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Low and high types in asymmetric first-price auctions

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Abstract

We study first-price auctions with n bidders where bidders' types (valuations for the object) are drawn independently according to heterogeneous distribution functions. We show a relation between the distributions of high types and their equilibrium bids. On the other hand, we show that there is no relation between the distributions of types and equilibrium bids of low types, i.e. the equilibrium bids of low types are invariable.

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JEL classification: D44; D72; D82

1. Introduction

We study first-price auctions with a single item and n bidders. Each bidder's type (valuation for the item) is private information to that bidder and is drawn independently according to a distribution function that is common knowledge. In the symmetric case where bidders' types are drawn from the same distribution function, Riley and Samuelson (1981) proved the existence of an equilibrium. In the asymmetric case where bidders' types are drawn from different distribution function, Maskin and Riley (2000b) proved the existence of an equilibrium under mild assumptions.

In this paper, we show that if bidders i and j have the same high type (a bidder with a high valuation for the item), and the distribution of bidder i's high types is weaker than the distribution of bidder j's high types, then in equilibrium bidder i bids more aggressively than bidder j. Therefore, if, for example, the bidders are allocated to several groups such that the bidders in the same group have

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the same type distribution, then for a sufficiently large number of bidders in each group, the winner of the auction is likely to come from the group in which the distribution of high types is weakest. However, contrary to high-type bidders, we show that, independent of the form of the type distributions, the equilibrium bids of low-type bidders are invariable. Furthermore, the equilibrium bids of low-type bidders are identical to the bids in the symmetric case where all types are uniformly distributed.

2. The model

Consider n players bidding for an indivisible object. Bidder i's type, $i = 1, 2, \ldots, n$, denoted by v_i, is private information to i, and is drawn independently from an interval [0,1][1] according to the distribution function F_i which is common knowledge. We assume that F_i has a continuous density $f_i = F_i' > 0$. We say that v_i is high if v_i is sufficiently close to 1, and v_i is low if v_i is sufficiently close to 0.

We denote the bid function of bidder i by $x_i = b_i(v_i)$, and assume that it is strictly monotonic and differentiable. Let $y_i = y_i(x_i)$ denote the inverse of b_i. Then, player i's maximization problem reads:

$$\max_x U_i(x) = \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} F_j(y_j(x)) \right)(v_i - x), \quad i = 1, \ldots n$$

Therefore, the bid functions are the solution of:

$$\frac{\partial U_i(x)}{\partial x} = (v_i - x) \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\prod_{\substack{k=1 \\ k \neq i,j}}^{n} F_k(y_k(x)) \right) f_j(y_j(x))\, y_j'(x) - \prod_{\substack{j=1 \\ j \neq i}}^{n} F_j(y_j(x)) = 0, \quad i = 1, \ldots, n$$

Substituting $y_i(x) = v_i$ yields:

$$(y_i(x) - x) \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\prod_{\substack{k=1 \\ k \neq i,j}}^{n} F_k(y_k(x)) \right) f_j(y_j(x))\, y_j'(x) - \prod_{\substack{j=1 \\ j \neq i}}^{n} F_j(y_j(x)) = 0, \quad i = 1, \ldots, n \tag{1}$$

The equilibrium strategies satisfy the following conditions (see Maskin and Riley, 2000a):

1. All the lowest-type bidders ($v = 0$) place a bid of zero, i.e. for every $i = 1, \ldots, n$, $b_i(0) = 0$. As a result, the initial condition for the system (1) is given by:

$$y_i(0) = 0, \quad i = 1, \ldots, n \tag{2}$$

2. All the highest-type bidders place the same bid denoted by $b_{\max} > 0$. Therefore:

$$y_i(b_{\max}) = 1, \quad i = 1, \ldots, n$$

[1] The choice of the interval [0,1] is a normalization.

In the symmetric case, where $F_i = F$, $i = 1, \ldots, n$, the inverse bid functions are symmetric, i.e. $y_i(x) = y(x)$ for all i. In that case the solution to Eq. (1) is given by:

$$x = b(v) = v - \frac{1}{F^{n-1}(v)} \int_0^v F^{n-1}(s)\,\mathrm{d}s \tag{3}$$

In particular, if bidders' types are uniformly distributed, i.e. $f_i = 1$, $i = 1, \ldots, n$, the equilibrium bids are given by:

$$b(v) = \frac{n-1}{n} v, \quad i = 1, \ldots, n$$

3. A rule of high types

In this section we show that a high-type bidder whose density is lower, bids more aggressively than a bidder with the same type whose density is higher.

Proposition 1. *If v is high and $f_i(1) > f_j(1)$, then, $b_i(v) < b_j(v)$, $i \neq j$, $i,j = 1, \ldots, n$.*

Proof. Rearranging (1) yields:

$$(y_i(x) - x) \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{f_i(y_i(x))\, y_j'(x)}{F_j(y_j(x))} = 1, \quad i = 1, \ldots, n \tag{4}$$

Note that for all $1 \leq j \leq n$, $F_j(y_j(b_{\max})) = 1$. Thus, the substitution $x = b_{\max}$ yields:

$$(y_i(b_{\max}) - b_{\max}) \sum_{\substack{k=1 \\ k \neq i}}^{n} f_k(y_k(b_{\max}))\, y_k'(b_{\max}) = 1, \quad i = 1, \ldots, n$$

Since $y_i(b_{\max}) = 1$ for all i we have:

$$\sum_{\substack{k=1 \\ k \neq i}}^{n} f_k(1)\, y_k'(b_{\max}) = \frac{1}{1 - b_{\max}}, \quad i = 1, \ldots, n$$

Subtracting the equation for i from the equation for j yields:

$$f_i(1) y_i'(b_{\max}) = f_j(1)\, y_j'(b_{\max}), \quad i, j = 1, \ldots, n$$

Hence, if $f_i(1) > f_j(1)$, then $y_i'(b_{\max}) < y_j'(b_{\max})$. Therefore $b_i'(1) > b_j'(1)$. This implies that if v is sufficiently close to 1, $b_i(v) < b_j(v)$. □

4. A rule of low types

In this section we show that independent of type distributions, the equilibrium bids of low type bidders are identical to the bids in the symmetric case where all types are uniformly distributed.

Proposition 2. *If v is low, then*

$$b_i(v) \approx \frac{n-1}{n} v, \quad i = 1, \ldots, n$$

Proof. Using L'Hospital's rule in (4) implies:

$$\begin{aligned} 1 &= \lim_{x \to 0} (y_i(x) - x) \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{f_j(y_j(x))\, y_j'(x)}{F_j(y_j(x))} = \sum_{\substack{j=1 \\ j \neq i}}^{n} f_j(0)\, y_j'(0) \lim_{x \to 0} \frac{(y_i(x) - x)}{F_j(y_j(x))} \overset{\text{L'Hospital}}{=} \\ &= \sum_{\substack{j=1 \\ j \neq i}}^{n} f_j(0)\, y_j'(0) \lim_{x \to 0} \frac{(y_i'(x) - 1)}{f_j(y_j(x))\, y_j'(x)} = \sum_{\substack{j=1 \\ j \neq i}}^{n} (y_i'(0) - 1) = (n-1)(y_i'(0) - 1) \end{aligned} \tag{5}$$

From (5), we obtain that $y_i'(0) = (n - 1/n)$, Thus, for v sufficiently close to 0 we have:

$$b_i'(v) \approx \frac{n-1}{n} \tag{6}$$

By (2) and (6) the result is obtained. □

5. Example

Consider a first price auction with two bidders. Bidder l's valuation is distributed according to $F_1(v) = 0.4v + 0.6v^2$ and bidder 2's valuation is distributed according to $F_2(v) = 1.5v - 0.5v^2$, where $v \in [0,1]$.

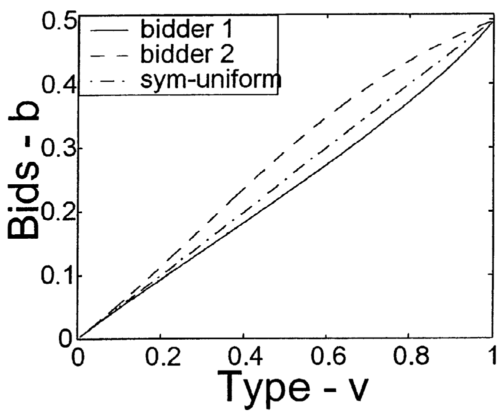

Fig. 1. The equilibrium bids of bidders 1 and 2 and the equilibrium bid in the symmetric uniform case.

Fig. 1 shows the equilibrium bid functions which we calculated by the numerical method of Marshall et al. (1994). By Proposition 1, since $f_1(1) > f_2(1)$, bidder 2's bid is higher than bidder l's bid for high types. By Proposition 2, even though $f_1(0) = 0.4 \neq f_2(0) = 1.5$, the bids of both bidders with low types are essentially the same and are almost identical to the bids in the symmetric uniform case.

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