On the collision rate of particles in turbulent flow with gravity

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(Received 10 May 2001; accepted 8 May 2002; published 8 July 2002)

Formula for the collision rate of inertial particles in a homogeneous isotropic stationary turbulent flow with gravity is derived. The obtained formula yields correct results for two special cases, where the formula by Saffman and Turner [J. Fluid Mech. **1**, 16 (1956)] fails. In the case of the particles with the same size, the collision rate predicted by Saffman and Turner is 29% higher than that predicted by the obtained formula. Turbulence effects on the collision rate are estimated for the conditions pertinent to droplets in atmospheric clouds.
[DOI: 10.1063/1.1490136]

The effect of turbulence on the collision rate of the particles was intensively investigated in the past. Saffman and Turner[1] obtained a formula for the collision rate in a homogeneous isotropic turbulent flow for a case when particles have inertia and gravity is involved. It was found that the effect of turbulence becomes comparable with that of gravity when the turbulent energy dissipation rate per unit mass $\epsilon \geqslant 2000\ \mathrm{cm}^2/\mathrm{s}^3$. After the study by Saffman and Turner,[1] many investigations addressed this problem. Most of them treated the case when gravity is neglected.[2–11] The case with gravity was considered by Abrahamson[12] and Wang *et al.*[9] Abrahamson[12] evaluated the lower limit of the particle size for which the derived formula is valid. For the atmospheric turbulent flows his formula is valid for large particles. Wang *et al.*[9] suggested a heuristic formula for the collision kernel. Their formula is based on the expression by Hu and Mei,[6] where gravity is neglected; the added gravity term is similar to that in the expression by Saffman and Turner,[1] with a slightly different coefficient.

A discussion of the formula derived by Saffman and Turner[1] continues to the present day.[8–11] One reason is that their formula does not recover the correct results in two special cases considered by them. This discrepancy is the result of the assumption that the relative velocities of the two colliding droplets are normally distributed. This assumption is reasonable when gravity is neglected and the problem is isotropic, while in the general case with gravity it is not substantiated because gravity violates the assumption about space isotropy.

In this study we considered collisions of particles in a turbulent flow with gravity. We represented the relative velocity as a sum of turbulent and gravity-induced components and assumed that only a turbulent component is normally distributed. Such an approach allowed us to derive a formula for particles collision rate, which recovers the correct results for two special cases, where the formula by Saffman and Turner[1] fails.

Consider two types of spherical particles moving in a homogeneous isotropic stationary turbulent flow. Particles of type 1 have radius r_1 and mass m_1, while radius and mass of particles of type 2 are r_2 and m_2, respectively. The particles are small compared with the Kolmogorov length scale and their Stokes response time is small compared with the Kolmogorov time scale. The material density of the particles is much larger than the density of the fluid. Motion of the particles is determined only by drag force and the force of gravity. Suppose that the particles number density is low so that particles move before collisions during a time interval that is much larger than the Stokes response time. Let $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ denote the velocities of the particles before the collision. The relative velocity of the colliding particles, $\mathbf{w} = \boldsymbol{\nu}_2 - \boldsymbol{\nu}_1$, in general, is not equal to zero. There are two reasons for this.

(1) Colliding particles move along different trajectories prior to the collision. Therefore, even if the particles have the same size and mass their velocities are different.
(2) Particles with different sizes have different mass and different settling velocity. Therefore, even if their locations are close their velocities are different.

Denote the mean number densities of particles of the first and the second types by n_1 and n_2, respectively. Our goal is to determine the collision rate N that is the mean number of collisions between particles of different types per unit volume per unit time.

Saffman and Turner considered two special cases. In the first case inertia of the particles and gravity are neglected and, therefore, every particle moves in a turbulent flow similarly to some fluid element. In this case the collision rate is as follows:

$$N = n_1 n_2 \sqrt{8\pi}\gamma r^3, \tag{1}$$

where $r = r_1 + r_2$; $\gamma^2 = \overline{(\partial u_x/\partial x)^2}$, and u_x is the x component of the fluid velocity. The magnitude of γ^2 can be evaluated[13] as $\gamma^2 = \epsilon/(15\nu)$, where ν is the kinematic viscosity of the fluid. In the second case particles move in stagnant air under gravity and velocity of every particle is equal to the terminal fall velocity $\boldsymbol{\nu} = \mathbf{g}\tau_p$, where $\mathbf{g}$ is the acceleration of gravity and τ_p is Stokes response time of the particle,

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$\tau_p = 2\rho_p r_p^2/(9\mu)$. Here μ is the dynamic viscosity of the fluid, ρ_p is material density of the particle, and r_p is its radius. Thus, the collision rate is as follows:

$$N = n_1 n_2 \pi r^2 g |\tau_1 - \tau_2|. \tag{2}$$

Saffman and Turner considered also a general case when particles have inertia and gravity is involved. Their formula for the collision rate in this case reads as

$$N = n_1 n_2 \sqrt{8\pi} r^2 [\tfrac{15}{9}\gamma^2 r^2 + \lambda^2 (\tau_1 - \tau_2)^2 + \tfrac{1}{3} g^2 (\tau_1 - \tau_2)^2]^{1/2}, \tag{3}$$

where $\lambda^2 = \overline{(Du_x/Dt)^2}$. The magnitude of λ^2 can be evaluated[14] as $\lambda^2 = 1.3\epsilon^{3/2}/\nu^{1/2}$. Let us examine Eq. (3) for the special cases considered above. When particles have no inertia, the last two terms in expression (3) vanish and the collision rate $N = n_1 n_2 \sqrt{8\pi}(\sqrt{15}/3)\gamma r^3$. Notably, the constant in the latter expression differs from the correct one in Eq. (1). In the second special case when particles move under gravity in a stagnant air, the first two terms in expression (3) vanish, and we obtain $N = n_1 n_2 \sqrt{8\pi/3} r^2 g |\tau_1 - \tau_2|$. Comparing this equation with Eq. (2) shows that in this case also the obtained constant is not correct.

The problem can be formulated as follows. Let us select any particle of type 1. Consider a spherical surface with a radius $r = r_1 + r_2$ centered at the center of the selected particle. Assume that this spherical surface moves together with the selected particle. When the marked particle collides with any particle of the second type, the center of the particle of type 2 is located at the spherical surface. The relative velocity $\mathbf{w}$ of two particles results in the flux of particles of type 2 through the spherical surface. The mean total flux of the particles of type 2 through the spherical surface is

$$\overline{Q} = n_2 \int_S \overline{|w_r|} dS,$$

where w_r is the radial component of the relative velocity $\mathbf{w}$ and S denotes the spherical surface. Since the turbulent flow is stationary the mean fluxes of the particles into the spherical surface and out of it are equal. Therefore the mean flux of the particles into the spherical surface reads as $\overline{Q_{in}} = 0.5\overline{Q}$ and the collision rate $N = n_1 \overline{Q_{in}}$. Introduce Cartesian and spherical coordinates centered at the center of the marked particle of the first type and with the z axis directed opposite to the gravity direction. Then

$$N = n_1 n_2 \pi r^2 \int_0^{\pi} \overline{|w_r|} \sin\theta d\theta, \tag{4}$$

where θ is an angle between the vector $\mathbf{r}_B$ and the z axis. The problem reduces to calculating the integral in Eq. (4). To this end we use the formula by Maxey,[15] which implies that the velocity of a small particle does not depend on its history and it is determined by the fluid velocity and fluid acceleration at the particle location only,

$$\boldsymbol{\nu} = \mathbf{u} - \vec{\psi}\tau_p + \mathbf{g}\tau_p, \tag{5}$$

where $\vec{\psi}(t, \mathbf{r}) = D\mathbf{u}/Dt$ is an acceleration of the fluid at time t at location $\mathbf{r}$. Note that the original expression obtained by Maxey includes the additional term $\mathbf{g}\cdot\vec{\nabla}\mathbf{u}\tau_p^2$. This term is of the second order with respect to τ_p and for small particles it can be neglected.

Let a particle of type 1 be located at the point A and a particle of type 2 be located at the point B. Then the relative velocity $\mathbf{w} = \boldsymbol{\nu}_2 - \boldsymbol{\nu}_1$ of two particles is $\mathbf{w} = (\mathbf{u}_{\mathrm{B}} - \mathbf{u}_{\mathrm{A}}) + (\vec{\psi}_{\mathrm{A}}\tau_1 - \vec{\psi}_{\mathrm{B}}\tau_2) + \mathbf{g}(\tau_2 - \tau_1)$. We assume that the radius of the sphere, r, and the Stokes response time of particles, τ_1 and τ_2, are small with respect to Kolmogorov's length and time, respectively. Then all three terms in the equation for $\mathbf{w}$ are of the first order of magnitude. Note that for small r vector $\vec{\psi}_{\mathrm{A}}$ is close to $\vec{\psi}_{\mathrm{B}}$. Thus, with the accuracy of the first-order terms including with respect to r, τ_1 and τ_2, one can write $\mathbf{w} = \mathbf{w}_f + \mathbf{w}_g$, where $\mathbf{w}_f = (\mathbf{u}_{\mathrm{B}} - \mathbf{u}_{\mathrm{A}}) + \vec{\psi}_{\mathrm{A}}(\tau_1 - \tau_2)$ and $\mathbf{w}_g = \mathbf{g}(\tau_2 - \tau_1)$. The radial component of the vector $\mathbf{w}$ is $w_r = \xi + h$, where the random variable $\xi = \mathbf{w}_f \mathbf{e}_{\mathrm{B}}$, the deterministic term $h = \mathbf{w}_g \mathbf{e}_{\mathrm{B}}$, and $\mathbf{e}_{\mathrm{B}}$ is a unit vector in the direction of vector $\mathbf{r}_{\mathrm{B}}$. In the homogeneous isotropic stationary turbulent flow the random variable ξ is the same for all points at the considered spherical surface. Let us place the center of the particle of type 2 at the point C, which is the point of intersection of the spherical surface with the x axis. In this case the radial and x components of the vector $\mathbf{w}_f$ coincide and $w_{f,\mathrm{C},r} = (u_{\mathrm{C},x} - u_{\mathrm{A},x}) + \psi_{\mathrm{A},x}(\tau_1 - \tau_2)$. A distribution of the random variable ξ is symmetrical and the mean value of ξ is equal to zero. Squaring the latter equation and averaging, we obtain

$$\sigma^2 = \overline{\xi^2} = \overline{(u_{\mathrm{C},x} - u_{\mathrm{A},x})^2} + \overline{\psi_{\mathrm{A},x}^2}(\tau_1 - \tau_2)^2 + 2\overline{(u_{\mathrm{C},x} - u_{\mathrm{A},x})\psi_{\mathrm{A},x}}(\tau_1 - \tau_2). \tag{6}$$

Consider the last term in Eq. (6). Note that $\psi_{\mathrm{A},x}$ differs from $\psi_{\mathrm{C},x}$ by the first-order terms. Therefore with the accuracy of the first-order terms $\overline{(u_{\mathrm{C},x} - u_{\mathrm{A},x})\psi_{\mathrm{A},x}} = \overline{(u_{\mathrm{C},x} - u_{\mathrm{A},x})\psi_{\mathrm{C},x}}$. On the other hand, the assumption about the spatial homogeneity implies that $\overline{(u_{\mathrm{C},x} - u_{\mathrm{A},x})\psi_{\mathrm{A},x}} = \overline{(u_{\mathrm{A},x} - u_{\mathrm{C},x})\psi_{\mathrm{C},x}}$. Comparing the last two equations, we find that with the accuracy of the first-order terms $\overline{(u_{\mathrm{C},x} - u_{\mathrm{A},x})\psi_{\mathrm{A},x}} = 0$. Hence, the last term in Eq. (6) is small compared with the others and can be omitted. Since r is small compared with the Kolmogorov length scale $u_{\mathrm{C},x} - u_{\mathrm{A},x} = r\partial u_x/\partial x$ and

$$\sigma^2 = \gamma^2 r^2 + \lambda^2 (\tau_1 - \tau_2)^2. \tag{7}$$

The vector $\mathbf{w}_g$ can be rewritten as $\mathbf{w}_g = -\mathbf{e}_z g(\tau_2 - \tau_1)$. Therefore $h = -g(\tau_2 - \tau_1)\mathbf{e}_z\mathbf{e}_{\mathrm{B}} = g(\tau_1 - \tau_2)\cos\theta$. Since $w_r = \xi + h$ and $h(\pi - \theta) = -h(\theta)$, we find that

$$|w_r(\pi - \theta)| = |\xi + h(\pi - \theta)| = |\xi - h(\theta)| = |-\xi + h(\theta)|.$$

Since the random variable $(-\xi)$ has the same distribution as ξ, the latter expression yields $\overline{|w_r(\pi - \theta)|} = \overline{|w_r(\theta)|}$. Thus, the integrand in Eq. (4) is symmetric with respect to $\theta = \pi/2$ and this equation can be rewritten as

$$N = n_1 n_2 2\pi r^2 \int_0^{\pi/2} \overline{|w_r|} \sin\theta d\theta. \tag{8}$$

Assume that ξ is normally distributed. Note that h has an arbitrary sign depending on the relative positions of the colliding particles. However, the mean value of $|w_r|$ does not

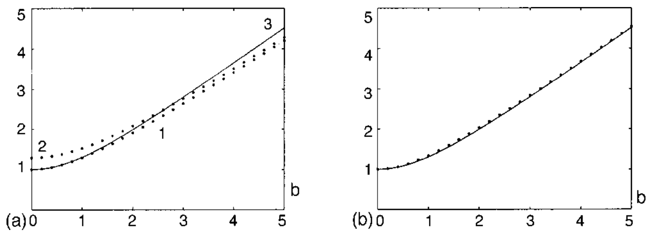

FIG. 1. (a) Plots of the functions: $1-f_1(b)$, $2-f_2(b)$, and $3-f(b)$; (b) plots of the functions $f(b)$ (line) and $f_W(b)$ (dots).

depend on the sign of h. Introduce the constant

$$h_+=g|\tau_1-\tau_2|\cos\theta,$$

which is positive when $0\leqslant\theta\leqslant\pi/2$ and find $\overline{|w_r|}$ from the equation $\overline{|w_r|}=\overline{|\xi+h_+|}$. Then

$$\overline{|w_r|}=\int_{-\infty}^{\infty}|\alpha|\frac{1}{\sigma\sqrt{2\pi}}e^{-(\alpha-h_+)^2/2\sigma^2}d\alpha$$
$$=\sigma\sqrt{2}\beta\,\mathrm{erf}\,\beta+\sigma\sqrt{2/\pi}e^{-\beta^2}, \tag{9}$$

where

$$\beta=\frac{h_+}{\sigma\sqrt{2}}=\frac{g|\tau_1-\tau_2|}{\sigma\sqrt{2}}\cos\theta,$$

and

$$\mathrm{erf}\,\beta=\frac{2}{\sqrt{\pi}}\int_0^\beta e^{-t^2}dt.$$

Equations (8) and (9) yield the following expression for the collision rate:

$$N=n_1n_2 2\pi r^2\frac{\sigma\sqrt{2}}{b}\int_0^b\left(\beta\,\mathrm{erf}\,\beta+\frac{1}{\sqrt{\pi}}e^{-\beta^2}\right)d\beta$$
$$=n_1n_2\sqrt{8\pi}r^2\sigma f(b), \tag{10}$$

where

$$b=g|\tau_1-\tau_2|/\sigma\sqrt{2}\quad\text{and}$$
$$f(b)=0.5\sqrt{\pi}(b+0.5/b)\,\mathrm{erf}\,b+0.5e^{-b^2}. \tag{11}$$

Let us verify the obtained solution for two special cases that were considered by Saffman and Turner. In the case of inertialess particles without gravity, Eqs. (7) and (11) yield $\sigma=\gamma r$, $b=0$, $f(b)=1$ and expression (10) becomes $N=n_1n_2\sqrt{8\pi}\gamma r^3$. The latter equation coincides with Eq. (1).

In the case without turbulence, Eqs. (7) and (11) yield $\sigma=0$, $b=\infty$. For large b, Eq. (11) becomes $f(b)=0.5\sqrt{\pi}b$ and expression (10) yields $N=n_1n_2\pi r^2g|\tau_1-\tau_2|$, which coincides with Eq. (2). Notably, in contrast to Eq. (3), Eq. (10) recovers the correct results for both two special cases.

Let us compare the obtained Eq. (10) with Eq. (3) obtained by Saffman and Turner. To this end we will transform Eq. (3) to the form similar to Eq. (10). Using Eqs. (7) and (11) Eq. (3) may be rewritten as

$$N=n_1n_2\sqrt{8\pi}r^2\sigma f_{\mathrm{ST}}(b,c), \tag{12}$$

where $c=\gamma r/\sigma$ and $f_{\mathrm{ST}}(b,c)=\sqrt{1+(b^2+c^2)2/3}$. Thus, instead of the function $f(b)$ in Eq. (10) the different function $f_{\mathrm{ST}}(b,c)$ appears in Eq. (12). Depending on the relationship between the two terms in Eq. (7), parameter c varies in the range $0\leqslant c\leqslant 1$. Therefore $f_1(b)\leqslant f_{\mathrm{ST}}(b,c)\leqslant f_2(b)$, where $f_1(b)=\sqrt{1+b^2 2/3}$ and $f_2(b)=\sqrt{1+(b^2+1)2/3}$. Figure 1(a) shows the plots of the three functions, $f_1(b)$, $f_2(b)$, and $f(b)$. One can see from this figure that the difference between the functions $f(b)$ and $f_{\mathrm{ST}}(b,c)$ is maximum when $b=0$ and $c=1$, that is, when the sizes of the colliding particles are equal. In this case $f(0)=1$ and $f_{\mathrm{ST}}(0,1)=1.29$. Thus, the error introduced by using function $f_{\mathrm{ST}}(b,c)$ may be as large as 29%.

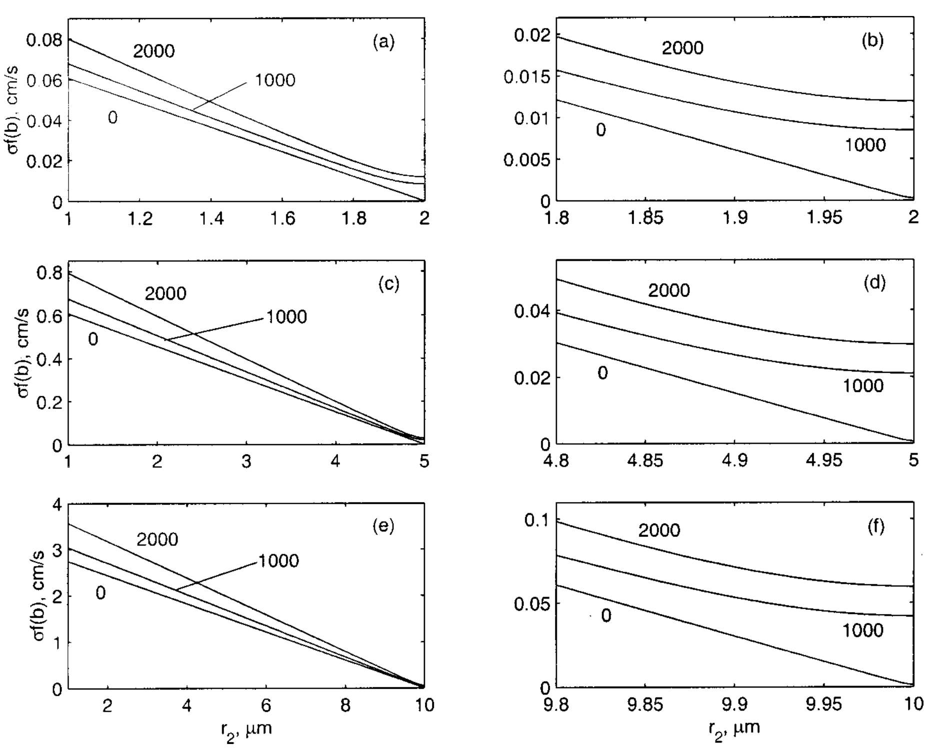

FIG. 2. $\sigma f(b)$ as a function of r_2 for different r and ϵ. (a), (b) $r=4$ μm; (c), (d) $r=10$ μm; (e), (f) $r=20$ μm. Numbers near the curves denote the magnitude of ϵ.

Compare now Eq. (10) with the heuristic formula suggested by Wang *et al.*[9] Using notations of this study the Wang *et al.*[9] formula reads as $N=n_1 n_2 \sqrt{8\pi} r^2 \sigma f_{\mathrm{W}}(b)$, where $f_{\mathrm{W}}(b)=\sqrt{1+b^2\pi/4}$ and the small term of the fourth order is neglected. Figure 1(b) shows the plots of functions $f(b)$ and $f_{\mathrm{W}}(b)$. The agreement between the plots is fairly good.

Now let us use Eq. (10) to analyze the effect of turbulence on the collision rate of droplets in atmospheric clouds. According to experimental data,[16,17] a turbulent energy dissipation rate ϵ lies in the range 1–2000 $\mathrm{cm}^2/\mathrm{s}^3$. Let us adopt the following parameters that are characteristic to atmospheric clouds: $\rho_p=1\ \mathrm{g/cm}^3$, $g=980\ \mathrm{cm/s}^3$, $\mu=1.8\times 10^{-4}\ \mathrm{g/(cm\cdot s)}$, $\rho_f=0.0012\ \mathrm{g/cm}^3$. One can see from Eq. (10) that for the fixed parameter r the effect of turbulence on the collision rate are determined by the term $\sigma f(b)$. Figure 2 shows $\sigma f(b)$ as a function of r_2 for different r and ϵ. Here r_2 is the radius of the smaller particle, $r_2 \leqslant r/2$. The applicability of Eq. (5) for the above values of ϵ and for the particles with the radius in the range 1–20 μm was validated by Dodin.[18] For each value of r two plots are shown in Fig. 2: the plot at the left is for the range 1 $\mu\mathrm{m} \leqslant r_2 \leqslant r/2$ and the plot at the right is for the range where colliding droplets are of the same sizes. The lower straight line in each plot corresponds to the case of particles moving under gravity in a stagnant air. In this case,

$$f(b)=0.5\sqrt{\pi} b \quad \text{and}$$

$$\sigma f(b)=\sqrt{\frac{\pi}{8}} g|\tau_1-\tau_2|=\frac{1}{9}\sqrt{\frac{\pi}{2}}\frac{g\rho_p}{\mu}(r^2-2rr_2).$$

An inspection of Fig. 2 and the analysis of the obtained formulas allow us to conclude that in contrast to the case of stagnant air, in turbulent flow the collision rate is not zero for all droplet sizes, including the case of the equal size droplets. The collision rate increases with the increase of a turbulent energy dissipation rate ϵ for all sizes of the colliding droplets. Like in the case of stagnant air for a given value of r, the collision rate is minimum for the particles of equal size, $r_2=r_1=r/2$. The turbulence effects on the collision rate are most pronounced in the vicinity of $r_2=r_1=r/2$.

ACKNOWLEDGMENTS

This study was partially supported by the German–Israeli Project Cooperation (DIP) administrated by the Federal Ministry of Education and Research (BMBF) and by INTAS (Grant No. 00-0309). The authors are grateful to one of the anonymous referees for helpful comments and for a comparison of formula (10) with the formula by Wang *et al.*[9]

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