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O.R. Applications

Location of terror response facilities: A game between state and terrorist

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Abstract

We study a leader follower game with two players: a terrorist and a state where the later one installs facilities that provide support in case of a terrorist attack. While the Terrorist attacks one of the metropolitan areas to maximize his utility, the State, which acts as a leader, installs the facilities such that the metropolitan area attacked is the one that minimizes her disutility (i.e., minimizes 'loss'). We solve the problem efficiently for one facility and we formulate it as a mathematical programming problem for a general number of facilities. We demonstrate the problem via a case study of the 20 largest metropolitan areas in the United States.

Keywords: Location; Terror; Game theory

1. Introduction

Since September 11 and the anthrax attack via the US postal office, terror has become a major threat. The September 11 terrorist attack has proved that there are individuals who are willing to take part in a mass murder of civilians. The major threat today seems to be attack against nuclear reactors, water reservoirs (chemical or biological), bioterror and spreading radioactive materials. Although these attacks are not the same, they have the potential to kill a huge number of people in a short period of time. According to Wein et al. (2003), a terrorist attack that spreads one kilogram of anthrax in a major city may cause about

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60,000 casualties even if the health care system could react to the attack according to procedures of the Center for Disease Control and Prevention (CDC) (the number is obtained from Fig. 4 in their work).

There is a common approach in the literature such as Kaplan et al. (2003) and Kaplan and Wein (2003) and the response to the latter by Marennikova (2003). This approach is based on similar tools to those used in Epidemic Theory and Mathematical Biology. However, there is a significant difference between regular epidemics and the one caused by a terrorist. While diseases seem to spread according to some mathematical rules, the terrorists are individuals that can behave rationally trying to maximize the damage caused. As we have learned from the September 11 attack, terrorists indeed behave rationally. They can execute complicated long term plans that involve recruiting many people, training and finding solutions to logistical and technical problems.

The Chernobyl nuclear disaster, although not a terror attack, demonstrates the potential damage that can be caused by a terror attack against a nuclear reactor. The nuclear power accident which occurred at Chernobyl killed 41 people, has increased significantly the number of cancer cases and caused the relocation of more than 150,000 people that resided in the surrounding 20-mile radius of the nuclear reactor.

In this paper we address the problem of locating facilities that contain the resources required to respond to a terror attack. We present a worse case scenario where the terrorist is sufficiently sophisticated to learn the location of the facilities and to attack the weak spots of the system. We therefore have a game with two players: the State and the terrorist. Given this approach, we let the State take into consideration the behavior of the terrorist and plan the facility locations such that the terrorist will cause minimum damage. The terrorist knows the State's move in the game and act accordingly. Moreover, in our model the State assume that the terrorist knows her move in the game.

We note that competitive location models are discussed in the literature in a different context. These problems arise when we consider where to locate facilities with respect to other competing facilities. Here, the competitors (e.g., department stores) try to attract as many customers as possible. For more on this type of competitive location model, readers can refer to Chapter 10 in Mirchandani and Francis (1990).

The paper is organized as follows. In Section 2 we introduce the model. In Section 3 we discuss the model when the State locates a single facility. In Section 4 we extend the analysis to multiple facilities. In Section 5 we introduce a case study of a terror attack on the US where the State installs up to four facilities. We analyze in detail two scenarios.

2. The model

2.1. The setting

Consider a network $G(N, L)$ where N ($|N| = n$) is the set of nodes and L is the set of links. The nodes represent cities (or metropolitan areas). Each city i is associated with a weight w_i which represents the expected damage (e.g., number of people infected) in case of a terrorist attack on that city.[1,2] There exists a link (i, j) between any $i, j \in N$ that has a length denoted by $d(i, j)$ (the average shortest travel distance between i and j) representing the delay of shipment of resources from the facility due to link (i, j).

There are two players in the game, the State and the terrorist. The State acts first and decides where to locate the facilities that contain the resources required in preparation for a terrorist attack. In addition, the

[1] Alternatively, we may assume that w_i is the mean of a random variable that represents the distribution of the damage in case of a terrorist attack on that city.

[2] The values of the w_is depends on the type of the attack. Different damage would result from different attack and the allocation of facilities storing vaccines in case of bioterror attack could be different from the allocation of troops or sanitation equipment in case of polluting water supply.

State may invest some resources on prevention. The Terrorist plays second, after the facilities are located. Given the location of the resources in the network, the Terrorist decides where he is going to strike. We assume a worse case scenario that the Terrorist knows the location of the resources.

2.1.1. Strategies

A Terrorist's strategy t in the game is simple. He chooses a node (city) i.e., the set of Terrorist's strategies is actually N ($t \in N$). We let the Terrorist use a mixed strategy i.e., randomize over the cities. In this case a Terrorist strategy is a vector of probabilities $t = (p_1, p_2, \ldots, p_N)$ where p_i is the probability that the Terrorist will attack city i. Obviously, for all i, $p_i \geqslant 0$ and $\sum_{i=1}^{N} p_i = 1$. We assume that there is no dependency between the cities so that a strike on one city has no effect on other cities.[3] This assumption may not be always valid. In the case of an anthrax attack, for example, this assumption is valid. However, in the case of an epidemic generated by a terrorist attack, such as smallpox, this assumption implies that the authority will succeed to seal the infected area before it spreads out to other cities in the network.

The State decision is over the location of the K facilities in the network (nodes or links). Locating a facility incurs a fixed cost $C > 0$ which is the setup cost of installing the facility. In addition, the State determines the level of resources c_{prev} to invest in preventing an attack. These resources are invested outside the network and are concerned with intelligence and security. The amount of resources spent on prevention dictates the probability that a terrorist attack will succeed. Namely, once the Terrorist selects the city he is planning to attack, there is a probability of $P(c_{\text{prev}})$ that the attack will succeed where P is a continuous decreasing convex function, $P(0) > 0$ and $\lim_{c_{\text{prev}} \to \infty} P(c_{\text{prev}}) = 0$.[4] The State strategy is a $K + 1$ vector of the K facilities locations and the level of prevention resources, $s = (x_1, x_2, \ldots, x_K, c_{\text{prev}})$ where x_k is the location of facility k, $k = 1, 2, \ldots, K$.

2.1.2. Utilities

Once the Terrorist attacks a city i and the attack succeeds, all the resources in the network may be available for use by the city. The amount of resources needed at any city i is assumed to be equal to the expected damage w_i.[5] The resources are sent through the network in a shortest travel path from the closest facility to city i. We assume that the cost for the State (or disutility) is a linear function of the delay in transferring of resources to the city attacked. This delay is the sum of the average delay at the city attacked $d(i)$ and the average delay time $d(k, i)$ between city i and the facility. The cost or disutility in case of a successful terrorist attack on city i is

$$f_i = (\alpha D(k,i) + \eta) w_i = (\alpha d(k,i) + \eta_i) w_i, \tag{1}$$

where k is the closest facility to city i, $D(k,i) = d(k,i) + d(i)$ and $\eta_i = \eta + \alpha d(i)$. The positive parameter α represent the cost of delaying unit of resource due to a unit of distance. The last component ηw_i stands for the disutility of an attack on a city, independently of the lack of resources where $\eta \geqslant 0$. The State expected utility is given by

$$U_S(t,s) = -P(c_{\text{prev}}) \sum_{i=1}^{n} p_i f_i - KC - c_{\text{prev}}. \tag{2}$$

[3] We may assume that w_i includes any damage that occurs to other cities in case of dependency.

[4] A simplified model that ignores the possibility of prevention may be obtained by assuming that P is a positive constant, possibly equal to 1. In this case, there is no reason for the State to spend any resources on prevention and we may omit the prevention arguments as the decision variable c_{pre} will always be set to zero.

[5] The resources needed at node i is a monotone function of w_i. For simplicity we assume that for every unit of damage we need a unit of resources. Alternatively, we may say that the damage is the resources required to return the attacked city to normal.

Notice that the utility is separable with respect to the prevention resources and the number of facilities and their locations. However, recall that the value of $\sum_{i=1}^{n} p_i f_i$ depends on the locations and thus, the decision with respect to c_{prev} is associated with the decision over the facilities.

In the case of a successful attack, the Terrorist's benefit is the damage he generates. This damage is a function of the lack of resources. We assume that he benefits similarly to the way the State suffers, i.e., his utility is

$$e_i = (\gamma d(k,i) + \delta_i) w_i, \tag{3}$$

where $\delta_i = \delta + \alpha d(i)$ and the component $\delta_i w_i$ represents the benefit from attacking a city with expected damage w_i, which is independent of any resources the State allocates for defence against attacks. Obviously, if δ is very large relative to γ, then the Terrorist's dominant strategy is to attack the city with the largest expected damage w_i independently of any decision made by the State. The Terrorist expected utility is given by $P(c_{\text{prev}}) \sum_{i=1}^{n} p_i e_i$. However, since $P(c_{\text{prev}})$ is a constant it is irrelevant to the Terrorist decision. Thus, we redefine the Terrorist utility as

$$U_T(t,s) = \sum_{i=1}^{n} p_i e_i. \tag{4}$$

Observe that we have a non-zero-sum game although part of the utilities resembles a zero-sum game.

It is important to note that in our model we assume that the State is informed about the values of the Terrorist's parameters γ and δ_i. This assumption may be supported by the knowledge we have about the terrorists' behavior. However, in practice the State may not be informed about the Terrorist utility since different types of terrorist may have different utilities. In this case we may modify the model by incorporating distributions over those parameters. Adding this component to the model will essentially result in a similar analysis but with more complex setting.

2.1.3. Equilibrium

When the locations of facilities are known, we have a leader-follower game in the sense of Stackelberg competition (on Stackelberg equilibrium see for example Rasmusen, 2001, p. 83). Since the State acts first and plays her strategy s and the Terrorist plays after he becomes aware of the State actions, his strategy $t(s)$ is a function of the State action. An equilibrium in this game is a pair $(t^*(\cdot), s^*)$ such that for every s, $U_S(t^*(\cdot), s) \leqslant U_S(t^*(\cdot), s^*)$ and for every function $t(\cdot)$, $U_T(t(\cdot), s^*) \leqslant U_T(t^*(\cdot), s^*)$.[6,7]

Before we begin the analysis of the model we prove the following useful result.

Lemma 1. *For every decision made by the State there is always a pure strategy that is the best response of the terrorist.*

[6] Obviously, this solution is a subgame perfect Nash equilibrium.

[7] However, there may be technical problems with immediate usage of the equilibrium concept. The difficulty arises when the State can locate facilities at any point in the network. As we will show later, for some locations, say point x, the Terrorist is indifferent between attacking two cities while the State prefers that the Terrorist will attack a specific one of the two cities and thus, she locates the facility at either $x - \xi$ or $x + \xi$ for $\xi > 0$ arbitrarily small. Since $\xi > 0$, by reducing ξ, the State will increase her utility. In this case, an equilibrium does not exist and we need to use the terminology of ϵ equilibrium or ϵ sub game perfect Nash equilibrium. We avoid this technical issue and use the notation x^- and x^+ for, respectively, $x - \xi$ and $x + \xi$. Therefore, when we talk about equilibrium in the single facility problem (where we allow a continuous decision space for the State), we mean ϵ equilibrium in the sense that any deviation from the equilibrium will lead to no more than ϵ gain by the State (see Radner, 1980). Thus, for every $\epsilon > 0$ we can define $\xi > 0$ such that the State cannot gain more than ϵ by deviating from the equilibrium strategy. This problem does not exist in the analysis of the problem with multiple facilities since we consider a discrete and finite decision space for the State. For further discussion of this problems see Myerson (1991, Section 3.13).

Proof. The Terrorist problem is

$$\max_{0\leqslant p_i\leqslant 1} \quad U_T(t,s)$$
$$\text{s.t.} \quad \sum_{i=1}^{n} p_i = 1.$$

This is a linear programming problem with one constraint. We can rewrite the Terrorist utility function (4) as

$$U_T(t,s) = \sum_{i=1}^{n} p_i[\gamma d(k(i),i) + \delta_i]w_i,$$

where $k(i)$ is the closest facility to node i. An optimal solution for the problem is $p_{e^*} = 1$ and $p_j = 0$ for $j \neq e^*$ where

$$e^* = \arg\max_i [\gamma d(k(i),i) + \delta_i]w_i. \qquad \square \tag{5}$$

3. A single facility

Assume that the State is planning to install only one facility. The location of the facility is not limited and any point on G is a possible location.

The State utility is $U_S(t,s) = -P(c_{\text{prev}})(\sum_{i=1}^{n} p_i[\alpha d(x,i) + \eta_i]w_i) - C - c_{\text{prev}}$. We ignore the installation cost C since it is a fixed cost. In this case, we can separate the location problem from the problem of prevention and redefine the State problem as

$$\min_{x\in G} U_S(t,x) = \sum_{i=1}^{n} p_i[\alpha d(x,i) + \eta_i]w_i$$

whereas the Terrorist problem given the location at x is

$$\max_{0\leqslant p_i\leqslant 1} U_T(t,x) = \sum_{i=1}^{n} p_i[\gamma d(x,i) + \delta_i]w_i.$$

Assume that x is a point on link $[a,b]$ at a distance of x from node a. Observe that for any node v

$$d(x,v) \stackrel{\text{def}}{=} \min\{x + d(a,v), l - x + d(b,v)\}.$$

Let $e^*(x)$ be the Terrorist's best response strategy against any x on (a,b). From Lemma 1,

$$e^*(x) = \arg\max_v \{\gamma d(x,v) + \delta_v\}w_v.$$

The function $UE(x) = \max_v\{\gamma d(x,v) + \delta_v\}w_v$ is the upper envelope of linear and piecewise linear concave functions and obviously, it is neither concave nor convex function of x. Define c_{ij} as a local center of $UE(x)$ if

$$\max_v \{\gamma d(c_{ij},v) + \delta_v\}w_v = \{\gamma d(c_{ij},i) + \delta_i\}w_i = \{\gamma d(c_{ij},j) + \delta_j\}w_j.$$

We note that the concept of a local center we use is similar to the concept of a local center in the context of the minimax problem (Chapter 7 in Mirchandani and Francis (1990)). Obviously at c_{ij}, $(\gamma d(c_{ij},i) + \delta_i)w_i = (\gamma d(c_{ij},j) + \delta_j)w_j$. Let $\mathscr{C}(a,b)$ be the set of all local centers on link $[a,b]$ including the points 0 and l which correspond to nodes a and b, respectively. We assume that the local centers are ordered such that $\mathscr{C}(a,b) = \{x_1, x_2, \ldots, x_t\}$ where $x_1 = 0 < x_2 < \cdots < x_t = l$. The next two lemmas are straightforward.

Lemma 2. *If for some r such that* $x_r, x_{r+1} \in \mathscr{C}(a,b)$, $x_r = c_{ij}$ *and* $x_{r+1} = c_{jk}$ *for some i, j, k then, for any* $x \in (x_r, x_{r+1})$, $e^*(x) = j$; $e^*(x_r) = i$ *or j and* $e^*(x_{r+1}) = j$ *or k.*

Lemma 3. *An optimal solution to the State problem on link* $[a,b]$ *is included in* $\mathscr{C}(a,b)$.

Consider the segment $[x_r, x_{r+1}]$ where $x_r = c_{ij}$ and $x_{r+1} = c_{jk}$. If the Terrorist attacks city j, the State considers the problem

$$\min_{x\in[x_r,x_{r+1}]} \alpha d(x,j)w_j + \eta_j w_j.$$

Since $d(x,j) = \min\{x + d(a,j), l - x + d(b,j)\}$, the minimum is achieved at either x_r or at x_{r+1}. Suppose that it is attained at x_r. From Lemma 2, $e^*(x_r^-) = i$ and $e^*(x_r^+) = j$. Therefore, in $[x_r, x_{r+1}]$ the optimal solution is

$$\arg\min\{\alpha d(x_r^-, i)w_i + \eta_i w_i, \alpha d(x_r^+, j)w_j + \eta_j w_j\}.$$

Denote by v_r^- and v_r^+ the two nodes that define the local center x_r (i.e., if $x_r = c_{ij}$ then, $v_r^- = i$ and $v_r^+ = j$). Let v_1^+ be the node whose distance from x_1^+ is a linear piece of $UE(x)$ and let v_t^- be the node whose distance from x_t^- is a linear piece of $UE(x)$. The optimal solution on link $[a,b]$ is

$$\arg\min\Bigg\{\left[\alpha d(x_1^+, v_1^+) + \eta_{v_1^+}\right]w_{v_1^+}, \left[\alpha d(x_t^-, v_t^-) + \eta_{v_1^-}\right]w_{v_t^-},$$
$$\min_{r=2,\dots,t-1}\left(\left[\alpha d(x_r^-, v_r^-) + \eta_{v_r^-}\right]w_{v_r^-}, \left[\alpha d(x_r^+, v_r^+) + \eta_{v_r^+}\right]w_{v_r^+}\right)\Bigg\}. \tag{6}$$

The optimal solution on the entire network is the best solution obtained when applying (6) to all links.

Example 1. Consider the following 3-node network in Fig. 1 where the numbers near the links are lengths and the numbers next to the nodes are weights. Suppose $\alpha = 5$, $\gamma = 1$,$d(i) = 0$, $\eta_i = \delta_i = 1$ so that $\eta_i = \eta$ and $\delta_i = \delta$ $i = 1,2,3$.

Let link $[a,b] = [2,3]$ and $x \in (2,3)$. Then,

$$(\gamma d(x,1) + \delta)w_1 = [\min\{x+3, 9-x\} + 1]15 = \begin{cases} 15x + 60; & 0 \leqslant x \leqslant 3, \\ -15x + 150; & 3 \leqslant x \leqslant 5, \end{cases}$$
$$(\gamma d(x,2) + \delta)w_2 = 20x + 20; \quad 0 \leqslant x \leqslant 5,$$
$$(\gamma d(x,3) + \delta)w_3 = -11x + 66; \quad 0 \leqslant x \leqslant 5.$$

In Fig. 2 we plot $(\gamma d(x,i) + \delta)w_i$ for $i = 1,2,3$ and $0 \leqslant x \leqslant 5$.

The function $\max_i\{\gamma d(x,i) + \delta_i\}w_i$ is the upper envelope (bold). The local centers are $c_{13} = 0.2308$ (the intersection of $15x + 60$ and $-11x + 66$) and $c_{12} = 3.7143$ (the intersection of $-15x + 150$ and $20x + 20$).

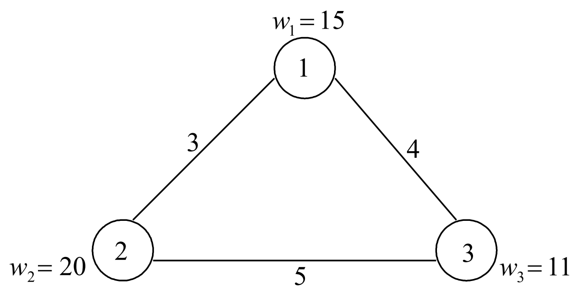

Fig. 1. The network in Example 1.

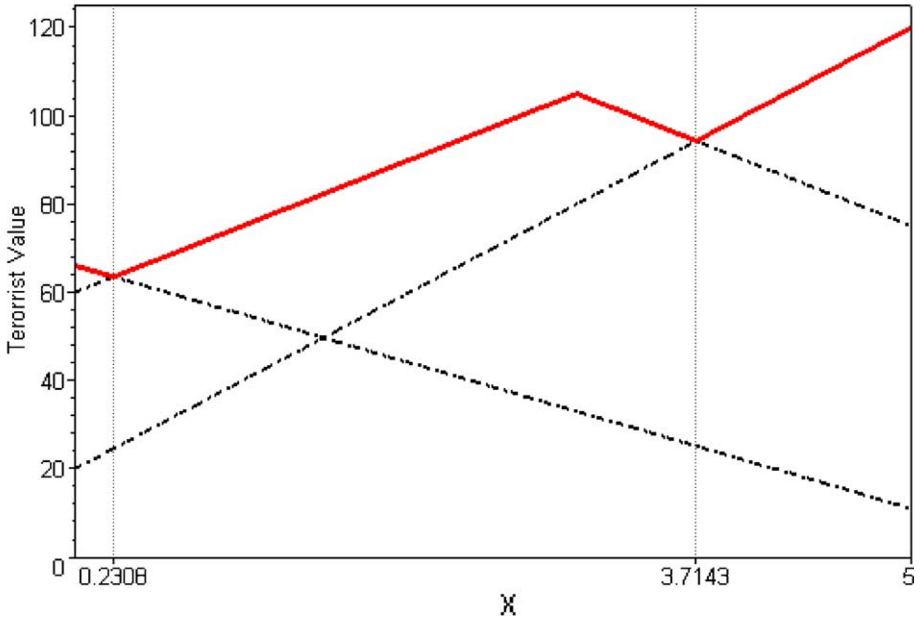

Fig. 2. Terrorist's utility as a function of the location on link [2, 3].

Therefore, $x_1 = 0$, $x_2 = 0.2308$, $x_3 = 3.7143$ and $x_4 = 5$. Since $v_1^+ = 3$, $v_2^- = 3$, $v_2^+ = 1$, $v_3^- = 1$, $v_3^+ = 2$ and $v_4^- = 2$, the solution of (6) is at $x_2^+ = 0.2308$ with an objective function value of 257.33.

It is interesting to consider the function that the State is minimizing over the link [2, 3]. Then

$$(\alpha d(x,1)+\eta)w_1 = [5\min\{x+3, 9-x\}+1]15 = \begin{cases} 75x+240; & 0 \leqslant x \leqslant 3, \\ -75x+690; & 3 \leqslant x \leqslant 5, \end{cases}$$

$$(\alpha d(x,2)+\eta)w_2 = 100x+20, \quad 0 \leqslant x \leqslant 5,$$

$$(\alpha d(x,3)+\eta)w_3 = -55x+286, \quad 3 \leqslant x \leqslant 5.$$

In Fig. 3 we plot $(\alpha d(x,i)+\eta)w_i$ for $i = 1, 2, 3$ and $0 \leqslant x \leqslant 5$.

Recall that the Terrorist's best response strategy over link [2, 3] is

$$e^*(x) = \begin{cases} 3; & 0 \leqslant x \leqslant 0.2308, \\ 1; & 0.2308 \leqslant x \leqslant 3.7145, \\ 2; & 3.7143 \leqslant x \leqslant 5. \end{cases}$$

The function the State minimizes is the bold curve. It has discontinuity at the two local centers of the Terrorist $x = 0.2308$ and $x = 3.7143$. From the figure, the best solution for the State on link [2, 3] is at $x_2^+ = 0.2308$ with utility value of 257.33 (as obtained by (6) earlier).

For link [1, 3], $\mathscr{C}(1,3) = \{0, 4\}$, $x_1 = 0$, $x_2 = 4$, $v_1^+ = v_2^- = 2$ and the solution of (6) is at $x_1 = 0$ with utility value of 320.

For link [1, 2], $\mathscr{C}(1,2) = \{0, 0.806, 3\}$, $x_1 = 0$, $x_2 = 0.806$, $x_3 = 3$, $v_1^+ = v_2^- = 2$, $v_2^+ = v_3^- = 3$. The solution of (6) is $x_2^- = 0.806$ with an objective function value of 239.14 and therefore it is the solution over the entire network.

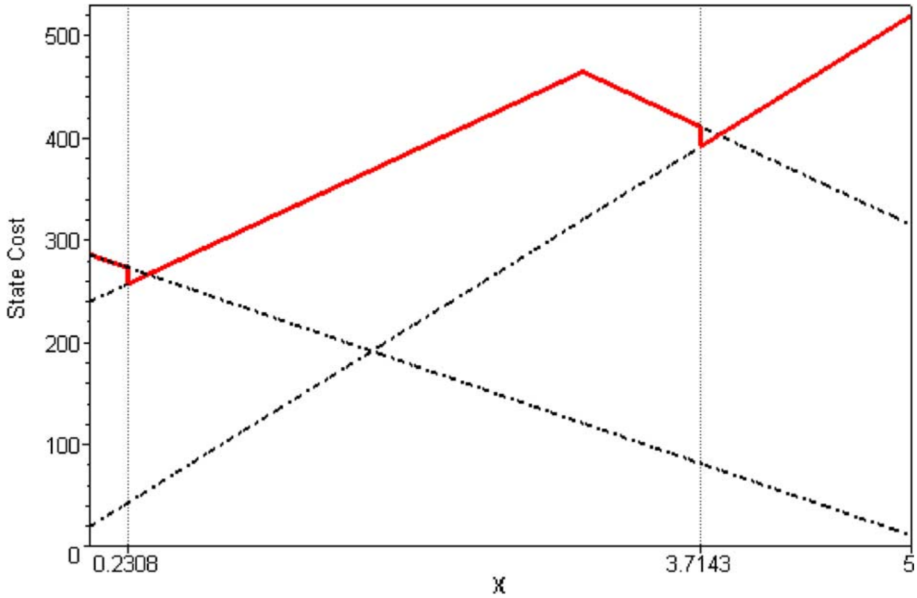

Fig. 3. Then State's disutility as a function of the location on link [2, 3].

3.1. Facility on nodes

Assume that the State is limited to install facilities only at a finite set of potential locations. Without any loss of generality we can assume that it is the set of nodes of the network. Following Lemma 1, the Terrorist best response against any node k chosen by the State is

$$e^*(k) = \max_{i \in N}(\gamma d(k,i) + \delta_i)w_i.$$

The State problem is

$$k^* = \arg\min_{k \in N}\left\{U_S(e^*(k),k) = (\alpha d(k,e^*(k)) + \eta_{e^*(k)})w_{e^*(k)}\right\}. \tag{7}$$

For the example above we have

$$e^*(k) = \begin{cases} 2 & k = 1, \\ 3 & k = 2, \\ 2 & k = 3. \end{cases}$$

Therefore, according to (7), $k^* = 2$.

3.2. Prevention cost

When the equilibrium solution with respect to the facility location by the State and the Terrorist location is determined, we can solve the problem of determining prevention resources. Let the location of the facility chosen by the State in equilibrium be denoted by x^* and the city attacked in equilibrium be denoted by i^*. The State utility in equilibrium is

$$U_S(x^*,i^*) = -P(c_{\text{prev}})[\alpha d(x^*,i^*) + \eta_{i^*}]w_{i^*} - C - c_{\text{prev}}.$$

By differentiation with respect to c_{prev}, the optimal prevention resources is the solution of

$$-P'(c_{\text{prev}})[\alpha d(x^*, i^*) + \eta_{i^*}]w_{i^*} = 1.$$

Namely, the optimal level of resources is when the change in the expected utility due to additional unit of resource is equal to the cost of a unit of resources.

4. Multiple facilities

4.1. Introduction to multiple facilities

When the number of possible facilities is not restricted to one, the problem becomes more complex and mathematically demanding and the computation of the equilibrium becomes more time consuming. However, as we will show later, the complexity can still be managed if the number of facilities is relatively small (which is probably the case in many practical problems). The inclusion of prevention cost becomes more crucial here since, as we explained in the previous section, the optimal number of facilities is not separable from the problem of how many resources the State should invest in prevention. We start with solving the problem when the number of facilities is fixed. Afterwards, we will address the problem of finding also the optimal number of facilities taking into consideration the prevention cost. In this section, we will use the notation $s_K = (x_1, x_2, \ldots, x_K)$ for the strategy of the State where x_k, $k = 1, 2, \ldots, K$ is the location of facility k. We assume again that N is the candidate set of potential locations for the facilities (namely, facilities on nodes).

4.2. Fixed number of facilities

We assume that the number of facilities is fixed. From Lemma 1, the Terrorist solution when the State chooses s_K is given by

$$e^*(s_K) = \arg \max_{i=1,2,\ldots,n} [\gamma d(s_K, i) + \delta_i]w_i, \tag{8}$$

where $d(s_K, i)$ is the shortest distance between the closest facility in s_K to city i. The State problem is

$$s_K^* = \arg \min_{s_K \subset N} (\alpha d(s_K, e^*(s_K)) + \eta_{e^*(s_K)})w_{e^*(s_K)}. \tag{9}$$

When $\delta_i = \eta_i = 0$ the problem is very easy to solve as shown in the following proposition.

Proposition 1. *When $\delta_i = \eta_i = 0$, for every values of α and γ, the State equilibrium solution is the (nodal) solution of the minimax problem and the Terrorist equilibrium solution is the node that determines the minimax solution for the State.*[8]

Proof. When $\delta_i = \eta_i = 0$, from (8) and (9) α and γ are irrelevant and for every s_K chosen by the State the Terrorist chooses $e^*(s_K) = \arg \max_{i=1,2,\ldots,n}[d(s_K, i)w_i]$ and therefore, from (9) $s_K^* = \arg\min_{s_K \subset N}(\max_{i=1,2,\ldots,n} d(s_K, i)w_i)$ which is the minimax nodal solution. □

[8] For discussion of the minimax problem see Chapter 7 of Mirchandani and Francis (1990).

The problem can be formulated as an integer program. We start by defining the decision variables. Let

$$y_{ij} = \begin{cases} 1 \text{ if city } i \text{ is assigned to facility } j, \\ 0 \text{ otherwise,} \end{cases} \quad i, j = 1, 2, \ldots, n,$$

$$x_j = \begin{cases} 1 \text{ if a facility is located in city } j, \\ 0 \text{ otherwise,} \end{cases} \quad j = 1, 2, \ldots, n,$$

$$z_i = \begin{cases} 1 \text{ if the Terrorist attacks city } i, \\ 0 \text{ otherwise,} \end{cases} \quad i = 1, 2, \ldots, n.$$

Define v to be the Terrorist utility in equilibrium. Obviously, in equilibrium we expect that

$$[\gamma d(i,j) + \delta_i] w_i y_{ij} \leqslant v, \quad i, j = 1, 2, \ldots, n \tag{10}$$

and since $\sum_{j=1}^{n} y_{ij} = 1$ and from Lemma 1, $\sum_{i=1}^{n} z_i = 1$, then

$$\sum_{i=1}^{n} \left[\gamma \sum_{j=1}^{n} y_{ij} d(i,j) + \delta_i \right] z_i w_i \leqslant v. \tag{11}$$

In fact, the inequality in (11) can be replaced by an equality since v will be minimized in the State's objective function as we will see next.

From (11) we can extract the delay between the city under attack and its supporting facility in terms of v namely,

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{n} y_{ij} d(i,j) \right) z_i w_i = \frac{v - \sum_{i=1}^{n} z_i w_i \delta_i}{\gamma}.$$

Thus, we can write the State utility as

$$\sum_{i=1}^{n} \left(\alpha \sum_{j=1}^{n} y_{ij} d(i,j) + \eta_i \right) z_i w_i = \alpha \sum_{i=1}^{n} \left(\sum_{j=1}^{n} y_{ij} d(i,j) \right) z_i w_i + \sum_{i=1}^{n} z_i w_i \eta_i \tag{12}$$

$$= \frac{\alpha}{\gamma} v + \sum_{i=1}^{n} z_i w_i \left(\eta_i - \frac{\alpha \delta_i}{\gamma} \right). \tag{13}$$

We can now formulate the problem

$$\min \quad \frac{\alpha}{\gamma} v + \sum_{i=1}^{n} z_i w_i \left(\eta_i - \frac{\alpha \delta_i}{\gamma} \right) \tag{14}$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_j = K, \tag{15}$$

$$\sum_{i=1}^{n} z_i = 1, \tag{16}$$

$$\sum_{j=1}^{n} y_{ij} = 1, \quad i = 1, 2, \ldots, n, \tag{17}$$

$$y_{ij} \leqslant x_j, \quad i, j = 1, 2, \ldots, n, \tag{18}$$

$$\sum_{k=1}^{n} y_{ik} d(i,k) + (M - d(i,j)) x_j \leqslant M, \quad i, j = 1, 2, \ldots, n, \tag{19}$$

$$[\gamma d(i,j) + \delta_i] w_i y_{ij} \leqslant v, \quad i, j = 1, 2, \ldots, n, \tag{20}$$

$$\sum_{i=1}^{n}\left[\gamma\sum_{j=1}^{n}y_{ij}d(i,j)+\delta_i\right]z_iw_i=v. \tag{21}$$

$$(\alpha d(i,j)+\eta_i)w_iy_{ij}\leqslant\frac{\alpha}{\gamma}v+\sum_{i=1}^{n}z_iw_i\left(\eta_i-\frac{\alpha\delta_i}{\gamma}\right),\quad i,j=1,2,\dots,n, \tag{22}$$

$$x_j=0,1,\quad z_i=0,1,\quad y_{ij}=0,1,\quad i,j=1,2,\dots,n, \tag{23}$$

where M is a sufficiently large number ($M\geqslant\max_{i,j\in N}d(i,j)$). In (15) we impose the location of K facilities. In (17) we ensure that each city is assigned to only one facility and in (18) we forbid an assignment of a city to a node that is not a location of a facility. In (16), (20) and (21) we make sure that the Terrorist objective is taken into account. Constraints (19) verify that for every configuration of facilities, the city under attack will be assigned to the closest facility in the sense of shortest path. In (22) we guarantee that the State objective is taken into consideration. Binary requirements are specified in (23).

4.2.1. Linearization

The problem is a non-linear programming since we have multiplications of the variables z_i and y_{ij} in constraint (21). We can linearize this constraint as follows. Let us define a new binary decision variable u_{ij} such that

$$u_{ij}=z_iy_{ij},\quad i,j=1,2,\dots,n,$$

where u_{ij} should satisfy the following constraints

$$z_i+y_{ij}-u_{ij}\leqslant 1,\quad i,j=1,2,\dots,n,$$
$$z_i+y_{ij}\geqslant 2u_{ij},\quad i,j=1,2,\dots,n.$$

In this case, constraint (21) becomes

$$\gamma\sum_{i=1}^{n}\sum_{j=1}^{n}u_{ij}d(i,j)w_i+\sum_{i=1}^{n}z_iw_i\delta_i=v$$

which is linear.

To demonstrate the problem with more than one facility we consider the following example.

Example 2. Consider again Example 1. Now we let the State install two facilities. Thus, the State strategies set is $S=\{(1,2),(2,3),(1,3)\}$. Using (8) the Terrorist best response to any strategy chosen by the State is given by

$$e^*(1,2)=\arg\max\{(0+1)15,(0+1)20,(4+1)11\}=3,$$
$$e^*(2,3)=\arg\max\{(3+1)15,(0+1)20,(0+1)11\}=1,$$
$$e^*(1,3)=\arg\max\{(0+1)15,(3+1)20,(0+1)11\}=2.$$

Using (9) the State chooses

$$\begin{aligned}s_2^*&=\arg\min\{U_S((1,2),e^*(1,2)),U_S((2,3),e^*(2,3)),U_S((1,3),e^*(1,3))\}\\&=\arg\min\{U_S((1,2),3),U_S((2,3),1),U_S((1,3),2)\}=\arg\min\{231,240,320\}=(1,2).\end{aligned}$$

Thus, in equilibrium, the State will install facilities in cities 1 and 2 while the Terrorist will attack city 3. Observe that when the State installs the facilities, she already knows that the Terrorist is going to attack a city without a facility. The "catch" in non-cooperative games here is that in equilibrium the State installs the facilities such that it forces the Terrorist to attack a city that minimizes her expected damage (in this example, it is also an 'unprotected' city).

4.3. Prevention cost and fixed number of facilities

When $K > 1$ is given, the optimal level of prevention cost is found in a similar way to that of a single facility. The equilibrium utility for the State is now given by

$$\widehat{U}_S(e^*(s_K^*), s_K^*) = -P(c_{\text{prev}})[\alpha d(s_K^*, e^*(s_K^*)) + \eta_{e^*(s_K^*)}]w_{e^*(s_K^*)} - c_{\text{prev}},$$

where $(e^*(s_K), s_K^*)$ are the equilibrium strategies. By differentiation with respect to c_{prev}, the optimal prevention resources is the solution of

$$-P'(c_{\text{prev}})[\alpha d(s_K^*, e^*(s_K)) + \eta_{e^*(s_K^*)}]w_{e^*(s_K^*)} = 1.$$

As before, the optimal level of resources is achieved when the change in the expected utility due to additional unit of resource is equal to the cost of a unit of resources. Observe that since the calculation is done after the equilibrium has already been found, the prevention cost does not add any complexity to the calculations.

4.4. Variable number of facilities

If K is a decision variable then, the problem incurs another level of complexity. We should consider all possible values of K and for every value calculate the equilibrium including the optimal prevention resources. Obviously if $C = 0$, then an optimal solution is $K = n$, namely, install a facility at every node. However, when C is sufficiently large we expect the optimal number of facilities to be below n. Denote by $c^*_{\text{prev}}(K)$ the optimal prevention resources invested by the State given a fixed number of facilities K. To show that the optimal number of facilities K^* is unique, we assume that the State can discard a facility if it does not increase her utility. Namely, the State can always use less facilities than K where discarding facility is with no cost. This assumption is not crucial as we will see later on.

Lemma 4. *The State utility in equilibrium without installment cost* $\widehat{U}_S(e^*(s_K^*), (s_K^*, c^*_{\text{prev}}(K))) = -P(c^*_{\text{prev}}(K))[\alpha d(s_K^*, e^*(s_K^*)) + \eta_i]w_{e^*(s_K^*)} - c^*_{\text{prev}}(K)$ *is a monotonic increasing function in the number of facilities K.*

Proof. By the assumption $[\alpha d(s_K^*, e^*(s_K^*)) + \eta_i]w_{e^*(s_K^*)}$ is a decreasing function of K (since we can always use less facilities than given) and thus, we have

$$\widehat{U}_S(e^*(s_K^*), (s_K^*, c^*_{\text{prev}}(K))) < \widehat{U}_S(e^*(s_{K+1}^*), (s_{K+1}^*, c^*_{\text{prev}}(K))) < \widehat{U}_S(e^*(s_{K+1}^*), (s_{K+1}^*, c^*_{\text{prev}}(K+1))). \qquad \square$$

Thus, we immediately have the following result.

Proposition 2. *There is a unique optimal number of facilities K^*.*

Proof. Since the State utility function including installation cost is $U_S = \widehat{U}_S - KC$ and since $\widehat{U}_S$ is increasing with K we have a single solution to the problem

$$\max_{K=1,2,\dots,n} \widehat{U}_S(e^*(s_K^*), (s_K^*, c^*_{\text{prev}}(K))). \qquad \square$$

The assumption of the possibility of discarding facilities in Lemma 4 is not crucial to our analysis since the number of possible facilities is bounded by n. The use of this Lemma is just for simplifying the computation by looking for a single K.

5. Case study

We solve the problem of multiple facilities introduced in the previous section with a case study of cities in the US. We selected the largest 20 metropolitan areas (by size of population) according to the US Census Bureau for 2000. The only change we made is omitting San Juan from the list (number 20) and adding Tampa (number 21). The damages are assumed to be proportional to the sizes of the metropolitan areas. Thus, the ratio between metropolitan area sizes are important and not the absolute sizes. The 20 metropolitan areas and their population sizes are given in Table A.1. Notice that the metropolitan areas (nodes) are numbered according to their sizes starting with the largest. The shortest distance matrix between the cities is included in Table A.2 and we assume that travel speed is one so that distance can replace travel times. We also assume that $d(i) = 0$ so that $\eta_i = \eta$ and $\delta_i = \delta\ \forall i = 1, \ldots, n$. We fixed the value of γ to $\gamma = 1$ and let α vary in $\{0.5, 1, 2, 3, 4, 5\}$. We chose $\eta = 2\delta$ (assuming that the State cares much more about avoiding damage than the Terrorist about causing damage) where δ varies in $\{0, 50, 125, 250, 500, 1000\}$. The number of facilities considered is between 1 and 4. For every combination of α, δ and η, we solved the problem for $K = 1, 2, 3, 4$. We did not optimize over K since there is no information available about the cost of facility installation.

The results of our study are summarized in Table 1 where for each combination of (recall that $\gamma = 1$) K, α, η and δ (first four columns) we report in the fifth column the equilibrium location of the facilities by the

Table 1
Equilibrium solutions for different combinations of the parameters

K	α	η	δ	Optimal locations	City attacked	State disutility	Terrorist utility
1	0.5	0	0	15	2	1,250,946	2,501,893
1	0.5	100	50	15	2	1,414,683	2,583,761
1	0.5	250	125	15	2	1,660,288	2,706,564
1	1	0	0	15	2	2,501,893	2,501,893
1	1	100	50	15	2	2,665,629	2,583,761
1	1	250	125	15	2	2,911,234	2,706,564
1	2	0	0	15	2	5,003,786	2,501,893
1	2	100	50	15	2	5,167,522	2,583,761
1	2	250	125	15	2	5,413,127	2,706,564
1	3	0	0	15	2	7,505,679	2,501,893
1	3	100	50	15	2	7,669,415	2,583,761
1	3	250	125	15	2	7,915,020	2,706,564
1	4	0	0	15	2	10,007,572	2,501,893
1	4	100	50	15	2	10,171,308	2,583,761
1	4	250	125	15	2	10,416,913	2,706,564
1	5	0	0	15	2	12,509,465	2,501,893
1	5	100	50	15	2	12,673,201	2,583,761
1	5	250	125	15	2	12,918,806	2,706,564
2	0.5	0	0	4 14	2	300,456	600,913
2	0.5	100	50	6 17	3	397,437	657,511
2	0.5	250	125	1 17	3	558,152	772,896
2	1	0	0	4 14	2	600,913	600,913
2	1	100	50	2 4	9	669,957	643,848
2	1	250	125	6 17	3	840,662	726,193
2	2	0	0	4 14	2	1,201,826	600,913
2	2	100	50	2 4	9	1,287,696	643,848
2	2	250	125	2 4	1	1,394,951	697,476
2	3	0	0	4 14	2	1,802,738	600,913
2	3	100	50	4 17	9	1,905,435	643,848

(continued on next page)

Table 1 (*continued*)

K	α	η	δ	Optimal locations	City attacked	State disutility	Terrorist utility
2	3	250	125	6 17	3	2,064,110	726,193
2	4	0	0	4 14	2	2,403,651	600,913
2	4	100	50	2 4	9	2,523,174	643,848
2	4	250	125	4 5	2	2,662,355	767,924
2	5	0	0	4 14	2	3,004,564	600,913
2	5	100	50	4 17	9	3,140,913	643,848
2	5	250	125	4 5	2	3,225,608	767,924
3	0.5	0	0	6 17 18	12	198,471	396,941
3	0.5	100	50	1 2 18	12	244,600	431,053
3	0.5	250	125	1 6 17	3	534,800	726,193
3	1	0	0	2 6 18	12	396,941	396,941
3	1	100	50	2 6 18	12	435,705	416,323
3	1	250	125	1 2 3	10	554,278	495,908
3	2	0	0	2 6 18	12	793,883	396,941
3	2	100	50	2 6 18	12	832,646	416,323
3	2	250	125	6 17 18	12	890,792	445,396
3	3	0	0	6 17 18	12	1,190,824	396,941
3	3	100	50	2 6 18	12	1,229,588	416,323
3	3	250	125	2 6 18	12	1,287,733	445,396
3	4	0	0	6 17 18	12	1,587,765	396,941
3	4	100	50	6 17 18	12	1,626,529	416,323
3	4	250	125	2 6 18	12	1,684,675	445,396
3	5	0	0	2 6 18	12	1,984,707	396,941
3	5	100	50	2 6 18	12	2,023,470	416,323
3	5	250	125	6 17 18	12	2,081,616	445,396
4	0.5	0	0	1 2 11 18	13	169,562	339,124
4	0.5	100	50	1 2 4 18	12	217,853	377,559
4	0.5	250	125	1 6 7 17	3	534,800	726,193
4	1	0	0	2 6 18 20	13	339,124	339,124
4	1	100	50	1 2 18 20	13	374,672	356,898
4	1	250	125	1 2 3 20	9	547,245	481,972
4	2	0	0	2 6 11 18	13	678,248	339,124
4	2	100	50	1 2 12 18	13	713,796	356,898
4	2	250	125	1 2 11 18	13	767,117	383,559
4	3	0	0	1 2 12 18	13	1,017,372	339,124
4	3	100	50	1 2 11 18	13	1,052,920	356,898
4	3	250	125	1 2 12 18	13	1,106,241	383,559
4	4	0	0	1 2 18 20	13	1,356,496	339,124
4	4	100	50	1 2 11 18	13	1,392,044	356,898
4	4	250	125	1 2 12 18	13	1,445,365	383,559
4	5	0	0	1 2 11 18	13	1,695,621	339,124
4	5	100	50	2 6 12 18	13	1,731,168	356,898
4	5	250	125	1 2 18 20	13	1,784,490	383,559
1	0.5	500	250	18	2	2,124,480	3,020,938
1	0.5	1000	500	3	2	3,069,240	3,682,433
1	0.5	2000	1000	8	2	4,901,451	4,890,808
1	1	500	250	15	2	3,320,575	2,911,234
1	1	1000	500	18	2	4,248,961	3,430,279
1	1	2000	1000	3	2	6,138,480	4,501,115
1	2	500	250	15	2	5,822,468	2,911,234
1	2	1000	500	15	2	6,641,150	3,320,575
1	2	2000	1000	18	2	8,497,922	4,248,961
1	3	500	250	15	2	8,324,361	2,911,234
1	3	1000	500	15	2	9,143,043	3,320,575

(*continued on next page*)

Table 1 (*continued*)

K	α	η	δ	Optimal locations	City attacked	State disutility	Terrorist utility
1	3	2000	1000	18	2	11,109,518	4,248,961
1	4	500	250	15	2	10,826,254	2,911,234
1	4	1000	500	15	2	11,644,936	3,320,575
1	4	2000	1000	18	2	13,721,115	4,248,961
1	5	500	250	15	2	13,328,147	2,911,234
1	5	1000	500	15	2	14,146,829	3,320,575
1	5	2000	1000	18	2	16,332,711	4,248,961
2	0.5	500	250	1 5	2	1,100,309	972,595
2	0.5	1000	500	6 17	1	2,202,666	1,225,352
2	0.5	2000	1000	1 17	1	4,239,973	2,119,987
2	1	500	250	1 2	3	1,116,304	887,366
2	1	1000	500	1 5	2	2,200,618	1,381,936
2	1	2000	1000	1 17	1	4,239,973	2,119,987
2	2	500	250	6 17	3	1,681,324	840,662
2	2	1000	500	1 17	3	2,232,608	1,116,304
2	2	2000	1000	1 2	1	4,239,973	2,119,987
2	3	500	250	2 6	3	2,293,048	840,662
2	3	1000	500	1 17	3	2,891,035	1,116,304
2	3	2000	1000	1 17	1	4,239,973	2,119,987
2	4	500	250	4 17	1	2,789,902	962,474
2	4	1000	500	1 2	3	3,549,463	1,116,304
2	4	2000	1000	6 7	1	4,901,409	2,285,345
2	5	500	250	6 17	3	3,516,495	840,662
2	5	1000	500	1 2	3	4,207,890	1,116,304
2	5	2000	1000	2 6	1	5,066,768	2,285,345
3	0.5	500	250	1 2 18	1	1,059,993	529,997
3	0.5	1000	500	1 11 17	1	2,119,987	1,059,993
3	0.5	2000	1000	1 6 17	1	4,239,973	2,119,987
3	1	500	250	1 2 18	1	1,059,993	529,997
3	1	1000	500	1 2 4	1	2,119,987	1,059,993
3	1	2000	1000	1 5 17	1	4,239,973	2,119,987
3	2	500	250	1 2 18	1	1,059,993	529,997
3	2	1000	500	1 4 17	1	2,119,987	1,059,993
3	2	2000	1000	1 15 17	1	4,239,973	2,119,987
3	3	500	250	1 2 3	10	1,546,095	554,278
3	3	1000	500	1 2 3	1	2,119,987	1,059,993
3	3	2000	1000	1 2 17	1	4,239,973	2,119,987
3	4	500	250	1 2 3	10	1,983,634	554,278
3	4	1000	500	1 2 18	1	2,119,987	1,059,993
3	4	2000	1000	1 15 17	1	4,239,973	2,119,987
3	5	500	250	1 2 3	10	2,421,173	554,278
3	5	1000	500	2 6 18	1	2,946,781	1,225,352
3	5	2000	1000	1 2 3	1	4,239,973	2,119,987
4	0.5	500	250	1 2 3 19	1	1,059,993	529,997
4	0.5	1000	500	1 8 17 20	1	2,119,987	1,059,993
4	0.5	2000	1000	1 2 5 17	1	4,239,973	2,119,987
4	1	500	250	1 2 5 18	1	1,059,993	529,997
4	1	1000	500	1 4 15 17	1	2,119,987	1,059,993
4	1	2000	1000	1 2 5 8	1	4,239,973	2,119,987
4	2	500	250	1 2 18 20	1	1,059,993	529,997
4	2	1000	500	1 2 4 9	1	2,119,987	1,059,993
4	2	2000	1000	1 5 6 17	1	4,239,973	2,119,987
4	3	500	250	1 4 17 18	2	1,388,485	599,275

(*continued on next page*)

Table 1 (*continued*)

K	α	η	δ	Optimal locations	City attacked	State disutility	Terrorist utility
4	3	1000	500	1 2 6 8	1	2,119,987	1,059,993
4	3	2000	1000	1 3 10 17	1	4,239,973	2,119,987
4	4	500	250	2 4 6 18	1	1,721,429	695,356
4	4	1000	500	1 2 18 20	1	2,119,987	1,059,993
4	4	2000	1000	1 2 3 4	1	4,239,973	2,119,987
4	5	500	250	2 6 18 20	1	1,886,788	695,356
4	5	1000	500	1 2 11 18	1	2,119,987	1,059,993
4	5	2000	1000	1 2 3 8	1	4,239,973	2,119,987

State, in the sixth column the city the Terrorist chooses to attack and in the last two columns the disutility of the State and the utility of the Terrorist. For example, when $K = 3$, $\alpha = 2$, $\eta = 500$ and $\delta = 250$ the equilibrium States facilities are at New York, Los Angeles and St. Louis. The Terrorist will attack New York. The disutility of the State is 1.06 millions and the utility of the terrorist is 0.53 millions. There are many scenarios that can be analyzed from Table 1. As an example we analyze two scenarios depicted in Figs. 4 and 5. In Fig. 6 we demonstrate the decline of the State's disutility as a function of the number of facilities K. The numbers are taken from Table 1. As can be seen from Fig. 6 the decline from K to $K+1$ is much larger than from $K+1$ to $K+2$.

In Fig. 4 we depict the equilibrium location of facilities and city attacked for $K = 1, 2, 3, 4$, $\delta = \eta = 0$. Recall that when $\delta = \eta = 0$ the equilibrium decision of the State is to locate the facilities at the *K—nodal*

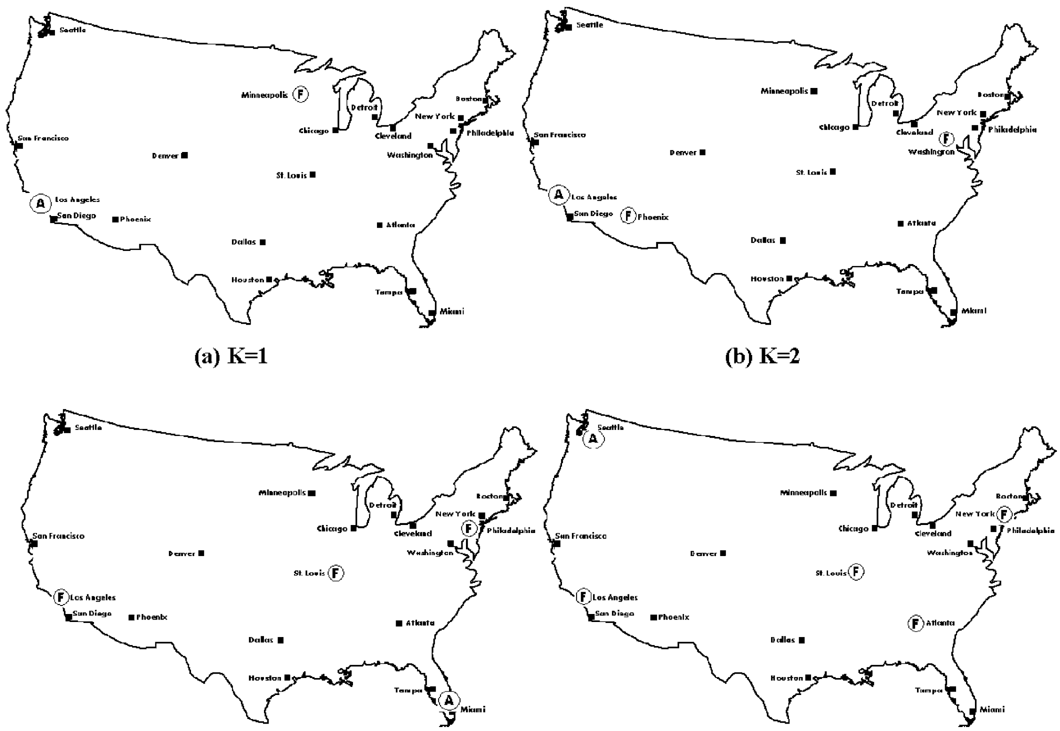

Fig. 4. The facilities location "F" and the city attached "A" in equilibrium for $\gamma = 1$, $\eta = \delta = 0$ and $K = 1, 2, 3, 4$.

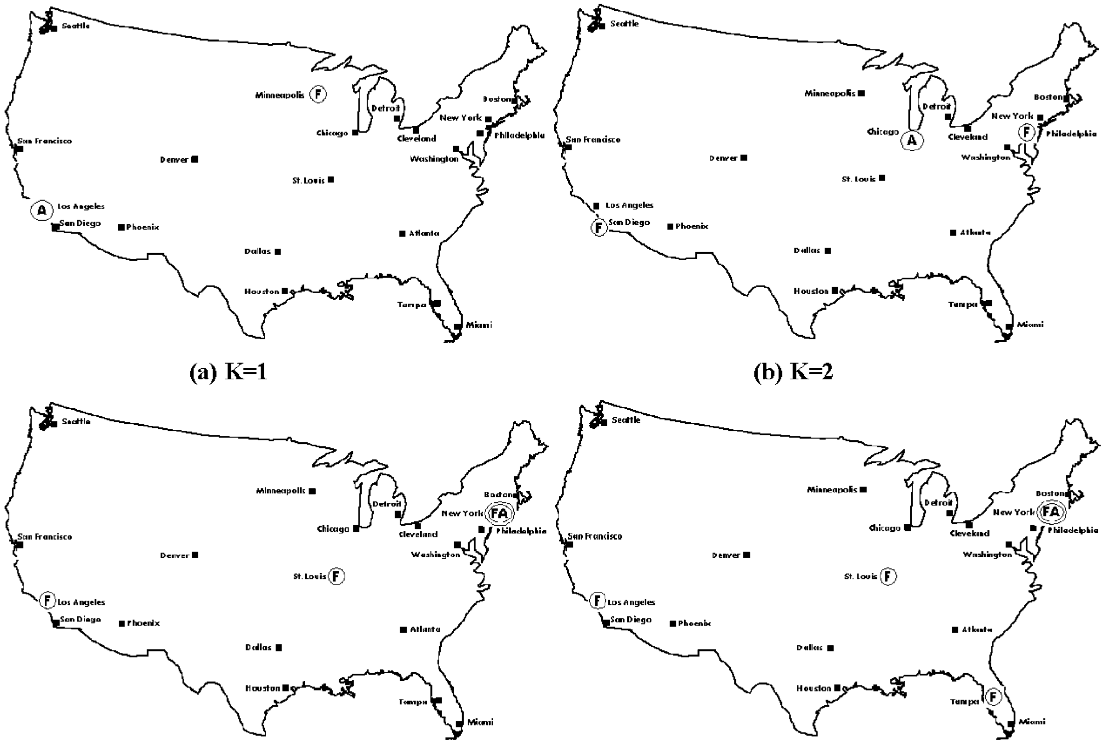

Fig. 5. The facilities location "F" and the city attached "A" in equilibrium for $\alpha = 2$, $\gamma = 1$, $\eta = 500$, $\delta = 250$ and $K = 1, 2, 3, 4$.

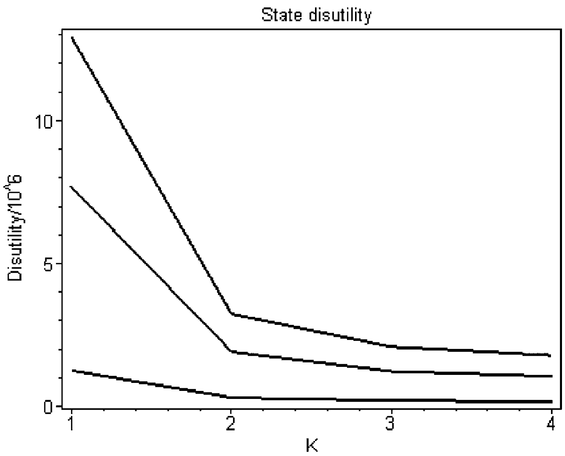

Fig. 6. State's disutility as function of K. The disutility axis is in millions. The lower curve is for $\alpha = 0.5$, $\eta = 0$, $\delta = 0$. The second curve is for $\alpha = 3$, $\eta = 100$, $\delta = 50$ and the upper curve is for $\alpha = 5$, $\eta = 250$, $\delta = 125$.

minimax solution and the city that determines the minimax solution is the city attacked regardless of the specific value of α. Also, the State disutility $= \alpha$ (Terrorist utility). When $K = 1$ the State locates the facility at Minneapolis and the city attacked is Los Angeles. By locating at Minneapolis the State deters the Terrorist from attacking New York, the largest city. Notice that a second facility does not change the location of the city attacked but by locating the two facilities at Washington DC and Phoenix the State reduces its disutility by 76%. With three facilities located at Los Angeles, Philadelphia and St. Louis, the Terrorist prefers to attack Miami instead of Los Angeles and thus, by locating 3 instead of 2 facilities the State reduces its disutility by 34%. With four facilities located at New York, Los Angeles, Atlanta and St. Louis, the Terrorist prefers to attack Seattle and consequently the State reduces its disutility due to locating 4 instead of 3 facilities by 14.6%.

In Fig. 5 we depict the equilibrium location of facilities and city attacked for $K = 1, 2, 3, 4$ when $\alpha = 2$, $\eta = 500$ and $\delta = 250$. When $K = 1$, the solution is identical to that of the case when $\delta = \eta = 0$ i.e., the State locates the facility at Minneapolis and the Terrorist attacks Los Angeles. When $K = 2$, the State locates the two facilities at Philadelphia and San Diego which are very close to the two largest cities—New York and Los Angeles, respectively, and the city attacked is Chicago which is the third largest city. Notice that the change of the solution compared to the case of $\eta = \delta = 0$ is due to the large values of η and δ which are the coefficients of the damage w_i (which is proportional to the size of the city). When $K = 3$, the State locates its facilities in New York, Los Angeles and St. Louis forcing the Terrorist to attack New York where a facility with all resources is located. When $K = 4$, the solution is identical to the case when $K = 3$ except for an additional facility located at Tampa. It is interesting to note that the State disutility and the Terrorist utility are the same when $K = 3$ and $K = 4$. The conclusion is that an additional facility is not required. In contrast, the disutility of the State decreases by 71% when $K = 2$ instead of $K = 1$ and by 37% if $K = 3$ instead of $K = 2$.

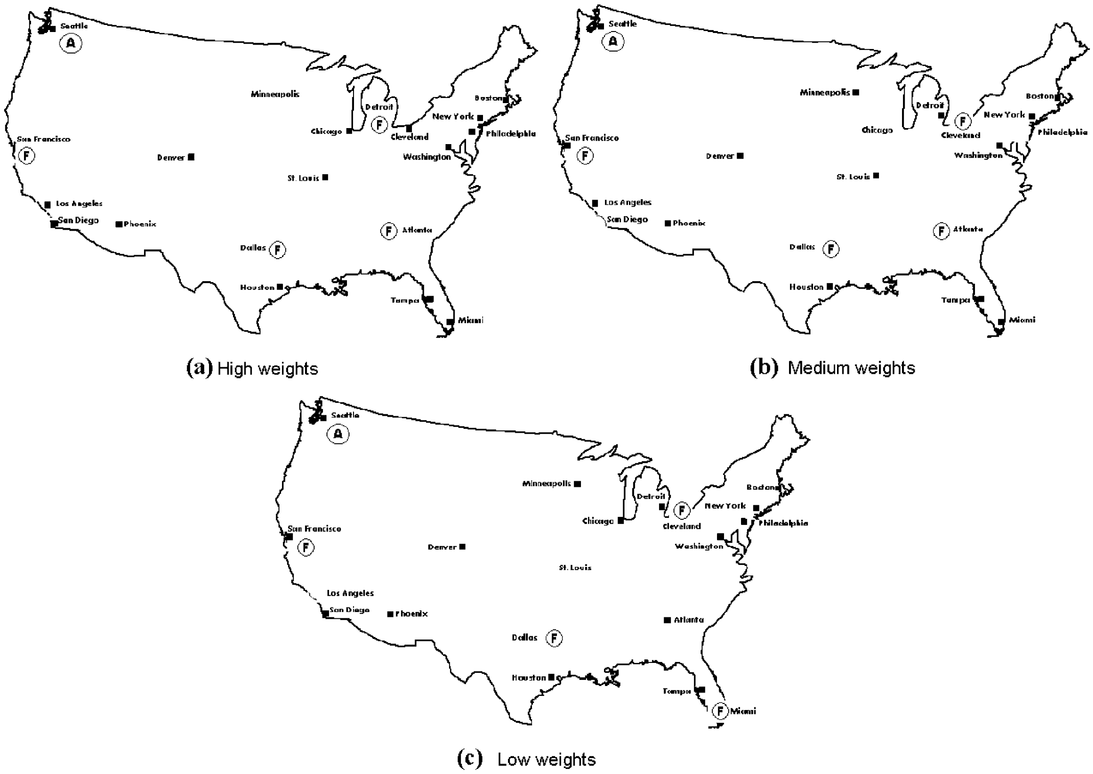

Fig. 7. Constant weights across cities with high, medium and low weights.

To check the sensitivity of the equilibrium solution to the variations of the cities' weights, we consider a version of the case study with identical weights across cities. We deviate from the original weights by choosing three scenarios. In the first scenario we take the cities' weight to be identical and equal to the weight of the largest city in Table A.1 namely, New York. In the second scenario all the cities have weights equal to 10th city, Houston. The third scenario consider all the weights identical to Tampa, the last city in the list. All the other parameters are similar to the last scenario considered above namely, $K = 4$, $\alpha = 2$, $\eta = 500$ and $\delta = 250$.

As we can see from Fig. 7, in equilibrium, Seattle is always the city under attack while the location of the facilities is similar. Changing the weights from high to medium shift the location of only one facility from Detroit to Cleveland. This change is insignificant since both cities are very close. Changing the weights from medium to low, shift the location of one facility from Atlanta to Miami. It follows that the equilibrium is not too sensitive to changes in the weights.

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Appendix

See Tables A.1 and A.2.

Table A.1
Metropolitans (source: US Census Bureau—April 2, 2001)

Node	Metropolitan	Population size
1	New York–Northern New Jersey–Long Island, NY–NJ–CT–PA	21,199,865
2	Los Angeles–Riverside–Orange County, CA CMSA	16,373,645
3	Chicago–Gary–Kenosha, IL–IN–WI CMSA	9,157,540
4	Washington–Baltimore, DC–MD–VA–WV CMSA	7,608,070
5	San Francisco–Oakland–San Jose, CA CMSA	7,039,362
6	Philadelphia–Wilmington–Atlantic City, PA–NJ–DE–MD CMS	6,188,463
7	Boston–Worcester–Lawrence, MA–NH–ME–CT CMSA	5,819,100
8	Detroit–Ann Arbor–Flint, MI CMSA	5,456,428
9	Dallas–Fort Worth, TX CMSA	5,221,801
10	Houston–Galveston–Brazoria, TX CMSA	4,669,571
11	Atlanta, GA MSA	4,112,198
12	Miami–Fort Lauderdale, FL CMSA	3,876,380
13	Seattle–Tacoma–Bremerton, WA CMSA	3,554,760
14	Phoenix–Mesa, AZ MSA	3,251,876
15	Minneapolis–St. Paul, MN–WI MSA	2,968,806
16	Cleveland–Akron, OH CMSA	2,945,831
17	San Diego, CA MSA	2,813,833
18	St. Louis, MO–IL MSA	2,603,607
19	Denver–Boulder–Greeley, CO CMSA	2,581,506
20	Tampa–St. Petersburg–Clearwater, FL MSA	2,395,997

Table A.2
Distance between cities

Cities	New York	Los Angeles	Chicago	Washington	San Francisco	Philadelphia	Boston	Detroit	Dallas	Houston	Atlanta	Miami	Seattle	Phoenix	Minneapolis	Cleveland	San Diego	St. Louis	Denver	Tampa
New York	0	2462	719	204	2582	78	190	489	1373	1421	748	1088	2413	2145	1023	408	2432	879	1629	1002
Los Angeles	2462	0	1749	2308		2402	2605	1987	1251	1382	1944	2348	954	367	1528	2057	116	1595	844	2162
Chicago	719	1749	0	598	1863	668	856	238	798	937	585	1186	1737	1447	354	312	1727	259	911	1001
Washington	204	2308	598	0	2449	126	394	400	1183	1221	544	924	2329	1980	934	304	2272	714	1487	817
San Francisco	2582	344	1863	2449	0	2530	2708	2095	1493	1651	2145	2601	679	658	1591	2175	460	1750	963	2407
Philadelphia	78	2402	668	126	2530	0	268	446	1300	1345	670	1024	2380	2081	985	358	2370	814	1573	931
Boston	190	2605	856	394	2708	268	0	618	1551	1607	938	1255	2496	2299	1126	552	2582	1042	1766	1182
Detroit	489	1987	238	400	2095	446	618	0	997	1106	599	1156	1935	1685	539	96	1965	456	1148	995
Dallas	1373	1251	798	1183	1493	1300	1551	997	0	224	717	1108	1683	888	860	1023	1184	544	660	915
Houston	1421	1382	937	1221	1651	1345	1607	1106	224	0	701	968	1891	1016	1054	1114	1301	678	874	792
Atlanta	748	1944	585	544	2145	670	938	599	717	701	0	605	2181	1589	905	553	1887	467	1204	416
Miami	1088	2348	1186	924	2601	1024	1255	1156	1108	968	605	0	2734	1981	1510	1086	2269	1062	1720	206
Seattle	2413	954	1737	2329	679	2380	2496	1935	1683	1891	2181	2734	0	1110	1396	2028	1058	1723	1026	2529
Phoenix	2145	367	1447	1980	658	2081	2299	1685	888	1016	1589	1981	1110	0	1274	1747	298	1267	586	1796
Minneapolis	1023	1528	354	934	1591	985	1126	539	860	1054	905	1510	1396	1274	0	632	1526	464	693	1315
Cleveland	408	2057	312	304	2175	358	552	96	1023	1114	553	1086	2028	1747	632	0	2030	494	1221	933
San Diego	2432	116	1727	2272	460	2370	2582	1965	1184	1301	1887	2269	1058	298	1526	2030	0	1558	834	2087
St. Louis	879	1595	259	714	1750	814	1042	456	544	678	467	1062	1723	1267	464	494	1558	0	788	861
Denver	1629	844	911	1487	963	1573	1766	1148	660	874	1204	1720	1026	586	693	1221	834	788	0	1516
Tampa	1002	2162	1001	817	2407	931	1182	995	915	792	416	206	2529	1796	1315	933	2087	861	1516	0

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