ELSEVIER

Physica A 260 (1998) 201–217

PHYSICA A

Stress distribution in sandpiles – A variational approach

T. Elperin *, A. Vikhansky

Pearlstone Center for Aeronautical Engineering Studies, Department of Mechanical Engineering, Ben-Gurion University of the Negev, P.O. Box 653, Beer-Sheva 84105, Israel

Received 7 April 1998

Abstract

Two variational closure models are proposed in order to describe stress distribution and arch formation in sandpiles located on a rigid base. The maximum strength and the maximum entropy closures are shown to yield the normal stress minimum below the pile apex. Stress distributions in sandpiles are determined numerically. The force fluctuations in a model regular granular assembly are studied by Monte Carlo simulation method. A dependence of the level of the fluctuations vs. the macroscopic stress state is determined. © 1998 Elsevier Science B.V. All rights reserved.

PACS: 46.10.+z; 05.60.+w; 83.70.Fn

1. Introduction

One of the most intriguing and controversial phenomenon in the statics of granular media is the frequently observed vertical stress minimum just below a sandpile apex. Although this problem was investigated quite extensively in the past and also recently (see, e.g., Refs. [1–10]) it remains generally unresolved.

It is of interest to suggest and to analyze the continuum mechanics models which can provide such a counterintuitive stress behavior. The authors of Refs. [3–6] postulated that vertical stresses in a granular material propagate down along fixed lines forming arch-like structures. Closure of the equation of sandpile equilibrium is achieved by adopting a constitutive relation whereby the direction of the principal axes of a stress tensor remains constant during pile formation. This direction is determined by an equilibrium condition at the free surface of a sandpile. This hypothesis is based on assumption that the sandpile is formed by dropping grain after grain from a point

* Corresponding author. Tel.: 972-1-472104; fax: 972-1-472990; e-mail: elperin@menix.bgu.ac.il.

PII: S0378-4371(98)00327-6

source on a rigid surface. Thus, no external disturbances and no particle rearrangements are allowed to occur inside an element of granular material after the moment of its formation. Thus, it is assumed that the only factor that affects a stress distribution in a sandpile is a history of pile formation. Notably, these models yield a high value of a pressure dip below the pile apex (about 30%) while the recent experiments show that the appearance and a magnitude of the dip depends on a type of a granular material [8].

In the present study we explore another possibility whereby external driving forces play an essential role. It is known that there is no chaotic thermal motion in a granular packing. Thus, only external disturbances can allow a granular system to sample all possible configurations in a phase space. Notably, it is known that even a small perturbation can induce extensive rearrangements in granular media as it was observed in experiments [11,12]. Therefore it is conceivable that stress distribution in piles is governed by some form of a variational principle similar to that in the equilibrium statistical mechanics.

Recently variational models were applied to describe sandpile evolution (see Refs. [13,14]). In these studies we demonstrated that under quite general conditions a variational principle imply the equation for a shape of a sandpile which is built on an arbitrary rigid support due to external distributed or a point source of granular material. The main goal of this study is to suggest two variational models of sandpiles which imply the occurrence of the vertical stress minimum below the sandpile apex and to present the results of numerical analysis of these models. These models employ the maximum load carrying capacity and the maximum entropy closures for continuum mechanics equation of sandpile equilibrium.

Another question related with the developed approach is the force fluctuations in granular assemblies. An original stochastic scalar model of weight redistribution in granular piling was proposed by Liu et al. [12]. Recently this model was modified to account for the vector nature of the intergranular forces [10]. In the present study we consider the dependence of the microscopic force fluctuations in a model regular granular packing vs. the macroscopic stress applied to the system. It is demonstrated that the set of mechanically acceptable force distributions is a convex polyhedron in a phase space of intergranular contact forces. An efficient stochastic algorithm is proposed to sample possible equiprobable states in the phase space.

2. Variational models of a pile and numerical simulations

We consider a two-dimensional "pile" (Fig. 1) consisting of $n\times(2n+1)$ rectangular granular blocks with a constant density ρ, where $\gamma=\tan(\phi)$ and the angle ϕ can be interpreted as the angle of inclination of the free surface of the sandpile. The stress tensor at the center of a block with integer coordinates (i,j) is

$$\boldsymbol{\sigma}=(\sigma_{zz}=p+\tau_1,\sigma_{xz}=\sigma_{zx}=\tau_2,\sigma_{xx}=p-\tau_1)\,,$$

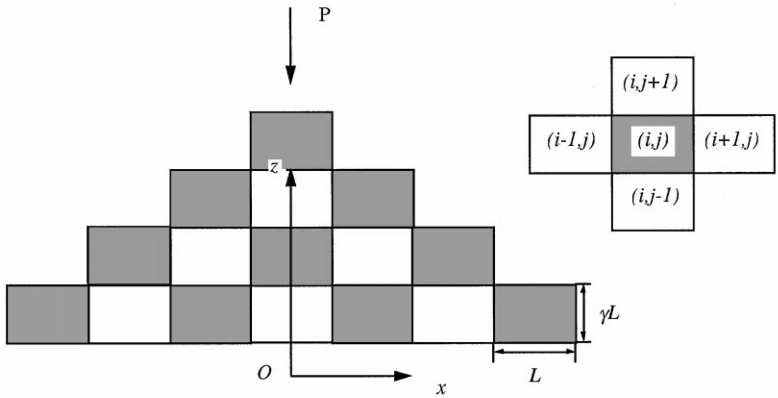

Fig. 1. Schematic view of a two-dimensional sandpile.

where p is the mean pressure and τ_i are components of the deviatoric part of the stress tensor. Interpolating the stress tensor between the centers of the blocks

$$\boldsymbol{\sigma}_{i+1/2,j} = \frac{1}{2}(\boldsymbol{\sigma}_{i+1,j} + \boldsymbol{\sigma}_{i-1,j}), \boldsymbol{\sigma}_{i,j+1/2} = \frac{1}{2}(\boldsymbol{\sigma}_{i,j+1} + \boldsymbol{\sigma}_{i,j-1}),$$

we arrive at the following equations of equilibrium of the block (i,j):

$$\begin{aligned}
&\frac{\gamma}{2L}[(p-\tau_1)_{i-1,j} - (p-\tau_1)_{i+1,j}] + \frac{1}{2L}[\tau_{2i,j-1} - \tau_{2i,j+1}] = 0\,,\\
&\frac{1}{2L}[(p+\tau_1)_{i,j-1} - (p+\tau_1)_{i,j+1}] + \frac{\gamma}{2L}[\tau_{2i-1,j} - \tau_{2i+1,j}] - \rho g = 0\,,
\end{aligned} \tag{1}$$

where g is the acceleration of gravity. Indeed, formula (1) is the approximation with the accuracy $O(\frac{1}{n})$ for the differential equation of sandpile equilibrium

$$\nabla \cdot \boldsymbol{\sigma} = -\rho g \mathbf{z}\,. \tag{1a}$$

The Coulomb yield criterion reads

$$\sqrt{\tau_1^2 + \tau_2^2} \leqslant p \cdot \sin\delta\,, \tag{2}$$

where $\delta = \tan^{-1}(\kappa)$ is an angle of internal friction of the granular material and κ is the friction coefficient. The angle of inclination of the free surface of the sandpile ϕ does not exceed the angle of internal friction, i.e., $\phi \leqslant \delta$. Equation (2) implies also that there are no tensile stress in the granular body, i.e., $p + \sqrt{\tau_1^2 + \tau_2^2}\cos 2\psi \geqslant 0$ along each line inclined at an arbitrary angle ψ with respect to the principal axis of the stress tensor.

Since all stresses at the free surface of the sandpile vanish, zero stresses are set for each block at the free surface:

$$p = \tau_1 = \tau_2 = 0\,.$$

At the rigid base of the sandpile the normal and tangential stresses satisfy the condition $\sigma_\tau \leqslant \tan(\delta)\sigma_n$. Thus, for the blocks at the bottom the Coulomb yield criterion reads

$$\tau_2 \leqslant \tan(\delta)(p + \tau_1) .$$

The well-known problem arising in modeling granular media is that the system of equations (1)–(2) is not closed. In the models we use to describe the stress state of the sandpile the above expressions are considered as the constraints in the optimization problem, i.e., we introduce an objective functional of the stress field $H(\sigma)$ that attains a maximum subjected to the constraints (1) and (2). In the present study the maximum load-carrying capacity and the entropy-like functional are used as an objective functional.

Consider a vertical force $\boldsymbol{P}$ applied to the top block of the sandpile. When $\boldsymbol{P}$ increases quasistatically the stress field inside the sandpile changes due to a rearrangement of granular packing at the microscopic level. We assume that stress changes in such a manner which provides the sandpile the best chance to survive during a loading process. Thus we arrive at the following maximization problem:

$$\boldsymbol{P} \xrightarrow[(\boldsymbol{\sigma}\ \text{satisfies (1) and (2)})]{} \max ,$$

i.e., $\boldsymbol{P}_{\max}$ is the maximum load that the pile can sustain. The constrained maximization problem was solved by the method of augmented Lagrangian (see, e.g. [Refs. 15, 18]), whereby at each iteration an unconstrained optimization problem is solved using a simple point relaxation method (see Appendix A). The detailed description of the numerical algorithm for solving optimization problem with nonsmooth constraints of type (2) can be found in our previous study [14]. Since Eqs. (1) contains variables with odd or even values of the index $i + j$ only the dashed blocks (see Fig. 1) were used in the computations.

The values of $\boldsymbol{P}_{\max}$ obtained from the computations were extremely small in comparison with the pile weight ($\boldsymbol{P}_{\max}/(\rho g \operatorname{Ln} \gamma) \xrightarrow[n\to\infty]{} 0$). The latter is in the agreement with the well-known fact that granular material cannot sustain a point load. However, the stress field induced by applying such a small force is of a greater physical interest.

In Fig. 2 we showed the dimensionless normal stress below the piles with different angles of inclination of the free surface ($\phi = \delta$). When $\phi = 33^\circ$ the normal stress has a maximum just below the pile apex, but when ϕ is gradually decreased the more pronounced minimum appears, i.e., arches are formed inside the pile in order to achieve the maximum strength. The fact that arch-like structures can sustain high loads is known for thousands of years. Thus the arching phenomenon can be explained as a result of the hardening processes inside a granular body whereby flimsy granular elements are destroyed during a loading process and more strong structures are formed instead.

Note that the obtained stress distribution below the flat sandpile (with an inclination angle ϕ less than some critical angle $\phi^* \approx 27^\circ$) is similar to the “echo” experiment proposed by Bouchaud et al. in Ref. [4], whereby the vertical stress in a granular

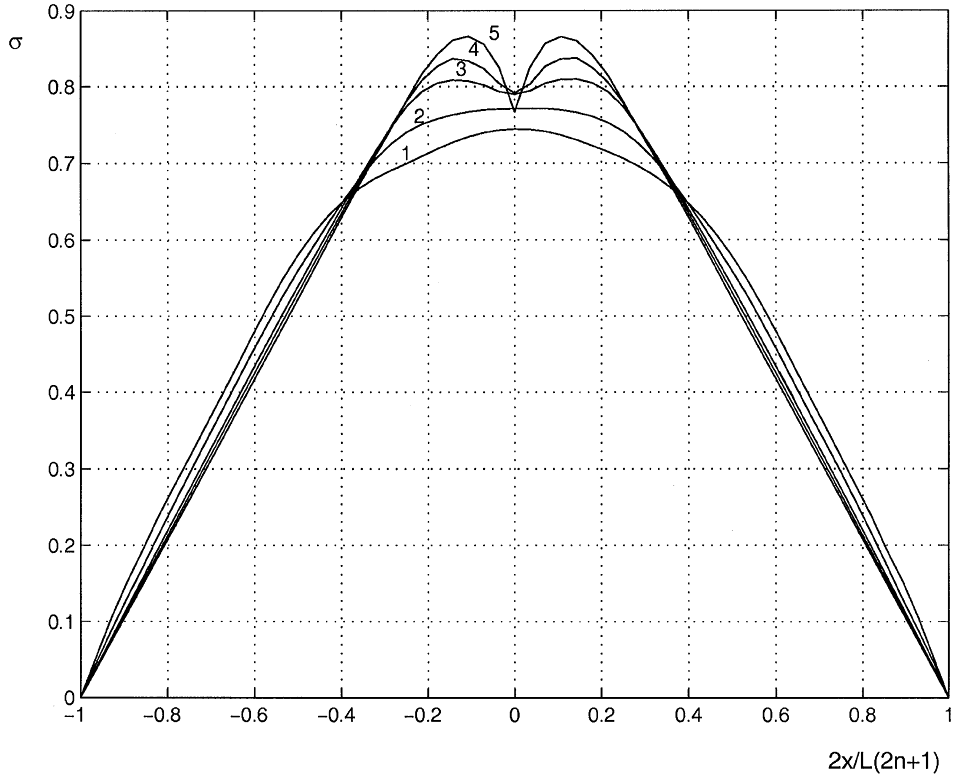

Fig. 2. Vertical stress distribution $((2n+1)L\sigma_{zz})/(\boldsymbol{P}_{\max}+\gamma\rho g n(n+1)L^2)$ at the base of a sandpile ($\phi=\delta$) under an external load for different angles of free surface inclination: (1) $\phi=33°$, (2) $\phi=27°$, (3) $\phi=21°$, (4) $\phi=18°$, (5) $\phi=15°$.

layer propagates along a "light cone" from a point source at the top to the bottom. In the stochastic simulations performed by Eloy et al. [10] the maximum amplitude of response to a localized force propagates along the lines inclined with respect to horizontal.

In the next numerical experiment no external load was applied to the sandpile. Instead an entropy-like objective function was employed. The arguments we used in order to construct the functional $H(\boldsymbol{\sigma})$ are as follows. The constitutive relations have a local character, i.e.,

$$H(\boldsymbol{\sigma}) = \int_{all\ the\ pile} S(\boldsymbol{\sigma}(x,z))\,dx\,dz\,,$$

where $S(\boldsymbol{\sigma})$ is an arbitrary function. Since $S(\boldsymbol{\sigma})$ must be invariant with respect to the coordinate transformations it can be only a function of the invariants of the stress tensor $\boldsymbol{\sigma}$:

$$S(p,\tau_1,\tau_2) = S(p,\sqrt{\tau_1^2+\tau_2^2})\,.$$

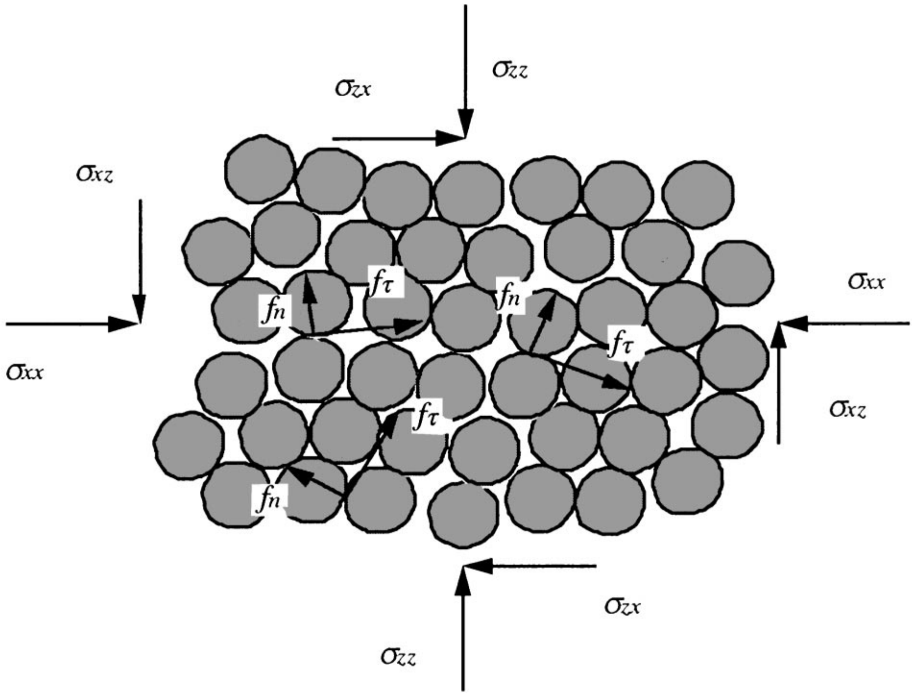

Fig. 3. An assembly of monosized disks.

Since the particles are hard (no characteristic stress) function S can depend only upon the ratio of the stress intensity and the mean pressure, i.e.,

$$S(\boldsymbol{\sigma}) = S\left(\frac{\sqrt{\tau_1^2 + \tau_2^2}}{p}\right).$$

The latter argument must be discussed in more details. Consider a two-dimensional granular assemble with a stress field $\boldsymbol{\sigma}$ (we will refer to it as a macroscopic state). The characteristic spatial scale of a stress field is assumed to be much larger than a grain size. At each contact point between the individual grains there are two forces $\boldsymbol{f}_n$ and $\boldsymbol{f}_\tau$ (see Fig. 3):

$$|\boldsymbol{f}_\tau| \leqslant \mu \boldsymbol{f}_n\,, \tag{3}$$

where $\mu = \tan(\chi)$ is a microscopic coefficient of friction and χ is an angle of inter-granular friction. Since the above formula has the form of inequality, the assemble is statically undetermined at the microscopic level, i.e., there exist a large number of microscopic states (arrangements of the individual grains) which correspond to a given macroscopic state $\boldsymbol{\sigma}$. Consider what happens when the ratio $\eta = \sqrt{\tau_1^2 + \tau_2^2}/p \sin\delta$ approaches unity. Then more and more contact points will reach the critical state, i.e., formula (3) will become an equality. Thus the shear forces reduce the degree of in-determinancy of the system. The latter argument can be interpreted as follows. For a given stress field $\boldsymbol{\sigma}$ the number of admissible microscopic states Ω is a decreasing function of η. It is conceivably suggest that the stress field in the sandpile provides the maximum value for the configuration entropy (see, e.g., [16]) $S \propto \ln(\Omega)$ subjected to the constraints (1)–(2). The latter assumption implies that during its construction a sandpile is allowed to sample various microscopic states. This sampling process can be described as follows. When a granular packing is subjected to an external forcing (vibrations of the support, thermal expansion of the grains due to temperature variations or kinetic energy of grains falling from a source with a sufficiently large intensity, etc.) the large force fluctuations occur inside the granular material. It is clear that contact points with η close to unity have less opportunity to survive during the stress rearrangement. Thus formation of a stress field with small values of η is more probable than with $\eta \approx 1$. Therefore the following optimization problem has to be solved in order to determine a stress field in a pile

$$\sum_{\text{all points}} S(\eta) \xrightarrow[(\boldsymbol{\sigma} \text{ satisfies (1) and (2)})]{} \max ,$$

where S is a decreasing function of η.

In the next section we present the results of statistical simulations of force fluctuations in a regular monodisperse granular assembly which support the above model. The intensity of shear stress is found to reduce the level of static indeterminacy of the system. The critical macroscopic stress state corresponds to zero microscopic stress fluctuations.

Recently the similar maximum entropy model was applied to the human grasping problem which is also statically undetermined [17]. It was demonstrated that the finger forces applied to a grasped object tend to maximize an information entropy functional.

We have no information about the concrete form of the entropy S. Several approximations for entropy were used in the simulations (power law, logarithmic, exponential etc.) and it was found that for $S = -(\eta)^{\omega}$ where ω varies in the range from 3 to 9, the normal stess has a dip beneath the sandpile apex (see Figs. 4–6). Notably, calculations showed that the stresses distribution is insensitive to variations of the exponent ω.

Fig. 7 demonstrates the distribution of η at the sandpile base. The stress tensor has a large deviatoric part near the edges while at the center of the pile the granular material is far from a critical state. Thus there are arch-like zones along the free surface of the pile at which the shear forces are fully mobilized.

3. Force fluctuations in a dense regular granular assemble

In the present section we introduce a stochastic model of force redistribution in a dense regular two-dimensional monodisperse static granular packing. Let N two-dimensional grains with radii 1 be arranged in triangular compact fashion (see Fig. 8) where each grain is in contact with six neighbors. Thus there are $(\frac{6}{2})N = 3N$ points

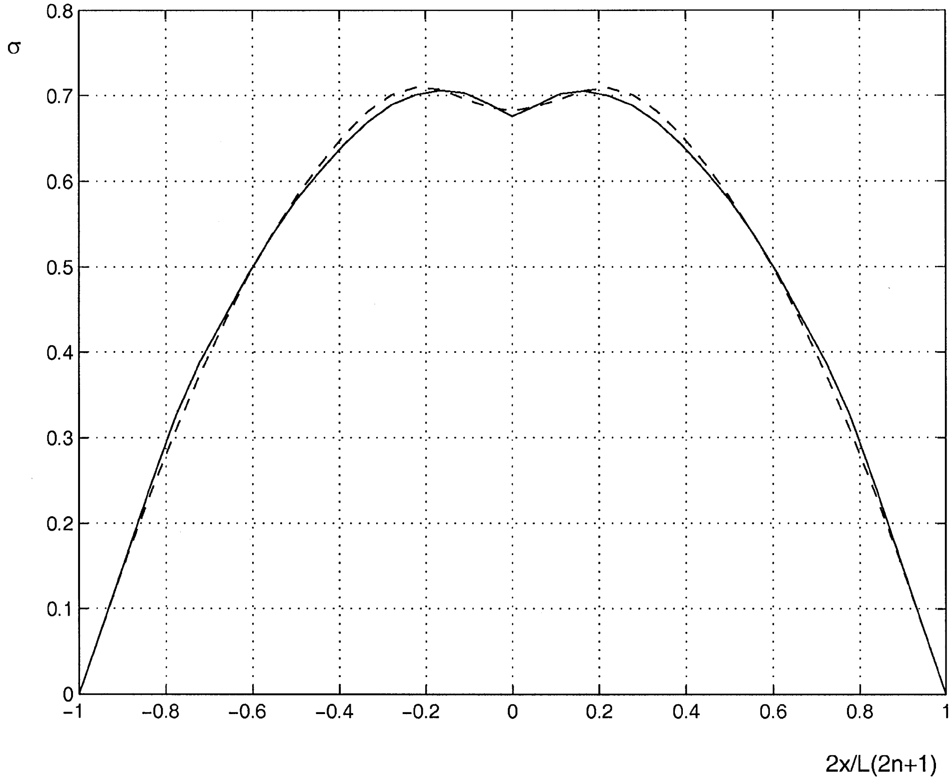

Fig. 4. Vertical stress distribution $\sigma_{zz}/\rho g\gamma\,\mathrm{Ln}$ at the base of sandpile ($\phi=\delta=33°$) for different values of the exponent ω: $\omega=4$ (solid line), $\omega=8$ (dashed line).

of contact and 6N normal and tangential forces. The equations of force and moment balance of a grain read

$$
\begin{aligned}
&(n_1-n_4)+(n_2-n_5-n_3+n_6)\cos 60°-(t_2-t_5+t_3-t_6)\sin 60°=0\,,\\
&(n_2-n_5+n_3-n_6)\sin 60°+(t_1-t_4)+(t_2-t_5+t_3-t_6)\cos 60°=0\,,\\
&t_1+t_2+t_3+t_4+t_5+t_6=0\,.
\end{aligned}
\tag{4}
$$

At each contact point the Coulomb law reads

$$
-\mu n_i \leqslant t_i \leqslant \mu n_i\,. \tag{5}
$$

The granular assemble is loaded by an external stress. The tensor of the macroscopic stress applied to the system is given by the folllowing relations:

$$
\sigma_{xx}=\left\langle\frac{(n_1+n_4)}{2}+\frac{(n_2+n_5+n_3+n_6)}{8}-\frac{\sqrt{3}(t_2+t_5-t_3-t_6)}{8}\right\rangle,
$$

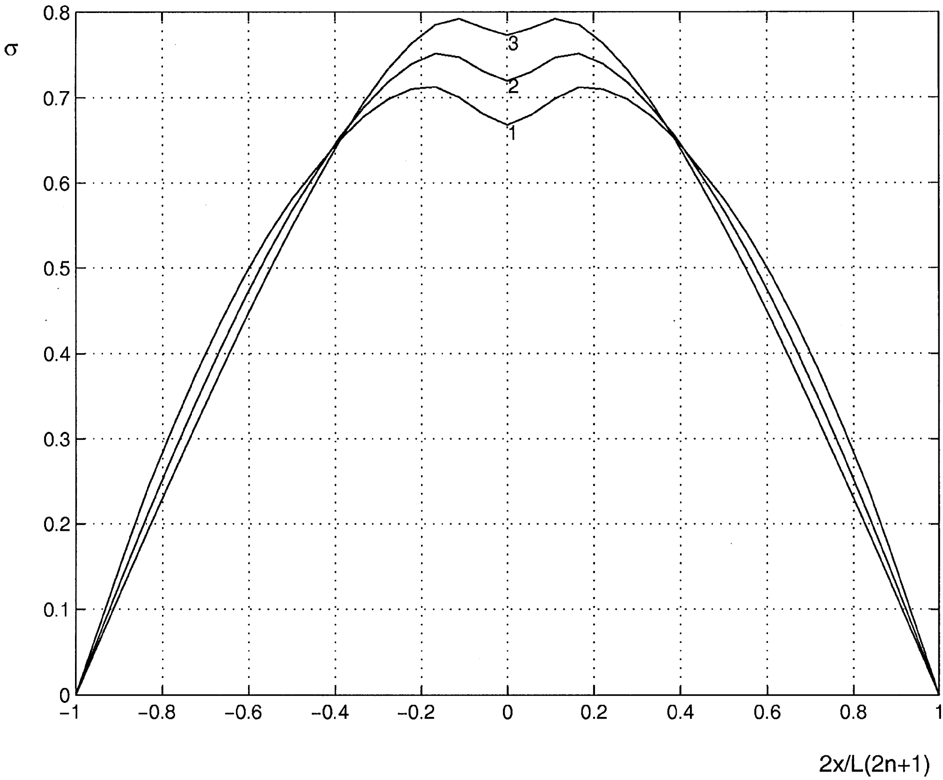

Fig. 5. Vertical stress distribution $\sigma_{zz}/\rho g \gamma$ Ln at the base of sandpile ($\delta = 33°, \omega = 6$) for different angles of free surface inclination (1) $\phi = 33°$, (2) $\phi = 27°$, (3) $\phi = 21°$.

$$\sigma_{zz} = \left\langle \frac{3(n_2 + n_5 + n_3 + n_6)}{8} + \frac{\sqrt{3}(t_2 + t_5 - t_3 - t_6)}{8} \right\rangle,$$

$$\sigma_{zx} = \sigma_{xz} = \left\langle \frac{\sqrt{3}(n_2 + n_5 - n_3 - n_6)}{8} - \frac{3(t_2 + t_5 + t_3 + t_6)}{8} \right\rangle, \tag{6}$$

where brackets $\langle \cdot \rangle$ denote an ensemble average over all the particles. The symmetry of the stress tensor follows from the equation of microscopic moment balance (for details see Appendix B). Periodic boundary conditions are assumed at the lateral contact points.

The following mean stress is applied to the system:

$$\sigma_{xx} = 1, \quad \sigma_{zz} = 1, \quad \sigma_{xz} = \tau, \tag{7}$$

i.e., the principal axis of the stress tensor is inclined at the constant (45°) angle with respect to the x- and z-axis and the mean pressure $p = 1$. The microscopic coefficient of friction μ and the intensity of shear stress τ are the only control parameters in the model. In our computations τ varies from 0 (hydrostatic pressure) to tan(30°) (critical

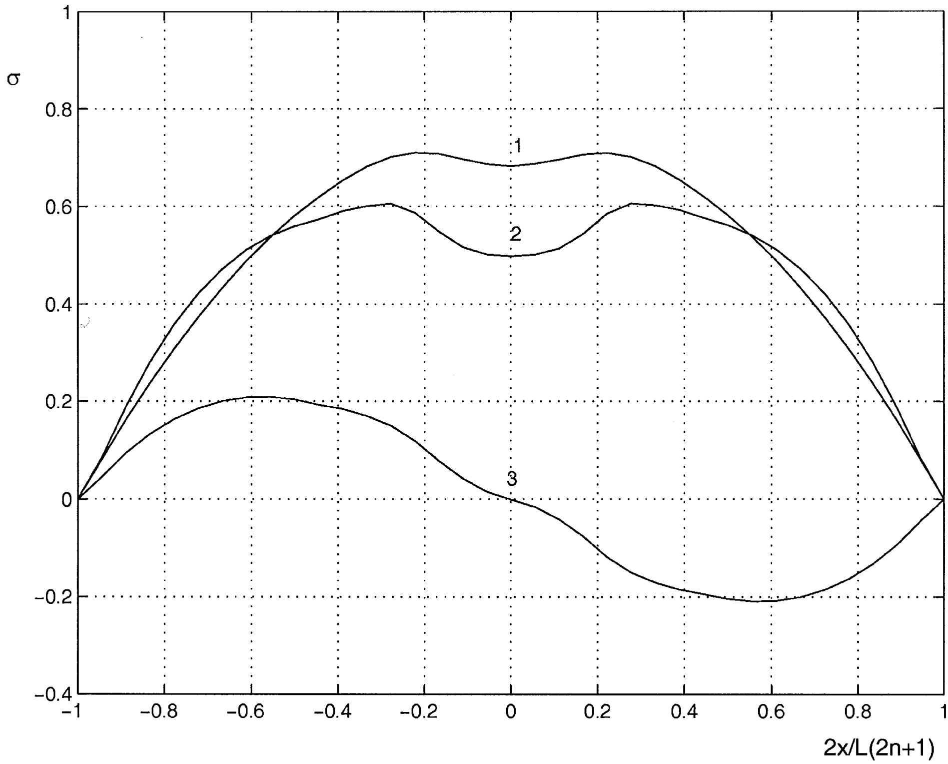

Fig. 6. Stress at the base of a sandpile $(\sigma_{zz},\sigma_{xx},\sigma_{xz})/\rho g\gamma\,\mathrm{Ln}$ for $\omega=6$, $\phi=\delta=33°$. (1) σ_{zz}, (2) σ_{xx}, (3) σ_{xz}.

shear), so that the macroscopic angle of internal friction δ is given by $\sin(\delta)=\tan(30°)$ and $\eta=\tau/\tan(30°)$.

Consider the set of mechanically acceptable states of the system in the space of parameters $\boldsymbol{x}=\{x_1,x_2,\dots,x_{6N}\}=\{n_i,t_i\}$. Denote the hyperplane of solutions satisfying the Eqs. (4) and (7) for a given η as $R(\eta)$. The inequalities (5) form in R the bounded convex polyhedron $G(\eta,\mu)$ of feasible solutions similar to the linear programming problem [18]. Each side of the polyhedron G corresponds to the case when one of the contacts is in a critical state, i.e., the Coulomb law in this point becomes an equality.

The computational algorithm which allows the granular system to explore the points in the phase space is presented schematically in Fig. 9 (see Appendix B). The initial state of the system is selected at a side of G and is updated at each step by moving in a random direction up to the next side of G. Thus, according to these simulation rules, the "molecule" $\boldsymbol{x}$ is in a "thermal equilibrium" with walls of the "container" G. Therefore, all feasible microscopic states have equal probability and the configuration entropy can be defined in terms of the microscopic stress as a logarithm of a volume of the polyhedron G: $S(\eta,\mu)\propto\ln[\int_{G(\eta,\mu)}\mathrm{d}\boldsymbol{x}]$.

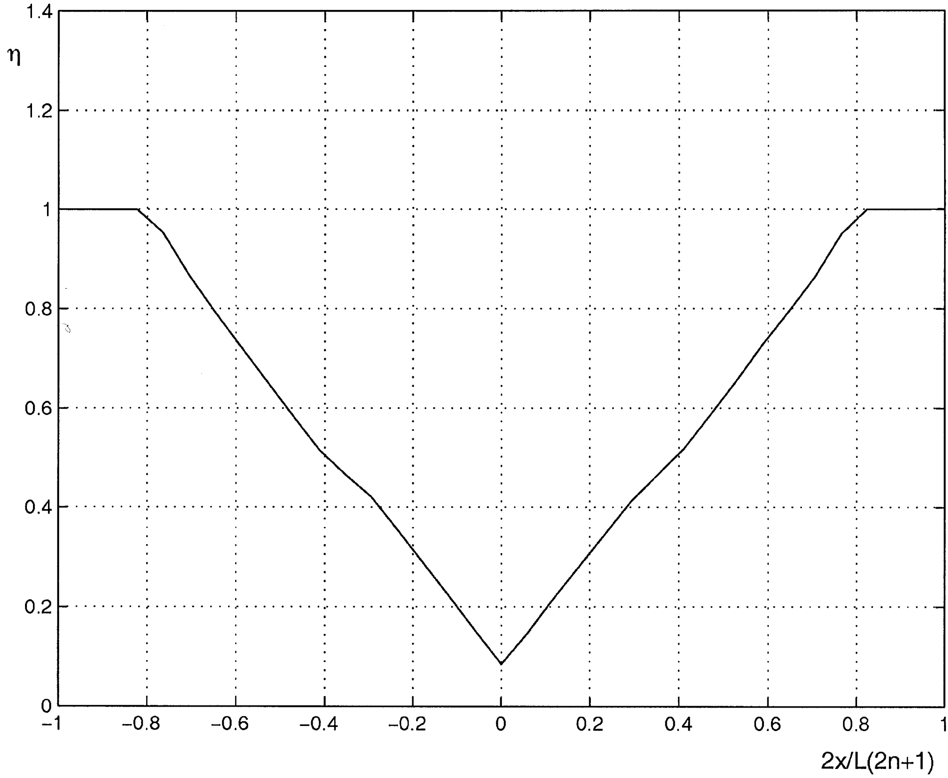

Fig. 7. The ratio $\eta = \sqrt{\tau_1^2 + \tau_2^2}/p \sin\delta$ for $\omega = 4$, $\phi = \delta = 33°$.

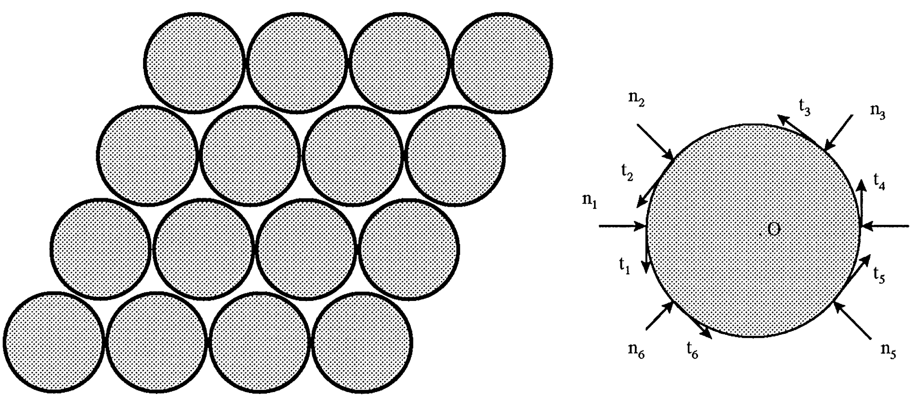

Fig. 8. Schematic view of a regular granular assembly of monosized disks and contact forces.

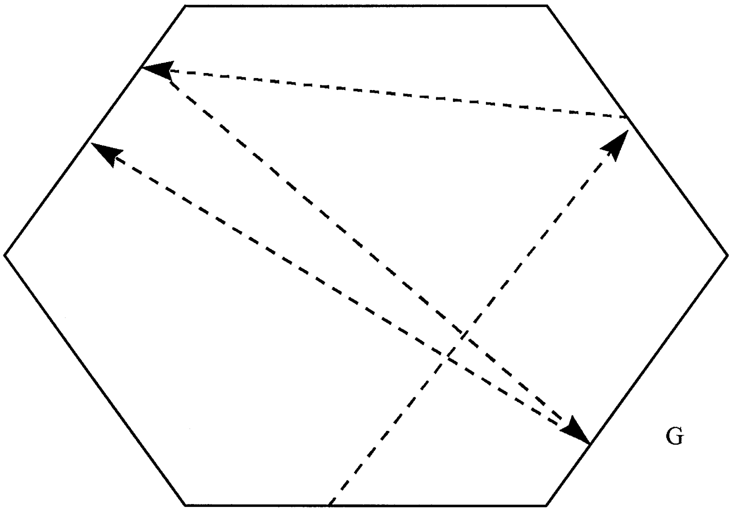

Fig. 9. Trajectory of the system in a phase space.

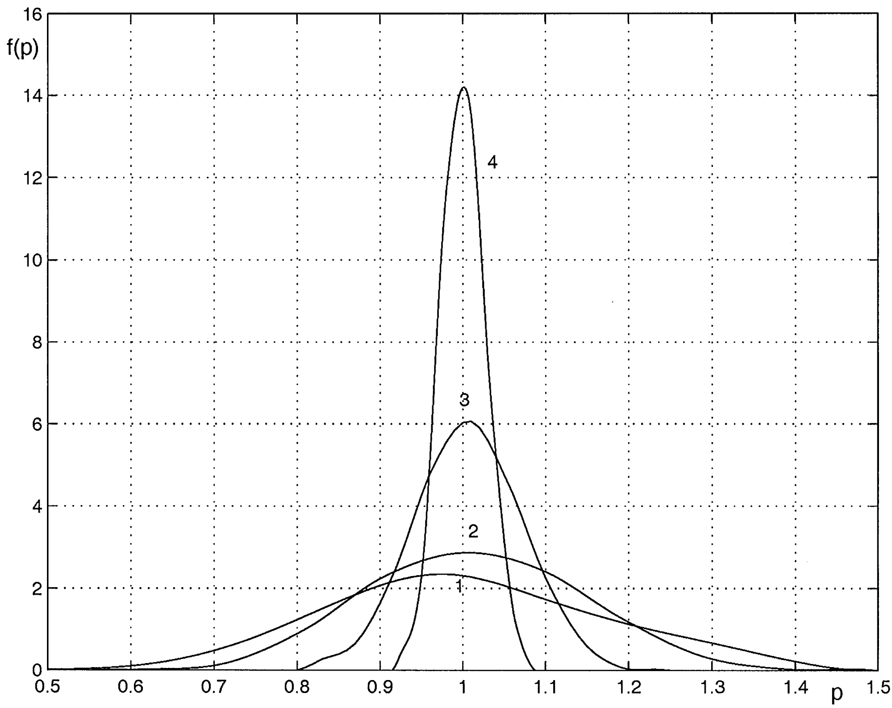

Fig. 10. Probability density function of the microscopic pressure $f(p)$ for different values of η, (1) $\eta = 0$, (2) $\eta = 0.35$, (3) $\eta = 0.7$, (4) $\eta = 0.87$ ($\mu = \tan 30°$).

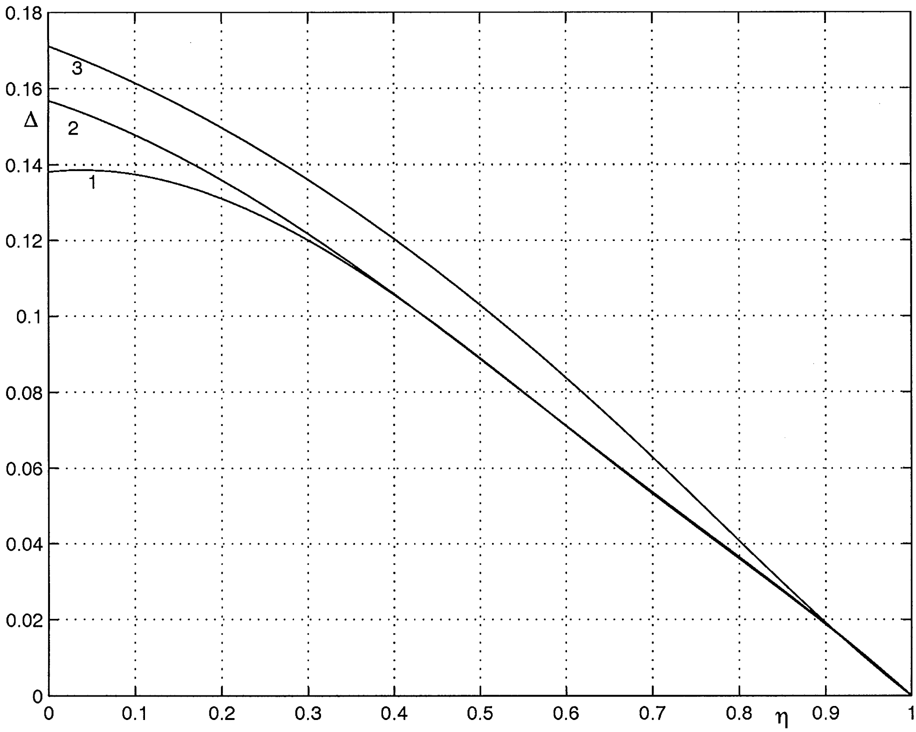

Fig. 11. Standard deviation of the pressure $\Delta = \left\langle (\bar{p} - p)^2 \right\rangle^{1/2}$ as a function of η for different values of $\mu = \tan(\chi)$, (1) $\chi = 3°$, (2) $\chi = 15°$, (3) $\chi = 30°$.

The above algorithm samples only the boundary points in the phase space because in a body embedded into a space of a large dimension most of the points belong to a thin boundary layer. Thus, e.g., volume of a multidimensional ball is equal to a volume of a thin spherical shell near its surface. In the other words in a sufficiently large granular assembly with the probability close to 1 at least one contact point is in a critical state. Thus for a sufficiently large N using the adopted algorithm we miss a negligibly small fraction of the phase space.

In Figs. 10 and 11 we have presented the results of the computations. The fluctuations of the microscopic pressure $p = (\sum_{i=1}^{6} n_i)/6$ in the granular system decrease with increasing the parameter η up to 0 for $\eta \to 0$. While the macroscopic stress is far from its critical value the assemble has a large number of degrees of freedom. When a large shear stress is applied to the system the set of possible microscopic states shrinks to a polyhedron with a zero volume. Thus it can be concluded that the ability of the granular system to explore all the points in the phase space which was assumed in the previous section, is strongly affected by the macroscopic stress which is applied to the granular body. Note that since each grain has six point of contact with its neighbors the shear forces play a minor role in the equations of equilibrium. Thus the force fluctuations are almost unaffected by the microscopic angle of friction χ (see Fig. 11).

4. Conclusions

We have suggested and investigated numerically two variational continuum mechanics closure models to describe the stress distribution in sandpiles, namely, maximum-load-carrying capacity and maximum of entropy. Both these models are based on the assumption that external forcing plays the role of a selection mechanism which chooses more stable structural elements from the space of all possible configurations. It was showed that the normal stress minimum does appear for the wide range of a model parameters. Force fluctuations in a model regular granular assemble were investigated by a statistic simulation algorithm. The results of the numerical simulation provides a partial support to the adopted hypothesis.

Acknowledgements

The work of A.V. was supported by Levy Eshcol Foundation administered by Israel Ministry of Science.

Appendix A. Augmented Lagrangian method for a solution of the constrained minimization problems

Write Eqs. (1) in the following form:

$$\varphi_1 = 0, \quad \varphi_2 = 0 .$$

Introduce the additional variable s according to the following formula:

$$\varphi_3 \triangleq s^2 - \tau_1^2 - \tau_2^2 = 0 .$$

Thus, Eq. (2) can be rewritten as

$$\varphi_4 \triangleq s - p\sin(\delta) - \zeta = 0, \quad \zeta \geqslant 0, \; s \geqslant 0 .$$

Then the augmented Lagrangian (see e.g. Refs. [15,18]) can be defined as follows:

$$L(p, s, \tau_i, \zeta, \lambda_i) = -H(p, s) + r\sum_{i=1}^{4} \varphi_i^2 + \sum_{i=1}^{4} \lambda_i \varphi_i ,$$

where r is a penalty parameter, ζ is slack variable and λ_i are Lagrangian multipliers. Therefore the constrained optimization problem can be solved by iterations as follows.

The direct variables p, s, τ_i, ζ are obtained from the following minimization problem for the fixed λ_i:

$$L \xrightarrow[p,s,\tau_i,\zeta]{} \min, \quad p \geqslant 0,\ s \geqslant 0,\ \zeta \geqslant 0 .$$

The latter problem can be solved by a point relaxation Newton method. After that the Lagrangian multipliers are recalculated according to the following formula:

$$\lambda_i = \lambda_i + r\varphi_i$$

and the computations are repeated until a convergence criterion is met.

Appendix B. Stochastic simulation method for determining contact forces fluctuations

In order to relate the distribution of the contact forces with the mean stress tensor $\boldsymbol{\sigma}$ let us define the continuous function $\bar{n}_1(x,z), \bar{n}_2(x,z), \bar{n}_3(x,z), \bar{t}_1(x,z), \bar{t}_2(x,z), \bar{t}_3(x,z)$. Thus the contact forces (see Fig. 8) can be expressed through the Taylor series expansion as

$$n_1 = \bar{n}_1(O) - \left|\frac{\partial \bar{n}_1}{\partial x}\right|_O, \quad n_4 = \bar{n}_1(O) + \left|\frac{\partial \bar{n}_1}{\partial x}\right|_O,$$

$$n_2 = \bar{n}_2(O) - \cos(30^\circ)\left|\frac{\partial \bar{n}_2}{\partial x}\right|_O + \sin(30^\circ)\left|\frac{\partial \bar{n}_2}{\partial z}\right|_O,$$

$$n_5 = \bar{n}_2(O) + \cos(30^\circ)\left|\frac{\partial \bar{n}_2}{\partial x}\right|_O - \sin(30^\circ)\left|\frac{\partial \bar{n}_2}{\partial z}\right|_O, \quad \text{etc}.$$

The above expressions are equivalent to

$$\bar{n}_1(O) = \tfrac{1}{2}(n_1 + n_4), \quad \bar{n}_2(O) = \tfrac{1}{2}(n_2 + n_5), \ \text{etc}.$$

Combining the above formulas with the equations of static equilibrium (4) we obtain the relations which are equivalent to the differential equations of pile statics:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0, \quad \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0, \quad \sigma_{xz} = \sigma_{zx},$$

where $\sigma_{xx}, \sigma_{xz}, \sigma_{zz}$ are given by Eq. (6). Thus these formally defined functions σ_{ij} can be identified as the components of the mean stress tensor. The equation of moment balance implies the symmetry of the stress tensor.

The simulation algorithm used is in this study can be interpreted as a combination of the simplex method [18] and the projection gradient method [19]. Rewrite the inequalities (5) as

$$t_i - \mu n_i + u_{1i} = 0, \quad u_{1i} \geqslant 0,$$

$$t_i + \mu n_i + u_{2i} = 0, \quad u_{2i} \geqslant 0,$$

where u_{ji} are slack variables. Thus

$$n_i = (u_{1i} + u_{2i})/2\mu, \quad t_i = (u_{1i} - u_{2i})/2 .$$

Eqs. (4)–(6) can be rewritten in terms of the slackness variables u_{ji} as

$$\boldsymbol{Au} = \boldsymbol{b}, \quad \boldsymbol{u} \geqslant 0 ,$$

where $\boldsymbol{A}, \boldsymbol{u}, \boldsymbol{b}$ are $3(N+1) \times 6N$ matrix, $6N$ and $3(N+1)$ vectors of variables and the right hand side parts of the equations, respectively. The solution algorithm can be described as follows. The initial state of the system is selected at a side of the polyhedron G and is updated at each step by random sampling a direction $\boldsymbol{k} \in R$ such that $\boldsymbol{u} + \theta \boldsymbol{k}$ also belongs to G for small θ. The length of the step θ is determined according to the following formula:

$$\theta = \min_{i|k_i \leqslant 0} \left| \frac{u_i}{k_i} \right| = \left| \frac{u_r}{k_r} \right| .$$

Thus, the system moves in the direction $\boldsymbol{k}$ up to the rth side of the polyhedron G. In order to choose a new direction a vector $\boldsymbol{k}$ is sampled randomly and is projected on the hyperplane R [19]:

$$\boldsymbol{k} = \boldsymbol{k} - \boldsymbol{A}^T [\boldsymbol{A}\boldsymbol{A}^T]^{-1} \boldsymbol{A}\boldsymbol{k}$$

If $k_r \leqslant 0$ then $\boldsymbol{k}$ is replaced by $-\boldsymbol{k}$.

All the simulations presented in this study were performed with 32×32 granular packing and were started from the point

$$n_1 = n_4 = \tfrac{2}{3}, \quad n_2 = n_5 = \tfrac{2}{3}(1 + \sqrt{3}\tau), \quad n_3 = n_6 = \tfrac{2}{3}(1 - \sqrt{3}\tau), \quad t_i = 0$$

i.e., without mean gradients and without microscopic tangential forces. It is clear that in this case the macroscopic shear stress can vary from $\tau = 0$ to $\tau = \tan(30°)$.

References

[1] J. Smid, J. Novosad, Proc. of 1981 Powtech. Conf. Ind. Chem. Eng. Symp. vol. 63, D3/V/1, 1981.
[2] V. Bucholtz, T. Poeschel, Phys. Rev. Lett. 71 (1993) 3963.
[3] S.F. Edwards, R.B. Oakeshott, Physica 38 D (1989) 88.
[4] J.-P. Bouchaud, M.E. Cates, P. Claudin, J. Phys. I (France) 5 (1995) 639.
[5] J.P. Wittmer, P. Claudin, M.E. Cates, J.-P. Bouchaud, Nature 382 (1996) 336.
[6] A. Watson, Science 273 (1996) 579.
[7] P. Claudin, J.-P. Bouchaud, Phys. Rev. Lett. 78 (1997) 231.
[8] R. Brockbank, J.M. Huntley, R.C. Ball, J. Phys. II (France) 7 (1997) 1521.
[9] P. Evesque, J. Phys. I (France) 7 (1997) 1305.
[10] C. Eloy, E. Clement, J. Phys. I (France) 7 (1997) 1541.
[11] C.H. Liu, S. Nagel, Phys. Rev. Lett. 68 (1992) 2301.
[12] C.H. Liu et al., Science 269 (1995) 513.
[13] T. Elperin, A. Vikhansky, Phys. Rev. E 53 (1996) 4536.
[14] T. Elperin, A. Vikhansky, Phys. Rev. E 55 (1997) 5785.
[15] R. Glowinski, J.L. Lions, R. Tremolieres, Numerical Analysis of Variational Inequalities, North-Holland, Amsterdam, 1981.

[16] P.M. Morse, Thermal Physics, W.A. Benjamin, inc. New York, Amsterdam, 1964.
[17] M. Hershkovictz, U. Tasch, M. Teboule, J. Robot. Systems 12 (1995) 249.
[18] M. Minoux, Programmation Mathematique, Bordas et C. N. E. T.-E. N. S. T., Paris, 1989.
[19] J.B. Rosen, SIAM J. 8 (1960) 181.