S_{k+1}}\right|_{S_j^*,F_j^*,j=1,\dots,K} > 0,$$

thus, $S_{k+1} = 0$ is not an optimal solution. It follows that $S^*_{k+1} > 0$ and that proves 1. From the first order condition for optimality,

$$\left.\frac{\partial V_1}{\partial S_{k+1}}\right|_{S_j^*,F_j^*,j=1,\dots,K} = 0,$$

we find that $B_k - S_k^* + F_k^* + V_{k+1} = 0$ and that proves part 2. □

We should prove that for every k, $F_k^* = 0$.

From Lemma A.3,

$$\frac{\partial V_1}{\partial F_k} = (1 - \alpha_k(e_k^*))\frac{\partial V_1}{\partial S_{k-1}} - \left(\prod_{j=1}^{k-1}\alpha_j(e_j^*)\right)\alpha'_k(e_k^*)e_k^{*'}[B_k - S_k + F_k + V_{k+1}].$$

There are two cases.

Case 1:

$$\left.\frac{\partial V_1}{\partial S_{k-1}}\right|_{S_j^*,F_j^*,j=1,\dots,K} = 0$$

(since $S^*_{k-1} > 0$). From Lemma A.5, $B_k - S_k^* + F_k^* + V_{k+1} = 0$. Thus,

$$V_k = \alpha_k(e_k^*)[B_k - S_k^* + F_k^* + V_{k+1}] - F_k = -F_k^*.$$

From the constraint $V_k \geqslant 0$ we find that $F_k^* = 0$.

Case 2:

$$\left.\frac{\partial V_1}{\partial S_{k-1}}\right|_{S_j^*,F_j^*,j=1,\dots,K} < 0$$

(since $S^*_{k-1} = 0$). From the constraint $V_k = \alpha_k(e_k^*)(B_k - S_k^* + F_k^* + V_{k+1}) - F_k^* \geqslant 0$, we find that $B_k - S_k^* + F_k^* + V_{k+1} \geqslant 0$, thus

$$\left.\frac{\partial V_1}{\partial F_k}\right|_{S_j^*,F_j^*,j=1,\dots,K} < 0$$

and, hence, $F_k^* = 0$. □

Proof of Theorem 1 and 2. To prove Theorems 1 and 2 we need the following lemmas.

Lemma A.6. *Let S_k^*, F_k^*, $k = 1,\dots,K$, be an optimal solution for the problem. For every ξ, $1 \leqslant \xi \leqslant K-1$, if $e_\xi^* > 0$ and $S_\xi^* > 0$ then*

1. $S_k^* = B_k \quad k = \xi + 1, \ldots, K.$
2. $V_k = 0 k = \xi + 1, \ldots, K.$

Proof. From Lemma A.5 and Proposition 5, $S_k^* = B_k + V_{k+1}$. Since $V_{K+1} = 0$ it follows that $S_K^* = B_K$. Substituting $S_K^* = B_K$ and $F_K^* = 0$ into V_K we find that $V_K = 0$. By induction from K to $\xi + 1$ we obtain the same results. □

Lemma A.7. *Let S_k^*, F_k^*, $k = 1, \ldots, K$, be an optimal solution for the problem. For every ξ, $1 \leqslant \xi \leqslant K - 1$, if $e_\xi^* > 0$ and $S_\xi^* > 0$ and $S_{\xi-1}^* = 0$ for the optimal solution, then*

$$V_1 = \sum_{j=1}^{\xi-1} \left(\prod_{i=1}^{i=j} \alpha_i(e_i^*) \right) B_j + \left(\prod_{i=1}^{i=\xi-1} \alpha_i(e_i^*) \right) \alpha_\xi(e_\xi^*)[B_\xi - S_\xi^*].$$

Proof. Substituting the results from Lemmas A.6, A.7 and Proposition 5 into V_1 yields the desired result. □

Proof of Theorem 1. Follows directly from Lemmas A.4 and A.6. □

Proof of Theorem 2. Follows directly from Lemma A.7. □

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