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Bid costs and endogenous bid caps

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We study contests where several privately informed agents bid for a prize. All bidders bear a cost of bidding that is an increasing function of their bids, and, moreover, bids may be capped. We show that regardless of the number of bidders, if bidders have linear or concave cost functions, then setting a bid cap is not profitable for a designer who wishes to maximize the average bid. On the other hand, if agents have convex cost functions (i.e., an increasing marginal cost), then effectively capping the bids is profitable for a designer facing a sufficiently large number of bidders. Furthermore, bid caps are effective for any number of bidders if the cost functions' degree of the convexity is large enough.

1. Introduction

■ In many sport competitions, audiences are thrilled most when several teams or individuals engage in close races. The bodies governing the competition rules in these events are interested in creating what they call a "competitive balance." In particular, this means increasing the expected performance of a league as a whole rather than the performance of the top team or individual.[1]

Entry in professional competitions is often restricted, and only contestants that have achieved a certain predefined minimum requirement are allowed to compete. Similarly, reserve prices and entry fees are often used to exclude players with low valuations from an auction. Such procedures can be beneficial for the seller in an auction (or for a contest designer) and have been amply discussed in the literature. On the other hand, commonsense intuition suggests that imposing upper bounds on bids will have a detrimental effect on the average bid level. Upper bounds will obviously constrain high-valuation bidders. As a consequence, the prize will not necessarily go to the agent who values it the most. This efficiency loss will, in turn, imply that the seller's share of the pie will be smaller.

But, contrary to the conventional wisdom sketched above, in many competitive situations we often observe severe constraints imposed on contestants. For example, in sports in which equipment plays a major role (e.g., sailing, motor racing, etc.) there are very strict, explicit or

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[1] See Szymanski (2001) for a review of issues arising in the design of sporting contests.

implicit technological constraints imposed on the allowed equipment.[2] Formula 1 racing cars must be constructed such that they cannot run faster than an absolute limit of 360 kilometers per hour. In addition, there are many stringent technical specifications whose main goal is to slow down the cars under various racing conditions.[3] It is well known that the constraints bind, and the Formula 1 competition is sometimes compared to a cat-and-mouse game among organizers and engineers. Similarly, the mandatory specifications for racing yachts cover pages of arcane technical detail.

In U.S. professional sports leagues (e.g., NBA, NFL), individual teams face annual caps on the sum of money they are allowed to spend on salaries.[4] The usual explanation is that the salary caps help even the competition between teams in large and small cities, since otherwise the big-city teams could afford to pay more, would buy the best talent, and destroy any semblance of competition in the league.[5] Salary caps[6] are clearly binding, and teams are sometimes forced to trade expensive players to make room for other needed team members.

To take some examples from other fields, consider first the much-discussed initiatives to cap both lobbying contributions (affecting the contests among lobbyists such as political action committees) and spending (directly affecting the contest among candidates) in U.S. electoral campaigns. An interesting question is whether these initiatives will induce the desired outcome, i.e., a decrease in aggregate campaign expenditure.

The various member countries in the European Union spend considerable resources (in the form of tax rebates, tax exemptions, etc.) to attract capital. This contest is the mirror image of a usual all-pay auction, since the prize goes to the lowest bidder. Various initiatives propose a harmonization of capital taxation in the European Union. In particular, this may mean the imposition of a minimum taxation of capital gains in all countries.[7] How will such a measure, if introduced, influence the entrepreneurial decisions and the welfare of each country?

In this article we provide a model that explains the effects and use of endogenous bidding constraints. Several results also have implications for auctions with financially constrained bidders where the constraints may be exogenous.

In Section 2 we describe the model. Several risk-neutral agents engage in a contest for a prize. Each bidder is privately informed about her valuation for the prize. The function governing the distribution of valuations is common knowledge, and valuations are drawn independently of each other. Each contestant makes a bid for the prize, but bids are constrained to be lower than a commonly known bid cap d that may be controlled by the designer. The contestant with the highest bid wins the prize,[8] but all contestants incur a cost that is a strictly increasing function of their bid (the cost function is common knowledge). This model is strategically equivalent to one in which the value of the prize is known and the same for all contestants, but each bidder is privately informed about an "ability parameter" influencing the cost functions (see Moldovanu and Sela, 2001a). Lower valuations correspond then to lower abilities, since a player with lower ability has higher costs to provide the same bid (or effort). We differentiate among the cases where the cost function is, respectively, linear, concave, or convex. There are several possible interpretations for these features:

[2] Antidoping rules play, in a sense, a similar role for sports in which external equipment is not crucial.

[3] Without going into very technical details, consider these rules: the cars have a maximal allowed engine capacity of 3,000 cubic centimeters, at most 10 pistons, at most 5 ventils per piston, and at most 7 gears; they must weigh at least 600 kilograms and have maximal size of 455 × 180 × 95 centimeters.

[4] For example, in the year 2000, NFL teams faced a salary cap of $62,172,000 per club. A club's top 51 salaried players count toward the cap.

[5] Others see the cap as a device by which team owners capture some of the rents, rather than having all the rents going to players. Fort and Quirk (1995) present empirical evidence that salary caps are nevertheless effective in restoring "the competitive balance."

[6] European leagues do not have salary caps, but they do impose indirect limitations such as the number of foreign (i.e., expensive) stars. These limitations often lead to curious citizenship awards and legal battles.

[7] At present Luxembourg, for example, does not tax capital gains at all.

[8] If several bidders make the same highest bid (a feature that arises here in equilibrium), then each of the high bidders has the same chance to get the prize.

(i) The agents are engaged in a contest where they all spend resources to win a prize. The cost of a bid is an increasing function of the bid, but it becomes infinite above a certain level that can be controlled by the designer.

(ii) The agents face an increasing cost of financing and an absolute budget constraint that cannot be exceeded, but the designer may provide additional financing.

Each contestant chooses his bid to maximize expected utility. The goal of the contest designer is to maximize the average bid at the contest (in an *ex ante* symmetric model this coincides with maximizing the expected sum of bids).

In Section 3 we analyze the case of linear cost functions and display the symmetric bidding equilibrium for contests with an effective bid cap d. In equilibrium, each bidder makes a bid that is a (weakly) increasing function of her valuation for the prize. Let $\tilde{b}$ denote the bid function in the symmetric equilibrium of an unconstrained contest (see Proposition 1), and let $\tilde{b}^{-1}(d)$ denote the valuation of the agent that makes a bid d in this equilibrium. In Proposition 2 we show that the equilibrium of the contest with bid cap d is characterized by a critical valuation $c = c(d) < \tilde{b}^{-1}(d)$ such that all lower types bid according to $\tilde{b}$, but all higher types make a bid equal to d. A bidder with the critical valuation c is exactly indifferent between bidding $\tilde{b}(c)$ and d. Since $c < \tilde{b}^{-1}(d)$, the equilibrium bidding function is not continuous at the critical valuation.

For all types in the interval $[c, \tilde{b}^{-1}(d))$, the equilibrium bid in the constrained contest is higher than the equilibrium bid in the contest without bid caps! This is the main "hidden" effect of bid caps. So to compute the overall effect of bid caps on the designer's revenue, it is necessary to compare this gain with the loss incurred because the constrained bid of all types higher than $\tilde{b}^{-1}(d)$ is lower than their unconstrained bids. *A priori*, it seems that the comparison depends on the exact shape of the equilibrium bid function, i.e., on such factors as the distribution of types and the number of bidders. Proposition 3 shows that with linear cost functions, the average loss due to bid caps is invariably higher than the average gain. Hence bid caps are disadvantageous for the designer. While we give a direct proof, this result can also be derived by Myerson's (1981) approach, which employs direct-revelation mechanisms. Note that in some cases (where a regularity condition on the function governing the distribution of valuations[9] is not satisfied), Myerson's revenue-maximizing auction does involve pooling and thus will also be inefficient. But as our result suggests, this optimal pooling cannot be of the form induced by bidding caps. In particular, the optimal auction never displays "distortion at the top."

In Section 4, Proposition 4, we display the symmetric equilibrium bid function for the case of strictly increasing (not necessarily linear) cost functions: this equilibrium is obtained by applying the inverse cost function to the unconstrained part of the bid function and to the critical value obtained for the linear case, respectively. Proposition 5 shows that bid caps decrease the designer's revenue when she faces bidders with concave cost functions. In contrast, Proposition 6 shows that for any strictly convex cost function, the introduction of effective bid caps strictly increases the designer's revenue if there are sufficiently many bidders or, alternatively, if the degree of the convexity of the cost functions, measured by the familiar Arrow-Pratt coefficient, is large enough. A (rough) intuition for this result is as follows: When there are sufficiently many contestants, the chances of getting the prize are slim, and only a small measure of types will make very high bids. Thus, the loss induced by capping the bids of high-valuation bidders is not too large.[10] On the other hand, with increasing marginal costs, a slight increase in bids is relatively less costly for a lower-valuation type, and more such types will find it optimal to increase their bid up to the allowed maximum. Hence the measure of types for which the cap leads to gains for the designer gets larger, and ultimately so large that it can dominate the losses sustained by capping the bids of high-valuation bidders.

At the end of this section we display an example showing that if bidders have convex cost

[9] Our analysis does not depend on any regularity assumptions.

[10] This intuition applies in general, no matter the shape of the cost function.

functions, setting a maximum bid is profitable for the designer even if she is allowed to impose a minimum bid.[11]

Concluding comments are gathered in Section 5. All proofs are relegated to the Appendix.

□ **Related literature.** The economic literature on contests and all-pay auctions is very large. All-pay auction models with incomplete information about the prize's value to different contestants include Weber (1985), Hillman and Riley (1989), and Krishna and Morgan (1997). Equilibrium uniqueness in such models with two players is treated in Amann and Leininger (1996) and Lizzeri and Persico (2000). All these articles study models with linear cost functions and unconstrained bidders.

Our article is closely related to several important contributions in the literature. Laffont and Robert (1996) show that an all-pay auction with a reserve price is a revenue-maximizing mechanism for selling one object to bidders that face linear costs and a common and common-knowledge fixed budget constraint. In addition, these authors show that the optimal reserve price for financially constrained bidders is lower than the one without constraints. Since in their interpretation the budget constraint is exogenously given, Laffont and Robert do not analyze what happens when this constraint varies.

Che and Gale (1998a) calculate the bidding equilibrium of a complete-information, all-pay contest with two bidders having different valuations for a prize and (using our terminology) linear cost functions. In contrast to our finding with linear cost functions, they show that a bid cap can increase the designer's revenue. Their result is due to the *ex ante asymmetry* in valuations. Che and Gale also make an application to political lobbying.[12]

Che and Gale (1998b) study standard auctions for one object where bidders are privately informed about their valuation and about their ability to pay (type spaces are two-dimensional). In their model, the symmetric equilibrium bid function depends continuously on the valuation and on the budget constraint. (Due to the technical complexity, Che and Gale do not analytically compute equilibria, and their arguments are indirect ones.) Their main intuition is that auction procedures that generate lower bids perform better because budget constraints will be binding for fewer types. In particular, an all-pay auction revenue dominates a first-price auction, and a first-price auction revenue dominates a second-price auction.[13] This last result is shown to generalize to frameworks where the winner incurs a bidding cost that is a convex function of her bid. As in Laffont and Robert (1996), the budget constraints are taken to be exogenous and not subject to variation.

Maskin (2000) focuses on the efficiency property of auctions in the presence of budget constraints. While no mechanism can be fully efficient, in a symmetric setting he shows that an all-pay auction is constrained efficient (i.e., maximizes expected welfare subject to incentive-compatibility and budget constraints). This result is also due to the fact that in the all-pay auction the bids are relatively low, and hence the budget constraint binds for fewer types (when the constraints bind, the allocation may be random, and hence inefficient).

Che and Gale (2000) describe the optimal mechanism for selling a good to a budget-constrained buyer who is privately informed about her valuation and about her ability to pay. This mechanism involves nontrivial price discrimination (whereas it reduces to a take-it-or-leave-it offer if the budget constraint is known).

Pitchik and Schotter (1988) study complete-information sequential auctions with two financially constrained bidders and two independent objects. In particular, they point out that the order of sale affects revenue. Benoit and Krishna (2001) extend this model to more than two bidders,

[11] Recall the $62,172,000 annual salary cap imposed on NFL teams. Interestingly, there is also a minimum salary requirement of $51,561,000.

[12] Baye, Kovenock, and deVries (1993) also study an asymmetric model with complete information and show that excluding some bidders may be advantageous for the beneficiary of the lobbying activities.

[13] Second-price auctions with two financially constrained bidders having affiliated signals are studied by Fang and Parreiras (2002). They show how budget constraints may attenuate the winner's curse, making unconstrained bidders more aggressive.

allowing for synergies among the objects and for budgets chosen by the bidders. They note that the seller may benefit from the budget constraints, and that this feature cannot occur in their model if only one object is auctioned. In their example, two objects are sold in a sequence of second-price auctions to two bidders. It is optimal for one of the bidders to force up the price of the first object because this depletes the budget of the other bidder, and the second object sells cheaply. The seller's revenue is higher than in the unconstrained auction.

Moldovanu and Sela (2001a, 2001b) study the effects of the bidding cost function in a contest model where the designer can split a fixed prize sum among several prizes and/or can split the agents in several subcontests. They show that organizing a grand contest with a unique prize is optimal if the contestants have linear or concave cost functions, but that organizing several parallel subcontests with multiple prizes can increase the designer's revenue if contestants face a convex cost function.

2. The contest model

■ We consider n agents bidding for an indivisible object. Bidder i's valuation for the object, denoted by v_i, is private information to i, $i = 1, 2, \ldots, n$. All bidders other than i perceive v_i as a random selection out of the interval $[0, 1]$, governed by the distribution function F, and independent of other valuations. We assume that F is continuously differentiable, and we denote by f the associated density function. We also assume that $f(v) > 0$ for all $v \in [0, 1]$.

Each bidder i submits a bid $x_i \leq d$, where $d \in [0, 1]$ is a commonly known bid cap. The cap can be exogenous (e.g., due to budget constraints) or controlled by the contest's designer. Bids are submitted simultaneously and independently of each other.

A bid x causes a cost $g(x)$, where $g : R_+ \to R_+$ is a strictly increasing function, twice continuously differentiable with $g(0) = 0$. The bidder with the highest bid wins the object,[14] while all bidders incur their respective bidding costs. That is, the payoff of bidder i who has valuation v_i and submits a bid x_i is either $v_i - g(x_i)$ if he wins the object, or $-g(x_i)$ if i does not win the object.

3. Linear cost functions

■ In this section we assume that the cost functions are linear, i.e., $g(x) = x$.

Proposition 1. Consider a contest where n bidders face linear cost functions and a bid cap $d > 1 - \int_0^1 F^{n-1}(y)dy$. Then the bid cap is not effective, and, in a symmetric equilibrium,[15] the bid function of every bidder is given by

$$\bar{b}(v) = vF^{n-1}(v) - \int_0^v F^{n-1}(y)dy, \qquad 0 \leq v \leq 1. \tag{1}$$

Proof. Well known and therefore omitted.

Proposition 2. Consider a contest where n bidders face linear cost functions and a bid cap d such that $0 < d \leq 1 - \int_0^1 F^{n-1}(y)dy$. In a symmetric equilibrium the bid function of every agent is given by

$$b(v) = \begin{cases} \bar{b}(v) & \text{if } 0 \leq v < c \\ d & \text{if } c \leq v \leq 1, \end{cases} \tag{2}$$

[14] If more than one bidder submits the highest bid, then the winner is randomly selected among the highest bidders (each one of them has the same chance to win the object).

[15] It can be shown that the symmetric equilibrium is unique.

where the critical value $c = c(d)$ is strictly monotonic increasing, and defined by

$$d = \frac{c(1 - F^n(c))}{n(1 - F(c))} - \int_0^c F^{n-1}(y)dy. \tag{3}$$

Proof. See the Appendix.[16]

Example 1. Assume that $n = 2$ and that $F(v) = v$ (uniform distribution on [0, 1]). Assuming that $d \leq 1/2$, the symmetric equilibrium bid function is given by

$$b(v) = \begin{cases} \frac{1}{2}v^2 & \text{if } 0 \leq v < 2d \\ d & \text{if } 2d \leq v \leq 1. \end{cases}$$

Figure 1 shows the bid functions with and without a bid cap.

It is worth mentioning that the standard auction types are not revenue equivalent here. For example, consider a second-price, sealed-bid auction. For any $d \leq 1$, the equilibrium is given by

$$b(v) = \begin{cases} v & \text{if } 0 \leq v < d \\ d & \text{if } d \leq v \leq 1. \end{cases}$$

In particular, there is no discontinuity in the bidding function, and no type bids more than in the unconstrained case. It can easily be checked that the all-pay auction dominates here both in terms of revenue and efficiency.

Proposition 3. With linear cost functions, the expected sum of bids is an increasing function of the bid cap d and of the number of bidders n.

In particular, the last result shows that regardless of the number of bidders, and for all distribution functions, setting an upper bound on bids is not profitable for a designer facing *ex ante* symmetric bidders with linear costs. This result should be contrasted with that of Che and Gale (1998a), who showed that with asymmetric and completely informed bidders, a bid cap may be advantageous for the designer. Although the effect they identify is clearly present in our framework for specific realizations of types, the previous proposition shows that no matter the distribution of valuations, the average loss associated with handicapping high-valuation agents dominates the average gain from having more aggressive middle-valuation agents.

4. Nonlinear cost functions

■ In this section we allow the cost function g to depend nonlinearly on the bid x. We denote by g^{-1} the inverse function of the strictly increasing function g.

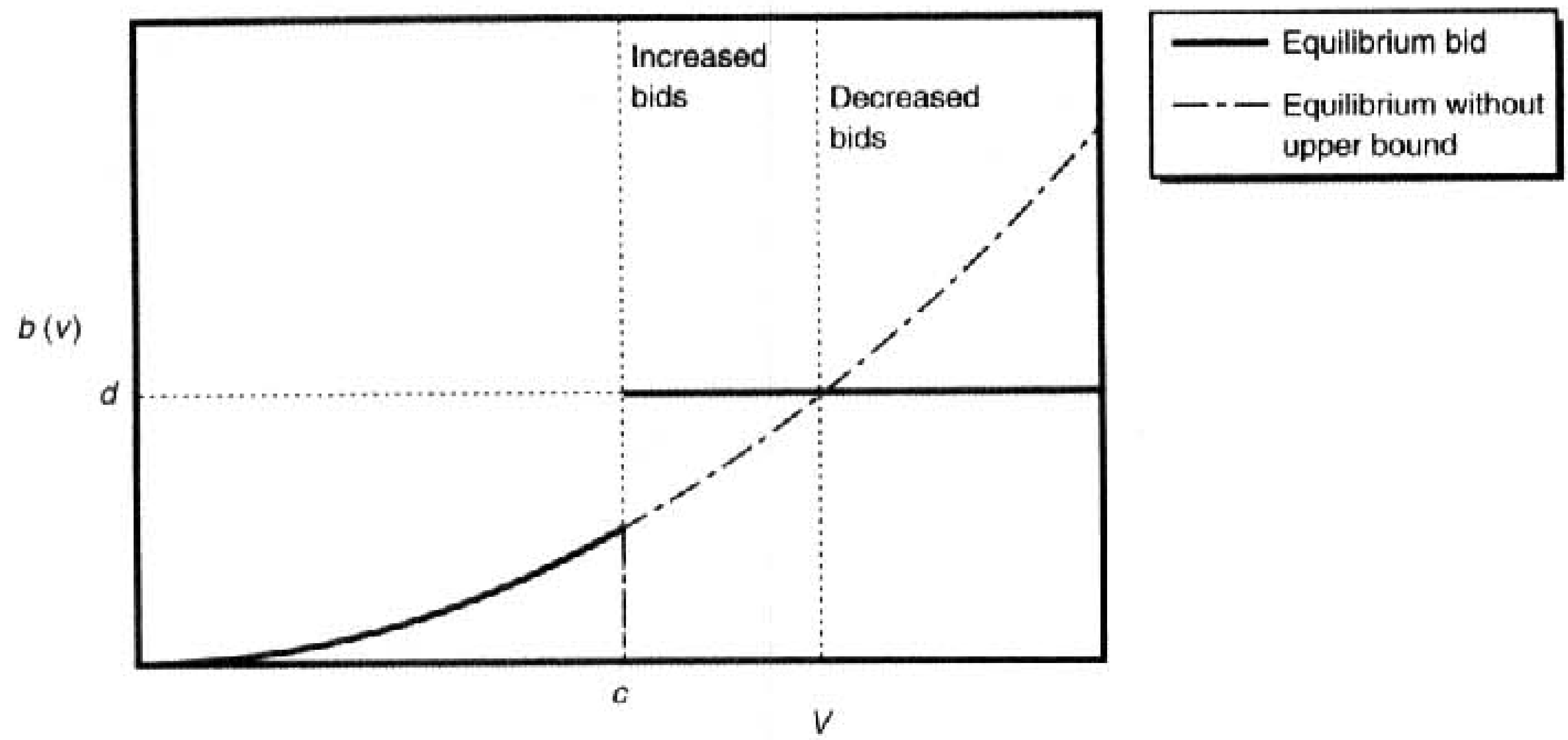
Proposition 4. Consider an all-pay auction where n bidders have a cost function g and face a bid cap d such that $0 < d \leq g^{-1}(1 - \int_0^1 F^{n-1}(y)dy)$.[17] In a symmetric equilibrium, the bid function of every agent is given by

$$b(v) = \begin{cases} g^{-1}\left(\tilde{b}(v)\right) & \text{if } 0 \leq v < c \\ d & \text{if } c \leq v \leq 1, \end{cases} \tag{4}$$

[16] Laffont and Robert (1996) use a direct-revelation approach to calculate, for each type, the equilibrium probability of getting the prize and the payment in a revenue-maximizing mechanism with a fixed budget constraint that is not subject to variation. Their approach (which employs an additional regularity condition on hazard rates) can be also used to derive the equilibrium here.

[17] As in the case of linear cost functions, higher bid caps have no influence on bidding behavior.

FIGURE 1



where the critical value c is defined by

$$d = g^{-1}\left(\frac{c(1 - F^n(c))}{n(1 - F(c))} - \int_0^c F^{n-1}(y)dy\right). \tag{5}$$

Proof. Analogous to that of Proposition 2.

We first show that, similar to the linear case, when bidders have concave cost functions, setting an effective bid cap is not profitable for the seller.

Proposition 5. With concave cost functions, the expected sum of bids is an increasing function of the bid cap d.

In contrast, the next result shows that capping the bids is optimal for a seller facing a large enough number of bidders with convex cost functions.

Proposition 6. In any contest where the bidders have convex cost functions, an effective bid cap d increases the designer's revenue if (1) the number of bidders n is large enough or (2) the cost functions' degree of convexity (measured by the Arrow-Pratt coefficient) is large enough.[18]

Example 2. Assume that $F(v) = v$ and that $g(x) = x^m$, $m > 1$. Let d, $0 < d < [(n-1)/n]^{1/m}$, be the bid cap. The symmetric equilibrium bid function is

$$b(v) = \begin{cases} \left(\frac{n-1}{n}v^n\right)^{1/m} & \text{if } 0 \leq v < c \\ d & \text{if } c \leq v \leq 1. \end{cases}$$

The critical value c as a function of d is given by

$$d = \left(\frac{c(1 - c^{n-1})}{n(1-c)}\right)^{1/m}.$$

The average bid of an agent is

$$U(c, n) = \int_0^c b(v)dv + (1-c)d = \left(\frac{n-1}{n}\right)^{1/m} \frac{c^{n/m}}{\frac{n}{m}+1} + \frac{1}{n^{1/m}}\left(c(1-c)^{m-1}(1-c^{n-1})\right)^{1/m}.$$

[18] For this result we assume that density f is once-differentiable and that the cost function g is three times continuously differentiable.

The optimal critical value c is obtained by the equation $dU(c, n)/dc = 0$. It can be verified that for large enough n, $dU(1, n)/dc < 0$, and therefore the optimal value is strictly less than one. That is, setting an upper bound $d > 0$ is profitable for the seller. If n is large enough, we can ignore terms that are exponentially small, and it can be verified that the optimal critical value is $c \approx 1/m$. Inserting this expression for d, we obtain that the optimal bid cap for large n is

$$d \sim \left(\frac{1}{n(m-1)}\right)^{1/m}.$$

Let $\tilde{b}^{-1}(d)$ be the type that places a bid of d in the symmetric equilibrium of the auction without bid caps. We have

$$\tilde{b}^{-1}(d) \sim \left(\frac{1}{n(m-1)}\right)^{1/n}.$$

Note that $\int_c^{\tilde{b}^{-1}(d)}(d - \tilde{b}(v))dv$ is the average gain of the seller in the auction with bid cap d relative to the same auction without a bid cap (where $\tilde{b}(v)$ is the symmetric equilibrium bid function in the absence of caps). Likewise, $\int_{\tilde{b}^{-1}(d)}^1(\tilde{b}(v) - d)dv$ is the average loss of the seller in the auction with bid cap d relative to the same auction without a bid cap.

Since the bid function $\tilde{b}$ is convex, the areas of the triangles $\Delta\ BAO$ and $\Delta\ TOS$ in Figure 2 satisfy

$$\Delta BAO = \frac{1}{2}(\tilde{b}^{-1}(d) - c)(d - \tilde{b}(c)) < \int_c^{\tilde{b}^{-1}(d)} \left(d - \tilde{b}(v)\right) dv,$$
$$\Delta TOS = \frac{1}{2}(\tilde{b}(1) - d)(1 - \tilde{b}^{-1}(d)) > \int_{\tilde{b}^{-1}(d)}^{1} \left(\tilde{b}(v) - d\right) dv, \tag{6}$$

We proceed to show that for n large enough, $\Delta BAO > \Delta TOS$. By (6), this shows that setting a bid cap is profitable for the seller.

Simple calculations (neglecting exponentially small components) yield

$$\Delta BAO \sim \left(\frac{1}{n}\right)^{1/m} C_1,$$
$$\Delta TOS \sim \left(1 - \left(\frac{1}{n(m-1)}\right)^{1/n}\right) C_2,$$

where C_1, C_2 are constants.

By L'Hôpital's rule, we obtain that

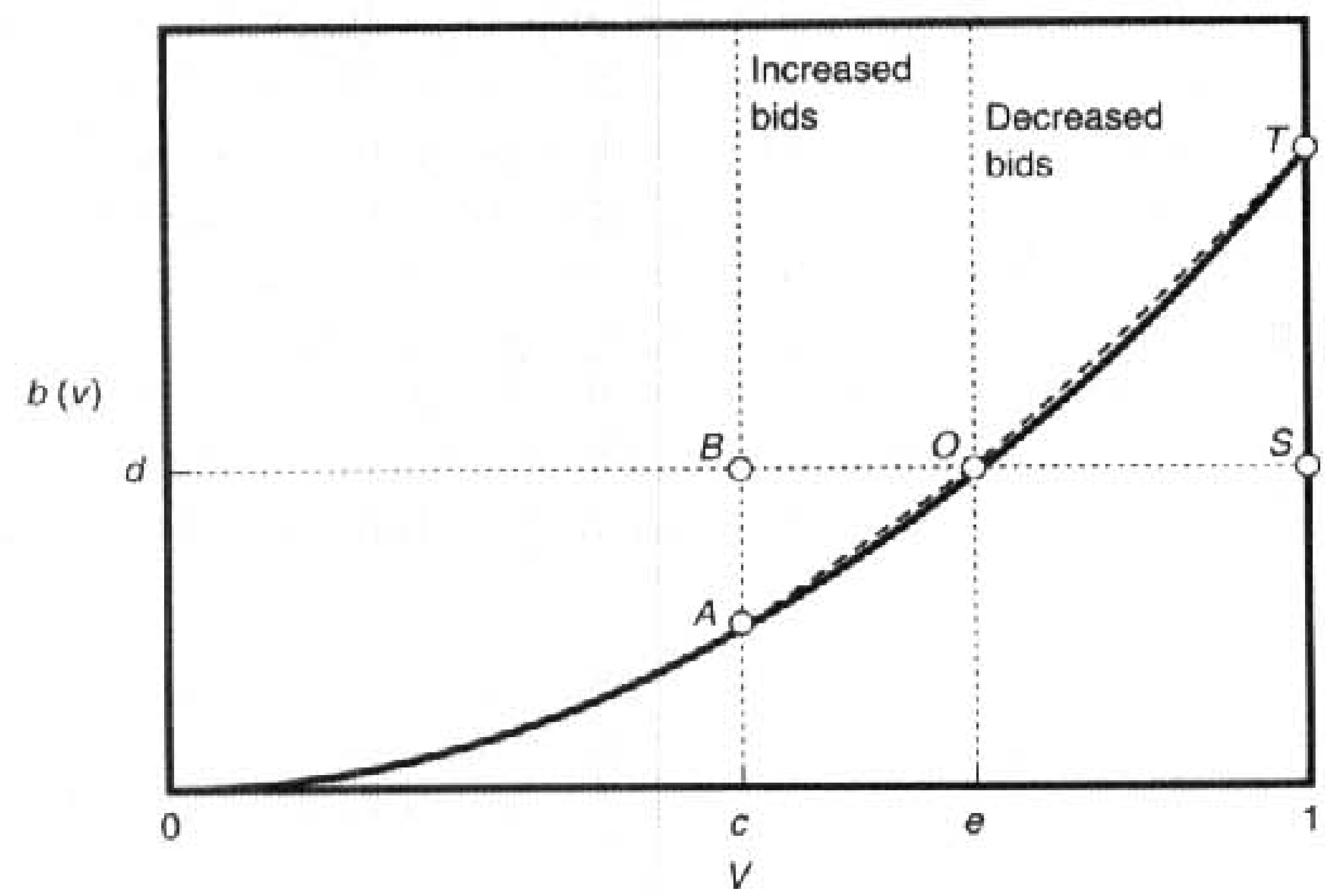
$$\lim_{n\to\infty} \frac{\Delta BAO}{\Delta TOS} \sim \lim_{n\to\infty} C_3 n^{1-1/m},$$

where C_3 is a constant.

For $m > 1$ (i.e., for convex cost functions), the ratio $\Delta BAO/\Delta TOS$ goes to infinity when n approaches infinity.

□ **Minimum bids and optimal mechanism design.** For a fixed and binding budget constraint d, Laffont and Robert (1996) have shown that an all-pay auction with a reserve price r is the revenue-maximizing mechanism. Note that in this framework, standard auctions do not necessarily allocate the good to the agent with the highest valuation, and they are not any more revenue equivalent. If the seller can vary d itself, the question arises whether bid caps remain effective when bidders with lower valuations can be excluded from the auction. The next example shows

FIGURE 2



that if bidders have convex cost functions, setting a maximum bid may be profitable for the designer even if she is allowed to impose a minimum bid.

Example 3. Assume that the seller imposes a minimum bid $r > 0$, and a bid cap $d > r$. Bidders have quadratic cost functions, i.e., $g(x) = x^2$, and uniformly distributed values over $[0, 1]$, i.e., $F(v) = v$.

Let $h = h(r)$ be the lowest type that makes a bid of at least r, and let c be the lowest type for whom placing a bid d is a best reply (this is the critical value from above).

The average bid is given by

$$U(c, h, n) = \int_h^c \sqrt{\frac{n-1}{n} v^n + \frac{h^n}{n}} dv + (1-c)\sqrt{\frac{c(1-c^{n-1})}{n(1-c)} + \frac{h^n}{n}}.$$

A numerical analysis reveals that $dU(c, h, n)/dh < 0$ for n large enough and for all $c < 1$. This implies that for n large enough, the optimal h equals zero, i.e., the optimal r also equals zero. Hence, the previous analysis applies and bid caps are optimal for sufficiently many bidders.

If the cap d is controlled by the designer, and if the *ex ante* symmetric contestants have linear cost functions, our Proposition 3 shows that a binding cap is detrimental for the designer. Myerson's (1981) revenue equivalence result and his optimality analysis show then that an all-pay auction without a bid cap but with a reserve price is revenue maximizing.

The revenue-maximizing mechanism for nonlinear cost functions is yet unknown, and may not be an all-pay auction.[19] Besides the advantage of bid caps displayed in the present article, Moldovanu and Sela (2001a, 2001b) show that a designer facing contestants with convex cost functions can improve her payoff by awarding several prizes or by organizing several subcontests. Even a combination of bid caps, several prizes, and several subcontests may be advantageous in some cases. The merit of each such procedure depends on the form of the function governing the distribution of valuations. In general, we conjecture that even with *ex ante* symmetric bidders, the precise form of the revenue-maximizing mechanism will depend on the distribution function, thus reducing the practical applicability of such a scheme. In contrast, in this article we identified a quite robust role for bid caps in a standard contest whose rules are fixed and do not depend on features unlikely to be known to the designer.

[19] For example, a referee suggested that a first-price auction should do better than an all-pay auction with concave cost functions (since having one bidder pay a large amount is cheaper than having many pay a small amount each).

5. Concluding remarks

■ We have studied a model of all-pay contests where the designer can restrict bids from above. This feature is often observed in real-life situations. In an *ex ante* symmetric model, we have related the effectiveness of bid caps to the form of the bidding costs borne by the contestants. Bid caps lower the bids of high-valuation (or high-ability) types but increase the bid of middle-valuation types. Moreover, caps increase the average bid if the contestants face increasing marginal costs, but they decrease it if the bidders face constant or decreasing marginal costs. These results also have several implications for auctions with financially constrained bidders where the seller can provide financing. A possible extension is the study of the interplay between exogenous constraints (such as budget limitations, which may be private information) and endogenous constraints, controlled by the designer.

Appendix

■ Proofs of Propositions 2, 3, 5, and 6 follow.

Proof of Proposition 2. Assume that the bid function of every bidder j, $j \neq i$, is given by

$$b(v) = \begin{cases} \tilde{b}(v) = vF^{n-1}(v) - \int_0^v F^{n-1}(y)dy & \text{if } 0 \leq v < c \\ d & \text{if } c \leq v \leq 1, \end{cases}$$

where the critical value c satisfies

$$d = \frac{c(1 - F^n(c))}{n(1 - F(c))} - \int_0^c F^{n-1}(y)dy. \tag{A1}$$

Note first that $c = 1$ solves equation (A1) for $d = 1 - \int_0^1 F^{n-1}(y)dy$, and $c = 0$ solves that equation for $d = 0$. Moreover, equation (A1) has a unique solution in the interval [0, 1] for each d, $0 < d \leq 1 - \int_0^1 F^{n-1}(y)dy$, since the function

$$H(c) = \left(\frac{c(1 - F^n(c))}{n(1 - F(c))} - \int_0^c F^{n-1}(y)dy\right) = \left[\frac{c}{n}(1 + F(c) + \cdots F^{n-1}(c)) - \int_0^c F^{n-1}(y)dy\right]$$

is strictly increasing on [0, 1]. For this last point, observe that for $c > 0$,

$$\begin{aligned} H'(c) &= \frac{c}{n}[f(c) + \cdots (n-1)F^{n-2}(c)f(c)] + \frac{1}{n}[1 + F(c) + \cdots F^{n-1}(c)] - F^{n-1}(c) \\ &> \frac{1}{n}[1 + F(c) + \cdots F^{n-1}(c)] - F^{n-1}(c) > 0. \end{aligned} \tag{A2}$$

The last inequality follows because $F^{n-1}(c) < F^k(c)$ for all $k = 0, \ldots, n-2$. In particular, this shows that the critical value $c = c(d)$ is a strictly increasing function of the bid cap d (as long as the cap binds).

We now show that $b_i(v) = b(v)$ is the best response of bidder i against the other bidders' strategies. The maximization problem of bidder i with valuation $v \leq c$ is given by

$$\max_x \left(vF^{n-1}(b^{-1}(x)) - x\right),$$

subject to $x \leq d$.

Assuming that $b_i(v)$ is continuously differentiable for all $v \leq c$, we obtain the first-order condition

$$v(n-1)F^{n-1}\left(b^{-1}(x)\right) f\left(b^{-1}(x)\right)\left(b^{-1}(x)\right)' - 1 = 0. \tag{A3}$$

It can be verified that $x = b(v)$ is a solution of the differential equation in (A3).

Consider now $v > c$. Obviously a bid x such that $b(c) < x < d$ is not a best response for bidder i, since every such bid is dominated by $x - \varepsilon$, where ε is a small positive number.

The expected payoff of bidder i with valuation c that submits the bid $b(c) = cF^{n-1}(c) - \int_0^c F^{n-1}(y)dy$ is

$$cF^{n-1}(c) - b(c) = \int_0^c F^{n-1}(y)dy. \tag{A4}$$

The probability of winning with a bid of d is

$$\begin{aligned}
\Pr(win) &= F^{n-1}(c) + \frac{1}{2}\binom{n-1}{1}F^{n-2}(c)(1-F(c)) + \frac{1}{3}\binom{n-1}{2}F^{n-3}(c)(1-F(c))^2 + \\
&\quad + \frac{1}{4}\binom{n-1}{3}F^{n-4}(c)(1-F(c))^3 + \cdots\cdots\cdots + \frac{1}{n}\binom{n-1}{n-1}(1-F(c))^{n-1} \\
&= \sum_{j=1}^{n}\frac{1}{j}\binom{n-1}{j-1}F^{n-j}(c)(1-F(c))^{j-1} \\
&= \frac{1}{n(1-F(c))}\sum_{j=1}^{n}\binom{n}{j}F^{n-j}(c)(1-F(c))^{j} \\
&= \frac{1-F^n(c)}{n(1-F(c))}.
\end{aligned}$$

Thus, the expected payoff of bidder i with valuation of c that submits a bid of d is

$$\frac{c(1-F^n(c))}{n(1-F(c))} - d. \tag{A5}$$

By setting (A1) in (A5), we obtain that bidder i with valuation $v = c$ is exactly indifferent between submitting $b(c) = cF^{n-1}(c) - \int_0^c F^{n-1}(y)dy$ and submitting the maximum allowed bid d. Similarly, all types $v > c$ strictly prefer a bid $b_i(v) = d$ to any lower bid.

Finally, we want to show that for any $d < 1 - \int_0^1 F^{n-1}(y)dy$, we have $c < \tilde{b}^{-1}(d)$, so that the constrained equilibrium bid function displays a discontinuity at the critical value c. For this it is enough to show that

$$\begin{aligned}
\tilde{b}(c) &< d \quad \Leftrightarrow \\
cF^{n-1}(c) - \int_0^c F^{n-1}(y)dy &< \frac{c(1-F^n(c))}{n(1-F(c))} - \int_0^c F^{n-1}(y)dy \Leftrightarrow \\
F^{n-1}(c) &< \frac{1}{n}\left(1 + F(c) + F^2(c) + \cdots F^{n-1}(c)\right).
\end{aligned}$$

The last inequality clearly holds, since $F^{n-1}(c) < F^k(c)$ for any $c < 1$ and for any k, $0 \leq k \leq n-2$. *Q.E.D.*

Proof of Proposition 3. Fix the number of bidders n. Given the bid cap d (or, alternatively, the critical value c), each agent's average bid is given by

$$\begin{aligned}
U(c,n) &= \int_0^c b(v)f(v)dv + d\int_c^1 f(v)dv \\
&= \int_0^c vF^{n-1}(v)f(v)dv - \int_0^c (\int_0^v F^{n-1}(y)dy)f(v)dv + d(1-F(c)) \\
&= \frac{vF^n(v)}{n}\bigg|_0^c - \frac{1}{n}\int_0^c F^n(v)dv - (F(v)\int_0^v F^{n-1}(y)dy \,|_0^c - \int_0^c F^n(v)dv) + d(1-F(c)) \\
&= \frac{c}{n} - \int_0^c F^{n-1}(v)dv + \frac{n-1}{n}\int_0^c F^n(v)dv.
\end{aligned}$$

Differentiating with respect to c gives

$$\frac{\partial U(c,n)}{\partial c} = \frac{1}{n} - F^{n-1}(c) + \frac{n-1}{n}F^n(c). \tag{A6}$$

Multiplying by n and recalling that $1 - s^n = (1-s)(1+s+s^2+s^3+\cdots+s^{n-1})$ yields

$$\frac{\partial[nU(c,n)]}{\partial c} = (1-F(c))(1+F(c)+F^2(c)+\cdots+F^{n-2}(c)-(n-1)F^{n-1}(c))$$

$$> (1 - F(c))((n-1)F^{n-2}(c) - (n-1)F^{n-1}(c)) > 0.$$

That is, the designer's expected revenue, $nU(c, n)$, increases in c.

We now analyze the dependence of the expected revenue on the number of bidders. For a fixed d, let $c(n)$ be the critical value (we now make the dependence on n explicit). Observe that $c(n)$ increases in n. Since $U(c, n)$ increases in c, we obtain

$$(n+1)U(c(n+1), n+1) \geq (n+1)U(c(n), n+1).$$

This yields

$$\begin{aligned}(n+1)U\,(c(n+1), n+1) - nU(c(n), n) &\geq \\ (n+1)U(c(n), n+1) - nU(c(n), n) &= \\ \int_0^c nF^{n-1}(v)(1 - F(v))^2\,dv &> 0.\end{aligned}$$

Thus, $nU(c(n), n)$ increases in the number of bidders n. *Q.E.D.*

Proof of Proposition 5. An agent's average bid as a function of the critical value and the number of bidders is given by

$$U(c, n) = \int_0^c b(v)f(v)dv + d(c)(1 - F(c)), \tag{A7}$$

where

$$b(v) = g^{-1}\left(vF^{n-1}(v) - \int_0^v F^{n-1}(y)dy\right)$$

and

$$d(c) = g^{-1}\left(\frac{c(1 - F^n(c))}{n(1 - F(c))} - \int_0^c F^{n-1}(y)dy\right).$$

Define

$$\begin{aligned} m = m(c) &\equiv \frac{c(1 - F^n(c))}{n(1 - F(c))} - \int_0^c F^{n-1}(y)dy = \frac{c}{n}\sum_{j=0}^{n-1} F^j(c) - \int_0^c F^{n-1}(y)dy \\ s = s(c) &\equiv \frac{c(1 - F^n(c))}{n(1 - F(c))} - cF^{n-1}(c). \end{aligned} \tag{A8}$$

Then, we can rewrite $d(c)$ and $b(c)$ as

$$d(c) = g^{-1}(m); \qquad b(c) = g^{-1}(m - s). \tag{A9}$$

Expanding $g^{-1}(m - s)$ to a second-order Taylor's series near m, we obtain

$$b(c) = g^{-1}(m) - s\left(g^{-1}(x)\right)'_{x=m} + \frac{1}{2}s^2\left(g^{-1}(x)\right)''_{x=y}, \tag{A10}$$

where $m - s < y < m$. Derivating $d(c)$ with respect to c yields

$$d'(c) = (g^{-1}(x))'_{x=m}\left(\frac{1 - F^n(c) - ncf(c)F^{n-1}(c)}{n(1 - F(c))} + \frac{cf(c)(1 - F^n(c))}{n(1 - F(c))^2} - F^{n-1}(c)\right). \tag{A11}$$

Differentiating with respect to c in (A7) and substituting (A9), (A10), and (A11), we obtain

$$\begin{aligned}\frac{\partial U(c, n)}{\partial c} &= (b(c) - d(c))f(c) + (1 - F(c))d'(c) \\ &= \frac{1}{n}\left(g^{-1}(x)\right)'_{x=m}\left[1 - F^n(c) - nF^{n-1}(c) + nF^n(c)\right] + \frac{1}{2}f(c)\left(g^{-1}(x)\right)''_{x=y}s^2.\end{aligned} \tag{A12}$$

Observe that $1 - F^n(c) = (1 - F(c))\left(1 + F(c) + F^2(c) + \cdots F^{n-1}(c)\right)$. Thus, we have

$$\frac{\partial U(c,n)}{\partial c} = \frac{1}{n}\left(g^{-1}(x)\right)'_{x=m}(1 - F(c))\left(1 + F(c) + F^2(c) + \cdots F^{n-1}(c) - nF^{n-1}(c)\right) + \frac{1}{2}f(c)\left(g^{-1}(x)\right)''_{x=y}s^2. \tag{A13}$$

Since g is increasing and concave, we have $(g^{-1}(x))' > 0$ and $(g^{-1}(x))'' > 0$. Therefore, the average bid is an increasing function of the critical value c, and it is never optimal to set an effective bid cap. *Q.E.D.*

Proof of Proposition 6. We assume here that f is continuously differentiable at $x = 1$, and that g^{-1} is three times continuously differentiable.

We will show that for any convex cost function g, we have $\partial U(c,n)/\partial c < 0$ near $c = 1$ for sufficiently large n. Hence the designer's revenue is decreasing in c near $c = 1$, and an effective bid cap (i.e., $c < 1$) is optimal.

The first equality in (A12) yields that $U'(1,n) = 0$. For the second derivative we have

$$\left.\frac{\partial^2 U(c,n)}{\partial c^2}\right|_{c=1} = \left[(b' - d')f + (b - d)f' - fd' + (1 - F)d''\right]\Big|_{c=1}. \tag{A14}$$

By (A9) we have

$$b'(1) = (g^{-1})'(1 - \int_0^1 F^{n-1}(y)dy)(n-1)f(1), \tag{A15}$$

and by (A8) and (A9) we have

$$\begin{aligned} d'(1) &= (g^{-1}(x)'_{x=m(1)}m'(1) = (g^{-1}(x)'_{x=m(1)}\left(\frac{1}{n}f(1)\sum_{j=1}^{n-1}j\right) \\ &= (g^{-1})'\left(1 - \int_0^1 F^{n-1}(y)dy\right)\frac{1}{2}(n-1)f(1). \end{aligned} \tag{A16}$$

That is, we obtain that

$$b'(1) = 2d'(1). \tag{A17}$$

Define now

$$t(c) \equiv cF^{n-1}(c) - \int_0^c F^{n-1}(y)dy.$$

Observe that $m(1) = t(1) = 1 - \int_0^1 F^{n-1}(y)dy$. Hence $\left(g^{-1}(m(1))\right)' = \left(g^{-1}(t(1))\right)'$ and $\left(g^{-1}(m(1))\right)'' = \left(g^{-1}(t(1))\right)''$. Since $b(c) = g^{-1}(t(c))$ and $d(c) = g^{-1}(m(c))$, we have

$$b''(1) = \left(g^{-1}(t(1))\right)''(n-1)^2f^2(1) + \left(g^{-1}(t(1))\right)'\left((n-1)f(1) + (n-1)(n-2)f^2(1) + (n-1)f'(1)\right) \tag{A18}$$

$$d''(1) = \frac{1}{4}\left(g^{-1}(m(1))\right)''f^2(1)(n-1)^2 + \left(g^{-1}(m(1))\right)'\left(\frac{1}{2}f'(1)(n-1) + \frac{f^2(1)}{n}\sum_{j=0}^{n-1}j(j-1)\right). \tag{A19}$$

Note that $F(1) = 1$ and that $d''(1) < \infty$. By (A14) and (A17), we now obtain that

$$\left.\frac{\partial^2 U(c,n)}{\partial c^2}\right|_{c=1} = \left[(b' - d')f + (b - d)f' - fd' + (1 - F)d''\right]\Big|_{c=1} = \left(b'(1) - 2d'(1)\right)f(1) = 0.$$

We now show that the third derivative is strictly negative at $c = 1$. By continuity, it must be negative for some $c < 1$. We have that

$$\frac{\partial^3 U(c,n)}{\partial c^3} = (b'' - d'')f + 2(b' - d')f' + (b - d)f'' - f'd' - 2fd'' + (1 - F)d'''.$$

It can be verified that $d'''(1)$ is finite. After rearranging terms and using (A17), we obtain

$$\left.\frac{\partial^3 U(c,n)}{\partial c^3}\right|_{c=1} = \left(b''(1) - 3d''(1)\right)f(1) + d'(1)f'(1).$$

Recalling that $\sum_{j=0}^{n-1} j(j-1) = (1/3)n(n-2)(n-1)$ and using (A16), (A18), and (A19), we finally obtain

$$\left.\frac{\partial^3 U(c,n)}{\partial c^3}\right|_{c=1} = (b''(1) - 3d''(1))f(1) + d'(1)f'(1) =$$
$$= \frac{1}{4}(n-1)^2 f^2(1)\left(g^{-1}(m(1))\right)'' + (n-1)\left(g^{-1}(m(1))\right)' f^2(1) =$$
$$= (n-1)f^2(1)\left[\frac{1}{4}(n-1)\left(g^{-1}(m(1))\right)'' + \left(g^{-1}(m(1))\right)'\right]. \tag{A20}$$

By the convexity of $g(x)$, we have $(g^{-1}(m(1))'' < 0$. Hence, for n large enough, we find that $(1/4)(n-1)(g^{-1}(m(1))'' + (g^{-1}(m(1))' < 0$ and thus $[\partial^3 U(c,n)/\partial c^3]|_{c=1} < 0$. Since $\partial^2 U(c,n)/\partial c^2]|_{c=1} = 0$, we obtain that $c = 1$ is a maximum of the function $\partial U(c,n)/\partial c$. Since $[\partial U(c,n)/\partial c]|_{c=1} = 0$, this yields $\partial U(c,n)/\partial c < 0$ for c in the vicinity of 1, as desired.

For the second part of the proposition, let n_0 be the lowest number that satisfies the inequality $\partial^3 U(c,n)/\partial c^3 < 0$. By (A20), the critical number n_0 is obtained by the equation

$$-\frac{\left(g^{-1}(m(1))\right)''}{\left(g^{-1}(m(1))\right)'} = \frac{4}{n-1}.$$

Thus, the larger the degree of convexity (as measured by the Arrow-Pratt coefficient), the smaller n_0 becomes. *Q.E.D.*

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