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Asymptotic revenue equivalence of asymmetric auctions with interdependent values

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ABSTRACT

We prove an asymptotic revenue equivalence among weakly asymmetric auctions with interdependent values, in which bidders have either asymmetric utility functions or asymmetric distributions of signals.

1. Introduction

A seller wishing to sell an object through an auction can choose from various auction mechanisms (first-price, second-price, English, etc.). A key criterion in the selection of an auction mechanism is the expected revenue for the seller (i.e., its revenue ranking). Myerson (1981) and Riley and Samuelson (1981) showed that all standard[1] symmetric private-value auctions with risk-neutral bidders in which bidders' values are independently distributed are revenue equivalent.[2] Bulow and Klemperer (1996) generalized this result to the case of symmetric auctions with interdependent values, in which bidders signals are independently distributed.[3] It is well known, however, that in most cases standard auctions are not revenue equivalent when bidders are asymmetric (see, Krishna (2002)).[4] Such an asymmetry can arise in auctions with interdependent values, either when bidders have asymmetric distributions of signals or when bidders have asymmetric utility functions of the signals. Since in many real-life auctions bidders are asymmetric, considerable research effort has been devoted to revenue ranking of asymmetric auctions (see, Krishna (2002)). Nevertheless, since analysis of asymmetric auctions is hard, relatively little is known about them at present.

Recently, Fibich et al. (2004) used an applied mathematics technique, known as perturbation analysis, to show that private-value auctions with bidders having weakly asymmetric distributions of (independent) values are asymptotically revenue equivalent. A natural question, is therefore, whether this result holds only for the special case of private-value auctions, or also in more general setups. In this paper we show that asymmetric auctions with interdependent values, in which bidders' signals are independently distributed, are also asymptotically revenue equivalent for the following two cases of asymmetry: (1) when bidders have asymmetric utility functions, and (2) when bidders have asymmetric distribution functions for their signals. In both cases we prove an asymptotic revenue equivalence result of the following type: Let ϵ be the asymmetry parameter and let $R(\epsilon)$ be the seller's expected revenue in equilibrium. Then, $R(\epsilon) = R(0) + \epsilon R'(0) + O(\epsilon^2)$, where both $R(0)$, the seller's expected revenue in the symmetric setup and $\epsilon R'(0)$, the leading-order effect of the asymmetry, are independent of the auction mechanism. Our results demonstrate that no matter which kind of asymmetry exists among the bidders, a weak asymmetry does not have a significant effect on revenue ranking in standard auctions. Furthermore, from

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[1] We say that an auction is standard if the rules of the auction dictate that the bidder with the highest bid wins the auction.

[2] We say that two auction mechanisms are revenue equivalent if both of them yield the same expected revenue to the seller.

[3] Standard symmetric auctions with interdependent values are in general not revenue equivalent when bidders' signals are affiliated. For example, the second-price auction generates more revenue than the first-price auction (Milgrom and Weber, 1982).

[4] Myerson (1981) showed that the revenue equivalence also holds for asymmetric auctions, provided that at any realization of the players' values/signals the probability of a player to win the object is independent of the auction mechanism. It can be easily verified, however, that Myerson's condition usually does not hold.

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the expression for $R'(0)$ it follows that the seller's revenue in weakly asymmetric auctions with interdependent values can be approximated, with $O(\epsilon^2)$ accuracy, with the revenue in the case of symmetric auctions in which the utility function (or distribution function) of the bidders is the arithmetic average of the original asymmetric utility functions (or distribution functions).

The paper is organized as follows. In Section 2 we provide a short literature review. In Section 3 we prove that auctions with interdependent values and asymmetric utility functions are asymptotically revenue equivalent. In Section 4 we prove that auctions with interdependent values and asymmetric distribution functions are also asymptotically revenue equivalent. Concluding remarks are in Section 5. The Appendix contains most of the proofs.

2. Literature review

Auction theory and practice has been the focus of research interest in economics, management, political-sciences, and more recently in operations management and revenue management (see Teich et al., 2004; Ağrali et al., 2008).

Auction theory began with the pioneering research of Vickrey (1961), who developed an analytical framework for analyzing auctions in a theoretical game setting. Basically, a simple auction consists of a seller who wishes to sell an object, and n players (bidders) who submit individual bids. Auction mechanisms vary according to the assumptions about bidders' valuations of the object such as risk-attitude, auction's rules, participation fee, etc. A common auction rule is the first-price sealed-bid auction, where the bidders simultaneously and independently submit their bids, the bidder with the highest offer wins the object and pays his bid, while all other $n-1$ bidders get nothing and pay nothing. Another common auction rule is the second-price auction, where bidders submit their bids simultaneously and independently, the bidder with the highest bid wins the object, and pays the second-highest bid. Yet another well-known auction is the open English auction, where the object price is increasing and is known to all bidders. A bidder decides when to drop out of the auction, depending on the current price, so that the last bidder wins the object and pays the last price offered.

One way to classify auctions is to distinguish between open and closed auctions. In open auctions the bidders are informed about the price offers of the other bidders, while in closed (sealed) auctions the bidders submit their bids without knowledge of the other players' bids. We may also classifying auctions according to players' valuations mechanism. In private-value auctions, each bidder determines his value of the object individually and independently of the other bidders. In contrast, in a common-value auction, the object value is the same for all bidders. However, the bidders differ in their beliefs about this unknown common-value. A typical example for common-value auctions is the mineral rights setting. If authorities offer mineral rights for oil, the value of the oil field is identical for all bidders, as it depends on the amount of oil in this field. However, the bidders may have different information (signals) about the amount of oil. In between private- and common-value settings, we may consider auctions with interdependent valuations, where the valuation of the object for each player depends on his private information (signal), and also on the other bidders signals.

The original setting in auction theory has been of private-value auctions that satisfy the following four assumptions:

1. Bidders are risk-neutral.
2. The valuation for the object for each player is drawn independently, according to a distribution function $F(v)$, which is the same for all players, and is known to all bidders. Thus, the problem involves symmetry of information, and is analyzed as a game with Bayesian players as established by Harsanyi (1967,1968).
3. The bidder with the highest bid wins the object.
4. The auction rules are anonymous, in the sense that the rule does not give any advantage to bidder according to their identity.

In this classic setting, the literature covers issues such as finding equilibrium bids and calculating the expected revenue for the seller and the expected payoff for the buyers. One of the most surprising result in this field established by Myerson (1981) and by Riley and Samuelson (1981), and is known as the revenue equivalence theorem. This important theorem states that under the four assumptions listed above, the expected revenue for the seller is independent of the auction mechanism. Thus, the revenue depends on the number of bidders and on the distribution of bidders' valuations, but not on auction's rules. The revenue equivalence was extended to the case of symmetric interdependent valuation environment by Bulow and Klemperer (1996).

Although the revenue equivalence theorem shows that the auction rules do not affect the expected revenue, in practice, in many situations sellers prefer one auction mechanism over the other. A possible explanation for this empirical observation is that some of the four classical assumptions are violated. Indeed, it is known that violation of the risk-neutrality assumption (see Holt, 1980) results is revenue differences among different auctions mechanisms. Griesmer et al. (1967) studied the case of asymmetry of valuations between bidders in first-price auction, and found equilibrium bids in the case of uniform distribution. Comparison of the expected revenue in first- and second-price auction shows that the revenue equivalence theorem is invalid and first-price auction yield higher expected revenues.

The problem of revenue ranking of asymmetric auctions is hard and still open. Fibich et al. (2004) considered the case of private-value auctions when bidders' valuations are weakly asymmetric. They found that the classical revenue equivalence theorem for symmetric auction can be replaced with an asymptotic revenue equivalence theorem for asymmetric auctions, which says that the revenue differences among different asymmetric auctions is of the second-order in the asymmetry parameter. In the current research we study whether this asymptotic revenue equivalence can be extended to weakly asymmetric auctions with interdependent valuations.

3. Asymmetric utility functions

Consider n risk-neutral bidders bidding for an indivisible object in a standard auction in which the highest bidder wins the object. Bidder $i, i = 1, \ldots, n$ receives a signal x_i which is independently drawn from the interval $[0,1]$ according to a common continuously differentiable distribution function $F(x_i)$, with a corresponding density function $f = F'$. The signal x_i is private information to i. We denote by $\mathbf{x}_{-i}$ the $n-1$ signals other than x_i. Bidder's i utility function (value) for the object, V_i, is a function of all the bidders' signals and is given by[5]

[5] Our results will remain unchanged if Eq. (1) is replaced with $V_i(x_i, \mathbf{x}_{-i}) = V(x_i, \mathbf{x}_{-i}) + \epsilon U_i(x_i, \mathbf{x}_{-i}) + O(\epsilon^2)$.

$$V_i(x_i, \mathbf{x}_{-i}) = V(x_i, \mathbf{x}_{-i}) + \epsilon U_i(x_i, \mathbf{x}_{-i}). \tag{1}$$

Thus, $\epsilon = 0$ is the case of a symmetric utility function V, and the parameter ϵ is the measure of the asymmetry among players' utility functions. In particular, $\epsilon \ll 1$ corresponds to the case of auctions with weakly asymmetric interdependent values.

We assume that V and U_i are continuous and monotonically increasing in all their variables, and satisfy the normalization condition $V(0,\ldots,0) = U_i(0,\ldots,0) = 0$. We also assume that V and U_i are symmetric in the $n-1$ components of $\mathbf{x}_{-i}$, i.e., from a bidder's point of view the signals of his opponents can be interchanged without affecting his value. We assume that the bidders' equilibrium strategies are monotonically increasing in each of the signals and are continuously differentiable with respect to ϵ. In particular, as ϵ approaches zero, the equilibrium bids approach the symmetric equilibrium bid in the symmetric case $\epsilon = 0$.

The assumption that the equilibrium strategies are continuously differentiable in the asymmetry parameter ϵ requires some conditions on the valuation functions. In some auction mechanisms (e.g., second-price auctions) such conditions can be easily derived, while in others (e.g., first-price auctions) such a derivation is considerably harder.[6] The following result shows a simple sufficient condition for differentiability in ϵ in second-price auctions[7]:

Lemma 1. *Consider a second-price auction with two bidders with valuation functions*

$$V_1(x_1, x_2) = V(x_1, x_2) + \epsilon U_1(x_1, x_2), \quad V_2(x_2, x_1) = V(x_2, x_1) + \epsilon U_2(x_2, x_1), \tag{2}$$

whose signals are symmetrically distributed with density function f. If for any x,

$$\frac{\partial V}{\partial x_1}(x, x) \neq \frac{\partial V}{\partial x_2}(x, x). \tag{3}$$

Then, for ϵ near zero there exist equilibrium bids $b_i = b_i(x;\epsilon), i = 1, 2$, such that $\lim_{\epsilon \to 0} b_i(x;\epsilon) = V(x, x)$, the symmetric equilibrium when $\epsilon = 0$. Moreover, the equilibrium bids are infinitely differentiable in ϵ.

Proof. See Appendix A.

The assumption that the utility functions are given by the forms (1) is not restrictive. Indeed, consider the case of n bidders with utility functions $\{V_i(x_i, \mathbf{x}_{-i})\}_{i=1}^n$, each of which is symmetric in the $n-1$ components of $\mathbf{x}_{-i}$. Let us first define the average (symmetric) utility function as

$$V(x_i, \mathbf{x}_{-i}) = \frac{1}{n}\sum_{k=1}^{n} V_k(x_k = x_i, \mathbf{x}_{-k} = \mathbf{x}_{-i}). \tag{4}$$

Let us also define

$$\epsilon = \max_i \max_{x_1,\ldots,x_n} \frac{|V_i - V|}{|V|}, \tag{5}$$

and

$$U_i(x_i, \mathbf{x}_{-i}) = \frac{V_i(x_i, \mathbf{x}_{-i}) - V(x_i, \mathbf{x}_{-i})}{\epsilon}. \tag{6}$$

Then, the V_i's are given by the form (1), with V, ϵ, and U_i given by (4)–(6). $\square$

Example 1. To illustrate that any group of asymmetric utility functions can be presented in the form (1), let us consider the case where the utility functions $\{V_i\}_{i=1}^n$ are weighted averages of the signals, i.e.,

$$V_i(x_i, \mathbf{x}_{-i}) = a_i x_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{i,j} x_j, \quad a_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{i,j} = 1.$$

Since V_i is symmetric in the last $n-1$ signals, it follows that

$$V_i(x_i, \mathbf{x}_{-i}) = a_i x_i + \frac{1 - a_i}{n-1}\sum_{\substack{j=1 \\ j \neq i}}^{n} x_j, \quad 0 < a_i < 1.$$

To bring the utility functions to the form (1), we first note that by (4), V is given by

$$V(x_i, \mathbf{x}_{-i}) = \bar{a} x_i + \frac{1 - \bar{a}}{n-1}\sum_{\substack{j=1 \\ j \neq}}^{n} x_j, \quad \bar{a} = \frac{1}{n}\sum_{i=1}^{n} a_i.$$

In addition, by (5), ϵ is equal to $\epsilon = \max_j \frac{|a_j - \bar{a}|}{|\bar{a}|}$, and by (6), the functions $\{U_i\}_{i=1}^n$ are given by

$$U_i(x_i, \mathbf{x}_{-i}) = \frac{a_i - \bar{a}}{\max_j \frac{|a_j - \bar{a}|}{|\bar{a}|}}\left(x_i - \frac{1}{n-1}\sum_{\substack{j=1 \\ j \neq i}}^{n} x_j\right).$$

[6] Recently, Lebrun (2009) proved rigorously the differentiability in ϵ of the equilibrium strategies of private-value asymmetric first-price auctions. It is reasonable to expect, therefore, that Lebrun's result can be extended to interdependent asymmetric first-price auctions.

[7] Since Condition (3) is not satisfied in the common-value case, our results do not apply to the case of the Wallet Game (see Klemperer, 1998).

Table 1
Seller's expected revenue (example in Section 3.2).

ϵ	R^{1st}	R^{2nd}	$R_{sym}\left[\frac{V_1+V_2}{2}\right]$	$\frac{R^{1st}-R^{2nd}}{R^{1st}}100\%$	$\frac{R^{1st}-R_{sym}\left[\frac{V_1+V_2}{2}\right]}{R^{1st}}100\%$
0.05	0.33749	0.33738	0.33750	0.03	0.003
0.1	0.34161	0.34120	0.34166	0.12	0.015
0.2	0.34979	0.34823	0.35000	0.46	0.06

3.1. Revenue equivalence

We recall that when $\epsilon = 0$, the case of a symmetric auction with utility function V, Bulow and Klemperer (1996) showed that regardless of the auction mechanism, the seller's expected revenue is given by

$$R_{sym}[V,F] = n(n-1)\int_{x=0}^{1} f(x)(1-F(x)) \times \left\{\int_{x_3=0}^{x}\cdots\int_{x_n=0}^{x} V(x_1=x, x_2=x, x_3,\dots,x_n)f(x_3)\cdots f(x_n)\,dx_3\dots dx_n\right\}dx. \tag{7}$$

We now prove an asymptotic revenue equivalence among all asymmetric auctions with interdependent values, under the same conditions used in Bulow and Klemperer (1996), except that we allow for a weak asymmetry among bidders' utility functions:

Theorem 1. *Consider any auction mechanism with n bidders that satisfies the following conditions:*

1. *All players are risk-neutral.*
2. *The signal of player i is private information to i and is drawn independently from a continuously differentiable distribution function $F(x)$ from a support [0,1] which is common to all players.*
3. *The object is allocated to the player with the highest bid.*[8]
4. *In equilibrium, any player i with the minimal signal $x_i = 0$ makes the same minimal bid $\underline{b}$ and expects a zero surplus.*

Let the utility function of player i be given by (1), and let $R_{sym}[V,F]$ be defined by Eq. *(7). Then, the seller's expected revenue is $R(\epsilon) = R(0) + \epsilon R'(0) + O(\epsilon^2)$, where $R(0) = R_{sym}[V,F]$ and*

$$R'(0) = R_{sym}\left[\frac{\sum_{i=1}^{n} U_i}{n}, F\right]. \tag{8}$$

Proof. See Appendix B. □

The revenue equivalence theorem for symmetric auctions with interdependent values (Bulow and Klemperer, 1996) says that $R(0)$ is independent of the auction mechanism. The novelty in Theorem 1 is, thus, that $\epsilon R'(0)$, the leading-order effect of asymmetry in the utility functions, is also independent of the auction mechanism. Hence, for a weak asymmetry the revenue difference among auctions with interdependent values is only second-order in ϵ. Indeed, in many cases these differences are only in the third or fourth digit, in which case the problem of revenue ranking is more of academic interest than of a practical value (see, e.g., Table 1).

The result of Theorem 1 can be rewritten as

$$R(\epsilon) = R_{sym}[V,F] + \epsilon R_{sym}\left[\frac{\sum_{i=1}^{n} U_i}{n}, F\right] + O(\epsilon^2) = R_{sym}\left[V + \epsilon\frac{\sum_{i=1}^{n} U_i}{n}, F\right] + O(\epsilon^2) = R_{sym}\left[\frac{\sum_{i=1}^{n} V_i}{n}, F\right] + O(\epsilon^2).$$

Therefore, an immediate consequence of Theorem 1 is that the seller's expected revenue in asymmetric auctions with n bidders can be well-approximated with the seller's expected revenue in the symmetric case with n bidders whose utility function is the arithmetic average of the n asymmetric utility functions:

Theorem 2. *Consider any auction mechanism that satisfies conditions 1–4 of* Theorem 1, *with n bidders having weakly asymmetric interdependent values $\{V_i\}_{i=1}^{n}$. Then, the seller's expected revenue is*

$$R[V_1,\dots,V_n] = R_{sym}\left[\frac{\sum_{i=1}^{n} V_i}{n}, F\right] + O(\epsilon^2),$$

where $\epsilon = \max_j \max_{x_1,\dots,x_n} |V_j - (\sum_{i=1}^{n} V_i)/n|$.

Proof. Apply Theorem 1 with V, ϵ, and U_i given by (4)–(6). Since $\sum_{i=1}^{n} U_i \equiv 0$, the result follows.

There is a delicate point which is probably worth clarifying. In Theorem 1 the expression for $R'(0)$ in (8) refers to a direct substitution of $V = \frac{\sum_{i=1}^{n} U_i}{n}$ in (7). This is not necessarily the same as the expected revenue when $V = \frac{\sum_{i=1}^{n} U_i}{n}$. For example, if $\frac{\sum_{i=1}^{n} U_i}{n} < 0$ then players would simply choose not to bid, so that the expected revenue would be zero, but the value of direct substitution in (7) would be negative. Of course, this distinction is not important in Theorem 2. □

[8] In the symmetric case, condition 3 is equivalent to the condition that the object is allocated to the player with the highest signal. This equivalence, however, does not hold for asymmetric auctions.

3.2. Example

Consider an auction with weakly asymmetric interdependent values and two bidders whose signals are independently uniformly distributed in [0,1], and whose utility functions are given by

$$V_1(x_1,x_2) = x_1, \quad V_2(x_2,x_1) = x_2 + \epsilon x_1 x_2. \tag{9}$$

In the following, we compare the seller's expected revenue in second-price auction, in first-price auction, and our explicit approximation $R_{\text{sym}}\left[\frac{\sum_{i=1}^n V_i}{n}, F\right]$.

For the second-price auction, an explicit calculation of the (exact) expected revenue for the seller (see Appendix C) gives

$$R^{2\text{nd}} = \frac{1}{2} - \frac{1}{2\epsilon} - \frac{1}{\epsilon^2} + \frac{\ln(1+\epsilon)}{\epsilon^2} + \frac{\ln(1+\epsilon)}{\epsilon^3}. \tag{10}$$

Taylor expansion of (10) gives

$$R^{2\text{nd}} = \frac{1}{3} + \frac{1}{12}\epsilon - \frac{1}{20}\epsilon^2 + \cdots. \tag{11}$$

By (4) and (9), the average utility function is[9] $\frac{1}{2}[V_1(x_1,x_2) + V_2(x_1,x_2)] = x_1 + 0.5\epsilon x_1 x_2$. In the case of two players, the symmetric revenue (7) is equal to $R_{\text{sym}}[V,F] = 2\int_0^1 V(x,x)(1-F(x))f(x)\,dx$. Substituting the average utility function in $R_{\text{sym}}[V,F]$ gives the symmetric approximation of the revenue $R_{\text{sym}}\left[\frac{V_1+V_2}{2}\right] = \frac{1}{3} + \frac{1}{12}\epsilon$, which, as expected, agrees with (11) up to $O(\epsilon^2)$. Finally, we note that while the expected revenue in the first-price auction cannot be calculated analytically, it can be calculated numerically (for details, see Appendix D).

As Table 1 shows, the differences among the seller's expected revenue in the first-price auction, the seller's expected revenue in the second price auction, and the symmetric approximation $R_{\text{sym}}\left[\frac{V_1+V_2}{2}\right]$ are only in the third or fourth digit. Indeed, even when the asymmetry level is $\epsilon = 20\%$, the revenue difference is less than 0.5%. Moreover, it is easy to see that, as predicted, the revenue differences scale like ϵ^2 (i.e., doubling the value of ϵ leads to a four-fold increase in the revenue difference).

Of course, one can ask whether one numerical example that shows that the predictions of the perturbation analysis are valid for ϵ which is only moderately small is typical, or a coincidence. To answer this question we tested several other examples (data not shown), and in all cases we observed that the predictions of Theorems 1 and 2 remain valid even when ϵ was only moderately small. This should not come as a surprise for people familiar with perturbation analysis. Indeed, more than 200 years of applications of perturbation analysis have shown that its predictions are usually valid not only for infinitesimally small ϵ, but also for moderately small ϵ.[10]

4. Asymmetric distribution functions

Consider n risk-neutral bidders bidding for an indivisible object in a standard auction where the highest bidder wins the object. Bidder $i, i = 1, \ldots, n$ receives a signal x_i which is private information to i and is independently drawn from the interval [0,1] according to a continuously differentiable distribution function[11]

$$F_i(x) = F(x) + \epsilon H_i(x) \tag{12}$$

where $F(0) = F_i(0) = 0, F(1) = F_i(1) = 1, H_i(0) = H_i(1) = 0$ and $|H_i| \leqslant 1$ in [0,1] for all i. Denote $h_i = H_i'$ and $f_i = F_i'$. The utility function $V(x_i, \mathbf{x}_{-i})$ is the same for all the bidders and is symmetric in the $n-1$ components of $\mathbf{x}_{-i}$, monotonically increasing in all its variables, and satisfies $V(0, \ldots, 0) = 0$.

The derivations in this section are also based on the implicit assumption that the equilibrium strategies are continuously differentiable in the asymmetry parameter ϵ. The following result suggests that the conditions under which the equilibrium strategies are differentiable with respect to ϵ in the case of asymmetric distribution functions may be simpler than in the case of asymmetric utility functions:

Lemma 2. *Consider a second-price auction with two bidders with valuation functions*

$$V_1(x_1,x_2) = V(x_1,x_2), \quad V_2(x_2,x_1) = V(x_2,x_1), \tag{13}$$

whose signals are asymmetrically distributed with density function $f_i = f + \epsilon h_i$. Then, the equilibrium bids are given by $b_1(x) = b_2(x) = V(x,x)$. In particular, the equilibrium bids are infinitely differentiable in ϵ.

Proof. See Appendix G. □

We recall that Fibich et al. (2004) showed that all private-value auctions in which bidders' values are distributed asymmetrically are asymptotically revenue equivalent. We now generalize this result for asymmetric auctions with interdependent values:

Theorem 3. *Consider any auction mechanism with n bidders that satisfies the following conditions:*

1. *All players are risk-neutral.*
2. *The signal x_i of player i is private information to i and is drawn independently by a continuously differentiable distribution function $F_i(x)$ from a support [0,1] which is common to all players.*

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[9] Note that $V_2(x_1,x_2) = x_1 + \epsilon x_2 x_1 \neq V_2(x_2,x_1) = x_2 + \epsilon x_1 x_2$.

[10] In fact, in many cases the predictions of the perturbation analysis remain valid even outside the domain where one expect them to be valid, i.e., for $\epsilon = O(1)$.

[11] The assumption that the distribution functions are of the form (12) is not restrictive. Similarly to what we have done in Section 3, we can bring any family of distribution functions $\{F_i\}_{i=1}^n$ to this form by defining $F = \frac{1}{n}\sum_{i=1}^n F_i$, $\epsilon = \max_i \max_v |F_i - F|/|F|$ and $H_i = (F_i - F)/\epsilon$.

3. *The object is allocated to the player with the highest bid.*
4. *In equilibrium, any player i with the minimal signal $x_i = 0$ makes the same minimal bid $\underline{b}$ and expects a zero surplus.*

Let the distribution function of the signal x_i of player i be given by (12), and let $R_{\text{sym}}[V, F]$ be defined by Eq. (7). *Then, the seller's expected revenue is given by $R(\epsilon) = R(0) + \epsilon R'(0) + O(\epsilon^2)$, where $R(0) = R_{\text{sym}}[V, F]$ and*

$$R'(0) = \frac{d}{d\epsilon} R_{\text{sym}} \left[V, F + \epsilon \frac{\sum_{i=1}^{n} H_i}{n} \right] \Bigg|_{\epsilon=0}.$$

Proof. See Appendix E. □

Remark 1. Theorem 3 generalizes the result of Fibich et al. (2004) for asymmetric private-value auctions. Indeed, it can be verified (see Appendix F) that in the special case of private-value $V(x_i, \mathbf{x_{-i}}) = x_i$, then $R'(0) = -(n-1) \int_0^1 F^{n-2}(1-F) \sum_{i=1}^{n} H_i \, dx$.

The revenue equivalence theorem for symmetric auctions with interdependent values (Bulow and Klemperer, 1996) says that $R(0)$ is independent of the auction mechanism. The novelty in Theorem 3 is, thus, that $\epsilon R'(0)$, the leading-order effect of asymmetry in the signal distribution functions, is also independent of the auction mechanism. As a result, the differences in revenues among the standard auctions are only of second order. Hence, as in the case of asymmetric functions, Theorem 3 implies that the seller's expected revenue in auctions with n bidders and asymmetric distribution functions can be well-approximated with the seller's expected revenue in the symmetric case with n bidders whose distribution function is the arithmetic average of the n asymmetric distribution functions.

Theorem 4. *Consider any auction mechanism that satisfies conditions 1–4 of* Theorem 3 *with distribution functions $\{F_i\}_{i=1}^{n}$. Let $F_{\text{avg}} = \frac{1}{n} \sum_{i=1}^{n} F_i$ and let $\epsilon = \max_i \max_v |F_i - F_{\text{avg}}|$ be small. Then, the seller's expected revenue is*

$$R[F_1, \ldots, F_n] = R_{\text{sym}}[V, F_{\text{avg}}] + O(\epsilon^2).$$

Proof. Apply Theorem 3 with $F_i = F_{\text{avg}} + \epsilon H_i$. Since $\sum_{i=1}^{n} H_i(x) \equiv 0$, it immediately follows that $R'(0) = 0$. □

5. Concluding remarks

In this study we considered asymmetric equilibria which bifurcate smoothly from the symmetric equilibrium. Therefore, it may seem that the asymptotic revenue equivalence results are immediate, since they follow from a continuity argument. This, however, is not the case. Indeed, if we consider the expected revenue R as a function of ϵ, and if we assume that the asymmetric equilibrium bids are smooth in ϵ, then it is indeed obvious that $R = R(\epsilon)$ is smooth in ϵ. Therefore, since the revenue equivalence theorem implies that $R(\epsilon = 0)$, the symmetric revenue, is independent of the auction mechanism, this immediately implies that the revenue differences among different auction mechanisms is $O(\epsilon)$ small. Our results, however, are much stronger, since we prove that $R'(\epsilon = 0)$ is also independent of the auction mechanism. Therefore, this implies that the revenue differences among different auction mechanisms is $O(\epsilon^2)$ small. Roughly speaking, if $\epsilon = 0.1$, the immediate continuity argument shows that the revenue differences among different auction mechanisms are on the order of 10%, whereas our asymptotic result shows that, in fact, that the revenue differences among different auction mechanisms are on the order of 1%.

The results of this paper demonstrate that regardless of the kind of asymmetry among the bidders, weak asymmetry does not have a significant effect on revenue ranking in standard auctions. Since analysis of asymmetric auctions is usually hard, this conclusion suggests that it is justified to neglect asymmetry when analyzing revenue ranking of auctions.

It is natural to ask, therefore, where this result can be generalized even further, so that any "$O(\epsilon)$ deviation" from the conditions of the classical revenue equivalence theorem would only result in a $O(\epsilon^2)$ effect on revenue ranking. It turns out that this is not the case. Indeed, Fibich et al. (2006) show that an $O(\epsilon)$ risk aversion generates $O(\epsilon)$ differences of revenues across standard auctions. Therefore, unlike asymmetry, risk aversion cannot be neglected in the analysis of revenue ranking of standard auctions.

Finally, we note that there are still many open questions which require further research. One open question is to find conditions under which one can rigorously justify the differentiability of the equilibrium bids in ϵ. Another open question to explicitly calculate the $O(\epsilon^2)$ effect of asymmetry on the revenue, in order to be able to find the revenue ranking of different auction mechanisms. This is probably an academic question when ϵ is small, but may be of a practical value when ϵ is not small. Another issue which was not considered in this study is the case when bidders signals are correlated (affiliated), rather than independent. While it is known that even in the symmetric case, there is no revenue equivalence when the signals are correlated (Milgrom and Weber, 1982), one could use similar perturbation techniques to study the effect of a *weak* correlation on the revenue ranking.

Acknowledgments

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Appendix A. Proof of Lemma 1

The expected utility of bidder i with signal x who makes a bid b is given by

$$EU_1(b,x) = \int_0^{b_2^{-1}(b)} (V_1(x,s) - b_2(s)) f(s) \, ds, \quad EU_2(b,x) = \int_0^{b_1^{-1}(b)} (V_2(x,s) - b_1(s)) f(s) \, ds.$$

The inverse equilibrium strategies $x_i(b) = b_i^{-1}(b)$ are determined from

$$\frac{\partial EU_1(b,x)}{\partial b} = \frac{\partial EU_2(b,x)}{\partial b} = 0,$$

leading to the system

$$H_i(x_1, x_2, b, \epsilon) = 0, \quad i = 1, 2,$$

where

$$H_1 = V_1(x_1, x_2) - b, \quad H_2 = V_2(x_2, x_1) - b. \tag{14}$$

At $\epsilon = 0$, this system has the symmetric solution $x_1(b) = x_2(b) = x_{\mathrm{sym}}(b)$, where $x_{\mathrm{sym}}(b)$ is the inverse function of $b_{\mathrm{sym}}(x) = V(x, x)$. In addition,

$$\left.\frac{\partial(H_1, H_2)}{\partial(x_1, x_2)}\right|_{x_1(b)=x_2(b)=x_{\mathrm{sym}}(b), \epsilon=0} = \begin{vmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \\ \frac{\partial V}{\partial x_2} & \frac{\partial V}{\partial x_1} \end{vmatrix} \neq 0.$$

Hence, the result follows from the implicit function theorem.

Appendix B. Proof of Theorem 1

Let us denote $B_{j,i}(x;\epsilon) = b_j^{-1}(b_i(x;\epsilon);\epsilon)$, where b_i is the equilibrium bid of bidder i and b_i^{-1} is the inverse equilibrium bid. Clearly, $B_{j,j}(x;\epsilon) = x$ and $B_{j,i}(x;\epsilon = 0) = x$. Let $E_i(x_i), P_i(x_i)$ and $S_i(x_i)$ be the expected payment, probability of winning and the expected surplus for bidder i with signal x_i at equilibrium. Then,[12]

$$S_1(x_1) = P_1(x_1) E_{\mathbf{x_{-1}}}[V_1(x_1, \mathbf{x_{-1}}) | 1 \text{ wins with signal } x_1] - E_1(x_1), \tag{15}$$

where $P_1(x_1) = \prod_{m=2}^n F(B_{m,1}(x_1;\epsilon)), \mathbf{x_{-1}} = (x_2, \ldots, x_n)$, and

$$E_{\mathbf{x_{-1}}}[V_1(x_1, \mathbf{x_{-1}}) | 1 \text{ wins with signal } x_1] = \frac{1}{P_1(x_1)} \int_{x_2=0}^{B_{2,1}(x_1;\epsilon)} \cdots \int_{x_n=0}^{B_{n,1}(x_1;\epsilon)} V_1(x_1, \mathbf{x_{-1}}) f(x_2) \cdots f(x_n)\, dx_2 \cdots dx_n \tag{16}$$

is the conditional expectation of the value for bidder 1, given that he wins with signal x_1. Applying a standard argument (see, e.g., Bulow and Klemperer, 1996; Klemperer, 1998), for any $\widetilde{x_1} \neq x_1$,

$$S_1(x_1) \geqslant S_1(\widetilde{x_1}) - P_1(\widetilde{x_1}) E_{\mathbf{x_{-1}}}[V_1(\widetilde{x_1}, \mathbf{x_{-1}}) - V_1(x_1, \mathbf{x_{-1}}) | 1 \text{ wins with signal } \widetilde{x_1}].$$

Therefore,

$$S_1(\widetilde{x_1}) - S_1(x_1) \leqslant P_1(\widetilde{x_1}) E_{\mathbf{x_{-1}}}[V_1(\widetilde{x_1}, \mathbf{x_{-1}}) - V_1(x_1, \mathbf{x_{-1}}) | 1 \text{ wins with signal } \widetilde{x_1}].$$

Substituting $\widetilde{x_1} = x_1 + dx$ with $dx > 0$, dividing both sides by dx and letting $dx \to 0$ gives

$$S_1'(x_1) \leqslant P_1(x_1) E_{\mathbf{x_{-1}}}\left[\frac{\partial V_1}{\partial x_1}\middle| 1 \text{ wins with signal } x_1\right].$$

Repeating this procedure with $dx < 0$ gives

$$S_1'(x_1) \geqslant P_1(x_1) E_{\mathbf{x_{-1}}}\left[\frac{\partial V_1}{\partial x_1}\middle| 1 \text{ wins with signal } x_1\right].$$

Hence,

$$S_1'(x_1) = P_1(x_1) E_{\mathbf{x_{-1}}}\left[\frac{\partial V_1}{\partial x_1}\middle| 1 \text{ wins with signal } x_1\right] = \int_{x_2=0}^{B_{2,1}(x_1;\epsilon)} \cdots \int_{x_n=0}^{B_{n,1}(x_1;\epsilon)} \frac{\partial V_1}{\partial x_1}(x_1, \mathbf{x_{-1}}) f(x_2) \cdots f(x_n)\, dx_2 \cdots dx_n. \tag{17}$$

Differentiating (15) with respect to x_1, substituting (17) and using (16) gives

$$\begin{aligned} E_1'(x_1) &= \frac{d}{dx_1}[P_1(x_1) E_{\mathbf{x_{-1}}}[V_1(x_1, \mathbf{x_{-1}}) | 1 \text{ wins with signal } x_1]] - S_1'(x_1) \\ &= \frac{d}{dx_1}\left[\int_{x_2=0}^{B_{2,1}(x_1;\epsilon)} \cdots \int_{x_n=0}^{B_{n,1}(x_1;\epsilon)} V_1(x_1, \mathbf{x_{-1}}) f(x_2) \cdots f(x_n)\, dx_2 \cdots dx_n\right] - S_1'(x_1) \\ &= \sum_{j=2}^n \frac{\partial B_{j,1}(x_1;\epsilon)}{\partial x_1} P_{1,-j}(x_1) E_{\mathbf{x_{-1,-j}}}\left[V_1(x_1, x_j = B_{j,1}(x_1;\epsilon), \mathbf{x_{-1,-j}}) | b_1(x_1) > \max_{m \neq 1,j} b_m(x_m)\right] f(B_{j,1}(x_1;\epsilon)), \end{aligned}$$

where $\mathbf{x_{-1,-j}}$ is $\mathbf{x_{-1}}$ without the x_j element, $P_{1,-j}(x_1) = \prod_{\substack{m=2 \\ m \neq j}}^n F(B_{m,1}(x_1;\epsilon))$ is the probability that player 1 with signal x_1 has a higher bid than bidders $2, \ldots, j-1, j+1, \ldots, n$, and

[12] To simplify the notations, we work with S_1 rather than S_i.

$$E_{\mathbf{x}_{-1,-j}}[V_1(x_1,x_j=B_{j,1}(x_1;\epsilon),\mathbf{x}_{-1,-j})|b_1(x_1)>ax_{i\neq 1,j}b_i(x_i)]$$
$$=\frac{1}{P_{1,-j}(x_1)}\int_{x_2=0}^{B_{2,1}(x_1;\epsilon)}\cdots\int_{x_{j-1}=0}^{B_{j-1,1}(x_1;\epsilon)}\int_{x_{j+1}=0}^{B_{j+1,1}(x_1;\epsilon)}\cdots\int_{x_n=0}^{B_{n,1}(x_1;\epsilon)}V_1(x_1,x_j=B_{j,1}(x_1;\epsilon),\mathbf{x}_{-1,-j})\left(\prod_{\substack{k=2\\k\neq j}}^{n}f(x_k)\,dx_k\right)$$

is the conditional expectation of the value for bidder 1 when he has a higher bid than bidders $2,\dots,j-1,j+1,\dots,n$ and when bidder j has signal $x_j=B_{j,1}(x_1;\epsilon)$. Similarly, for player i,

$$E_i'(x_i)=\sum_{\substack{j=1\\j\neq i}}^{n}\frac{\partial B_{j,i}(x_i;\epsilon)}{\partial x_i}P_{i,-j}(x_i)E_{\mathbf{x}_{-i,-j}}[V_i(x_i,x_j=B_{j,i}(x_i;\epsilon),\mathbf{x}_{-i,-j})|\ b_i(x_i)>\max_{m\neq i,j}b_m(x_m)]f(B_{j,i}(x_i;\epsilon)), \tag{18}$$

where $\mathbf{x}_{-i,-j}$ is $(x_1,x_2,\dots,x_n)$ without the x_i and x_j elements.

Let $R_i(\epsilon)$ be the expected payments of player i averaged across her signals. Then,

$$R_i(\epsilon)=\int_0^1E_i(x)f(x)\,dx=E_i(x)F|_0^1-\int_0^1E_i'(x)F(x)\,dx=E_i(1)-\int_0^1E_i'(x)F(x)\,dx=E_i(0)+\int_0^1E_i'(x)(1-F(x))\,dx. \tag{19}$$

We now show that

$$E_i(0)=E_i(x_i=0;\epsilon)=0. \tag{20}$$

Indeed, from (15) we have that

$$E_i(x_i=0)=P_i(0)E_{\mathbf{x}_{-1}}[V_1(x_1=0,\mathbf{x}_{-1})|1\text{ wins with signal }0]-S_1(x_1=0).$$

>From Condition 4 it follows that for all $i\neq j$,

$$B_{j,i}(0;\epsilon)=b_j^{-1}(\underline{b}(\epsilon);\epsilon)=0,$$

where $\underline{b}(\epsilon)$ is the minimal bid. Therefore, $P_i(0)=0$.

In addition, from Condition 4 we have that $S_1(x_1=0)=0$. Therefore, we proved (20).

Substitution of (18), (20) in (19) gives

$$R_i(\epsilon)=\int_0^1E_i'(x)(1-F(x))\,dx$$
$$=\int_0^1\left\{\sum_{\substack{j=1\\j\neq i}}^{n}\frac{\partial B_{j,i}(x;\epsilon)}{\partial x}P_{i,-j}(x)E_{\mathbf{x}_{-i,-j}}[V_i(x_i=x,x_j=B_{j,i}(x;\epsilon),\mathbf{x}_{-i,-j})|b_i(x)>\max_{m\neq i,j}b_m(x_m)]f(B_{j,i}(x;\epsilon))(1-F(x))\right\}dx. \tag{21}$$

The seller's expected revenue is given by $R(\epsilon)=\sum_{i=1}^{n}R_i(\epsilon)$. In the symmetric case $\epsilon=0$ we have that $B_{j,i}(x;0)=x, V_i=V$, that $b_i>b_m\iff x_i>x_m$, and that $P_{i,-j}(x_i)=F^{n-2}(x_i)$. Therefore, the expected revenue in the symmetric case is given by $R(0)=R_{\text{sym}}[V,F]$, where

$$R(0)=nR_1(0)=n\int_0^1\sum_{j=2}^{n}\left(E_{\mathbf{x}_{-1,-j}}\left[V(x_1,x_j=x_1,\mathbf{x}_{-1,-j})|x_1>\max_{m\neq 1,j}x_m\right]\right)F^{n-2}(x_1)f(x_1)(1-F(x_1))\,dx_1$$
$$=n(n-1)\int_0^1E_{\mathbf{x}_{-1,-2}}\left[V(x_1,x_2=x_1,\mathbf{x}_{-1,-2})|x_1>\max_{m\neq 1,2}x_m\right]F^{n-2}(x_1)f(x_1)(1-F(x_1))dx_1,$$

and

$$E_{\mathbf{x}_{-1,-2}}\left[V(x_1,x_2=x_1,\mathbf{x}_{-1,-2})|x_1>\max_{m\neq 1,2}x_m\right]=\frac{1}{F^{n-2}(x)}\int_{x_3=0}^{x_1}\cdots\int_{x_n=0}^{x_1}V(x_1,x_2=x_1,\mathbf{x}_{-1,-2})\left(\prod_{k=3}^{n}f(x_k)\,dx_k\right)$$

is the conditional expectation of the value for bidder 1 given that his signal is equal to that of bidder 2 and is higher than the other $(n-2)$ signals.

We now proceed to calculate $R'(0)=\sum_{i=1}^{n}R_i'(0)$. Since $B_{j,i}$ and $V_i=V+\epsilon U_i$ depend on ϵ, differentiating (21) and setting $\epsilon=0$ gives that $R_i'(0)=I_{i,1}+I_{i,2}$, where

$$I_{i,1}=\frac{d}{d\epsilon}\left[\int_0^1\sum_{\substack{j=1\\j\neq i}}^{n}\frac{\partial B_{j,i}(x;\epsilon)}{\partial x}P_{i,-j}(x_i)E_{\mathbf{x}_{-i,-j}}\left[V(x_i=x,x_j=B_{j,i}(x;\epsilon),\mathbf{x}_{-i,-j})|x_i>\max_{m\neq i,j}B_{i,m}(x_m;\epsilon)\right]f(B_{j,i}(x;\epsilon))(1-F(x))\,dx\right]_{\epsilon=0},$$
$$I_{i,2}=\int_0^1\sum_{\substack{j=1\\j\neq i}}^{n}E_{\mathbf{x}_{-i,-j}}\left[U_i(x_i=x,x_j=x,\mathbf{x}_{-i,-j})|x_i>\max_{m\neq i,j}x_m\right]F^{n-2}(x)f(x)(1-F(x))dx.$$

The proof follows from the fact that

$$\sum_{i=1}^{n}I_{i,1}=0. \tag{22}$$

Indeed, in that case

$$R'(0) = \sum_{i=1}^{n} I_{i,2} = R_{\text{sym}}\left[\frac{\sum_{i=1}^{n} U_i}{n}, F\right].$$

To prove (22), first note that $\frac{\partial b_j^{-1}}{\partial b}\Big|_{\epsilon=0} = \left(b_{sym}^{-1}\right)'$, where $b_{sym}(x)$ is the equilibrium bid in the symmetric case $\epsilon = 0$. Therefore,

$$\frac{\partial B_{j,i}}{\partial \epsilon}\bigg|_{\epsilon=0} = (b_{sym}^{-1})' \frac{\partial b_i}{\partial \epsilon}\bigg|_{\epsilon=0} + \frac{\partial b_j^{-1}}{\partial \epsilon}\bigg|_{\epsilon=0}.$$

Differentiating the identity $x = b_j^{-1}(b_j(x;\epsilon);\epsilon)$ with respect to ϵ and substituting $\epsilon = 0$ gives

$$0 = \left(b_{sym}^{-1}\right)' \frac{\partial b_j}{\partial \epsilon}\bigg|_{\epsilon=0} + \frac{\partial b_j^{-1}}{\partial \epsilon}\bigg|_{\epsilon=0}.$$

Hence,

$$\frac{\partial B_{j,i}}{\partial \epsilon}\bigg|_{\epsilon=0} = \left(b_{sym}^{-1}\right)' \left(\frac{\partial b_i}{\partial \epsilon}\bigg|_{\epsilon=0} - \frac{\partial b_j}{\partial \epsilon}\bigg|_{\epsilon=0}\right)$$

and

$$\frac{\partial}{\partial \epsilon}[B_{i,j} + B_{j,i}]_{\epsilon=0} = 0. \tag{23}$$

Since $I_{i,1}$ can be written as

$$I_{i,1} = \int_0^1 \left[G_1(x) \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\partial B_{j,i}(x;\epsilon)}{\partial \epsilon}\bigg|_{\epsilon=0} + G_2(x) \frac{\partial}{\partial x} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\partial B_{j,i}(x;\epsilon)}{\partial \epsilon}\bigg|_{\epsilon=0} \right] dx,$$

with the functions $G_1(x)$ and $G_2(x)$ being independent of index i, application of (23) proves (22).

Appendix C. Derivation of Eq. (10)

The equations for the bid functions are (see, Krishna, 2002)

$$V_1(x_1(b_1), x_2(b_1)) = x_1 = b_1 \quad V_2(x_2(b_2), x_1(b_2)) = x_2 + \epsilon x_1 x_2 = b_2,$$

which gives the inverse equilibrium bids

$$x_1 = b_1 \text{ and } x_2 = \frac{b_2}{1 + \epsilon b_2}. \tag{24}$$

The distribution of the second-highest bid b is

$$\begin{aligned} F^{2\text{nd}}(b) &= \Pr(\min(b_1, b_2) \leqslant b) = \Pr(\{b_1 \leqslant b\} \cup \{b_2 \leqslant b\}) \\ &= \Pr(b_1 \leqslant b) + \Pr(b_2 \leqslant b) - \Pr(b_1 \leqslant b, b_2 \leqslant b) \\ &= \Pr(x_1 \leqslant b_1^{-1}(b)) + \Pr(x_2 \leqslant b_2^{-1}(b)) - \Pr(x_1 \leqslant b_1^{-1}(b), x_2 \leqslant b_2^{-1}(b)) \\ &= F(x_1(b)) + F(x_2(b)) - F(x_1(b)F(x_2(b)). \end{aligned} \tag{25}$$

Therefore, the seller's expected revenue in the second-price auction is

$$R = \int_0^{\bar{b}} b\, dF^{2\text{nd}}(b) = bF^{2\text{nd}}(b)|_0^{\bar{b}} - \int_0^{\bar{b}} F^{2\text{nd}}(b)\, db = \bar{b} - \int_0^{\bar{b}} F^{2\text{nd}}(b)\, db = \bar{b} - \int_0^{\bar{b}} [F(x_1(b)) + F(x_2(b)) - F(x_1(b))F(x_2(b))] db,$$

where $\bar{b}$ is the maximal price, or the second-highest bid, in equilibrium. Since $\bar{b} = 1$ and given the inverse bids (24) the exact expected revenue is

$$R = 1 - \int_0^1 \left(b + \frac{b}{1+\epsilon b} - \frac{b^2}{1+\epsilon b}\right) db,$$

which leads to Eq. (10).

Appendix D. Expected revenue in first-price auctions

In the case of a first-price auction, the expected utility of bidder 2 is

$$U_2(x_2, b) = \int_0^{x_1(b)} [x_2 + \epsilon x_1 x_2 - b] f(x_1) dx_1 = F(x_1(b))(x_2 - b) + \epsilon x_2 \int_0^{x_1(b)} x_1 f(x_1) dx_1,$$

where $x_i(b)$ is the inverse bid function of player i. Differentiating U_2 with respect to b and substituting $x_2 = x_2(b)$ gives

$$x_1'(b) = \frac{F(x_1(b))}{f(x_1(b))} \frac{1}{x_2(b) + \epsilon x_1(b) x_2(b) - b}. \tag{26}$$

Repeating this procedure for bidder 1 gives

$$x_2'(b) = \frac{F(x_2(b))}{f(x_2(b))} \frac{1}{x_1(b) - b}. \tag{27}$$

The ordinary-differential Eqs. (26) and (27) for the inverse equilibrium bids, together with the initial conditions $x_1(0) = x_2(0) = 0$ and the boundary condition $x_1(\bar{b}) = x_2(\bar{b})$, where $\bar{b}$ is the (unknown) maximal bid in equilibrium, are solved using a shooting method (Marshall et al., 1994). Unlike (Marshall et al., 1994), however, we do not calculate the seller's expected revenue using Monte Carlo methods. Rather, following Fibich and Gavious (2003), we first note that the distribution of the highest bid is

$$F_{1st}(b) = \Pr(\max(b_1(x_1), b_2(x_2)) \leqslant b) = \Pr(b_1(x_1) \leqslant b)\Pr(b_2(x_2)) \leqslant b) = F(x_1(b))F(x_2(b)).$$

Therefore, the seller's expected revenue is given by

$$R^{1st} = \int_0^{\bar{b}} bF_{1st}'(b)\,db = bF_{1st}(b)|_0^{\bar{b}} - \int_0^{\bar{b}} F_{1st}(b)\,db = \bar{b} - \int_0^{\bar{b}} F(x_1(b))F(x_2(b))\,db.$$

Let us define the auxiliary equation

$$y'(b) = F(x_1(b))F(x_2(b)), \quad y(\bar{b}) = \bar{b}. \tag{28}$$

Since $R^{1st} = y(0)$, the expected revenue is easily calculated by integrating Eq. (28) backwards, once (26) and (27) have been solved.

Appendix E. Proof of Theorem 3

We use here the same notations and approach as in Appendix B. The expected surplus for bidder 1 with signal x_1 at equilibrium is given by

$$S_1(x_1) = P_1(x_1)E_{\mathbf{x_{-1}}}[V(x_1, \mathbf{x_{-1}})|1 \text{ wins with signal } x_1] - E_1(x_1), \tag{29}$$

where

$$E_{\mathbf{x_{-1}}}[V(x_1, \mathbf{x_{-1}})|1 \text{ wins with signal } x_1] = \frac{1}{P_1(x_1)} \int_{x_2=0}^{B_{2,1}(x_1;\epsilon)} \cdots \int_{x_n=0}^{B_{n,1}(x_1;\epsilon)} V(x_1, \mathbf{x_{-1}}) f_2(x_2) \cdots f_n(x_n)\,dx_2 \cdots dx_n. \tag{30}$$

Repeating the derivation of (17) in Appendix B (with V_1 replaced with V) gives that

$$S_1'(x_1) = P_1(x_1)E_{\mathbf{x_{-1}}}\left[\frac{\partial V}{\partial x_1}\middle| 1 \text{ wins with signal } x_1\right]. \tag{31}$$

Differentiating (29) with respect to x_1, substituting (31) and using (30) gives

$$\begin{aligned} E_1'(x_1) &= \frac{d}{dx_1}[P_1(x_1)E_{\mathbf{x_{-1}}}[V(x_1, \mathbf{x_{-1}})|1 \text{ wins with signal } x_1]] - S_1'(x_1) \\ &= \sum_{j=2}^n \frac{\partial B_{j,1}(x_1;\epsilon)}{\partial x_1} P_{1,-j}(x_1) E_{\mathbf{x_{-1,-j}}}\left[V(x_1, x_j = B_{j,1}(x_1;\epsilon), \mathbf{x_{-1,-j}})|b_1(x_1) > \max_{m \neq 1,j} b_m(x_m)\right] f_j(B_{j,1}(x_1;\epsilon)). \end{aligned}$$

Similarly, for player i,

$$E_i'(x_i) = \sum_{\substack{j=1 \\ j\neq i}}^n \frac{\partial B_{j,i}(x_i;\epsilon)}{\partial x_i} P_{i,-j}(x_i) E_{\mathbf{x_{-i,-j}}}\left[V(x_i, x_j = B_{j,i}(x_i;\epsilon), \mathbf{x_{-i,-j}})|\ b_i(x_i) > \max_{m \neq i,j} b_m(x_m)\right] f_j(B_{j,i}(x_i;\epsilon)). \tag{32}$$

Let $R_i(\epsilon)$ be the expected payments of player i averaged across her signals. Then,

$$\begin{aligned} R_i(\epsilon) &= \int_0^1 E_i(x) f_i(x)\,dx = E_i(x)F_i|_0^1 - \int_0^1 E_i'(x)F_i(x)dx = E_i(1) - \int_0^1 E_i'(x)F_i(x)dx = E_i(0) + \int_0^1 E_i'(x)(1 - F_i(x))dx \\ &= \int_0^1 E_i'(x)(1 - F_i(x))dx, \end{aligned} \tag{33}$$

where in the last equality we used the identity $E_i(0) = 0$, the proof of which is identical to that of (20). Substitution of (32) in (33) gives

$$R_i(\epsilon) = \int_0^1 \left(\sum_{\substack{j=1 \\ j\neq i}}^n \frac{\partial}{\partial x}[B_{j,i}(x;\epsilon)] P_{i,-j}(x) E_{\mathbf{x_{-i,-j}}}\left[V(x_i = x, x_j = B_{j,i}(x;\epsilon), \mathbf{x_{-i,-j}})|x_i > \max_{m\neq i,j} B_{i,m}(x_m;\epsilon)\right] f_j(B_{j,i}(x;\epsilon))(1 - F_i(x)) \right) dx.$$

The seller's expected revenue is given by $R(\epsilon) = \sum_{i=1}^n R_i(\epsilon)$. In the symmetric case $\epsilon = 0$ we have that $B_{j,i}(x;0) = x, F_i = F, f_j = f$, and that $P_{i,-j}(x) = F^{n-2}(x)$. Therefore, the expected revenue in the symmetric case is given by $R(0) = nR_1(0) = R_{\text{sym}}[V, F]$.

We now proceed to calculate $R'(0) = \sum_{i=1}^n R_i'(0)$. We have that $R_i'(0) = I_{i,1} + I_{i,2}$, where

$$I_{i,1} = \frac{d}{d\epsilon}\left[\int_0^1 \left\{ \sum_{\substack{j=1 \\ j\neq i}}^n \frac{\partial B_{j,i}(x;\epsilon)}{\partial x} P_{i,-j}(x) E_{\mathbf{x_{-i,-j}}}\left[V(x_i = x, x_j = B_{j,i}(x;\epsilon), \mathbf{x_{-i,-j}})|x_i > \max_{m\neq i,j} B_{i,m}(x_m;\epsilon)\right] f(B_{j,i}(x;\epsilon))(1 - F(x)) \right\} dx \right]_{\epsilon=0},$$

$$I_{i,2} = \int_0^1 \left\{ \sum_{\substack{j=1 \\ j \neq i}}^n E_{\mathbf{x}_{-\mathbf{i},-\mathbf{j}}} \left[V(x_i = x, x_j = x, \mathbf{x}_{-\mathbf{i},-\mathbf{j}}) | x_i > \max_{m \neq i,j} x_m \right] F^{n-2}(x)(h_j(x)(1-F(x)) - f(x)H_i(x)) \right\} dx.$$

Therefore,

$$R'(0) = \sum_{i=1}^n I_{i,1} + \sum_{i=1}^n I_{i,2}. \tag{34}$$

To calculate $\sum_{i=1}^n I_{i,1}$, we first note that

$$\begin{aligned}
&\frac{d}{d\epsilon}\left[P_{1,-2}(x)E_{\mathbf{x}_{-\mathbf{1},-\mathbf{2}}}(V(x_1 = x, x_2 = x, \mathbf{x}_{-\mathbf{1},-\mathbf{2}}) | b_i(x_i) > \max_{m \neq 1,2} b_m(x_m))\right]_{\epsilon=0} \\
&\quad = \frac{d}{d\epsilon}\left[\int_{x_3=0}^{B_{3,1}(x;\epsilon)} \cdots \int_{x_n=0}^{B_{n,1}(x;\epsilon)} V(x,x,\mathbf{x}_{-\mathbf{1},-\mathbf{2}}) f_3(x_3)\cdots f_n(x_n)\, dx_3 \cdots dx_n\right]_{\epsilon=0} \\
&\quad = \sum_{k=3}^n \int_{x_3=0}^x \cdots \int_{x_n=0}^x V(x,x,\mathbf{x}_{-\mathbf{1},-\mathbf{2}}) h_k(x_k) \prod_{\substack{m=3 \\ m\neq k}}^n f(x_m)\, \mathbf{dx}_{-\mathbf{1},-\mathbf{2}} \\
&\qquad + \sum_{k=3}^n \left.\frac{\partial B_{k,1}(x;\epsilon)}{\partial \epsilon}\right|_{\epsilon=0} f(x) \int_{x_4=0}^x \cdots \int_{x_n=0}^x V(x,x,x,\mathbf{x}_{-\mathbf{1},-\mathbf{2},-\mathbf{3}}) f(x_4)\cdots f(x_n)\, dx_4 \cdots dx_n,
\end{aligned}$$

where in the last equality we utilized the symmetry of V. Therefore,

$$\begin{aligned}
I_{i,1} &= \int_0^1 \widetilde{G}_1(x) \sum_{\substack{j=1 \\ j\neq i}}^n \left.\frac{\partial B_{j,i}(x;\epsilon)}{\partial \epsilon}\right|_{\epsilon=0} dx + \int_0^1 \widetilde{G}_2(x) \frac{\partial}{\partial x} \sum_{\substack{j=1 \\ j\neq i}}^n \left.\frac{\partial B_{j,i}(x;\epsilon)}{\partial \epsilon}\right|_{\epsilon=0} dx \\
&\quad + \int_0^1 \sum_{\substack{j=1 \\ j\neq i}}^n \sum_{\substack{k=1 \\ k\neq i,j}}^n \left(\int_{\mathbf{x}_{-\mathbf{i},-\mathbf{j}}=0}^x V(x,x,\mathbf{x}_{-\mathbf{i},-\mathbf{j}}) h_k(x_k) \prod_{\substack{m=1 \\ m\neq i,j,k}}^n f(x_m)\, \mathbf{dx}_{-\mathbf{i},-\mathbf{j}} \right) f(x)(1-F(x))\, dx,
\end{aligned}$$

where $\widetilde{G}_1(x)$ and $\widetilde{G}_2(x)$ are independent of index i. Since application of (23) gives

$$\sum_{i=1}^n \sum_{\substack{j=1 \\ j\neq i}}^n \left.\frac{\partial B_{j,i}(x;\epsilon)}{\partial \epsilon}\right|_{\epsilon=0} = 0,$$

we get that

$$\sum_{i=1}^n I_{i,1} = \int_0^1 \sum_{i=1}^n \sum_{\substack{j=1 \\ j\neq i}}^n \sum_{\substack{k=1 \\ k\neq i,j}}^n \left(\int_{\mathbf{x}_{-\mathbf{i},-\mathbf{j}}=0}^x V(x,x,\mathbf{x}_{-\mathbf{i},-\mathbf{j}}) h_k(x_k) \prod_{\substack{m=1 \\ m\neq i,j,k}}^n f(x_m)\, \mathbf{dx}_{-\mathbf{i},-\mathbf{j}} \right) f(x)(1-F(x))\, dx. \tag{35}$$

To simplify (35), we first note that if $k \neq i,j$, then

$$\begin{aligned}
&\int_{\mathbf{x}_{-\mathbf{i},-\mathbf{j}}=0}^x V(x_i = x, x_j = x, \mathbf{x}_{-\mathbf{i},-\mathbf{j}}) h_k(x_k) \prod_{\substack{m=1 \\ m\neq i,j,k}}^n f(x_m)\, \mathbf{dx}_{-\mathbf{i},-\mathbf{j}} \\
&\quad = \int_{x_k=0}^x h_k(x_k) \left(\int_{\mathbf{x}_{-\mathbf{i},-\mathbf{j}-\mathbf{k}}=0}^x V(x_i = x, x_j = x, x_k, \mathbf{x}_{-\mathbf{i},-\mathbf{j}-\mathbf{k}}) \prod_{\substack{m=1 \\ m\neq i,j,k}}^n f(x_m)\, \mathbf{dx}_{-\mathbf{i},-\mathbf{j},-\mathbf{k}} \right) dx_k = \int_{t=0}^x h_k(t) T(x,t) dt,
\end{aligned}$$

where we changed the integration variable from x_k to t and where

$$\begin{aligned}
T(x,t) &= \int_{\mathbf{x}_{-\mathbf{i},-\mathbf{j}-\mathbf{k}}=0}^x V(x_i = x, x_j = x, x_k = t, \mathbf{x}_{-\mathbf{i},-\mathbf{j}-\mathbf{k}}) \prod_{\substack{m=1 \\ m\neq i,j,k}}^n f(x_m)\, \mathbf{dx}_{-\mathbf{i},-\mathbf{j},-\mathbf{k}} = \int_{x_4=0}^x \cdots \int_{x_n=0}^x V(x_1 = x, x_2 = x, x_3 \\
&= t, x_4, \ldots, x_n) f(x_4) \ldots f(x_n)\, dx_4 \ldots dx_n,
\end{aligned}$$

is identical for all i,j,k because of the symmetry of V. Therefore,

$$\begin{aligned}
\sum_{i=1}^n I_{i,1} &= \int_0^1 \sum_{i=1}^n \sum_{\substack{j=1 \\ j\neq i}}^n \sum_{\substack{k=1 \\ k\neq i,j}}^n \left[\int_{t=0}^x h_k(t) T(x,t)\, dt\right] f(x)(1-F(x)) dx = \int_0^1 \left[\int_{t=0}^x T(x,t) \sum_{i=1}^n \sum_{\substack{j=1 \\ j\neq i}}^n \sum_{\substack{k=1 \\ k\neq i,j}}^n h_k(t)\, dt\right] f(x)(1-F(x))\, dx \\
&= (n-1)(n-2) \int_0^1 \left[\int_{t=0}^x T(x,t) \sum_{i=1}^n h_i(t) dt\right] f(x)(1-F(x)) dx.
\end{aligned} \tag{36}$$

To calculate $\sum_i I_{i,2}$, we first utilize the symmetry of V in the last $n-1$ signals to get that

$$M(x) = F^{n-2}(x)E_{\mathbf{x}_{-\mathbf{i},-\mathbf{j}}}\left[V(x_i = x, x_j = x, \mathbf{x}_{-\mathbf{i},-\mathbf{j}})|x_i > \max_{m\neq i,j} x_m\right] = F^{n-2}(x)E_{\mathbf{x}_{-\mathbf{1},-\mathbf{2}}}\left[V(x_1 = x, x_2 = x, \mathbf{x}_{-\mathbf{1},-\mathbf{2}})|x_1 > \max_{m\neq 1,2} x_m\right]$$
$$= \int_{x_3=0}^{x} \cdots \int_{x_n=0}^{x} V(x_1 = x, x_2 = x, x_3, \ldots, x_n) f(x_3) \ldots f(x_n)\, dx_3 \ldots dx_n.$$

Therefore,

$$\sum_{i=1}^{n} I_{i,2} = (n-1)\int_0^1 M(x)\sum_{i=1}^{n}(h_i(x)(1-F(x)) - f(x)H_i(x))\, dx. \tag{37}$$

Combining (34), (36), and (37) gives

$$R'(0) = (n-1)(n-2)\int_0^1 \left[\int_{t=0}^{x} T(x,t)\sum_{i=1}^{n} h_i(t)\, dt\right] f(x)(1-F(x))\, dx + (n-1)\int_0^1 M(x)\sum_{i=1}^{n}(h_i(x)(1-F(x)) - f(x)H_i(x))dx. \tag{38}$$

To complete the proof, we note that if we expand $R\left[V, F + \epsilon \frac{1}{n}\sum_{i=1}^{n} H_i\right]$ in ϵ, we get that

$$R\left[V, F + \epsilon\frac{1}{n}\sum_{i=1}^{n} H_i\right] = R[V,F] + \epsilon R'(0) + O(\epsilon^2),$$

where $R'(0)$ is given by (38).

Appendix F. Private-value auctions

In the special case of private-value case $V(x_i, \mathbf{x}_{-\mathbf{i}}) = x_i$, we have $M(x) = xF^{n-2}(x)$ and $T(x,t) = xF^{n-2}(x)$. Substitution in (38) gives

$$R'(0) = (n-1)(n-2)\sum_{i=1}^{n}\int_0^1 xF^{n-3}H_i f(1-F)\, dx + (n-1)\sum_{i=1}^{n}\int_0^1 xF^{n-2}[h_i(1-F) - fH_i]\, dx$$
$$= (n-1)\sum_{i=1}^{n}\int_0^1 [xF^{n-2}[h_i(1-F) - fH_i] + x(F^{n-2})'H_i(1-F)]\, dx$$
$$= (n-1)\sum_{i=1}^{n}\int_0^1 [xF^{n-2}[h_i(1-F) - fH_i] - F^{n-2}[x(H_i f(1-F))' + H_i(1-F)]]\, dx = -(n-1)\sum_{i=1}^{n}\int_0^1 F^{n-2}H_i(1-F)\, dx.$$

Appendix G. Proof of Lemma 2

A similar derivation to Eq. (14) shows that the equilibrium bids satisfy the system

$$V(x_1, x_2) = b, \quad V(x_2, x_1) = b.$$

Therefore, the result follows.

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