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Price–quality relationship in the presence of asymmetric dynamic reference quality effects

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Abstract The purpose of this study is to examine the relationship between price and reference quality and their combined effect on profits. An analytical modeling approach aimed at solving the optimal solution for the profit maximization problem under these conditions is developed, enabling the exact path of the optimal price and quality over time to be depicted. Based on separating the effects of price and reference quality on demand, this analysis also provides insight into the contribution of these two effects to the steady-state solution through elasticities. Our results show that a monotonic inverse relationship exists between price and quality, such that a steady-state level is obtained where the quality–price ratio is lower when reference quality effects exist than when such effects do not exist. In other words, consumers obtain higher quality for a higher price but with a lower price per unit of quality. Overall, accounting for reference quality effects will increase a firm's profits.

Keywords Reference quality · Price elasticity · Optimal pricing · Differential games · Equilibrium

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1 Introduction

When designing a positioning strategy for its products, a firm has to decide, among other things, about the price and quality level that will meet its objectives, satisfy consumer preferences, and beat its competition. Given this task, a firm can decide on a low-price, low-quality strategy, a high-price, high-quality strategy, or any mid-range relationship between price and quality.

One important task in this context is to decide the exact level of the two dimensions such that the selected relationship between price and quality reflects the value that consumers ascribe to the firm's products. Thus, a low-price, low-quality strategy should result in a low-price/quality ratio (i.e., a low price for a unit of quality) and a high-price, high-quality strategy should result in a high-price/quality ratio (i.e., a high price for a unit of quality). This task becomes even more complex because these ratios can be achieved at both ends of this price–quality relationship.

Consider the case of price and quality in paper towels. In an analysis of the quality of various brands conducted by Consumer Reports in 2009, an overall score is given to each product to indicate its quality, along with a market price for 100 ft^2 of the product. There is a substantial variation in price and quality among the 20 different products in the category. Though perceived quality measures are more commonly used to assess product quality, we note that objective measures are also used for that purpose in the literature (e.g., Gerstner 1985; Hardie et al. 1993). For example, the 365 Everyday Value brand (by Whole Foods) costs $2.42 and provides 25 points of quality while the Brawny brand costs $3.61 and provides 72 points of quality. Calculating the ratio between price and quality tells us which brand offers a higher value or a lower cost for one unit of quality. In this case, the ratio results in a price of $0.0968 for the 365 Everyday Value and $0.0501 for Brawny. It is interesting to note that the higher price, higher quality product provides greater value to the consumer. Managers can decide to provide different values for their products at either the lower end or higher end of the price–quality relationship continuum. For example, 365 Everyday Value could lower its price, raise its quality or do both in order to increase its value (depicted by the ratio of price to quality) and, similarly, Brawny could take some measures to change its value. As noted earlier, this decision is a matter of positioning and part of a strategic decision about the firm's objectives. A similar ratio of price to quality could be achieved by high-quality, high-price products and low-quality, low-price products. This decision, however, should also take into account the expected effects on the potential profits of the firm that result from this decision.

Looking at the data on the paper towels, which might be viewed as a case of steady state in this product category, we see a relationship between price and quality, as illustrated in Fig. 1. A simple linear regression reveals a positive relationship between price and quality ($b = 0.01586$, t value $= 1.819$, and $R^2 = 0.171$), which is not surprising based on the above explanation. Regressing the price/quality ratio against quality (verifying whether there is some tendency towards one of the ends of the price–quality relationship as can be seen in

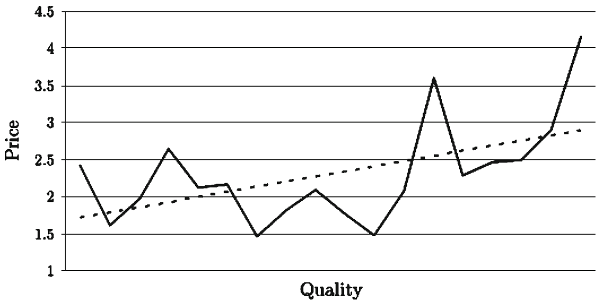

Fig. 1 The relationship between price and quality

Fig. 2), reveals that an inverse relationship between this ratio and quality does exist. A significant −0.00074 quality coefficient and an R^2 of 0.499 indicate that as quality increases, the value (i.e., the price/quality ratio) decreases.

While various explanations can be found in the literature for this type of real-life relationship, such as diminishing returns for quality (demand side) or diminishing returns for the cost of producing quality (supply side), there might be other alternative explanations. For example, what would be the effect of consumers' expectations about the future quality of the product on the price level and on the price–quality ratio? It is interesting, therefore, to investigate whether an inverse relationship between the price and quality, and a strategy about quality is also optimal in terms of profit maximization. Figure 2 demonstrates that marketers certainly engage in such practices. At the same time, we can investigate whether other alternative explanations for this relationship exist. Exploring this issue can be accomplished in a more comprehensive framework that takes into account both the firm's goal of maximizing profit and consumer behavior.

In order to study such an alternative explanation and to examine the profitability of this strategy, we introduce a dynamic demand model that is based on price and reference quality into a profit maximization problem of a repeat purchase product. Using the reference formation process for quality that is based on previous experiences with the product and asymmetric evaluations of gains and losses of quality from a base, or expected, level (Kahneman

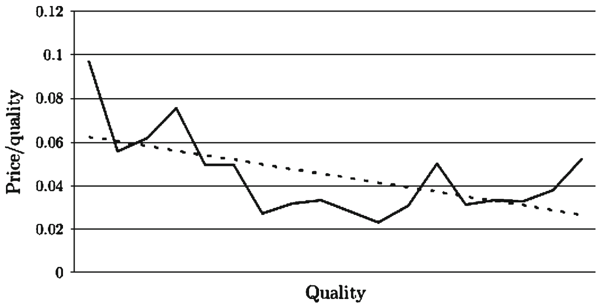

Fig. 2 The relationship between the price–quality ratio and quality

and Tversky 1979), we identify the optimal price and quality level under these conditions by isolating the effect of reference quality on demand. Our results indicate that these levels are higher (in terms of both price and quality) compared with the results of such a model that does not include reference effects in a monopoly case. Furthermore, the price to quality ratio is lower when reference effects are taken into account than when they are ignored. Thus, we show that models that include consideration of consumers' past experience in evaluating the quality of a product will produce a higher price–quality relationship, but at a lower cost per unit of quality. We also present a case of Bertrand competition for this problem and provide indications that the qualitative results presented in the monopoly case might also hold in the competitive case.

A special feature of our analytical modeling approach that differentiates it from previous studies (e.g., Kopalle and Winer 1996) is the ability to separate the effects of reference quality and price on demand through reference quality formation. This type of analysis, therefore, enables us to examine the contribution of each of these components to steady-state levels of the optimal solution. Furthermore, we analyze this contribution by studying the price and quality elasticities at steady state.

This analysis reveals that when reference quality effects exist, consumers' quality and price elasticities increase. The higher the quality, the higher the price at a steady-state level in the presence of reference quality effects compared with the no reference effects.

We first review the literature about the relationship between price and quality. Then, we propose a method for analyzing the reference quality formation process, which is followed by a demand function formulation. Next, we set up the profit maximization problem. Both the demand and profit maximization problems are analyzed first with no reference quality effects. Then these effects are added to depict the exact nature of the effect of reference quality on demand and optimal pricing strategy. This analysis is followed by a Bertrand competition model that will help us understand the strength of the results obtained in the monopoly case. We conclude with a discussion about the proposed analytical approach and its implications.

2 Background

The relationship between price and quality has been the focus of numerous research efforts that have examined different factors that might affect this relationship. Studies exploring external factors (e.g., industry structure and competitors' actions) and internal factors (e.g., cost structure) affecting price and quality can be found in the operations, production, marketing, and economics literature (e.g., Paulson Gjerde and Slotnick 2004). Studies that examine consumer factors also abound in the behavioral literature of both recent times and the distant past (e.g., Monroe and Dodds 1988) and have taken many forms and directions.

An expansion of the traditional consumer evaluation process (the utility derived from absolute valuations of product attributes) is the inclusion of relative evaluations through reference points in this process. Based on a substantial body of behavioral literature that examines the relationship between new points of information (e.g., current product attribute level) and established reference points of information (e.g., reference attribute level), we can infer that consumers use their own experience with a product to form expectations about a product's attributes that affects their purchasing behavior. The results of these studies (e.g., Winer 1986) demonstrate that consumers do behave as if they have a reference point (price; RP) to which they compare the actual shelf price P (Winer 1988). In addition, they have a different evaluation for gains (i.e., RP $>$ P) and losses (i.e., RP $<$ P), such that losses loom larger than gains, as posited by the Prospect Theory (Kahneman and Tversky 1979). Furthermore, dynamic optimal pricing strategies accounting for reference price effects show that in steady-state conditions the optimal price is higher than the price level without reference effects. This dynamic modeling approach also shows higher profits for such an optimal pricing strategy. All of these effects are mainly attributed to the idea that consumers update their reference price in light of new price information, where different possible reference price formation processes might take place (see Lowengart 2002, for a review).

The notion of reference points and the marketing mix has been extended beyond those related to price. Lattin and Bucklin (1989) introduce reference promotion into an empirical investigation of consumers' choice and find significant reference effects. Hardie et al. (1993) use reference quality (objective) in a choice model that is based on a generalized prospect theory value function (Tversky and Kahneman 1991) and show that this effect is significant. Kopalle and Winer (1996) introduce reference quality into the modeling area of reference effects together with reference price effects. Narasimhan and Mendez (2001) use a certain type of reference process to describe quality, under the assumption that price and quality are held constant, and examine the convergence toward steady-state conditions in a diffusion process with repurchases (Bass 1969).

An intuitive and interesting avenue to explore within this reference point framework is the dynamic relationship between reference quality and price. Can we gain a better understanding of this relationship when consumers develop a mental reference point about quality that is based on their experience with the product? The purpose of this article is to expand such an understanding by developing a dynamic analytical model of profit optimization with demand that is based on reference quality and price.

As noted earlier, previous research in this area is rather scarce. The work of Kopalle and Winer (1996) is a pioneering effort in this field, as it analyzes reference effects on both price and quality. The reference quality formation process is dependent not only on differences between quality and reference quality, but also on the gap between the price and the reference price. The rate of updating reference quality effects, therefore, is linked to the rate of updating reference price effects. As such, this structure is somewhat less flexible in the identification of the relative effect of price and quality. Kopalle and Winer use a numerical analysis to obtain solutions for the optimization problem for various sets of parameters. In Narasimhan and Mendez (2001), profit maximization and the resulting optimal pricing strategies are not analyzed.

Our modeling approach to analyzing this problem is different, in that it takes an analytical approach towards the profit maximization problem. Furthermore, we relax the structured dependency between quality and price, as in Kopalle and Winer (1996), and separate the effect of reference quality from that of price. As a result, we are able to isolate the effect of each variable, examine its impact on profit maximization and obtain additional insights into the relative effects of price and quality.

The proposed analytical modeling approach is not limited to a specific set of parameters but rather provides a general solution for this problem. As a result, we can analyze optimal price and quality levels and other properties of the optimal solution that cannot be obtained otherwise. For example, it is possible to find the relationship between reference quality and demand at steady-state conditions or its effect on price. Furthermore, the proposed approach enables us to identify the reference quality and price elasticities that can not be obtained otherwise.

Further insights into the relationship between optimal price and optimal quality can also be obtained using this modeling approach. Analyzing the optimal price and quality levels at a steady state as well as in the introductory phase of a new product launch yield some interesting results that cannot be achieved otherwise. For example, the trajectories of both price and quality over time can be depicted, as can the relationship between them. In fact, our results indicate that at steady-state conditions, both price and quality will be greater than when there are no reference quality effects. In other words, when consumers use their experience to make predictions about the quality of future purchases of the product, they will enjoy higher quality products, but at a higher price. Furthermore, the initial reference quality level compared with the steady-state level determines the quality's trajectory path. When the initial reference quality level is above the steady-state solution, a low initial quality level that increases over time in the introductory phase will occur. Surprisingly, a decrease in the initial high-quality level of the product over time is the optimal solution if the initial reference quality level is below the steady-state solution. In addition, we calculate the price and quality elasticities of demand. The findings demonstrate that when reference quality is taken into account, elasticities are greater than in the case of no reference quality.

We begin the analysis with the reference quality formation process, followed by the formulation of a demand function and then the setting up of the profit maximization problem. Both the demand function and the profit maximization problem are analyzed first with no reference quality effects. Then these effects are added to depict the exact nature of the effect of reference quality on demand and optimal pricing strategy.

3 Reference quality formation

We adopt the commonly used reference point formation process as it has been applied in the case of pricing (e.g., Sorger 1988) and quality (Kopalle and Winer 1996). It is assumed that consumers' expectations with respect to quality evolve over time and are based on their previous experience with the product. Assuming that distant interactions with the product are recalled less, the functional form of this process is operationalized with an exponential smoothing process of historical quality that represents a decaying memory effect as follows:

$$r_q(t) = e^{-\beta t}\left[r_q^0 + \beta \int_0^t e^{\beta s} q(s)\,\mathrm{d}s\right], \tag{1}$$

where $r_q(t)$ is the reference quality at time t, $q(t)$ is the product quality at time t, β is the memory parameter, and r_q^0 is the initial reference quality at $t = 0$. As demonstrated in Fibich et al. (2005), we can identify the time scale of the reference effects and denote its parameter $\hat{T}_{\mathrm{RQ}}$ such that $\hat{T}_{\mathrm{RQ}} = 1/\beta$. This time scale parameter provides an intuitive way to understand the time frame in which reference quality has an effect on consumer behavior. For example, if a product has a constant quality level, $q(t) = q$, and then a single change in quality level occurs, a gap between the reference quality and the new quality level is created. If the new quality level is kept constant for a long time, it will take consumers several $\hat{T}_{\mathrm{RQ}}$'s time frames (3–4) to adjust their expectations about the product's quality to the new level, $r_q(t) \approx q$ for $t > 3\hat{T}_{\mathrm{RQ}}$.[1] Note that β is the decaying memory parameter, representing consumers' use of older information that can be estimated from empirical data using the relationship $\beta = \ln(1/\eta)/T_{\mathrm{interpurchase}}$ where $T_{\mathrm{interpurchase}}$ is the average time that elapsed between a consumer's exposure to information about quality (e.g., the average interpurchase time of the product that is a proxy for the consumers' exposure to price information) and η is the weight a consumer gives to current quality experiences relative to the weight given to the reference quality (Kalyanaram and Little 1994; Lattin and Bucklin 1989). For example, if $\eta = 0.57$ and the interpurchase time between consecutive purchases is 11.4 weeks, then the time scale, $\hat{T}_{\mathrm{RQ}}$, is 20.28 weeks. In such a case (holding the quality level constant after it was changed), it will take about 60 weeks for the new quality level to

[1] If the quality is held constant, $q(t) = q$, then the reference quality can be written as $r(q) = q + \left(r_q^0 - q\right) e^{-\beta t}$ and the exponential term $e^{-\beta t}$ reaches a very small value if $t > 3/\beta = 3\hat{T}_{\mathrm{RQ}}$ because $e^{-3} \approx 0.05$.

be fully utilized in the reference quality formation process. This time scale of reference effects has an effect on calculating price elasticity (Fibich et al. 2005) and price promotion (Fibich et al. 2007). By differentiation of Eq. 1, we can find the relationship

$$\frac{\mathrm{d}r_q}{\mathrm{d}t} = \beta\left[q(t) - r_q(t)\right] \tag{2}$$

with the initial reference quality condition $r_q(0) = r_q^0$.

Next, we consider the demand formation process and the profit maximization problem. We start with a demand function with no reference quality effects and then extend the analysis to a demand with symmetric reference effects. We later present a demand function with asymmetric reference quality effects and analyze the result of optimal profits in a steady state.

3.1 Demand formation and profit maximization with no reference quality effects

We begin with a basic linear demand function with no reference effects

$$Q_{\text{no-ref}} = a + \delta_1 q - \delta_2 p \tag{3}$$

where $Q_{\text{no-ref}}$ is the demand with no reference quality effects, q represents product quality, p price, $\delta_1 > 0$ and $\delta_2 > 0$ are the quality and price parameters, respectively. Next, we consider the profit maximization problem. We assume that the production cost is constant per unit $c \geq 0$ (constant marginal cost).[2] Like Kopalle and Winer (1996), we assume that the producer bears an instantaneous cost of quality, which represents the resources needed to support a specific level of quality at every time t where the coefficient of this cost is rescaled to 1. The cost of quality is made up of the resources and efforts invested in the entire process of quality management and quality control. Furthermore, this cost is independent of other variables in the optimization problem and is affected by organizational knowledge and the technology of quality. We let the cost of quality follow a simple quadratic form (i.e., increasing marginal cost), and the cost parameter is scaled to 1 (i.e., the cost of quality is Cq^2 where $C = 1$). This assumption about the parameter C does not affect the results. Moreover, the cost of quality depends only on the level of quality chosen by the firm. The static maximization problem in the case of no reference quality effects is:

$$\underset{p,q}{\text{MAX}}(p - c)\,Q_{\text{no-ref}} - q^2.$$

[2] Hereafter, we assume the regularity condition $a - \delta_2 c > 0$.

The static optimal price and optimal quality are given by

$$p_{\text{no-ref}}^{\text{optimal}} = \frac{2a + \left(2\delta_2 - \delta_1^2\right)c}{4\delta_2 - \delta_1^2} \quad ,$$

$$q_{\text{no-ref}}^{\text{optimal}} = \frac{(a - \delta_2 c)\,\delta_1}{4\delta_2 - \delta_1^2} \;. \tag{4}$$

4 Demand formation and profit maximization with reference quality effects

In this section, we consider the case of demand with reference quality effects. Based on the Prospect Theory (Kahneman and Tversky 1979), which asserts that losses loom larger than gains, we use an asymmetric reference quality effect on demand. The following demand function represents such an asymmetric effect of reference quality on the optimal quality and pricing strategy.

In order to analyze the effect of reference quality on demand, we compare two stylized models with and without reference effects. As in most of the literature on analytical modeling, the analysis of such a situation enables us to isolate such effects and obtain a comprehensive characterization of the phenomenon at hand.

The demand rate in this case is given by

$$Q = a + \delta_1 q(t) - \delta_2 p(t) + \gamma\left(q(t) - r_q(t)\right) \;. \tag{5}$$

Here $p(t)$ and $q(t)$ are product price and quality, respectively, $r_q(t)$ is reference quality, γ is given by

$$\gamma = \begin{cases} \gamma_{\text{loss}} & q \le r_q \\ \gamma_{\text{gain}} & q > r_q \end{cases} \qquad \text{where} \qquad \gamma_{\text{gain}} \le \gamma_{loss} \;, \tag{6}$$

and reference quality formation is governed by Eq. 2. The non-smooth optimization problem is to find $p(t)$ and $q(t)$ that maximize the total discounted profit between $t = 0$ and $t = \infty$,

$$\underset{p(t),q(t)}{\text{MAX}}\, \Pi[p(t), q(t)] = \int_0^\infty e^{-\alpha t}\Big[(p(t) - c)\,Q(t) - q^2(t)\Big]\,\mathrm{d}t \;, \tag{7}$$

subject to Eqs. 2, 5, and 6 and to an initial reference quality

$$r_q(0) = r_q^0 \;, \tag{8}$$

where $q^2(t)$ represents the instantaneous cost of quality. We note that in the absence of reference quality effects (i.e., $\gamma = 0$) the solution of the dynamic problem is identical to the static problem presented in the previous section. To find the optimal strategy in the presence of asymmetric reference quality effects, we apply the two-stage method as in Fibich et al. (2003). First, in stage I, we solve the symmetric optimization problem with $\gamma_{\text{gain}} = \gamma_{\text{loss}}$. Then in stage II, we generalize to the asymmetric case when $\gamma_{\text{gain}} < \gamma_{\text{loss}}$.

4.1 Profit maximization—the symmetric case

The optimal solution to the problem in Eq. 7 with the conditions Eqs. 2, 5, and 8 is given in the following proposition.

Proposition 1

$$q_{\text{optimal}}(t) = q^{\text{ss}}_{\text{optimal}} + \left(1 - \frac{m}{\beta}\right)\left(r^0_q - q^{\text{ss}}_{\text{optimal}}\right) e^{-mt}, \tag{9}$$

$$p_{\text{optimal}}(t) = p^{\text{ss}}_{\text{optimal}} + \frac{1}{2\delta_2}\left(\delta_1\left(1 - \frac{m}{\beta}\right) - \frac{\gamma m}{\beta}\right)\left(r^0_q - q^{\text{ss}}_{\text{optimal}}\right) e^{-mt}, \tag{10}$$

where the exponent m is given by

$$m = -\frac{\alpha}{2} + \frac{1}{2}\sqrt{\alpha^2 + 4\beta\frac{(\alpha+\beta)(4\delta_2 - \delta_1^2) - \alpha\gamma\delta_1}{4\delta_2 - (\delta_1 + \gamma)^2}}$$

and

$$p^{\text{ss}}_{\text{optimal}} = \frac{2a + \left(2\delta_2 - \delta_1^2 - \delta_1\gamma\frac{\alpha}{\alpha+\beta}\right)c}{4\delta_2 - \delta_1^2 - \delta_1\gamma\frac{\alpha}{\alpha+\beta}}, \quad q^{\text{ss}}_{\text{optimal}} = \frac{(a - \delta_2 c)\left(\delta_1 + \gamma\frac{\alpha}{\alpha+\beta}\right)}{4\delta_2 - \delta_1^2 - \delta_1\gamma\frac{\alpha}{\alpha+\beta}}. \tag{11}$$

Proof See Appendix A. □

Observe that $m > 0$ indicates the convergence of the whole system to a steady-state condition (not only reference quality) under an optimal pricing strategy. The steady-state price and quality levels illustrate a specific dependency on the model parameters that is related to the dynamic aspects of the optimization problem: α, β, and, γ. In other words, all three parameters appear in the form of $\gamma\frac{\alpha}{\alpha+\beta}$ indicating that the impact of reference effects is strongly affected by the time-scale parameters of the optimization problem, α and β . Note that when $\gamma = 0$, there are no reference effects. Thus, $p^{\text{ss}}_{\text{optimal}}$ and $q^{\text{ss}}_{\text{optimal}}$ are the same as in the model that does not include reference effects (Eq. 4).

This result is attributed to the fact that when the discounting factor is very low, retailers can ignore reference effects because these effects have an impact primarily at the introductory stage. The cumulative effect on the optimal solution is relatively low compared with the overall long-term profits. In addition, note that the ratio $\frac{q_{\text{optimal}}(t) - q^{\text{ss}}_{\text{optimal}}}{p_{\text{optimal}}(t) - p^{\text{ss}}_{\text{optimal}}}$ is constant, indicating that at any time t the gap between the current quality and its steady-state value is linearly proportional to the gap between the current price and the steady-state price. Moreover, $p^{\text{ss}}_{\text{optimal}}$ and $q^{\text{ss}}_{\text{optimal}}$ are independent of the initial reference quality r^0_q , implied from Eqs. 10 and 9, such that the optimal price and quality trajectories depend on the distance between the initial reference quality

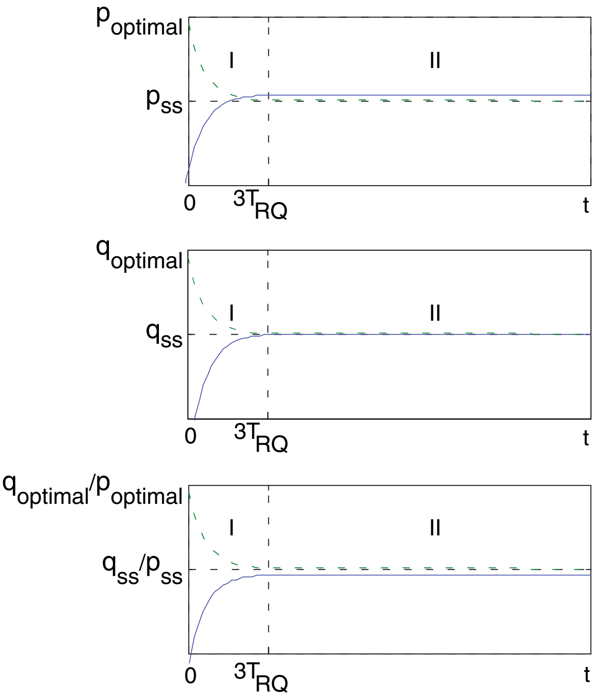

Fig. 3 **a–c** Optimal price, quality, and price–quality ratio trajectories for low (*broken line*) and high (*solid line*) initial reference quality levels

and the steady-state quality level, $(r_q^0 - q_{\text{optimal}}^{\text{ss}})$. This relationship, therefore, captures the important effect of the initial reference quality on the optimal quality strategy.

In Fig. 3, we illustrate the pattern of the optimal price and quality paths as well as the relationship between quality and price. Both the optimal price and quality follow a pattern of an introductory period with either decreasing or increasing levels until they reach the second stage, where they stabilize over time.[3] Figure 1a, b represents the price and quality paths. We also show the relationship between quality and price in Fig. 1c, which follows a similar pattern as well. The pattern of a relatively short introductory period and a steady-state period is in line with the findings of Fibich et al. (2005), where the introductory phase lasts for about $3T_{\text{RQ}} = 1/m$,[4] before quality and price reach a stable level. In all three figures, the initial decrease patterns (broken lines) represent the cases where the initial reference quality, r_q^0, is lower than the steady-state level, and the initial increase patterns (solid lines) represent the cases where the initial reference quality, r_q^0, is higher than the steady-state level. In other words, when the initial reference quality is high, retailers can

[3]The parameters we use are: $a = 10, \gamma = 0.7, \delta_1 = 0.5, \delta_2 = 1, c = 1, \beta = 2, \alpha = 0.1, r_q^0$ takes the values 3.4 (lower curves) or 0.1 (upper curves).

[4]The time unit of convergence towards a steady-state condition (see footnote 1)

use this initial expectation to increase profits by capitalizing on the adjustment process of the reference quality. The initial difference between r_q^0 and q is not immediately internalized by consumers; only a fraction of it is (depending on memory and time scale). As initial expectations are higher than at the steady-state level, it is beneficial to educate consumers about lower quality through a low initial quality level that is produced at a lower cost and is sold at a lower price. This educational period lasts until the reference effects are eliminated. A reverse phenomenon is observed when the initial beliefs are lower than the steady-state level. Using the same adjustment mechanism, retailers can educate consumers about higher quality levels at steady state by starting with a high-quality level and capitalizing on the gains it will create in consumers' behavior. Higher prices (due to higher quality) and increased demand through quality gains will offset higher costs. As we will show later, this process of decreasing (increasing) initial quality levels is augmented by a price decrease (increase) that creates an overall decrease in the quality–price ratio. When the initial reference quality r_q^0 is low, the firm holds quality $q(t)$ above reference quality $r_q(t)$ and thus benefits from consumers' gain effects for every time period, t. However, at the same time consumers adapt to the higher level of quality and their expectations for quality become higher. Increasing the quality increases demand, but at the same time, increases costs, because the producer must pay the costs of maintaining higher quality. Nevertheless, the firm can increase the price and benefit from higher profits. The same arguments hold for the other case when the initial reference quality r_q^0 is high.

Figure 3 shows that the dynamics of the convergence process toward a steady price and quality is exponential and that the decrease (increase) in the price and quality follows the same pattern. Proposition 1 indicates that the rate of convergence towards a steady state of both these terms is constant. At any time t, the ratio between the difference in price and its steady-state level and the difference between quality and its steady-state level is constant. From Proposition 1, we can calculate the optimal reference quality path.

Corollary 1 *The optimal reference quality path is given by*

$$r_q(t) = q_{\text{optimal}}^{\text{ss}} + \left(r_q^0 - q_{\text{optimal}}^{\text{ss}}\right) e^{-mt}. \tag{12}$$

Proof By substitution of Eq. 9 into Eq. 1. □

Since $r_q(t) - q_{\text{optimal}}(t) = \frac{m}{\beta}(r_q^0 - q_{\text{optimal}}^{\text{ss}})e^{-mt}$, the gap in quality between the actual and reference levels is proportional to the difference between the initial reference quality, r_q^0, and the steady-state quality level. This quality gap diminishes at an exponential rate.

4.2 Implications

The parameter m in Proposition 1 has important implications. Since the optimal solution Eqs. 9 and 10 has an exponential structure, we can conclude that the gap between the initial price and quality levels and their steady-state values decreases at an exponential rate. The time scale for stabilizing near the steady-state values is determined by the quantity $T_{\mathrm{RQ}} = 1/m$.[5] In $3-4$ time units of T_{RQ} (see footnote 1), the optimal quality will approximately reach its steady-state level, $q_{\mathrm{optimal}}(t) \approx q^{\mathrm{ss}}_{\mathrm{optimal}}$ and the same holds for price, $p_{\mathrm{optimal}}(t) \approx p^{\mathrm{ss}}_{\mathrm{optimal}}$. As a result, when changing the quality of a product, firms should not expect an immediate consumer reaction, as their adjustment process to the new quality level is affected by T_{RQ}. Observe that in the solution Eqs. 10 and 9 there is a dependency on m/β, where this relationship is always greater than 1 because $\beta < m$. The magnitude of the change between the initial price and quality, and the steady-state levels depends on this ratio, which is a ratio between two time scale parameters. The first is the time scale for consumers' reference quality adjustments, β, and the second m, incorporates other relevant parameters such as the discounting factor. Next, we compare the impact of reference effects on price and quality to those with no reference effects.

Corollary 2 *The optimal price and quality in steady-state $q^{\mathrm{ss}}_{\mathrm{optimal}}$ and $q^{\mathrm{optimal}}_{\mathrm{no\text{-}ref}}$ are increasing with γ, in particular*

$$p^{\mathrm{ss}}_{\mathrm{optimal}} \geq p^{\mathrm{optimal}}_{\mathrm{no\text{-}ref}}$$

and

$$q^{\mathrm{ss}}_{\mathrm{optimal}} \geq q^{\mathrm{optimal}}_{\mathrm{no\text{-}ref}}$$

Proof The result obtained by showing that $\frac{d}{\mathrm{d}\gamma} q^{\mathrm{ss}}_{\mathrm{optimal}} > 0$ and $\frac{d}{\mathrm{d}\gamma} p^{\mathrm{ss}}_{\mathrm{optimal}} > 0$. □

This important result indicates that both the optimal price and quality are higher in the case of reference quality effects than in the case where these effects are absent. Consumers benefit from higher quality products but at a higher price. The overall effect of using previous experience with the products benefits consumers, because the ratio between quality and price increases in the presence of reference quality effects:

Proposition 2 *The ratio of quality to price in steady-state $\frac{q^{\mathrm{ss}}_{\mathrm{optimal}}}{p^{\mathrm{ss}}_{\mathrm{optimal}}}$ is increasing with γ, in particular*

$$\frac{q^{\mathrm{ss}}_{\mathrm{optimal}}}{p^{\mathrm{ss}}_{\mathrm{optimal}}} \geq \frac{q^{\mathrm{optimal}}_{no\text{-}ref}}{p^{optimal}_{no-ref}}\,.$$

[5] Observe that since $m > \beta$, it follows that $T_{\mathrm{RQ}} < \hat{T}_{\mathrm{RQ}}$ where $\hat{T}_{\mathrm{RQ}} = 1/\beta$ is the time scale of the reference effects defined below Eq. 1.

Proof The result obtained by showing that $\frac{d}{\mathrm{d}\gamma}\frac{q^{\mathrm{ss}}_{\mathrm{optimal}}}{p^{\mathrm{ss}}_{\mathrm{optimal}}} > 0$. □

A question of interest for managers is the impact of the change in price or quality on demand (i.e., elasticity). Fibich et al. (2005) studied the impact of reference prices on the elasticity of demand. In the current model, we address the issue of elasticity of demand with respect to reference quality and price in a similar manner. In steady-state conditions, $r^{\mathrm{ss}}_q = \lim_{t\to\infty} r_q(t) = q^{\mathrm{ss}}_{\mathrm{optimal}}$ because all reference effects have vanished due to the length of time that has elapsed since the last quality change, and the reference quality has converged toward steady-state conditions. Thus, the rate of sales is given by substituting $q^{\mathrm{ss}}_{\mathrm{optimal}}$ and $p^{\mathrm{ss}}_{\mathrm{optimal}}$ in Q that is given in Eq. 5. Using the new relationship, we find that $Q^{\mathrm{ss}} = \lim_{t\to\infty} Q(t) = Q_{\text{no-ref}}(q^{\mathrm{ss}}_{\mathrm{optimal}}, p^{\mathrm{ss}}_{\mathrm{optimal}})$. This solution might lead us to the incorrect conclusion that changing the product's quality or price will have the same effect on the elasticity of demand, as in the case of no reference quality effects. Note, however, that any change in quality will immediately trigger the reference effects process with its consequential impact on demand, which differs significantly from a corresponding change where no reference effects are assumed. To demonstrate this phenomenon, we analyze the elasticities in these two cases. The elasticities of demand with respect to price and quality are defined by

$$\varepsilon_p = \frac{\partial Q}{\partial p}\frac{p}{Q}, \varepsilon_q = \frac{\partial Q}{\partial q}\frac{q}{Q}$$

Thus, the ratio between the elasticities of quality and price is:

$$\frac{\varepsilon_q}{\varepsilon_p} = \frac{\frac{\partial Q}{\partial q}}{\frac{\partial Q}{\partial p}}\frac{q}{p}.$$

The steady-state elasticities under the optimal price and quality conditions with reference effects, given our linear model in Eq. 5, are given by

$$\varepsilon^{\mathrm{ss}}_p = -\delta_2 \frac{p^{\mathrm{ss}}_{\mathrm{optimal}}}{Q_{\text{no-ref}}\left(q^{\mathrm{ss}}_{\mathrm{optimal}}, p^{\mathrm{ss}}_{\mathrm{optimal}}\right)}, \varepsilon^{\mathrm{ss}}_q = (\delta_1 + \gamma)\frac{q^{\mathrm{ss}}_{\mathrm{optimal}}}{Q_{\text{no-ref}}\left(q^{\mathrm{ss}}_{\mathrm{optimal}}, p^{\mathrm{ss}}_{\mathrm{optimal}}\right)}. \tag{13}$$

Simple algebra shows that both elasticities increase with γ and thus, under optimal decision making, $\varepsilon^{\mathrm{ss}}_p > \varepsilon^{\text{no-ref}}_p$ and $\varepsilon^{\mathrm{ss}}_q > \varepsilon^{\text{no-ref}}_q$.

Proposition 2 demonstrates that the presence of reference quality effects will lead to a higher ratio of quality to price or in other words, higher value for consumers. Again, we compare the ratio of quality to price in the models with and without reference quality effects. Relating this proposition to Fig. 2, we note that this figure is based on real data and represents a steady-state situation, not the dynamics of this market. As we noted in the introductory section, the decrease in the ratio of price to quality (resulting in cheaper products) as the absolute quality level increases can result from various reasons (e.g., diminishing returns on quality, lower production costs per unit of quality). However, this decline may also be a result of firms who are using reference quality effects in their decisions to optimize profits and firms that ignore such effects. Such outcomes go beyond positioning issues, as they may be the result of this decision. Thus, firms that take reference quality effects into account will have higher quality products at a higher price, as in Fig. 1, but with a lower price per unit of quality, as in Fig. 2.

Furthermore, Proposition 2 provides us with the additional, valuable insight, which follows from Corollary 2, that quality increases with the reference quality parameter, γ. In other words, the stronger the reference quality effect on demand (heterogeneity in its effect across brands in the category - γ parameter that is product dependent—for example), the higher the optimal quality level, and the ratio between quality and price (follows from Proposition 2). Thus, another explanation for the steady-state scenario presented in Fig. 2 might be attributed to such differences.

To gain a better understanding of the relative sensitivity between the two elasticities, we calculate the ratio between quality and price elasticities as in the following proposition:

Proposition 3 *The elasticities ratio* $\left|\frac{\varepsilon_q^{ss}}{\varepsilon_p^{ss}}\right|$ *is increasing with* γ*, in particular*

$$\left|\frac{\varepsilon_q^{ss}}{\varepsilon_p^{ss}}\right| > \left|\frac{\varepsilon_q^{\text{no-ref}}}{\varepsilon_p^{\text{no-ref}}}\right|$$

Proof By Eq. 13 we have,

$$\left|\frac{\varepsilon_q^{ss}}{\varepsilon_p^{ss}}\right| = \frac{\delta_1 + \gamma}{\delta_2} \frac{q_{\text{optimal}}^{ss}}{p_{\text{optimal}}^{ss}}.$$

Following Proposition 2 observe that the elasticities ratio is increasing with γ and in particular, substituting $\gamma = 0$ we get $\frac{\delta_1}{\delta_2} \frac{q_{\text{no-ref}}^{\text{optimal}}}{p_{\text{no-ref}}^{\text{optimal}}} = \left|\frac{\varepsilon_q^{\text{no-ref}}}{\varepsilon_p^{\text{no-ref}}}\right|$. □

This proposition provides an important insight into consumer sensitivity to price and quality. Our results show that consumers are relatively more sensitive to changes in quality than in price when reference quality effects are present. Even though reference effects vanish in steady-state conditions (reference levels converged to actual quality levels), the elasticities for the reference quality case are not the same as in the case of no reference quality effects. The reference quality will affect the elasticities, and any deviation from the quality level at steady state will lead to greater market reaction. Furthermore, demand is more sensitive to changes in quality than price when reference effects exist. The phenomenon observed in the last proposition follows from the impact that reference quality has on demand, which affects the latter directly through the $\delta_2 q$ component and indirectly through the reference effects $\gamma(q(t) - r_q(t))$.

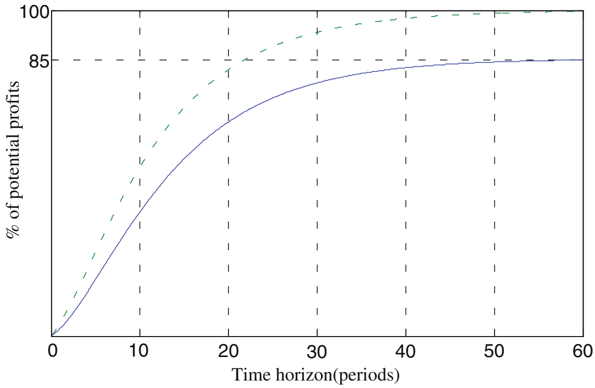

Fig. 4 The ratio between the cumulative profits at time T and the maximum possible profits under the optimal pricing strategy for $T = \infty$. The *broken line* represents the optimal pricing strategy with reference quality effects, and the *solid line* represents pricing with no reference effects

Next, we examine the relative effect of reference quality on profits compared with the case where retailers ignore reference effects even though they exist. Note that we are examining the differences between the two models analytically to determine the impact of the additional component in the model—reference quality effects.

In Fig. 4, we present the ratio between the profits when the planning horizon T is finite and the maximal potential profits when the retailer uses the optimal price and quality (as in Proposition 1) under the optimal pricing strategy for $T = \infty$. The x-axis is the planning horizon, T. The broken line (upper) is the ratio between the profit under a pricing strategy that takes reference quality effects into account and the maximal profits under the optimal pricing strategy in an infinite time horizon. The solid line (lower) is the ratio between the profit when the retailer ignores reference quality effects (even though they exist) and uses the optimal price and quality that is given in Eq. 4 and the maximal optimal profit for $T = \infty$. [6]

As can be seen in Fig. 4, there is a significant difference in the potential profits between the two different pricing concepts. Ignoring reference quality effects, based on the analytical analysis of the isolated contribution of reference quality effects, might lead to a significant loss in potential profits (the difference between the broken and solid lines).

As these calculations are parameter dependent, we note that the importance of reference quality effects depends on the memory parameter β. When this parameter is small, consumers tend to remember distant occurrences, so the effects of reference quality will last longer and have a greater impact on profits. In Table 1, we present a sensitivity analysis of the ratio of profits between optimal price with and without reference quality effects for different memory lengths, β (the higher the value, the faster the decaying process), and discount factors, α, under an infinite planning horizon.

[6] The parameters we use are: $a = 10$, $\delta_1 = 1.5$, $\delta_2 = 2$, $\gamma = 1.5$, $c = 1$, $\beta = 0.1$, $\alpha = 0.1$ and $r_q^0 = 3.5$.

Table 1 Decrease in profits under a pricing strategy without reference effects

β	α	Profit with no reference effects Optimal profit
0.1	0.1	85.2%
0.5	0.1	97%
1	0.1	98.5%
0.1	0.2	73.2%
0.5	0.2	93.9%
1	0.2	97 %

4.3 Profit maximization—the asymmetric case

As noted earlier, positive and negative deviations from a reference point have different effects on consumer behavior, such that gains (i.e., $r_q^0 < q$) have a smaller effect on demand than losses (i.e., $r_q^0 > q$). In order to solve the optimization problem in the asymmetric case (stage II), we first observe that from Eq. 9 and $r_q(t) = q^{\text{ss}} + (r_q^0 - q^{\text{ss}})e^{-mt}$ it follows that $(q_{\text{optimal}} - r_q) = m/\beta(r_q^0 - q^{\text{ss}})e^{-mt}$ and does not change its sign during the planning horizon. If we consider q^{ss} as a function of γ, then by Corollary 2 $q^{\text{ss}}(\gamma)$ increases monotonically in γ. Therefore, $\gamma_{\text{gain}} < \gamma_{\text{loss}}$ implies that $q^{\text{ss}}(\gamma_{\text{gain}}) < q^{\text{ss}}(\gamma_{\text{loss}})$. Since all of the above holds, we are in a position to apply the two-stage method as in Fibich et al. (2003).

Proposition 4 *The optimal strategy in the asymmetric case Eq. 6 subject to Eqs. 2, 5 and 8, is given by Eq. 9, in which the value of γ is given by:*

$$\gamma = \begin{cases} \gamma_{\text{loss}} & r_q^0 > q^{\text{ss}}(\gamma_{\text{loss}}) \\ \gamma_{\text{gain}} & r_q^0 < q^{\text{ss}}(\gamma_{\text{gain}}) \end{cases},$$

where $q^{\text{ss}}(\gamma)$ is the steady-state quality of the product if the effect of reference quality is γ. If $q^{\text{ss}}(\gamma_{\text{gain}}) \leq r_q^0 \leq q^{\text{ss}}(\gamma_{\text{loss}})$ then, $q(t) \equiv r_q^0$ and $p(t) \equiv (a + \delta_1 r_q^0 + \delta_2 c)/(2\delta_2)$.

Proof See Appendix B. □

The results of the asymmetric case indicate that there is no qualitative difference between these results and those that were obtained in the symmetric case. Our intuition about the results with respect to the optimal quality path and elasticities, therefore, remains the same in this case as well. The firm maintains the quality above the reference quality if the initial reference quality is low and is smoothly increased to the steady-state level. A similar process takes place when the initial reference quality is high, and the firm reduces the quality level slightly. There is an important difference with respect to the middle range where the producer maintains the quality of the product at the level of the initial reference quality. In this case, the producer avoids the cost of increasing the product's quality or giving consumers an inferior product.

In the last proposition, we consider one type of asymmetry where $\gamma_{\text{gain}} < \gamma_{\text{loss}}$ as proposed by the Prospect Theory (Kahneman and Tversky 1979). There is, however, empirical evidence with regard to reference price indicating that gains can loom larger than losses, $\gamma_{\text{gain}} > \gamma_{\text{loss}}$, in aggregate analyses (e.g., Greenleaf 1995). Our analysis does not consider this case specifically, but we use previous research to make inferences about the solution. This type of asymmetry where $\gamma_{\text{gain}} > \gamma_{\text{loss}}$ has been considered previously in analytical (Fibich et al. 2007) and numerical (Kopalle et al. 1996) analyses. Incorporating such effects in profit maximization problems in the presence of reference price effects result in unstable solutions. In isolating the effect of a single price change (equivalent to a single quality change), Fibich et al. (2007) find that an increase in price might be more profitable than a price decrease. Furthermore, in the case of multiple price changes the phenomenon of chattering appears. Kopalle et al. (1996) obtain similar results in a discrete model where the price does not converge into a fixed optimal price level but rather fluctuates. Like the results in Fibich et al. (2007), a single quality change (reduction), as in our study, might be more profitable than an increase in quality, but an optimal quality level that is converging to a steady state will not be obtained, making this result almost impossible to implement.

5 Competition

In this section, we extend the model to capture competition between firms that is based on price. Toward this end, we use a formal model for Bertrand competition. For clarity of presentation, we present the case of duopoly where two firms compete in the market and offer similar (substitute) products with possibly different prices and qualities. As a result of this approach, we depart from the optimal control method used in the monopoly case. Let us assume, for simplicity, a case of two symmetric firms (the same parameters for both firms), $i = 1, 2$, each facing a demand function of the form

$$Q_i = a + \delta_1 q_i(t) - \delta_2 p_i(t) - h_1 q_j(t) + h_2 p_j(t) + \gamma \left(q_i(t) - r_q(t)\right) ,\quad i = 1, 2, \quad j \neq i \tag{14}$$

where $h_1, h_2 > 0$ and $h_1 p_j(t), h_2 q_j(t)$ are the cross effects of price and quality set by the competing firm, and the reference quality $r_q(t)$ is identical across all consumers in the market, meaning there is no heterogeneity. We assume that the reference quality is evolving based on the average of the quality produced by the firms. That is,

$$r_q(t) = e^{-\beta t}\left[r_q^0 + \beta \int_0^t e^{\beta s} \frac{q_1(s) + q_2(s)}{2}\,\mathrm{d}s\right]$$

or

$$\frac{\mathrm{d}r_q}{\mathrm{d}t} = \beta\left(\frac{q_1(t) + q_2(t)}{2} - r_q(t)\right) \tag{15}$$

with the initial condition $r_q(t) = r_q^0$. For simplicity, let us further assume identical discounting factors $\alpha > 0$, production costs $c > 0$, and quality costs q^2 for the two firms. The firm's profit function, therefore, can be given by

$$\Pi_i[p_1(t), q_1(t), p_2(t), q_2(t)] = \int_0^\infty e^{-\alpha t}\left[(p_i(t) - c)Q_i(t) - q_i^2(t)\right] \mathrm{d}t\,.$$

When considering a dynamic game with discrete time steps (a multistage game), a common solution for this problem is a subgame perfect Nash equilibrium. When considering a continuous time game, we have a differential game in which an appropriate equilibrium concept for solving this problem is known as a feedback (or closed loop) solution that is similar to the sub game perfect Nash equilibrium. Calculating this equilibrium requires solving a system of differential equations that governs this equilibrium, a problem for future research. Other studies of similar problems, however, show that the qualitative results of the equilibrium in such a competitive model are the same as in the case of a monopoly (see Fibich et al. 2003). At equilibrium, firms follow the pattern of skimming or penetration depending on the initial reference quality level. Since the extension we present here from a monopoly case to a competitive case follows the same pattern as in Fibich et al. (2003), it is logical to infer that the results obtained in the monopoly case will remain the same. To make sure that this pattern of skimming or penetration (namely, price and quality increasing or decreasing towards a steady state) is indeed a viable solution in the competitive model, we present the structure of the feedback equilibrium. We show that this pattern is valid in equilibrium without calculating all of the coefficients needed to determine the equilibrium strategies. The Hamilton–Jacobi–Bellman equations governing the equilibrium are given by (Kamien and Schwartz 1991; Starr and Ho 1969)

$$\alpha V^i\left(r_q\right) = \max_{p_i, q_i}\left[\left(p_i(t) - c)Q_i(t) - q_i^2(t)\right) + \frac{\mathrm{d}V^i(r_q)}{\mathrm{d}r_q}\beta\left(\frac{q_1 + q_2}{2} - r_q\right)\right], \quad i = 1, 2 \tag{16}$$

where $V^i(r_q)$ is firm i's value function (i.e., the current value of profits in equilibrium when the initial reference quality is r_q). We limit our analysis here to a symmetric solution where $p_1 = p_2, q_1 = q_2$ and avoid proving uniqueness. Substituting Eqs. 14 and 15 in Eq. 16 and maximizing the right hand side of this equation with respect to p_i, q_i gives

$$p_i = A_1 + A_2 r_q + A_3 \frac{d}{\mathrm{d}r_q} V^i\left(r_q\right), \quad i = 1, 2,$$

$$q_i = B_1 + B_2 r_q + B_3 \frac{d}{\mathrm{d}r_q} V^i\left(r_q\right), \quad i = 1, 2, \tag{17}$$

where $A_k, B_k, k = 1, 2, 3$ are constants, depending on the problem's parameters. Substituting Eq. 17 in Eq. 16 and using the symmetry that implies $V^1 = V^2$ gives the differential equations

$$\begin{aligned}\alpha V^i(r_q) = &\left(A_1 + A_2 r_q + A_3 \frac{d}{\mathrm{d}r_q} V^i\left(r_q\right) - c\right)\\ &\times\left[a + (\delta_1 - h_1 + \gamma)\left(B_1 + B_2 r_q + B_3 \frac{d}{\mathrm{d}r_q} V^i\left(r_q\right)\right)\right.\\ &\left.\quad - (\delta_2 - h_2)\left(A_1 + A_2 r_q + A_3 \frac{d}{\mathrm{d}r_q} V^i\left(r_q\right)\right) - \gamma r_q(t)\right]\\ &-\left(B_1 + B_2 r_q + B_3 \frac{d}{\mathrm{d}r_q} V^i\left(r_q\right)\right)^2\\ &+\frac{d}{\mathrm{d}r_q} V^i(r_q)\beta\left(B_1 + B_2 r_q + B_3 \frac{d}{\mathrm{d}r_q} V^i\left(r_q\right) - r_q\right), \qquad i = 1, 2.\end{aligned}$$

Rearranging these terms yields

$$\begin{aligned}\alpha V^i\left(r_q\right) = &M_0 + M_1 r_q + M_2 r_q^2 + M_3 r_q \frac{d}{\mathrm{d}r_q} V^i\left(r_q\right)\\ &+M_4 \frac{d}{\mathrm{d}r_q} V^i\left(r_q\right) + M_5\left(\frac{d}{\mathrm{d}r_q} V^i\left(r_q\right)\right)^2.\end{aligned}$$

These types of nonlinear differential equations are common in differential games (e.g., Fershtman and Kamien 1987; Fibich et al. 2003). The solutions to these equations have a quadratic structure,

$$V^i(r_q) = C_1 + C_2 r_q + C_3 r_q^2$$

where $C_k, k = 1, 2, 3$ are constants. Thus, we can conclude that p_i and q_i are linear functions of the reference quality r_q. Substituting q_i in Eq. 15 gives a first order linear differential equation $\frac{d}{\mathrm{d}t} r_q(t) + D_1 r_q(t) + D_2 = 0$ (where $D_k, k = 1, 2$ are constants) that yields a solution for the pattern of the reference quality in equilibrium with exactly the same structure as the solution we obtained in the case of the monopoly (Proposition 1).

Based on this discussion, we can infer that the qualitative results obtained in the monopoly case still hold in the competitive scenario.

6 Conclusions

The purpose of this study is to expand the rich literature about the relationship between price and quality through an examination of the effect of reference quality and price on profits. The analytical solution developed in this study enables us to depict the exact path of the optimal price and quality of a product over time. We relax some restrictive assumptions about the relationship between price and quality in other modeling approaches to create a model that would provide us with more understanding about this issue. As a result, insights about the sensitivity of consumers to price and quality are derived through elasticities. Such results cannot be obtained without this type of analysis and have not been discussed in the literature.

The nature of the solution allows us to draw several managerial conclusions to further enhance a firm's profits. For example, the existence of reference quality increases consumer elasticities to both price and quality compared with the case of no reference quality. This increase in elasticities, however, comes with increased steady-state levels of price and quality. In other words, the optimal product will have a higher quality and a higher price (relative to the case of no reference effects), but overall the cost of that quality (per unit) will be lower. This result implies a tradeoff between increased quality and increased price through two different mechanisms (a greater increase in quality elasticity than in that of price elasticity). The combined impact of these two effects is a lower price per unit of quality at steady state. Managers, therefore, can indicate this price–quality relationship in their marketing strategy. In addition, we provide indications that these results are also valid for a competitive scenario where firms compete on price.

We also create a Bertrand model to capture competition in such reference effects and indicate that the qualitative results obtained in the monopoly case will also be preserved in the competitive case. Thus, our results can explain the differences in price and the relationship between price and quality in the case of the paper towels presented at the beginning of this paper. As Fig. 1 demonstrates, there is a positive relationship between price and quality. However, as Fig. 2 shows, the price per unit of quality is lower.

Another conclusion that can be drawn is the penetration and skimming pattern of product quality. Our analysis indicates that counter intuitively, a firm can reduce the quality of its products over time to raise overall profits. This will happen when initial consumer quality beliefs are at a higher level than the steady state.

Further research can include reference price effects as well as reference promotion effects in the framework developed here to examine the combined impact of all of these factors. Another avenue for future research can include provisions for other product types such as durables or services and empirically investigating the use of perceptions about quality to demonstrate this pattern.

Appendix A: Proof of Proposition 1

The current value Hamiltonian is given by

$$H = (p(t) - c)\, Q(t) - q^2(t) + \lambda\beta \left(q(t) - r_q(t)\right) .$$

Therefore, the first-order conditions for optimality are

$$\frac{\partial H}{\partial p} = a + \delta_1 q - 2\delta_2 p + \gamma \left(q - r_q\right) + \delta_2 c = 0 \,, \tag{18}$$

$$\frac{\partial H}{\partial q} = (p - c)\,(\delta_1 + \gamma) - 2q + \lambda\beta = 0 \,, \tag{19}$$

$$\frac{d\lambda}{\mathrm{d}t} = \alpha\lambda - \frac{\partial H}{\partial r_q} = (\alpha + \beta)\,\lambda + \gamma(p - c),$$

$$\frac{\mathrm{d}r_q}{\mathrm{d}t} = \beta\left(q - r_q\right) \,,$$

with the initial condition $r_q(0) = r_q^0$ and the assumption that $\lim_{t\to\infty} p(t) < \infty$. In steady-state $\frac{d\lambda}{\mathrm{d}t} = 0$ and thus, λ in steady-state $\lambda_{\mathrm{ss}} = -\frac{\gamma}{\alpha+\beta}(p_{\mathrm{optimal}}^{\mathrm{ss}} - c)$. Furthermore, in steady-state $q_{\mathrm{optimal}}^{\mathrm{ss}} = r_q^{\mathrm{ss}}$. Substituting λ_{ss} in Eq. 19 and solving with Eq. 19 gives Eq. 11. The above equations are reduced to a linear differential equation that can be easily solved to get Eqs. 9 and 10. However, to avoid excessive detail, we note that substitution of Eqs. 9 and 10 in the above equations shows that Eqs. 9 and 10 is indeed the solution.

Appendix B: Proof of Proposition 4

This proof will show that under asymmetric conditions, optimal price and quality paths will not create a situation in which consumers' evaluation of these variables will switch from gains to losses or vice versa. We use the technique introduced in Fibich et al. (2003) for that purpose. Throughout this proof, the notations $q_{\mathrm{optimal}}(t)$, $p_{\mathrm{optimal}}(t)$ represent the optimal solution for the symmetric problem given by Eqs. 9 and 12. To prove this proposition, we will need the following three lemmas.

Lemma 1 *The steady-state quality, $q_{\mathrm{optimal}}^{\mathrm{ss}}$, given in Eq. 11 is a monotonic increasing function of γ.*

Proof Differentiating $q_{\mathrm{optimal}}^{\mathrm{ss}}(\gamma)$ with respect to γ gives

$$\frac{d}{\mathrm{d}\gamma} q_{\mathrm{optimal}}^{\mathrm{ss}}(\gamma) = \frac{4\,(a - \delta_2 c)\,\frac{\alpha}{\alpha+\beta}\delta_2}{\left[4\delta_2 - \delta_1^2 - \delta_1\gamma\frac{\alpha}{\alpha+\beta}\right]^2} > 0.$$

□

Lemma 2 $\left(q_{\mathrm{optimal}}(t) - r_q(t)\right)$ *does not change this sign for any t.*

Proof From Eqs. 9 and 12, we have

$$q_{optimal}(t) - r_q(t) = -\frac{m}{\beta}\left(r_q^0 - q^{ss}\right)e^{-mt}$$

which has a constant sign. □

Let us denote the profits under any quality–price strategy $q(t)$, $p(t)$ by

$$g(\gamma_{\text{gain}}, \gamma_{\text{loss}}; q(t), p(t)) := \int_0^\infty e^{-\alpha t}\left[(p(t) - c)\,Q(t) - q^2(t)\right]\mathrm{d}t\,,$$

where Q is given by Eq. 5 and γ is given by Eq. 6 and define

$$G\left(\gamma_{\text{gain}}, \gamma_{\text{loss}}\right) := \sup_{q(t),p(t)} g\left(\gamma_{\text{gain}}, \gamma_{\text{loss}}; q(t), p(t)\right)\,. \tag{20}$$

Lemma 3

$$G\left(\gamma_{\text{gain}}, \gamma_{\text{loss}}\right) \leq G\left(\tilde{\gamma}, \tilde{\gamma}\right) \quad \textit{for all } \gamma_{\text{gain}} \leq \tilde{\gamma} \leq \gamma_{\text{loss}}. \tag{21}$$

Proof The function $g(\gamma_{\text{gain}}, \gamma_{\text{loss}}; q(t), p(t))$ increases monotonically in γ_{gain} and decreases monotonically in γ_{loss}. Thus,

$$g\left(\gamma_{\text{gain}}, \gamma_{\text{loss}}; q(t), p(t)\right) \leq g\left(\tilde{\gamma}, \tilde{\gamma}; q(t), p(t)\right) \quad \text{for all } \gamma_{\text{gain}} \leq \tilde{\gamma} \leq \gamma_{\text{loss}}. \tag{22}$$

Then, by Eq. 22 we have that

$$\sup_{q(t),p(t)} g\left(\gamma_{\text{gain}}, \gamma_{\text{loss}}; q(t), p(t)\right) \leq \sup_{q(t),p(t)} g\left(\tilde{\gamma}, \tilde{\gamma}; q(t), p(t)\right) \quad \text{for all } \gamma_{gain} \leq \tilde{\gamma} \leq \gamma_{\text{loss}}, \tag{23}$$

1. and since the right hand side of Eq. 23 is $G(\gamma_{\text{gain}}, \gamma_{\text{loss}})$ and the left hand side of Eq. 23 is $G(\tilde{\gamma}, \tilde{\gamma})$ we get

$$G\left(\gamma_{\text{gain}}, \gamma_{\text{loss}}\right) \leq G(\tilde{\gamma}, \tilde{\gamma}) \quad \text{for all } \gamma_{\text{gain}} \leq \tilde{\gamma} \leq \gamma_{\text{loss}}.$$

□

We now turn to proving the proposition in which we have three possible cases with respect to the relationship between the initial reference quality r_q^0 and the steady-state quality level q^{ss}:

1. We start with the first case and show that if $r_q^0 < q^{ss}(\gamma_{\text{gain}})$ then, $q_{\text{optimal}}(t)$, $p_{\text{optimal}}(t)$ given by Eqs. 9 and 12 with $\gamma = \gamma_{\text{gain}}$ is the optimal solution for the asymmetric case. As we show by Lemma 2, $\left(q_{\text{optimal}}(t) - r_q(t)\right)$ does not

change its sign and thus, substituting $\gamma = \gamma_{\text{gain}}$ in $q_{\text{optimal}}(t)$ we have that $q_{\text{optimal}}(t) \geq r_q(t)$ for all $t \geq 0$. Therefore,

$$g\left(\gamma_{\text{gain}}, \gamma_{\text{loss}}; q_{\text{optimal}}(t), p_{\text{optimal}}(t)\right) = g\left(\gamma_{\text{gain}}, \gamma_{\text{gain}}; q_{\text{optimal}}(t), p_{\text{optimal}}(t)\right)$$
$$= G\left(\gamma_{\text{gain}}, \gamma_{\text{gain}}\right) \quad (24)$$

where the last equality follows because $g(\gamma_{\text{gain}}, \gamma_{\text{gain}}; q_{\text{optimal}}(t), p_{\text{optimal}}(t))$ is symmetric in γ. Thus, $q_{\text{optimal}}(t), p_{\text{optimal}}(t)$ is the optimal strategy that maximizes g. On the other hand, from Eq. 20 and the monotonicity of $G(\gamma_{\text{gain}}, \gamma_{\text{loss}})$ given by Lemma 3 we have that

$$g\left(\gamma_{\text{gain}}, \gamma_{\text{loss}}; q_{\text{optimal}}(t), p_{\text{optimal}}(t)\right) \leq G\left(\gamma_{\text{gain}}, \gamma_{\text{loss}}\right) \leq G\left(\gamma_{\text{gain}}, \gamma_{\text{gain}}\right). \quad (25)$$

Combining Eqs. 24 and 25 yields

$$g\left(\gamma_{\text{gain}}, \gamma_{\text{loss}}; q_{\text{optimal}}(t), p_{\text{optimal}}(t)\right) = G\left(\gamma_{\text{gain}}, \gamma_{\text{loss}}\right),$$

which proves that the symmetric solution $q_{\text{optimal}}(t), p_{\text{optimal}}(t)$ with γ_{gain} is the optimal solution for the asymmetric problem.

2. The second case is where $r_q^0 > q^{ss}(\gamma_{\text{gain}})$. Then, $q_{\text{optimal}}(t), p_{\text{optimal}}(t)$ given by Eqs. 9 and 12 with $\gamma = \gamma_{\text{loss}}$ is the optimal solution for the asymmetric case is proved similarly.
3. Last, we show that for the case where $q^{ss}_{\text{optimal}}(\gamma_{\text{loss}}) \leq r_0 \leq q^{ss}_{\text{optimal}}(\gamma_{\text{gain}})$, then $q(t) \equiv r_q(t) \equiv r_q^0$ and $p(t) = p^{ss}_{\text{optimal}}(\tilde{\gamma})$ for specific $\tilde{\gamma}$ is the optimal solution for the asymmetric case. Let $\tilde{\gamma}$ be the solution to $q^{ss}_{\text{optimal}}(\tilde{\gamma}) = r_q^0$. By the monotonicity of $q^{ss}_{\text{optimal}}(\gamma)$ given in Lemma 1 there exists $\tilde{\gamma}$, $\gamma_{\text{gain}} \leq \tilde{\gamma} \leq \gamma_{\text{loss}}$ that solves this equation. Therefore, when we use $q(t) = r_q^0$ and $p(t) = p^{ss}_{\text{optimal}}(\tilde{\gamma})$ there is no loss or gain and thus,

$$g\left(\gamma_{\text{gain}}, \gamma_{\text{loss}}; r_q^0, p^{ss}_{\text{optimal}}(\tilde{\gamma})\right) = g\left(\tilde{\gamma}, \tilde{\gamma}; r_q^0, p^{ss}_{\text{optimal}}(\tilde{\gamma})\right) = G(\tilde{\gamma}, \tilde{\gamma}).$$

On the other hand, $g(\gamma_{\text{gain}}, \gamma_{\text{loss}}; r_q^0, p^{ss}_{\text{optimal}}(\tilde{\gamma})) \leq G(\gamma_{\text{gain}}, \gamma_{\text{loss}}) \leq G(\tilde{\gamma}, \tilde{\gamma})$. Combining the last two relationships gives $g(\gamma_{\text{gain}}, \gamma_{\text{loss}}; p_{\text{optimal}}(t)) = G(\gamma_{\text{gain}}, \gamma_{\text{loss}})$.

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References

Bass, F. M. (1969). A new product growth model for consumer durables. *Management Science, 15*, 215–227.

Consumer Reports (2009). Available at: http://www.consumerreports.org/cro/magazine-archive/december-2009/home-garden/paper-towels/overview/paper-towels-ov.htm.

Fershtman, C., & Kamien, M. I. (1987). Dynamic duopolistic competition with sticky prices. *Econometrica, 55*, 1151–1164.

Fibich, G., Gavious, A., & Lowengart, O. (2003). Explicit solutions of optimization models and differential games with nonsmooth (asymmetric) reference-price effects. *Operations Research, 51*(5), 721–734.

Fibich, G., Gavious, A., & Lowengart, O. (2005). The dynamics of price elasticity of demand in the presence of reference price effects. *Journal of The Academy of Marketing Science, 33*(1), 66–78.

Fibich, G., Gavious, A., & Lowengart, O. (2007). Optimal price promotion in the presence of asymmetric reference-price effects. *Managerial and Decision Economics, 28*, 1–9.

Gerstner, E. (1985). Do higher prices signal higher quality? *Journal of Marketing, 22*, 209–215.

Greenleaf, E. A. (1995). The impact of reference price effects on the profitability of price promotions. *Marketing Science, 14*, 82–104.

Hardie, B. G. S., Johnson, E. J., & Fader, P. S. (1993). Modeling loss aversion and reference dependence effects on brnad choice. *Marketing Science, 12*, 378–394.

Kahneman, D., & Tversky, A. (1979). Prospect theory: An analysis of decision making under risk. *Econometrica, 47*, 263–91.

Kalyanaram, G., & Little. J. D. C. (1994). An empirical analysis of latitude price acceptance in consumer package goods. *Journal of Consumer Research, 21*, 408–418.

Kamien, M. I., & Schwartz N. L. (1991). *Dynamic optimization: The calculus of variations and optimal control in economics and management*. New York: Elsevier.

Kopalle, P. K., Rao, A. C., & Assuncao, J. L. (1996). Asymmetric reference price effects and dynamic pricing policies. *Marketing Science, 15*, 60–85.

Kopalle, P. K., & Winer, R. S. (1996). A dynamic model of reference price and expected quality. *Marketing Letters, 7*(1), 41–52.

Lattin, J. M., & Bucklin, R. E. (1989). Reference effects of price and promotion on brand choice behavior. *Journal of Marketing Research, 26*, 229–310.

Lowengart, O. (2002). Reference price conceptualizations: An integrative framework of analysis. *Journal of Marketing Management, 18*(1–2), 145–171.

Monroe, K. B., & Dodds, W. B. (1988). A research program for establishing the validity of the price–quality relationship. *Journal of the Academy of Marketing Science, 16*(1), 151–168.

Narasimhan, R., & Mendez, D. (2001). Strategic aspects of quality: A theoretical analysis. *Production and Operations Management, 10*(4), 514–526.

Paulson Gjerde, K. A., & Slotnick, S. A. (2004). Quality and reputation: The effects of external and internal factors over time. *International Journal of Production Economics, 89*, 1–20.

Sorger, G. (1988). Reference price formation and optimal pricing strategies. In G. Feichtinger (Ed.) *Optimal control theory and economic analysis 3* (pp. 97–120). Amsterdam: Elsevier (North Holland).

Starr, A. W., & Ho, Y. C. (1969). Nonzero-sum differential games. *Journal of Optimization Theory and Applications, 3*, 184–208.

Tversky, A., & Kahneman, D. (1991). Loss aversion and riskless choice: A reference dependence model. *Quarterly Journal of Economics. 106*, 1039–1061.

Winer, R. S. (1986). Reference price model of brand choice for frequently purchased products. *Journal of Consumer Research, 13*, 250–256.

Winer, R. S. (1988). Behavioral perspective on pricing: buyers' subjective perceptions of price revisited. In T. M. Devinney (Ed.), *Issues in pricing: Theory and research.*. Lexington: Lexington Books.