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Physica A 303 (2002) 48–56

PHYSICA A

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Variational model of granular flow in a three-dimensional rotating container

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Received 23 August 2001

Abstract

We considered a flow of a cohesionless granular material in a partially filled three-dimensional rotating container. A variational continuum mechanics model is suggested to describe the profile of the free surface of the granular material. A combination of a level set method and a variational form of energy conservation equation allows us to describe flow of granular material in arbitrary-shaped containers. © 2002 Elsevier Science B.V. All rights reserved.

PACS: 46.10.+z; 05.60.+w; 83.70.Fn

Keywords: Granular material; Free surface problem; Rotating drum

Dynamics, mixing and separation of granular materials in partially filled containers have been the subject of numerous experimental and theoretical investigations (see, e.g., Refs. [1–10]). Different aspects of dynamics of granular material in two-dimensional rotating drums were studied in the past. However, a problem of a continuous free-surface flow of granular material in a three-dimensional rotating container was not addressed in spite of its theoretical and technological importance. The only available results on the dynamics of a free surface of granular material in a three-dimensional rotating drum were obtained numerically using discrete elements method [11] or experimentally (see, e.g., Ref. [9]) while the continuum mechanics approach was not explored before. Note, that while in a two-dimensional slowly rotating drum the free surface of a granular material has a nearly flat profile with a nearly constant angle of inclination,

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PII: S0378-4371(01)00488-5

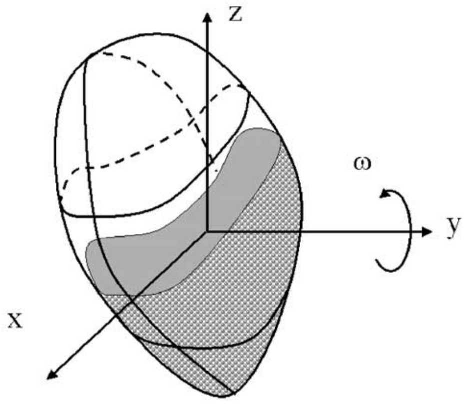

Fig. 1. Schematic view of the flow of granular material in a rotating container and the stationary coordinate system.

in a three-dimensional case the situation changes drastically. The free surface evolves with time in a nontrivial manner that depends on the shape of the drum, and also on the way the mixer is tumbled. In a two-dimensional case the axis of rotation is always directed normally to the drum while in a three-dimensional case it can be, e.g., wobbled which causes an additional axial flow of granular material [2,10]. In our previous study [10], we considered flow of granular material in an ellipsoidal mixer. Due to the simple analytical form of the container the problem of the free-surface evolution was reduced to a system of two ordinary differential equations. It is of interest to describe the dynamics of the free surface of granular material in an arbitrary-shaped rotating drum using continuum mechanics approach. In the present investigation, we suggested a combination of variational form of energy conservation principle with a level set method [12]. The developed numerical procedure allows us to describe a free-surface flow of granular material in arbitrary-shaped rotating containers.

Let us consider a three-dimensional container (Fig. 1) that rotates with a constant angular velocity ω around y-axis. The drum is partially filled with a granular material with a constant bulk density ρ and a constant angle of repose μ. We assume that the angular velocity ω is sufficiently large to cause continuous avalanches while the inertial forces are much smaller than the gravity and friction forces, e.g., Froude number $Fr^2 \equiv \omega^2 L/g \ll 1$ where L is a characteristic size of the drum. The later assumption is not too restrictive since it is well known that Froude numbers in industrial tumbling mixers are quite low (see, e.g., Ref. [13]). Thus, the only characteristic time of the process is $1/\omega$, and for simplicity we set the angular velocity equal 1.

The flow of granular material in the drum can be described as follows. Particles rotate with the bulk of the granular material and fall down into a thin cascading layer when they reach the free surface $z = h(t, x, y)$ which has a constant angle of inclination μ with respect to the horizontal:

$$(\nabla h)^2 = \tan^2(\mu) = \gamma^2 . \quad (1)$$

Certainly, the latter assumption is an approximation whereby Eq. (1) can be viewed as a zero order term in the expansion of the momentum conservation equation in Froude number series (see, e.g., Ref. [1]). In the present study, we adopted the model of sandpile evolution suggested in Refs. [14–17]. The brief description of this model is presented in the following.

Assume that the flow of the granular material occurs only in a very thin boundary layer and it does not involve the stationary bulk of the material. Denote the horizontal projection of the mass flux density per unit area by $\rho\vec{J}(t,x,y)$. Since inertial forces are small, the material flux is directed toward the steepest decent of the free surface:

$$\rho\vec{J} = -k\rho\nabla h\,, \tag{2}$$

where the flow rate $k(t,x,y) \geqslant 0$ is the unknown scalar function. Thus, the equation of mass balance for granular material reads:

$$\frac{\partial h}{\partial t} + \vec{u}_{||}\cdot\nabla h - u_{\perp} = \nabla\cdot(k\nabla h)\,, \tag{3}$$

where $\vec{u}_{||} = (h,0)$ and $u_{\perp} = -x$ are horizontal and vertical components of the rotationally induced bulk velocity at the free surface.

A horizontal projection of an intersection of the free surface with the walls of the container is an unknown closed curve Γ on the x,y-plane. Since the walls are impermeable, the boundary condition for Eq. (3) reads:

$$\vec{n}\cdot\vec{J}|_{\Gamma} = 0\,, \tag{4}$$

where unity vector $\vec{n}$ is normal to the boundary. Eqs. (1)–(4) with respect to the unknown functions $k(t,x,y)$, $h(t,x,y)$ and Γ provide a closed mathematical formulation of the problem.

Note, that the direct solution of the above equations for an arbitrary rotating container is a very complicated problem. Indeed, there is no explicit equation for the flow rate k, and it has to be obtained from the complementary condition (1). The shape of the container enters the problem implicitly through the boundary conditions (4) posed at the unknown curve Γ. The latter also complicates a solution. In this study, we introduce a variational formulation of the problem that allows a convenient numerical realization.

Consider a dimensionless density function $\phi(t,x,y,z)$ in the laboratory frame of reference that is subjected to the following constraints:

$$\phi \leqslant 1, \qquad \phi \geqslant -1\,. \tag{5}$$

From symmetry considerations function ϕ is defined such that the region where $\phi = 1$ is filled by granular material, while $\phi = -1$ corresponds to particles-free space. Below we will demonstrate that $-1 < \phi < 1$ in a layer of zero thickness. Thus, the level set $\phi = 0$ represents the free surface of the granular material. Introduce the expression that is formally similar to the potential energy of the granular material in the container:

$$U = \rho g \int z\phi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,, \tag{6}$$

where z is vertical coordinate in the laboratory frame of reference. The level set function ϕ and the three-dimensional vector of the volumetric flux $\vec{q}(t,x,y,z)$ are related by the continuity law:

$$\frac{\partial\phi}{\partial t} + \nabla\cdot\vec{q} = 0\,. \tag{7}$$

When the granular material flows down the slope, the force that compensates the gravity force is dry friction, i.e., the friction force is rate-independent:

$$\vec{F} = \gamma\rho g\frac{\vec{q}_{||}}{|q_{||}|}\,.$$

Hereafter the subscripts $||$ and $\perp$ denote horizontal and vertical components of a vector, respectively. The equations for the dissipation rate can be written in the form that was proposed in Refs. [16,17]:

$$\varepsilon = \gamma\rho g\frac{\vec{q}_{||}}{|q_{||}|}\cdot\vec{q} = \gamma\rho g|q_{||}|\,.$$

Note that the vertical component of the friction force is zero, i.e., the granular material can move vertically down without friction.

Using the synchronous variations of dimensionless density $\delta\phi$ and of the horizontal component of the integral volumetric flux $\delta\vec{Q}$ (where $\partial\vec{Q}/\partial t = \vec{q}$), the variational form of the energy conservation equation can be written as follows:

$$\rho g\int z\delta\phi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = -\int\gamma\rho g\frac{\vec{q}_{||}}{|q_{||}|}\cdot\delta\vec{Q}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,. \tag{8}$$

The variations of ϕ and $\vec{Q}$ are not independent and are related by the continuity equation:

$$\delta\phi = -\nabla\cdot\delta\vec{Q}\,. \tag{9}$$

Multiplying the constraints (5) by Lagrangian multipliers $\rho g\mu$ and $\rho g\lambda$, respectively, substituting Eq. (9) into Eq. (8) and using Gauss' theorem we obtain:

$$\begin{aligned}
&\rho g\int\left\{z\delta\phi + \gamma\rho g\frac{\vec{q}_{||}}{|q_{||}|}\cdot\delta\vec{Q} + (\mu-\lambda)\delta\phi\right\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\quad = -\rho g\int\left\{z(\nabla\cdot\delta\vec{Q}) + \gamma\rho g\frac{\vec{q}_{||}}{|q_{||}|}\cdot\delta\vec{Q} - (\mu-\lambda)(\nabla\cdot\delta\vec{Q})\right\}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\quad = \rho g\int\left\{\nabla z + \gamma\rho g\frac{\vec{q}_{||}}{|q_{||}|} + \nabla(\mu-\lambda)\right\}\cdot\delta\vec{Q}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0\,.
\end{aligned}$$

Therefore,

$$\nabla z + \gamma\rho g\frac{\vec{q}_{||}}{|q_{||}|} + \nabla(\mu-\lambda) = 0\,.$$

The vertical and horizontal components of the above equation read:

$$\frac{\partial(\mu-\lambda)}{\partial z} = -1\,, \tag{10}$$

$$\nabla_{||}(\mu-\lambda) = -\gamma\frac{\vec{q}_{||}}{|q_{||}|}\,. \tag{11}$$

Since the vertical component of the friction force is always zero, the granular material can consolidate under the action of gravity at the lower part of the container and the interface between the bulk of the material and the void space is of zero thickness. Lagrangian multipliers μ and λ are nonzero if and only if $\phi=-1$ (void space) or $\phi=1$ (bulk of the material), respectively. At an interface, that is at the free surface of the bulk, both Lagrangian multipliers are equal to zero. Integration of Eq. (10) yields:

$$\mu-\lambda = h - z\,,$$

where $h(t,x,y)$ is height of the free surface of the granular material in the laboratory frame of reference. Substituting the above formula into Eq. (11) we obtain:

$$\nabla_{||}h = -\gamma\frac{\vec{q}_{||}}{|q_{||}|}\,. \tag{12}$$

Eq. (12) implies that the variational problem (8) is equivalent to Eqs. (1) and (2) in the sense that the flow of granular material occurs if and only if $|\nabla_{||}h|=\gamma$, i.e., when the free surface of the material is inclined by an angle of repose. Therefore, the two-dimensional problem of sandpile evolution is imbedded into the three-dimensional problem (7).

Note, that according to Eq. (12) the three-dimensional model does not have a unique solution, i.e., flow of granular material is possible at any depth below the free surface. In spite of this nonuniqueness the variational problem (8) can be implemented in numerical algorithm as follows. We perform all the calculations in a coordinate system attached to the container. After discretization in time, mass conservation law and formula for $\delta\vec{Q}$ at time $k\Delta t$ read:

$$\delta\vec{Q}_k = \delta\vec{q}_k\Delta t\,,$$

$$\phi_k = \phi_{k-1} - \nabla\cdot\vec{q}_k\Delta t\,. \tag{13}$$

Rotation affects the flow of the granular material only through the orientation of the gravity force with respect to the drum. Denote upward directed unity vector in the container frame of reference as $\vec{\kappa}(t)$. After simple manipulations variational equation (8) yields the following minimization problem at time $k\Delta t$:

$$\frac{1}{\Delta t}\int(\vec{r}\cdot\vec{\kappa}(t_k))\phi_k\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \gamma\int|q_{||}|_k\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \to \min\,, \tag{14}$$

where $\vec{r}=(x,y,z)$ is the radius vector in the coordinate system attached to the drum and Δt is time increment. The free surface of granular material corresponds to a zero level of the level-set function ϕ. At each time step of the solution we rotate the granular material with a container as a solid body by a small angle Δt and after that allow the

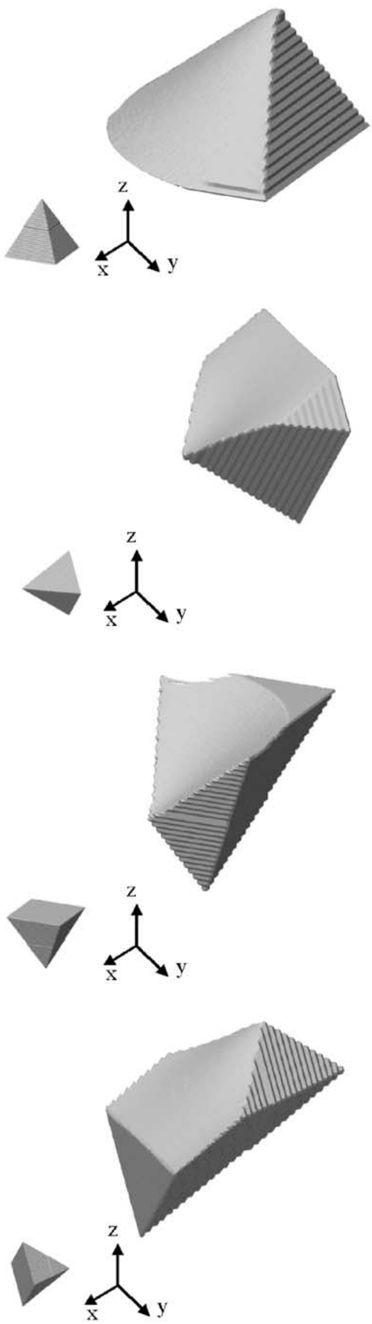

Fig. 2. Time evolution of the free surface of granular material in a half-filled pyramidal container that is rotated around the diagonal of the base ($\gamma = 0.7$). The orientation of the container and the coordinate system are shown in the left lower corner of the picture ($\gamma = 0.7$).

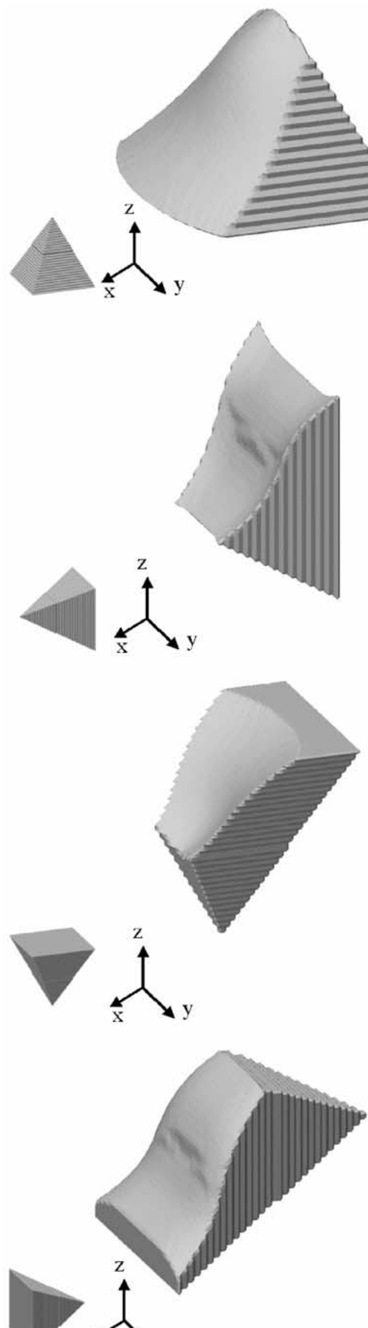

Fig. 3. Time evolution of the free surface of granular material in a half-filled rotated pyramidal container that is rotated around the edge of the base.

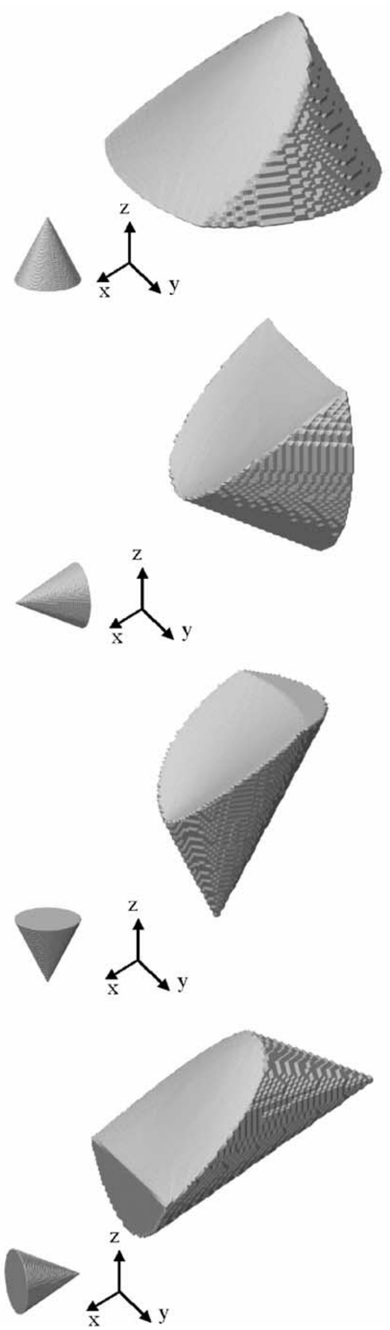

Fig. 4. Time evolution of the free surface of granular material in a half-filled rotated conical container ($\gamma = 0.7$).

system to relax by solving the variational problem (14) under constraints (5) and (13). Only the points near the zero level of ϕ have to be updated at each time step of the numerical procedure, i.e., the problem is effectively two-dimensional. The minimization problem (14) was solved by an augmented Lagrangian method which is a combination of the duality and penalty methods. The method is similar to that used for solution of Bingham fluid flow problems (see Ref. [18, pp. 402–404]), and application of this method to the solution of problem of sandpile evolution was suggested in Refs. [16,17]. In calculations, we used a staggered $60 \times 60 \times 60$ nonbody-fitted square mesh and the time increment $\Delta t = \pi/600$.

In Figs. 2–4, we demonstrate the results of the computations. Inspection of these figures shows that the form of the free surface at different moments of time depends on the shape of the container. Since the vector of material flux is directed toward the steepest descent of the free surface, the free surface is convex when flow of the granular material converges and is concave when the flow diverges. Note that initially the shape of the free surface depends on the manner that the container was filled on, but after a few revolutions the system reaches a steady periodic regime.

In summary, we have analyzed surface flow of granular material in a three-dimensional arbitrary-shaped slowly rotating container. The problem of free-surface evolution was formulated as a variational form of energy conservation equation. The derived variational inequalities were solved numerically.

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